An integral test for the transience of a Brownian path with limited local time

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Abstract.

We study a one-dimensional Brownian motion conditioned on a self-repelling behaviour. Given a nondecreasing positive function $f(t), t \geq 0$, consider the measures μ_t obtained by conditioning a Brownian path so that $L_s \leq f(s)$, for all $s \leq t$, where L_s is the local time spent at the origin by time s. It is shown that the measures μ_t are tight, and that any weak limit of μ_t as $t \to \infty$ is transient provided that $t^{-3/2} f(t)$ is integrable. We conjecture that this condition is sharp and present a number of open problems.

Résumé.

Etant donnée une fonction croîssante $f(t), t \geq 0$, considérons la mesure μ_t obtenue lorsqu'on on conditionne un mouvement brownien de sorte que $L_s \leq f(s)$, pour tout $s \leq t$, où L_s est le temps local accumulé au temps s à l'origine. Nous montrons que les mesures μ_t sont tendues, et que toute limite faible de μ_t lorsque $t \to \infty$ est la loi d'un processus transient si $t^{-3/2} f(t)$ est intégrable. Nous conjecturons que cette condition est également nécessaire pour la transience et proposons un certain nombre de questions ouvertes.

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1 Introduction

Let $(X_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$. It is well-known that $d = 2$ is a critical value for the recurrence or transience of X. In this paper, we show however that even in dimension 1, a very small perturbation of the Brownian path may result in the transience of the process. Let $f : [0,\infty) \to [0,\infty)$ be a given nonnegative Borel function such that $f(0) > 0$, and consider the event

$$\mathcal{K}_t = \{L_s \leq f(s), \quad \text{for all } s \leq t\}.$$

Here $(L_s, s \geq 0)$ is a continuous determination of the local time process of X at the origin $x = 0$. Our goal in this paper is to analyse the limiting behaviour of the Wiener measure conditioned on $\mathcal{K}_t$, as $t \to \infty$. Since $(L_s, s \geq 0)$ is almost surely nondecreasing, we may and will assume without loss of generality that f is nondecreasing. (Otherwise one may always consider $\hat{f}(t) = \inf_{s \geq t} f(s)$). Note that by Brownian scaling, L_t is of order $\sqrt{t}$ for an unconditional Brownian path. Hence when $f(t) \leq t^{1/2}$, this constraint is of a self-repelling nature, since it forces the Brownian path to spend less time than it would naturally want to at the origin. We will thus assume that f is nondecreasing and $t^{-1/2}f(t)$ is non-increasing.

Our main result is that if f is only logarithmically smaller than $t^{1/2}$, then X becomes transient almost surely in the limit $t \to \infty$. Here, a probability measure $\mathbb{P}$ on the space $\mathcal{C}$ of continuous sample paths is called transient (almost surely) if $\mathbb{P}(\lim_{t\to\infty} |X_t| = +\infty) = 1$. If $\limsup_{t\to\infty} X_t = +\infty$ and $\liminf_{t\to\infty} X_t = -\infty$ with $\mathbb{P}$-probability 1, then $\mathbb{P}$ is called recurrent. We equip $\mathcal{C}$ with the topology of uniform convergence on compact sets, which turns $\mathcal{C}$ into a Polish space, and discuss weak convergence of probability measures on $\mathcal{C}$ with respect to this topology.

Theorem 1. *Let $\mathbb{W}$ denote the Wiener measure on $\mathcal{C}$, and let $\mathbb{W}_t = \mathbb{W}(\cdot|\mathcal{K}_t)$. Then $\{\mathbb{W}_t, t \geq 0\}$ is a tight family. Assume further that*

$$\int_1^\infty \frac{f(t)}{t^{3/2}} dt < \infty. \tag{1}$$

Then for any weak subsequential limit $\mathbb{P}$ of $\mathbb{W}_t$ as $t \to \infty$, $\mathbb{P}$ is transient almost surely.

In particular, if $f(t) \sim \sqrt{t}(\log t)^{-\gamma}$ with $\gamma \geq 0$, then $\mathbb{P}$ is transient as soon as $\gamma > 1$. We believe, but have not succeeded in proving, that condition (1) is sharp, in the following sense.

Conjecture 1. If

$$\int_1^\infty \frac{f(t)}{t^{3/2}} dt = \infty, \tag{2}$$

then any weak limit $\mathbb{P}$ of $\mathbb{W}_t$ as $t \to \infty$ is recurrent almost surely.

It should be noted that this problem is open even in the basic case where $f(t) \sim \sqrt{t}$ as $t \to \infty$, for whch it is still the case that $\mathbb{W}(\mathcal{E}_t) \to 0$ as $t \to \infty$.

Figure 1 below illustrates this result.

In the recurrent regime, i.e. if (2) holds, we further believe that the local time process of X is well-defined almost surely under $\mathbb{P}$, but that X is "far away from breaking the constraint $\mathcal{K}_t$", in the following sense:

Conjecture 2. Assume (2), and let $\mathbb{P}$ be any weak limit of $\mathbb{W}_t$. Then there exists a nonnegative deterministic function $\omega(t) \to \infty$ as $t \to \infty$ such that

$$\mathbb{P}\left(L_t \leq \frac{f(t)}{\omega(t)}\right) \to 1 \tag{3}$$

as $t \to \infty$.

Furthermore, when $f(t) \sim t^{1/2}(\log t)^{-\gamma}$ as $t \to \infty$, with $0 < \gamma < 1$, we believe that

$$L_t \leq f(t)\exp(-C(\log t)^{\gamma}), \tag{4}$$

with probability asymptotically 1, for some $C > 0$. It may seem surprising at first that, in the recurrent regime, the process shouldn't use its full allowance of local time. This phenomenon is related to entropic effects, which cause the process to stay far away from breaking the constraint to allow for more fluctuations. In [2], we already observed a similar behaviour in the case where the local time profile of the process is conditioned to remain bounded at every point, and have called this phenomenon "Brownian entropic repulsion". This aspect is actually crucially exploited in our proof, which relies on considering a suitable softer constraint $\mathcal{K}'_t$ (easier to analyse, because more "Markovian"), but which nevertheless turns out to be equivalent to that of $\mathcal{K}_t$.

Discussion and relation to previous works. The integral test (1) is reminiscent of classical integral tests on Brownian motion. Indeed, an indication that this test provides the right answer follows from a pretty basic calculation. This calculation is carried out in Lemma 4, where the probability of hitting 0 during the interval $[t/2, t]$ given $\mathcal{K}_t$ is estimated. However we stress that one of the major difficulties of this problem is to control the long-range interactions induced by conditioning far away into the future, and to show that this propagates down to an arbitrarily large but finite window close to the origin in a manageable way. It is this long-range interaction, inherent to the study of self-interacting processes, which is at the source of our difficulties in proving Conjecture 1 and 2.

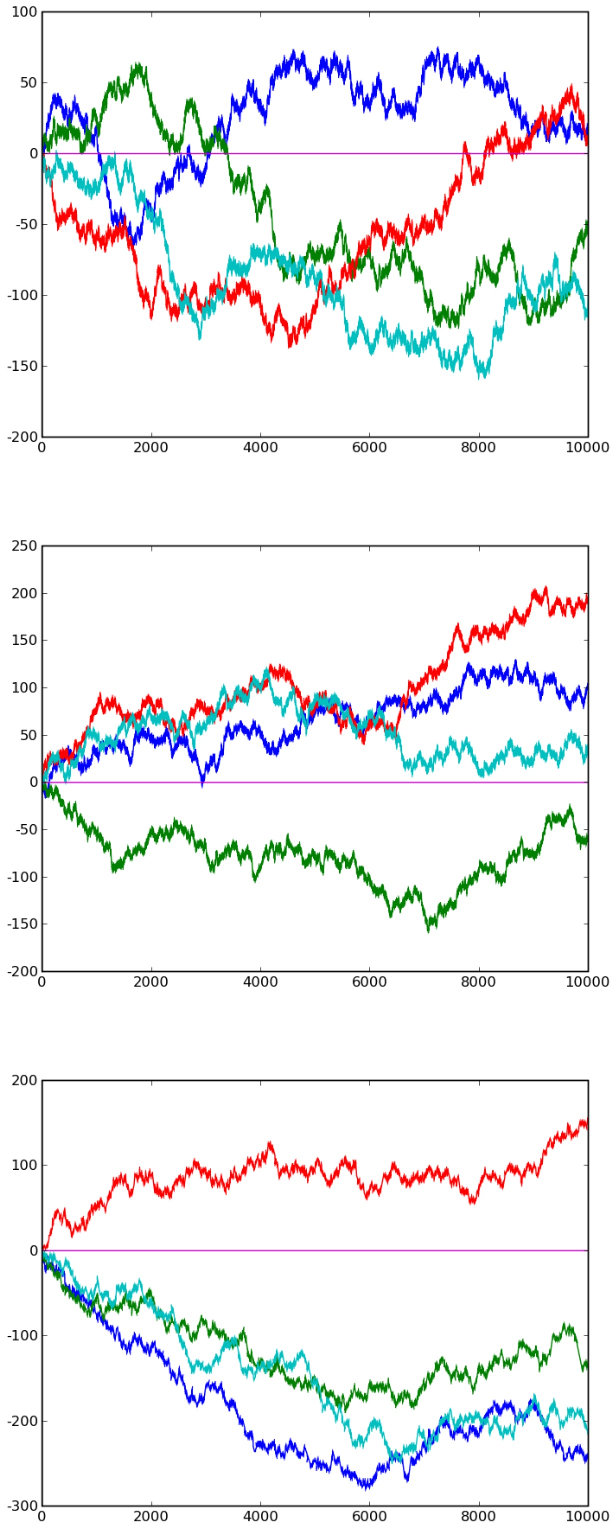

Figure 1: Simulations of trajectories up to time $t = 10^4$. From top to bottom: $\gamma = 0.5$, $\gamma = 0.9$, and $\gamma = 1.1$.

Putting a bound on the local time can be viewed as introducing some form of self-repellence of the process. The problems studied here are reminiscent of some problems arising in the mathematical study of *random polymers*, of which an excellent review can be found in [7]. Our work is also somewhat related in spirit to a series of papers by Roynette et al. (see, e.g., [13], or the forthcoming monograph by Roynette and Yor [14]) and Najnudel [9] , although our goals and methods are quite different.

Theorem 1 establishes tightness of the measures $\mathbb{P}_t$ but not weak convergence. One possible approach to prove uniqueness would be to identify the limiting process as the unique solution to a certain stochastic differential equation. We note that this is related to the work of Barlow and Perkins [1], which describes the behaviour of Brownian motion near a typical slow point, i.e., a time t near which the growth of B satisfies

$$\limsup_{h \to 0} h^{-1/2} |B_{t+h} - B_t| = 1.$$

Blowing up the trajectory near this slow point, their Theorem 3.3 gives precisely a description of the process as a solution to a certain stochastic differential equation.

Organization of the paper. In section 2, we prove some preliminary results which contain results interesting for their own sake. Namely, it is shown in Theorem 2 that a Brownian motion conditioned on having a local time at the origin bounded by 1 is transient, and that the total local time accumulated by this process is a uniform random variable on $(0, 1)$. Note that this is smaller than 1 almost surely, so here again the process doesn't use its full allowance of local time. Also, in Theorem 3 that a Brownian motion conditioned on the event $\mathcal{E}_t = \{L_t \leq f(t)\}$ is recurrent in the limit $t \to \infty$ as soon as $f(t) \to \infty$, no matter how slowly.

In section 3, we give a proof of the main result (Theorem 1). This is based partly on Theorem 3 and on a general result which shows that any conditioning of the Brownian motion based on its zero set cannot grow faster than diffusively (Lemma 1) and, in the case of $\mathcal{K}_t$, this is matched by a lower bound of the same order of magnitude (Lemma 2). These various ingredients are put together using a coupling method, which then gives the proof of the result.

Finally in section 4, we study a slightly different but related problem, where a Brownian path is conditioned to spend no more than one unit of time in the negative half-line. It is shown there again that the measures converge weakly to a limiting process, which is (unsurprisingly) transient, and also that the total amount of time spent in the forbidden region by this process is equal to U^2, where U is a uniform random variable on (0,1). Hence here again, the process does not use its full allowance, another expression of the entropic repulsion principle.

2 Preliminaries

2.1 Brownian motion with bounded local time.

It will be convenient to define various processes on the same space, but governed by different probability measures on this space. We take for this common space the space $\mathcal{C} = C([0,\infty),\mathbb{R})$ = the space of continuous functions from $[0,\infty)$ into $\mathbb{R}$. X_s will denote the s-th coordinate function on $\mathcal{C}$; we shall also write $X(s)$ for X_s occasionally when s is a complicated expression. In this setup Brownian motion is obtained by putting the Wiener measure $\mathbb{W}$ on $\mathcal{C}$; $\mathbb{W}$ is concentrated on the paths which start at $X(0) = 0$ and makes increments over disjoint intervals independent with suitable Gaussian distributions. We now take $L(\cdot,\cdot)$ as a jointly continuous local time of the Brownian motion. This is a continuous function $L(s,x)$ which satisfies

$$\big\{|\{s \le t : X_s \in B\}|\big\} = \int_0^t I[X_s \in B]ds = \int_{x\in B} L(t,x)dx \tag{5}$$

$\mathbb{W}$-almost surely simultaneously for all Borel sets B and $t \ge 0$ ([8] , Sect 3.4). Under the measure $\mathbb{W}$ there a.s. exists such a jointly continuous function, and it is clearly unique for any sample function for which it exists.

As a first step towards the proof of Theorem 1, we prove the following simple result. Assume that $f(t) = 1$, so that

$$\mathcal{K}_t := \{L_t \le 1\}.$$

Our first theorem describes the weak limit of $\mathbb{W}_t$ as $t \to \infty$. This description involves a Bessel-3 process, a description of which can be found, for instance, in [11].

Theorem 2. *The measures $\mathbb{W}_t$ converge weakly on $\mathcal{C}$ to a measure $\mathbb{P}$. Under $\mathbb{P}$, the process X is transient and $\mathbb{P}$ can be described as follows: On some probability space let $U, \{B(s), s \ge 0\}, \epsilon$ and $\{B^{(3)}(s), s \ge 0\}$ respectively be a random variable with a uniform distribution on $[0,1]$, a Brownian motion, a random variable uniform on $\{-1,1\}$, and a Bessel-3 process, and assume that these four random elements are independent of each other. Define*

$$\tau = \sup\{v : L(v,0) < U\} \tag{6}$$

(where L is the local time of B), and

$$Y(t) = \begin{cases} B(t) & \textit{if } t \le \tau \\ \epsilon B^{(3)}(t-\tau) & \textit{if } t > \tau. \end{cases}$$

Then $\mathbb{P}$ is the distribution of $\{Y(t), t \ge 0\}$.

Somewhat informally, the theorem says that under $\mathbb{P}$, X can be described by first drawing an independent uniform random variable U. Then X is the standard Brownian motion until it has accumulated a local time at 0 equal to U, and performs a three-dimensional Bessel process afterwards.

It is well known that a Bessel-3 process starting at the origin diverges to infinity almost surely. This is of course the reason why the process governed by $\mathbb{P}$ is transient. However, we can say more. It is also well known that L_t, the local time at 0 can change only at times t when $X_t = 0$. This fact is also clear from (5). Together with the description of the process under $\mathbb{P}$ this implies that L_t is a.s. constant on $t \geq \tau$ at which it takes the value U (by definition and continuity of the inverse local time τ). Thus, the theorem implies

$$L_\infty = L_\tau = U. \tag{7}$$

Since $U < 1$ almost surely, this shows that under $\mathbb{P}$, X does *not* use its full allowance of local time, which is another expression of the entropic repulsion principle.

Proof of Theorem 2.

Step 1. In this step we shall give a representation of Brownian motion by means of excursions. This will turn out to be useful for the proof. Readers familiar with this sort of things are encouraged to skip this step and go to step 2. To help with the intuition, consider the set $Z := \{t : X_t = 0\}$. If X_t is a continuous function of t, then Z is a closed set, and its complement, $\mathbb{R} \setminus Z$ is a countable union of maximal open intervals. On each such interval $X \neq 0$. The piece of the path of X on such an interval is called an *excursion* of X. One can now try to construct a process equivalent to X by first picking excursions on some probability space and according to a suitable distribution, and then putting these excursions together. For X a Brownian motion, this can be done rather explicitly. The following description can be found in a number of references (see, e.g., [8, Section III.4.3], [11, Chapter XII]). The excursions are elements of $\mathcal{W}$ which is the collection of continuous functions $w : [0, \infty) \to \mathbb{R}$ such that $w(0) = 0$, and for which there exists a $\zeta(w) > 0$ such that $w(t) > 0$ or $w(t) < 0$ for all $0 < t < \zeta(w)$ and $w(t) = 0$ for $t \geq \zeta(w)$. $\zeta(w)$ is called the *length of the excursion* w, or sometimes its duration.

Itô's fundamental result about the excursions of a Brownian motion states that there exists a σ-finite measure ν on the space $\mathcal{W}$, called *Itô's measure*, such that the Brownian motion, viewed in the correct time-scale, can be seen as a Point process of excursions with intensity measure ν. To state this result precisely, note that X has only countably many excursions (since there are only finitely many excursions above $1/n$ for each finite $n \geq 1$ in any compact time-interval). Let

$(e_i)_{i=1}^{\infty}$ be an enumeration of these excursions. Since L_t only increases on the zero set of X, let ℓ_i be the common value of L_t throughout the excursion e_i, for $i \geq 1$. Then Itô's theorem states that:

$$M := \sum_{i \geq 1} \delta_{(\ell_i, e_i)} \tag{8}$$

is a Poisson point process on $(0, \infty) \times \mathcal{W}$, with intensity measure $d\lambda \otimes d\nu$, where $d\lambda$ is the Lebesgue measure on $(0, \infty)$. That is, for any Borel set in $(0, \infty) \times \mathcal{W}$, if

$$\mathcal{P}(B) := (\text{the number of points of this process in the set } B),$$

then $\mathcal{P}(B)$ has a Poisson distribution with mean $\int_B d\lambda d\nu$, and for disjoint sets B_i in $(0, \infty) \times \mathcal{W}$, the $\mathcal{P}(B_i)$ are independent. With a slight abuse of notation, we say that $(l, e) \in M$ if $M(l, e) = 1$. Note that the collection of points (ℓ_i, e_i) entirely determines the path of X. [Indeed, if we define, for all $u > 0$,

$$\tau(u) = \sum_{\substack{(\ell, e) \in M \\ \ell \leq u}} \zeta(e),$$

then for all $i \geq 1$, the function $\tau(u)$ has an upward jump of size $\zeta(e_i)$ at time s_i, and these are the only jumps of τ. If $t > 0$, let $s = \inf\{u \geq t : \Delta\tau(u) > 0\}$, and let e be the excursion associated with the jump of τ at time u. Then it is easy to check that we have the formula

$$e(t - \tau(u^-)) = X_t$$

where X is the original process that we started with, and for all $u > 0$,

$$\tau(u) = \inf\{t > 0 : L_t > u\}. \tag{9}$$

Thus the excursions can easily be put together.]

A well-known description of Itô's measure (see, e.g., [11], XII.4), which we will use in this proof, is the following determination of the "law" of the duration of an Itô excursion:

$$\nu(\zeta > t) = \int_t^{\infty} \frac{ds}{\sqrt{2\pi s^3}} = \sqrt{\frac{2}{\pi t}}. \tag{10}$$

Step 2. Consider the event $\mathcal{E}'_t$ that by time $\tau(1)$ (defined by (9)), there is an excursion of duration greater than t. That is, formally:

$$\mathcal{E}'_t = \{\text{there exists } (\ell, e) \in M \text{ such that } \ell \leq 1 \text{ and } \zeta(e) > t\}$$

Observe that on the one hand, $\mathcal{E}'_t \subset \mathcal{E}_t$. Indeed, $\mathcal{E}_t$ can be written as:

$$\mathcal{E}_t = \{\tau(1) > t\}$$

and it is clear that this occurs on $\mathcal{E}'_t$. On the other hand, we claim that the two events have asymptotically the same probability. Indeed, if $B = \{e \in \mathcal{W} : \zeta(e) > t\}$, then $\mathcal{E}'_t$ is the event that by time 1, at least one point of M has fallen in B. Since the number of such points is Poisson with a parameter given in (10), we obtain:

$$P(\mathcal{E}'_t) = 1 - \exp\left(-\sqrt{\frac{2}{\pi t}}\right)$$

from which we deduce:

$$P(\mathcal{E}'_t) \sim \sqrt{\frac{2}{\pi t}} \tag{11}$$

as $t \to \infty$. To compute $P(\mathcal{E}_t)$, we appeal to Lévy's reflection principle. Let

$$S_t = \sup_{s \leq t} X_s.$$

Then Lévy showed that under the Wiener measure $\mathbb{W}$, the two processes $(L(t, 0), t \geq 0)$ and $(S_t, t \geq 0)$ have the same distribution. On the other hand, for fixed $t \geq 0$, by the standard reflection principle, S_t has the same distribution as $|X_t|$ (see, e.g., Durrett [4], Section 7.4, or [11] Sections III.3.7 and VI.2.3). Consequently,

$$\mathbb{W}\{\mathcal{E}_t\} = \mathbb{W}\{|X_t| \leq 1\} = \frac{1}{\sqrt{2\pi t}} \int_{x \in [-1,1]} e^{-x^2/2t} dx \sim \sqrt{2/(\pi t)}. \tag{12}$$

Let $A \in \mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. It follows from the above that

$$\mathbb{W}(A|\mathcal{E}_t) \underset{t\to\infty}{\longrightarrow} \mathbb{P}(A) \text{ if and only if } \mathbb{W}(A|\mathcal{E}'_t) \underset{t\to\infty}{\longrightarrow} \mathbb{P}(A). \tag{13}$$

It thus suffices to prove Theorem 2 when the event we condition on is $\mathcal{E}'_t$, rather than $\mathcal{E}_t$. In fact, for the same reason, one can condition on the event $\mathcal{E}^{(2)}_t$ that there is exactly one excursion of duration greater than t prior to time $\tau(1)$. Indeed $\mathcal{E}^{(2)}_t \subset \mathcal{E}'_t$ and we also have $P(\mathcal{E}^{(2)}_t) \sim P(\mathcal{E}_t)$ since the probability that there are two or more such excursion is $O(t^{-1})$.

Now, for $B \subset \mathcal{W}$, let $(N^B_u, u \geq 0) = \mathcal{P}([0, u) \times B)$ be the number of points of M that have fallen in B by time u, and take B to be the set $B = \{e \in \mathcal{W} : \zeta(e) > t\}$ defined above (11). Note that $N^B_\cdot$ is a Poisson process with rate $\nu(B)$. It is well-known that, conditionally on the number of jumps of a Poisson process by time 1, the jump times have the distribution of the uniform order statistics. In particular, since

$$\mathcal{E}_t^{(2)} = \{N_1^B = 1\},$$

we see that conditional upon $\mathcal{E}_t^{(2)}$, there exists a uniform random variable U in $(0,1)$ and $e \in B$ such that $(U,e) \in M$. Moreover e is independent of U and is distributed according to $\nu(\cdot|B)$. That is,

$$P(e \in \cdot) = \frac{\nu(e \in \cdot)}{\nu(\zeta > t)}. \tag{14}$$

Consider now the conditional distribution of $\sum_{i\geq 1: e_i \in B^c} \delta_{(\ell_i, e_i)}$ given $\mathcal{E}_t^{(2)}$. By the independence property of Poisson point processes in disjoint sets, this distribution is simply equal to the unconditional distribution of the restriction of M to $(0,\infty)\times B^c$. Therefore, let M' be an independent realization of M, and let

$$\tilde{M} = M'|_{(0,1]\times B^c} + \delta_{(U,e)} + M|_{(1,\infty]\times \mathcal{W}} \tag{15}$$

Let $\tilde{X}$ be the process obtained by reconstructing the path from the point process $\tilde{M}$. The above reasoning shows that for a set $A \in \mathcal{F}_s$, where $s > 0$ is fixed (while $t > s$ tends to infinity,)

$$\begin{aligned} P(X \in A|\mathcal{E}_t) &\sim P(X \in A|\mathcal{E}_t^{(2)}) \\ &\sim P(\tilde{X} \in A) \end{aligned}$$

Thus it suffices to show that $\tilde{X}$ converges in distribution to the law Q of the process Y in Theorem 2. Note that $(\tilde{X}_t, 0 \leq t \leq \tau(U))$ depends only on the points of M', and is thus independent of (U,e). Moreover, provided M' did not have any point in B on the time-interval $[0,1]$ (an event of probability $1-o(1)$), $(\tilde{X}_t, 0 \leq t \leq \tau(U)$ has the same distribution as $(X_t, 0 \leq t \leq \tau(U))$. Since e is also independent from U, it thus suffices to prove that

$$\nu(\cdot|\zeta > t) \underset{t\to\infty}{\longrightarrow} P(\epsilon B^{(3)} \in \cdot) \tag{16}$$

weakly. There are many ways to prove (16), and we propose one below. Let us postpone the proof of this statement for a few moments and finish the proof of Theorem 2. What we have proved is that for every $s > 0$,

$$\mathbb{W}(A|\mathcal{E}_t) \to \mathbb{P}(A), \quad A \in \mathcal{F}_s, \quad s > 0 \tag{17}$$

where $\mathbb{P}$ is the measure described in the statement of Theorem 2. It is not hard (but not immediate) to deduce weak convergence of $\mathbb{W}(\cdot|\mathcal{E}_t)$ towards $\mathbb{P}$. The problem is that one cannot directly apply the $\lambda - \pi$ system theorem of Dynkin to conclude that (17) holds for all $A \in \mathcal{F}_\infty$. Instead, note that, as in the proof of Lemma 6 in [2], (17) implies tightness (all events involved in the verification of tightness are measurable with respect to $\mathcal{F}_s$ for some $s > 0$), and furthermore any weak limit must be identical to $\mathbb{P}$, because for instance of the convergence of the finite-dimensional distributions. Thus $\mathbb{W}_t$ converges weakly towards $\mathbb{P}$.

Turning to the proof of (16), which is well-known in the folklore (but we haven't been able to find a precise reference), we propose the following simple argument. First note that under ν, sign(e) is uniform on $\{-1, +1\}$ and is independent from $(|e(x)|, x \geq 0)$, which has a "distribution" equal to ν^+, the restriction of ν to positive excursions. Let $\nu^+(\cdot|\zeta = t)$ denote the law of a positive Itô excursion conditioned to have duration equal to t, that is, the weak limit of

$$\frac{\nu^+(\cdot;\zeta \in (t, t+\varepsilon))}{\nu^+(\zeta \in (t, t+\varepsilon))} \tag{18}$$

as $\varepsilon \to 0$. Since

$$\nu^+(A|\zeta > t) = \int_{s>t} \nu(A|\zeta = s)\frac{ds}{\sqrt{2\pi s^3}}, \tag{19}$$

it suffices to prove

$$\nu^+(\cdot|\zeta = t) \underset{t\to\infty}{\longrightarrow} P(B^{(3)} \in \cdot) \tag{20}$$

It is not hard to show (see, e.g., Pitman [10], formula (28)), that a Brownian excursion conditioned to have duration equal to t is equal in distribution to a 3-dimensional Bessel bridge of duration t, that is, can be written as

$$e_u = \sqrt{b_{1,u}^2 + b_{2,u}^2 + b_{3,u}^2}, \quad 0 \leq u \leq t, \tag{21}$$

where $(b_{i,u}, 0 \leq u \leq t)_{i=1}^3$ are three independent one-dimensional Brownian bridges. Now, it is easy to check (see, e.g., Yor [15], Section 0.5) that if $\mathbb{W}^{(t)}$ is the law of a one-dimensional bridge of duration t, then for $s < t$,

$$\frac{d\mathbb{W}^{(t)}}{d\mathbb{W}}|_{\mathcal{F}_s} = \left(\frac{t}{t-s}\right) \exp\left(-\frac{X_s^2}{2(t-s)}\right). \tag{22}$$

Letting $t \to \infty$ and $s > 0$ fixed, we see that the above Radon-Nikodyn derivative converges to 1. This means that the restrictions of $(b_{i,u}, 0 \leq u \leq t)_{i=1}^3$ to $\mathcal{F}_s$ converge to three independent Brownian motions. By (21), it follows that the restriction of $(e_u, 0 \leq u \leq t)$ converges to $(\sqrt{X_{1,u}^2 + X_{2,u}^2 + X_{3,u}^2}, 0 \leq u \leq s)$ in distribution, where $(X_{i,u}, u \geq 0)_{i=1}^3$ are three independent Brownian motions. The law of this process is, of course, the same as $\mathbb{P}|_{\mathcal{F}_s}$, and hence Theorem 2 is proved.

2.2 Slowly growing local time.

We now consider a problem which may be considered the basic building block for the proof of Theorem 1. Let f be a nonnegative nondecreasing function, and let

$$\mathcal{E}_t = \{L_t \leq f(t)\}. \tag{23}$$

Let

$$\mathbb{Q}_t(\cdot) = \mathbb{W}(\cdot|\mathcal{E}_t) \tag{24}$$

with $\{X_t, t \geq 0\}$ distributed according to the Wiener measure $\mathbb{W}$. Note the difference between $\mathcal{E}_t$ and $\mathcal{K}_t$, where the conditioning concerns the entire growth of the local time profile up to time t, whereas $\mathcal{E}_t$ concerns only the value of L at time t.

Theorem 3. *Assume that $f(t) \to \infty$ as $t \to \infty$. Then $\mathbb{Q}_t$ converges weakly to the standard Wiener measure $\mathbb{W}$ on $\mathcal{C}$.*

This result may be a little surprising at first: even if f grows as slowly as $\log\log(t)$ we still obtain a recurrent process, indeed a Brownian motion in the limit. What is going on is that the effect of the conditioning is to create one very long excursion, but whose starting point escapes to infinity as $t \to \infty$. As a result the conditioning becomes trivial in the weak limit.

Proof. The strategy for the proof of Theorem 3 is similar to that used in the proof of Theorem 2, and we will thus give fewer details. We start again by noticing that the event $\mathcal{E}_t = \{L_t \leq f(t)\}$ is equivalent to the event $\{\tau(f(t)) > t\}$. If we define the event $\mathcal{E}'_t$ that there is at least one excursion of duration greater than t by time $\tau(f(t))$, then we have again $\mathcal{E}'_t \subset \mathcal{E}_t$, and letting $\lambda = f(t)\nu(\zeta > t) = f(t)\sqrt{2/(\pi t)}$,

$$\mathbb{W}(\mathcal{E}'_t) = 1 - e^{-\lambda} \sim f(t)\sqrt{\frac{2}{\pi t}} \tag{25}$$

while once again, by Lévy's identity and the reflection principle,

$$\mathbb{W}(\mathcal{E}_t) = \mathbb{W}(|X_t| \leq f(t)) = \frac{1}{\sqrt{2\pi t}} \int_{|x| \leq f(t)} e^{-x/(2t)} dx \sim f(t)\sqrt{\frac{2}{\pi t}} \tag{26}$$

Therefore, this time again, if $A \in \mathcal{F}_\infty$ then $\mathbb{W}(A|\mathcal{E}_t) \to \mathbb{P}(A)$ as $t \to \infty$ if and only if $\mathbb{W}(A|\mathcal{E}'_t) \to \mathbb{P}(A)$. Thus let $M = \sum_{i\geq 1} \delta_{(\ell_i, e_i)}$ be Itô's excursion point process, and let $\tilde{M}$ be a realisation of M conditionally given $\mathcal{E}'_t$. Then reasoning as in (15), we may write

$$\tilde{M} = M'|_{(0,f(t)]} + \delta_{(Uf(t),e)} + M|_{(f(t),\infty)\times\mathcal{W}} \tag{27}$$

where M' is an independent realisation of M, U is an independent uniform random variable on $(0,1)$, and e has the distribution (14). Letting $\tilde{X}$ be the path reconstructed from the points of $\tilde{M}$, we see that the distribution of $\tilde{X}$ up to time $\tau(Uf(t))$ is that of a standard Brownian motion. (We also have that $\tilde{X}_t = e(t - \tau(Uf(t)))$ for $\tau(Uf(t)) \leq t \leq \tau(Uf(t)) + \zeta(e)$, and that the process e still converges to a 3-dimensional Bessel process as $t \to \infty$ with a random sign, but as we will see this is irrelevant). Observe that now the random variable $\tau(Uf(t))$ tends to infinity almost surely since $f(t) \to \infty$ as $t \to \infty$. In particular this convergence holds in probability. Fix any $s \geq 0$ and $A \in \mathcal{F}_s$. Then

$$|\mathbb{P}((\tilde{X}_u, 0 \leq u \leq s) \in A) - \mathbb{W}(A)| \leq 2\mathbb{P}(\tau(Uf(t)) > s) \to 0$$

as $t \to \infty$. This proves that the law of $(X_u, 0 \leq u \leq s)$ conditionally given $\mathcal{E}'_t$, converges weakly as $t \to \infty$ (or indeed in total variation) towards $\mathbb{W}|_{\mathcal{F}_s}$. Therefore, by (25) and (26), the law of $(X_u, 0 \leq u \leq s)$ given $\mathcal{E}_t$ also converges weakly towards $\mathbb{W}|_{\mathcal{F}_s}$. This proves Theorem 3. □

In words, the conditioning "becomes invisible" asymptotically: the effect of conditioning on $\mathcal{E}_t$ is to add a three-dimensional Bessel which starts far away in the future. Observe that, as a byproduct, given $\mathcal{E}_t$,

$$\frac{L_t}{f(t)} \xrightarrow[t\to\infty]{} U \tag{28}$$

in distribution, where U is a uniform random variable on $(0,1)$.

3 Proof of the main result

3.1 Preliminary estimates

To begin with the proof of Theorem 1, we first divide the positive half-line into dyadic blocks. Let $t_j = 2^j$, and let $I_j = [t_{j-1}, t_j)$. We use as a shorthand notation $K_n = \mathcal{K}_{t_n}$. Our strategy will consist in studying a less stringent constraint for which the analysis is easier. Let C'_n be the following modified constraint:

$$C'_n = \{L_s - L_{t_{n-1}} \leq f(s), \quad \text{for all } s \in I_n\}, \tag{29}$$

and define analogously

$$K'_n = \bigcap_{j=1}^{n} C'_j \tag{30}$$

Note that K'_n is obviously a "weaker" constraint, in the sense that $K_n \subset K'_n$. The difference between K_n and K'_n is that K'_n is "more Markovian" and hence easier to analyse. We will show however that K_n has a positive probability to occur given K'_n (bounded away from zero), which means that any result which is true for K'_n with probability 1, will also be true for K_n. Note also that K'_n may be realised as the event $\mathcal{K}_{t_n}$ associated with a modified function $\hat{f}(t)$, which stays constant on each interval I_j and taks the value $f(t_j)$ on this interval.

Let $H_j = \{X_s = 0 \text{ for some } s \in I_j\}$, and let $A'_j = H_j \cap C'_j$.

We start the proof with the following general result which we will extensively use as an upper-bound on the growth of X given $\mathcal{K}_t$. For a function $\omega \in \mathcal{C}$, let $Z(\omega) = \{x \geq 0 : \omega(x) = 0\}$ be the set of its zeros; we view Z as a random variable under the probability measure $\mathbb{W}$. That is, we equip the set Ω of all closed subsets of the real nonnegative half-line with the σ-field $\mathcal{B}$ defined by $Z \in \mathcal{B}$ if and only if $\{\omega \in \mathcal{C} : Z(\omega) \in \mathcal{Z}\} \in \mathcal{F}$, where $\mathcal{F}$ is the Borel σ-field on $\mathcal{C}$ generated by the open sets associated with local uniform convergence. Let $\mathcal{Z}$ be a set of closed subsets of $\mathbb{R}_+$, such that $\mathbb{W}(Z \in \mathcal{Z}) > 0$. Let $\preceq$ be the order on Ω defined by $\omega \preceq \omega'$ if and only if $\omega(x) \leq \omega'(x)$ for all $x \geq 0$. Finally, let $P^{(\text{Bess})}$ denote the law of a three-dimensional Bessel process, which is obtained when one considers, e.g., the Euclidean norm of a Brownian motion in $\mathbb{R}^3$.

Lemma 1. *Conditionally on $\{Z \in \mathcal{Z}\}$, the $|X|$ is dominated by a three-dimensional Bessel process. More precisely, for any continuous bounded functional F on $\mathcal{C}$, which is nondecreasing for the order $\preceq$,*

$$\mathbb{W}(F(|X_s|, s \geq 0) | Z \in \mathcal{Z}) \leq P^{(\text{Bess})}(F(X_s, s \geq 0)), \tag{31}$$

where $P^{(\text{Bess})}$ denote the law of a 3-dimensional Bessel process.

Proof. Let $(R_s, s \geq 0)$ be a three-dimensional Bessel process. Recall that $(R_s, s \geq 0)$ is solution of the stochastic differential equation:

$$dR_s = dB_s + \frac{1}{R_s} ds. \tag{32}$$

We work conditionally on $\{Z(\omega) = z\}$ for a given $z \in \mathcal{Z}$. z being closed, its complement is open and defines open intervals which are referred to as excursion intervals. Given $\{Z = z\}$, the law of X can be described as a concatenation of independent processes whose laws are precisely Itô excursions conditioned on their duration ζ, where ζ is the length of the excursion interval of z under consideration. More formally, for $\zeta > 0$, let n_ζ denote the law of a brownian excursion conditioned to have duration ζ. Note that this *a priori* only makes sense Lebesgue-almost everywhere (see, e.g., V.10 in Feller [6]). However, the Brownian scaling property implies that n_ζ is weakly continuous for the topology induced by local uniform convergence, so that n_ζ is defined unambiguously for all $\zeta > 0$. Furthermore, it is not hard to show (see, e.g., Pitman [10], formula (28)), that under n_1, X is a solution to the stochastic differential equation:

$$dX_s = dB_s + \frac{1}{X_s}ds - \frac{X_s}{1-s}ds, \tag{33}$$

for which there is strong uniqueness. By Brownian scaling, under n_ζ, X is thus the unique in law solution to the stochastic differential equation:

$$dX_s = dB_s + \frac{1}{X_s}ds - \frac{X_s}{\zeta - s}ds \tag{34}$$

Start with a small parameter $\delta > 0$. If $X \in \mathcal{C}$ is a continuous function, let $X^{(\delta)}$ denote the element of $\mathcal{C}$ obtained by removing all the excursions of X of duration smaller than δ (alternatively, retaining all the excursions of length greater than δ). Since there are always no more than a finite number of excursions longer than δ on any compact interval, there is no problem in ordering these excursions chronologically. Likewise, let $X_{(\delta)}$ denote the process X where all the excursions longer than δ have been removed. Then Itô's excursion theorem tells us that under $\mathbb{W}$, the processes $X^{(\delta)}$ and $X_{(\delta)}$ are independent. Furthermore, the law of $X^{(\delta)}$ can be obtained by concatenating an i.i.d. sequence of excursions conditioned to have length greater than δ. As a consequence, let $Z^{(\delta)}$ be the zero set of $X^{(\delta)}$, and for $z \in \Omega$ let $z^{(\delta)} \in \Omega$ be the closed subset of $\mathbb{R}_+$ where all intervals of z^c of length smaller than δ. Then, given $\{Z(\omega) = z\}$, $X^{(\delta)}$ may be described as a concatenation of independent excursions of respective laws $n_{\zeta_1}, \ldots$, together with independent random signs, where $\zeta_1, \ldots$ are the chronologically ordered interval lengths of $(z^{(\delta)})^c$. Since $\mathbb{W}(Z \in \mathcal{Z}) > 0$, it follows that

$$E[F(|X^{(\delta)}|)|Z \in \mathcal{Z}] = \frac{1}{\mathbb{W}(Z \in \mathcal{Z})} \int_\Omega \mathbf{1}_{\{z \in \mathcal{Z}\}} \mu(dz) E[F(|X^\delta|)|Z = z] \tag{35}$$

By the above two observations, given $\{Z = z\}$, $X^{(\delta)}$ may be obtained as a concatenation of independent solutions to the stochastic differential equations (34). We may thus construct a coupling of $X^{(\delta)}$ with a Bessel process $(R_s, s \geq 0)$ as follows. Fix a Brownian motion $(B_s, s \geq 0)$. Let $\zeta_1, \zeta_2, \ldots$ denote the lengths of the excursions of $z^{(\delta)}$, ordered chronologically. We first construct $(X^{(\delta)}_s, 0 \leq s \leq \zeta_1)$ and $(R_s, 0 \leq s \leq \zeta_1)$ on the same probability space by solving (32) and (34) using the same Brownian motion $(B_s, 0 \leq s \leq \zeta_1)$ (this is possible by strong uniqueness), and by giving $X^{(\delta)}$ a random sign on that interval. Since (34) contains an additional negative drift term compared to (32), it follows that

$$|X_s^{(\delta)}| \leq R_s \tag{36}$$

holds pointwise on $[0, \zeta_1]$. By the Markov property of Brownian motion at time ζ_1, there is no problem in extending this construction on the interval $[\zeta_1, \zeta_1 + \zeta_2]$, this time using the Brownian motion $(B_s - B_{\zeta_1}, \zeta_1 \leq s \leq \zeta_1 + \zeta_2)$ for both (32) and (34). Since $R_{\zeta_1} \geq |X_{\zeta_1}^{(\delta)}|$ by (36), and since (34) still contains an additional negative drift term compared to (32), we conclude that the comparison (36) still holds true on the interval $[\zeta_1, \zeta_1 + \zeta_2]$. It is trivial to extend this construction on all of $\mathbb{R}_+$ by induction, and to obtain (36) pointwise on all of $\mathbb{R}_+$. Since F is nondecreasing, we deduce that

$$E[F(|X^\delta|)|Z = z] \leq E^{(\mathrm{Bess})}[F(X)].$$

Plugging this into (35), we get

$$E[F(|X^{(\delta)}|)|Z \in \mathcal{Z}] \leq E^{(\mathrm{Bess})}[F(X)] \tag{37}$$

On the other hand, F is by assumption continuous for the local uniform convergence, and it is easy to see that $X^{(\delta)}$ converges almost surely to X in the local uniform convergence. Since F is furthermore bounded, we conclude by the dominated convergence theorem that

$$E[F(|X|)|Z \in \mathcal{Z}] \leq E^{(\mathrm{Bess})}[F(X)]$$

as requested. □

Lemma 1 has the following concrete and useful consequence. Observe that the event K_n may be written as $\{Z(\omega) \in \mathcal{Z}_n\}$ for some set $\mathcal{Z}_n$. Indeed, recall that almost surely for every $t \geq 0$:

$$L_t = \lim_{\delta \to 0} \left(\frac{\pi}{2}\delta\right)^{1/2} N_t^\delta$$

where N_t^δ is the number of excursions longer than δ by time t. (See, e.g., Proposition (2.9) of chapter XII in [11] for a proof.) Thus we obtain in particular:

$$\mathbb{W}(F(|X_s|, s \geq 0)|K_n') \leq P^{(\mathrm{Bess})}(F(X_s, s \geq 0)). \tag{38}$$

Remark 1. An easy modification of the above proof shows that if $X_0 = x \neq 0$ almost surely (i.e., if $\mathbb{W}$ is replaced by the law of $(x + B_t, t \geq 0)$ in (31)), then the same result holds where R is a Bessel process also started at x.

Corresponding to this upper-bound on the growth of X given $\mathcal{K}_t$, we will prove a matching lower-bound and show that given $\mathcal{K}_t$, $|X|$ dominates a reflecting Brownian motion. in terms of reflecting Brownian motion.

Lemma 2. *For all $T \geq 2$,*

$$\mathbb{W}_T(F(|X_s|, s \geq 0)) \leq \mathbb{W}(F(|X_s|, s \geq 0)).$$

Proof. It is well known that a conditioned Brownian motion can be seen as an h-transform of the process and hence as adding a drift to the Brownian motion. It suffices to show that this drift is always the same sign as the current position of the process. Formally, fix $T \geq 0$ and let

$$\mathcal{A} = \{\omega \in \Omega : L_s(\omega) \leq f(s) \text{ for all } 0 \leq s \leq T\}. \tag{39}$$

For $t \geq t_0$, and $0 \leq \ell \leq f(t)0$, define:

$$\mathcal{A}(t, \ell) = \{\omega \in \Omega : L_s \leq f(s+t) \text{ for all } 0 \leq s \leq T - t\}. \tag{40}$$

That is, $\mathcal{A}$ is the initial constraint and $\mathcal{A}(t, \ell)$ is what remains of that constraint after t units of time and having already accumulated of local time at zero of ℓ by that time. Let

$$h(x, t, \ell) = \mathbb{W}_x(\mathcal{A}(t, \ell)). \tag{41}$$

Then the conditioned process $\mathbb{P}_T$ may be described by using Girsanov's theorem, to get that under $\mathbb{P}_T$, then X is a solution to

$$dX_t = dW_t + \nabla_x \log h(X_t, t, L_t) \tag{42}$$

where L is the local time at the origin of X and W is a one-dimensional Brownian motion. Details can be found for instance in [12, IV.39] in the case where the constraint depends only on the position of X (and not on its local time as well) but the proof remains unchanged in this case as well. Assume to simplify that $x \geq 0$, and let us show that $\nabla_x \log h(x, t, \ell) \geq 0$ for all $t \geq t_0$ and $0 \leq \ell \leq f(t)$. It suffices to prove that $(\partial h/\partial x)(x, t, \ell) \geq 0$. Thus it suffices to prove that for all $y \geq x$ sufficiently close to x,

$$h(y, t, \ell) \geq h(x, t, \ell). \tag{43}$$

In other words, we want to show that the probability of $\mathcal{A}(t, \ell)$ is monotone in the starting point. We use a coupling argument similar to the one we used in [2, Lemma 8]. Let X be a standard Brownian motion started at x and let Y be another Brownian motion, started at y. Consider the stopping time:

$$\tau = \inf\{t > 0 : Y_t = |X_t|\} \tag{44}$$

and construct the process Z defined by $Z_t = Y_t$ for $t \leq \tau$ and for $t > \tau$:

$$Z_t = \begin{cases} X_t & \text{if} X_\tau = Y_\tau \\ -X_t & \text{if} X_\tau = -Y_\tau \end{cases}$$

Then Z_t has the law of Brownian motion started at y. Moreover, we claim that for any $s > 0$, in this coupling:

$$L_s(Z) \leq L_s(X).$$

Indeed, note that for any $s \leq \tau$ we have $L_s(Z) = 0$ since Z cannot hit 0 before τ. Afterwards, the local time of Z increases exactly as that of X, hence in general, for all $s \geq 0$:

$$L_s(Z) = (L_s(X) - L_\tau(X))_+. \tag{45}$$

From (45), we also see that the increment of $L_s(Z)$ over any time-interval is smaller or equal to the increment of $L_s(X)$ over the same time-interval. Therefore, if X satisfies $\mathcal{A}(t, \ell)$, then so does Z. As a consequence,

$$\mathbb{W}_x(\mathcal{A}(t, \ell)) \leq \mathbb{W}_y(\mathcal{A}(t, \ell))$$

which is the same as (43). Thus Lemma 2 is proved. □

A first consequence of these two dominations is a simple proof that the conditioned processes form a tight family of random paths.

Lemma 3. *The family of measures $\{\mathbb{P}_t\}_{t \geq 0}$ is tight.*

Proof. By classical weak convergence arguments (see, e.g., Billingsley [3]), it suffices to prove that for each fixed A, the following two limit relations hold:

$$\lim_{b \to \infty} \limsup_{T \to \infty} \mathbb{P}_T \left(\sup_{0 \leq s \leq A} |X(s)| > b \right) = 0, \tag{46}$$

and for each $\eta > 0$

$$\lim_{\varepsilon \downarrow 0} \limsup_{T \to \infty} \mathbb{P}_T \left(\sup_{0 \leq s, s' \leq A, |s - s'| < \varepsilon} |X(s) - X(s')| > \eta \right) = 0. \tag{47}$$

(46) is a direct consequence of Lemma 1, so we move to the proof of (47). Basically, the domination above and below by a Bessel process and a reflecting Brownian motion give us a uniform Kolmogorov estimate:

$$\mathbb{E}_T(|X_t - X_s|^4) \leq C(t - s)^2$$

for some $C > 0$ and all $t, s > 0$. For instance, to get a bound on $\mathbb{E}((X_t - X_s)_+^4)$, it suffices to note that if $s < t$ without loss of generality, then conditionally on $\mathcal{F}_s$, the process $(X_{s+u}, u \geq 0)$ under $\mathbb{W}_T$ may be realised again as a Brownian motion conditioned upon satisfying a constraint of the form $\mathcal{K}_{T-s}$, and hence the domination by a 3-dimensional Bessel process started from X_s holds. The bound on $\mathbb{E}((X_t - X_s)_-^4)$ follows similarly.

Thus, if n is an integer and $0 \leq k \leq A2^n$, by Markov's inequality:

$$\mathbb{W}_T\left(\sup_{0\leq k\leq A2^n} |X_{(k+1)2^{-n}} - X_{k2^{-n}}| \geq 2^{-n/8}\right) \leq A2^n C 2^{n/4-2n} = CA2^{-3n/4}.$$

Thus, summing over $n \geq N$ for some $N \geq 1$:

$$\mathbb{W}_T\left(\exists\, n \geq N, \sup_{0\leq k\leq A2^n} |X_{(k+1)2^{-n}} - X_{k2^{-n}}| \geq 2^{-n/8}\right) \leq CA2^{-3N/4}. \tag{48}$$

Now, let s, t be two dyadic rationals such that $s < t$ and $|s-t| < 2^{-N}$, and assume that the complement of the event in the left-hand side of (48) holds. Let $r \geq N$ be the least integer such that $|s-t| > 2^{-r-1}$. Then there exists an integer $0 \leq k \leq A2^r$, as well as two integers ℓ, m such that

$$\begin{aligned} s &= k2^{-r} - \varepsilon_1 2^{-r-1} - \ldots - \varepsilon_\ell 2^{-r-\ell};\\ t &= k2^{-r} + \varepsilon_1' 2^{-r-1} + \cdots + \varepsilon_m 2^{-r-m}; \end{aligned}$$

with $\varepsilon_i, \varepsilon_i' \in \{0,1\}$. For $0 \leq i \leq \ell$ and $0 \leq j \leq m$ let

$$\begin{aligned} s_i &= k2^{-r} - \varepsilon_1 2^{-r-1} - \ldots - \varepsilon_\ell 2^{-r-i};\\ t_j &= k2^{-r} + \varepsilon_1' 2^{-r-1} + \cdots + \varepsilon_m 2^{-r-j}. \end{aligned}$$

Thus by the triangular inequality:

$$\begin{aligned} |X_t - X_s| &\leq |X_{t_0} - X_{s_0}| + \sum_{i=1}^{\ell} |X_{s_i} - X_{s_{i-1}}| + \sum_{j=1}^{m} |X_{t_j} - X_{t_{j-1}}|\\ &\leq 2^{-r/8} + \sum_{i=1}^{\ell} 2^{-(r+i)/8} + \sum_{j=1}^{m} 2^{-(r+j)/8}\\ &\leq \frac{3}{1-2^{-1/8}} 2^{-r/8} \leq c|t-s|^{1/8} \end{aligned}$$

with $c = 3/(1-2^{-1/8})2^{1/8}$. Therefore, by the density of dyadic rationals in $[0, A]$, it follows that

$$\mathbb{W}_T \left(\sup_{s,t\in[0,A];|s-t|\leq\epsilon} |X_s - X_t| > c|s-t|^{1/8} \right) \leq CA\varepsilon^{3/4}.$$

where $\varepsilon = 2^{-N}$. Since the right-hand side does not depend on T, we may take the limsup as $T \to \infty$, and (47) follows directly. □

3.2 Proof of the Theorem 1.

Lemma 4. *Let $\gamma > 0$. There exists $C > 0$ such that for all n large enough,*

$$\mathbb{W}(A'_j|K'_{j-1}) \leq C\frac{f(t_j)}{\sqrt{t_j}}. \tag{49}$$

Proof. We use an *a priori* rough upper-bound, based on the local time accumulated at the end of the interval I_j. Let $\tau_j = \inf\{t > t_{j-1} : X_t = 0\}$. Conditionally on H_j and $\mathcal{F}_{\tau_j}$, $(X_{\tau_j+t}, t \geq 0)$ is a Brownian motion started at 0 by the strong Markov property. Thus by (25),

$$\mathbb{W}(C'_j|\mathcal{F}_{\tau_j}; K'_{j-1}, H_j) \leq C\frac{f(t_j)}{\sqrt{t_j - \tau_j}}.$$

Now, note that if T^0 denote the hitting time of zero and $\mathbb{W}^x$ denote the Wiener measure started from x and $S_t = \sup_{s\leq t} X_s$, then by translation invariance and the reflection principle, we have for all $t > 0$ and $x \neq 0$,

$$\begin{aligned}
\frac{\mathbb{W}^x(T^0 \in dt)}{dt} &= \frac{d}{dt}\mathbb{W}(S_t > x) \\
&= \frac{d}{dt} 2\int_{x/\sqrt{t}}^{\infty} e^{-v^2/2}\frac{dv}{\sqrt{2\pi}} \\
&= ct^{-3/2}xe^{-x^2/2t} \\
&\leq ct^{-1}
\end{aligned} \tag{50}$$

where $c > 0$ is a universal constant independent of t and x. As a consequence, conditioning on $\mathcal{F}_{t_{j-1}}$ and letting $x = X_{t_{j-1}}$,

$$\begin{aligned}
\mathbb{W}(A'_j|\mathcal{F}_{t_{j-1}}, X_{t_{j-1}} = x) &\leq \int_{t_{j-1}}^{t_j} \mathbb{W}(\tau_j \in ds|\mathcal{F}_{t_{j-1}}, X_{t_{j-1}} = x)\frac{Cf(t_j)}{\sqrt{t_j - s}} \\
&\leq Cf(t_j)\int_{t_{j-1}}^{t_j} \frac{1}{\sqrt{t_j - s}}\frac{ds}{s} \\
&\leq C\frac{f(t_j)}{\sqrt{t_{j-1}}}\int_1^2 \frac{1}{\sqrt{2-u}}\frac{du}{u} \leq C\frac{f(t_j)}{\sqrt{t_j}}
\end{aligned} \tag{51}$$

Taking the expectation of both sides, we deduce the result. $\square$

We now show how the above lemma can be extended to show that the probability of hitting zero during the interval I_j is uniformly small as $n \to \infty$. This is the key lemma of the proof, since it precisely deals with the way the self-interaction of the process "propagates" down from $+\infty$ to the finite window $[t_{j-1}, t_j]$ we are considering .

Lemma 5. *There exists $C > 0$ and $j_0 \geq 1$ such that*

$$\mathbb{W}(H_j | K'_n) \leq C \frac{f(t_j)}{\sqrt{t_j}}, \tag{52}$$

uniformly in $n \geq j+1$ and $j \geq 1$.

Proof. We first note that for all $n \geq j+1$,

$$\mathbb{W}(H_j | K'_n) = \mathbb{W}(A'_j | K'_{j-1}) \frac{\mathbb{W}(K'_n | K'_j \cap H_j)}{\mathbb{W}(K'_n | K'_j)}. \tag{53}$$

In view of Lemma 4, it thus suffices to prove that

$$\mathbb{W}(K'_n | K'_j \cap H_j) \leq C \mathbb{W}(K'_n | K'_j) \tag{54}$$

for all $j \geq j_0$ and $n \geq j+1$, for some uniform $j_0, C > 0$. Consider a process $(X_1(t), t \geq 0)$, (resp. $(X_2(t), t \geq 0)$), which is a Brownian motion conditioned on $K'_j \cap H_j \cap \{|X_1(t_j)| \leq \sqrt{t_j}\}$ (resp. $K'_j \cap \{|X_2(t_j)| > \sqrt{t_j}\}$). Assume further that X_1 and X_2 are independent. Note that conditionally on $X_1(t_j)$ and $X_2(t_j)$, both processes evolve after time t_j as independent Brownian motions started from their respective positions at this time. Construct a process $\hat{X}$ as follows. Let $\tau := \inf\{t > t_j : |X_2(t)| = |X_1(t)|\}$, and let $\epsilon = X_2(t)/X_1(t) \in \{-1, 1\}$. Define

$$\hat{X}_t = \begin{cases} X_2(t) & \text{for } t \leq \tau \\ \epsilon X_1(t) & \text{for } t \geq \tau. \end{cases}$$

Then by the Markov property, $\hat{X}$ has the same distribution as X_2. Moreover, note that $|X_1(t_j)| \leq |X_2(t_j)|$ holds almost surely, and hence

$$|X_1(t)| \leq |\hat{X}(t)|,$$

for all $t \geq t_j$. Recall that by (5), if $(B_u, u \geq 0)$ is a Brownian motion, then for all $s \leq t$, $L_t(B) - L_s(B) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{\{|B_u| \leq \varepsilon\}} du$ almost surely. It follows immediately from this and the above that

$$L_t(\hat{X}) - L_{t_j}(\hat{X}) \leq L_t(X_1) - L_{t_j}(X_1), \quad t \geq t_j.$$

Therefore, if $X_1 \in K_n'$ then automatically $\hat{X} \in K_n'$. Hence we deduce that

$$P(X_1 \in K_n') \le P(\hat{X} \in K_n'). \tag{55}$$

Now, observe that

$$\begin{aligned} P(\hat{X} \in K_n') &= \mathbb{W}(K_n' | K_j' \cap \{|X(t_j)| > \sqrt{t_j}\}) \\ &\le \frac{\mathbb{W}(K_n'; K_j')}{\mathbb{W}(K_j')\mathbb{W}(|X(t_j)| > \sqrt{t_j}|K_j')} \\ &\le \frac{1}{p_1}\mathbb{W}(K_n'|K_j') \end{aligned} \tag{56}$$

where $\mathbb{W}(|X_1(t_j)| > \sqrt{t_j}|K_j') \ge p_1 > 0$ does not depend on n, by Lemma 2.

Similarly,

$$\begin{aligned} P(X_1 \in K_n') &= \mathbb{W}(K_n' | K_j' \cap H_j \cap \{|X(t_j)| \le \sqrt{t_j}\}) \\ &\ge \frac{\mathbb{W}(K_n'; K_j' \cap H_j; |X(t_j)| \le \sqrt{t_j})}{\mathbb{W}(K_j' \cap H_j)} \\ &\ge \mathbb{W}(K_n'|K_j' \cap H_j)\mathbb{W}(|X(t_j)| \le \sqrt{t_j}|K_j' \cap H_j) \\ &\ge p_2\mathbb{W}(K_n'|K_j' \cap H_j), \end{aligned} \tag{57}$$

where $\mathbb{W}(|X(t_j)| \le \sqrt{t_j}|K_j' \cap H_j) \ge p_2 > 0$ does not depend on n by Lemma 1. Putting together (55), (56), and (57), we find that

$$\mathbb{W}(K_n'|K_j' \cap H_j) \le \frac{1}{p_1 p_2}\mathbb{W}(K_n'|K_j')$$

which is precisely what was required. □

Lemma 6. *Assume that $\gamma > 1$. There exists $c_1, c_2 > 0$ which does not depend on n such that*

$$c_1 t_n^{-1/2} \le \mathbb{W}(K_n) \le \mathbb{W}(K_n') \le c_2 t_n^{-1/2}, \tag{58}$$

for all n sufficiently large.

Proof. The first inequality is a trivial consequence of the observation that K_n occurs automatically if $\tau_1 > t_n$ and $L_{t_1} < f(t_1)$, where $\tau_j = \inf\{t \ge t_j : X_t = 0\}$. The second inequality is also trivial since $K_n \subset K_n'$. It thus remains to show the third inequality. Let $G_{k,n} = \bigcap_{j=k}^n H_j^c = \{\tau_j > t_n\}$. Then by Lemma 5, letting $g(t) = f(t)/\sqrt{t}$, which is non-increasing by assumption, we have that

$$\mathbb{W}(G_{k,n}^c|K_n') \le C\sum_{j=k}^n g(t_j) \le C\sum_{j=k}^\infty g(t_j).$$

Since g is non-increasing, note that

$$\int_{j-1}^{j} g(2^t)dt \geq g(t_j)$$

and thus

$$\sum_{j=k}^{\infty} g(t_j) \leq \int_{k-1}^{\infty} g(2^t)dt = \frac{1}{\ln 2}\int_{t_{k-1}}^{\infty} \frac{g(u)}{u}du = \frac{1}{\ln 2}\int_{t_{k-1}}^{\infty} \frac{f(u)}{u^{3/2}}du.$$

Since f satisfies the condition (1), then the integral in the right-hand side is finite and we can find J large enough that $\mathbb{W}(G_{k,n}^c|K_n') \leq 1/2$ when $k \geq J$. Thus we get,

$$\begin{aligned}\mathbb{W}(K_n') &= \mathbb{W}(K_n'; \tau_J > t_n) + \mathbb{W}(K_n'; \tau_J \leq t_n)\\ &\leq \mathbb{W}(\tau_J > t_n) + \frac{1}{2}\mathbb{W}(K_n').\end{aligned}$$

Thus

$$\frac{1}{2}\mathbb{W}(K_n') \leq \mathbb{W}(\tau_J > t_n) \sim c_2 t_n^{-1/2} \tag{59}$$

as $n \to \infty$, for some $c_2 > 0$ that does not depend on n. The proof follows immediately. □

As a consequence of Lemma 6, we see that $\mathbb{W}(K_n|K_n') \geq c_1/c_2 > 0$, uniformly in n. In particular, applying Lemma 5, we get the following estimate valid for all $t \geq t_j$:

$$\mathbb{W}(H_j|\mathcal{K}_t) \leq C' g(t_j) \tag{60}$$

for some $C' > 0$ which does not depend on t. It is now easy to conclude that any weak limit of $\mathbb{W}_t$ is transient:

Proof of Theorem 1. Fix $\varepsilon > 0$. Observe that by (60), we can find $J \geq 1$ such that $\mathbb{W}(\tau_J > t|K_n) \geq 1 - \varepsilon$ for all $t \geq t_j$. Let $A > 0$ be an arbitrarily large number, and fix $u < v$. Let

$$R = \left\{ \inf_{s\in[t_J+u, t_J+v]} |X_s| \leq A \right\}.$$

$$\begin{aligned}\mathbb{W}(\mathcal{K}_t; R) &= \mathbb{W}(\mathcal{K}_t; R; \tau_J < t) + \mathbb{W}(K_n; R, \tau_J \geq t)\\ &\leq \mathbb{W}(\mathcal{K}_t)\varepsilon + \mathbb{W}(\tau_J > t)\mathbb{W}(R|\tau_J > t)\end{aligned}$$

Now, by Lemma 6 we see that dividing by $\mathbb{W}(\mathcal{K}_t)$ we find

$$\mathbb{W}(R|\mathcal{K}_t) \le \varepsilon + c_3 \mathbb{W}(R|\tau_J > t).$$

Now, observe that given $\tau_J > t$, and given $\mathcal{F}_{t_j}$, the process $\{X(t), t \ge t_j\}$ converges weakly to a 3-dimensional Bessel process started from $X(t_j)$ as $t \to \infty$. Note that $f \mapsto \inf_{[u,v]} |f|$ is a continuous functional on the space $\mathcal{C}$ of continuous sample paths (equipped with the topology of local uniform convergence). Since a Bessel process started from $x > 0$ always dominates a Bessel process started at 0, we deduce that

$$\limsup_{t\to\infty} \mathbb{W}(R|\tau_J > t) \le P^{(\mathrm{Bess})}(R')$$

where $P^{(\mathrm{Bess})}$ denotes the law of a 3-dimensional Bessel process started at 0, and

$$R' = \left\{ \inf_{s\in[u,v]} |X_s| \le A \right\}.$$

Since $P^{(\mathrm{Bess})}$ is almost surely transient, it follows that we can find u large enough such that *for all* fixed $v > u$, $P^{(\mathrm{Bess})}(R') \le \varepsilon$ (independently of v), in which case we obtain

$$\limsup_{t\to\infty} \mathbb{W}(R|\mathcal{K}_t) \le \varepsilon(1 + c_3).$$

Using once again the fact that the infimum over a compact set is a continuous functional, we see that for any weak subsequential limit $\mathbb{P}$ of $\mathbb{W}(\cdot|\mathcal{K}_t)$,

$$\mathbb{P}(R) \le \varepsilon(1 + c_3),$$

for all $v > 0$ arbitrarily large. Letting $v \to \infty$ and changing ε into $\varepsilon/(1 + c_3)$, we obtain by monotone convergence:

$$\mathbb{P}(\inf_{[u,\infty)} |X_s| \le A) \le \varepsilon.$$

Thus if $\Lambda = \sup\{t \ge 0 : |X_t| \le A\}$, we have $\mathbb{P}(\Lambda > u) \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have proved that $\Lambda < \infty$, $\mathbb{P}$-almost surely. Since A is arbitrary, this proves that $\mathbb{P}$ is transient almost surely. □

4 Brownian motion with bounded negative part

Until now, we have only considered conditionings which involve the local time of a Brownian motion at a specified point. The next result studies the case where the forbidden region is a semi-infinite interval. Let $(X_t, t \geq 0)$ be a one-dimensional Brownian motion and let A_t be the additive functional of X defined by

$$A_t = \int_0^t \mathbf{1}_{\{X_s<0\}} ds. \tag{61}$$

A_t is known as the negative part of Brownian motion, and is nothing else than the time spent by X in the negative half-axis. Let

$$\mathcal{E}_t = \{A_t \leq 1\}$$

and let $\mathbb{Q}_t$ be the measure defined by conditioning the Wiener measure on the event $\mathcal{E}_t$.

Theorem 4. *As $t \to \infty$, $\mathbb{W}_t$ converges weakly to a measure $\mathbb{P}$ which is transient almost surely. Moreover, under $\mathbb{P}$ we have*

$$A_\infty \stackrel{d}{=} \mathbf{U}^2,$$

where $\mathbf{U}$ is a uniform random variable on $(0,1)$.

An explicit description of the measure $\mathbb{Q}$ of the limiting process is given in the proof. In a way that is analogous to Theorem 2, the process is also made up of several independent pieces glued together at a certain random time. The proof of Theorem 4 uses explicit descriptions of the Brownian path and precise distributional results (such as Paul Lévy's arcsine law). At this point it is not clear how to extend this result to time-dependent conditionings in the manner of Theorem 3 or Theorem 1.

Proof. We start by recalling Paul Lévy's second arcsine law, which states that $A_t =_d g_t$, where

$$g_t = \sup\{s \leq t : X_s = 0\}$$

and both of these random variables have the arcsine law:

$$\mathbb{W}(\frac{A_t}{t} \in dx) = \frac{1}{\pi\sqrt{x(1-x)}} \mathbf{1}_{\{x\in(0,1)\}} dx. \tag{62}$$

(See, e.g., Theorem 2.7 in Chapter VI, and (3.20) in Chapter III of [11].) It is also well-known (see, for instance, Theorem 4.1 of [15]) that there is the following decomposition of the sample path of Brownian motion at g_t.

- $(B_s, s \leq g_t)$ and $(B_{g_t+s}, s \geq 0)$ are independent.

- $(\frac{1}{\sqrt{g_t}} B_{sg_t}, 0 \le s \le 1)$ is a standard Brownian bridge of duration 1.

Therefore, by independence between the two pieces, the event that $\{g_t \le 1\}$ and the excursion containing 1 is positive has probability:

$$\mathbb{W}(g_t \le 1 \text{ and } X_1 > 0) = (1/2)\mathbb{W}(A_t \le 1)$$

It follows that:

$$\mathbb{W}(g_t \le 1 \text{ and } X_1 > 0 | A_t \le 1) = 1/2 \tag{63}$$

On this event it is clear that X is transient, so transience of W occurs with probability at least $1/2$. In fact a more precise argument allows one to to generalize this and get transience with probability 1. We will show that there is a constant that for every $A > 1$

$$\lim_{t\to\infty} \mathbb{W}(g_t > A | \mathcal{E}_t) = 1/(2\sqrt{A}) \tag{64}$$

where by definition $\mathcal{E}_t = \{A_t \le 1\}$ is the event that we condition on. To see why (64) holds, we start by remarking that due to the Arscine law:

$$\mathbb{W}(\mathcal{E}_t) = \int_0^{1/t} \frac{dx}{\pi\sqrt{x(1-x)}} \sim \frac{2}{\pi\sqrt{t}} \tag{65}$$

Observe on the other hand that

$$\mathbb{W}(g_t \le A | \mathcal{E}_t) = \frac{1}{\mathbb{W}(\mathcal{E}_t)} \mathbb{W}(g_t \le A; A_{g_t} \le 1; B_t > 0). \tag{66}$$

Also, yet another result of P. Lévy tells us that if $(b_s, 0 \le s \le 1)$ is a standard Brownian bridge, then

$$\int_0^1 \mathbf{1}_{\{b_s \le 0\}} ds \stackrel{d}{=} \mathbf{U} \tag{67}$$

a uniform random variable in $(0,1)$. (See, e.g, (3.9) in Chapter XII of [11]). Putting these pieces together and using (66), it follows that for $A > 1$,

$$\begin{aligned}
\mathbb{W}(g_t \le A; A^-_{g_t} \le 1; B_t > 0) &= \frac{1}{2}\int_0^{A/t} \frac{dx}{\pi\sqrt{x(1-x)}} \cdot (1 \wedge \frac{1}{xt}) \\
&= \frac{1}{2}\int_0^{1/t} \frac{dx}{\pi\sqrt{x(1-x)}} + \frac{1}{2}\int_{1/t}^{A/t} \frac{dx}{\pi\sqrt{x(1-x)}} \frac{1}{xt} \\
&\sim \frac{1}{\pi\sqrt{t}} + \frac{1}{2t}\int_{1/t}^{A/t} \frac{dx}{\pi x^{3/2}} \\
&\sim \frac{1}{\pi\sqrt{t}}(2 - 1/\sqrt{A})
\end{aligned}$$

(64) now follows immediately from (66) and (65). On the other hand, a similar computation yields for $0 < A < 1$,

$$\mathbb{W}(g_t < A|\mathcal{E}_t) \sim \sqrt{A}/2. \tag{68}$$

It is easy to see from (64) that the conditional law of B given $\mathcal{E}'_t$ converges to the law of a transient process. This process can be described via the following sample path decomposition. Let g be a random variable such that:

$$P(g \in dy) = \begin{cases} \frac{1}{4} y^{-1/2} & \text{if } 0 < y \leq 1 \\ \frac{1}{4} y^{-3/2} dy & \text{if } y > 1. \end{cases} \tag{69}$$

This is obtained by differentiating respectively (68) and (64). Let $(b_s, 0 \leq s \leq 1)$ be an independent Brownian bridge conditioned so that

$$\int_0^g \mathbf{1}_{\{b_{t/g} \leq 0\}} dt \leq 1 \tag{70}$$

Then for $t \leq g$ we put $X_t = g^{1/2} b_{t/g}$, and after time g, we glue an independent 3-dimensional Bessel process. Note that with probability $1/2$, $g < 1$, so in that case, the conditioning (70) is trivial, and the total time accumulated by $(X_t, t \geq 0)$ in the negative half-line is uniform on $(0, g)$. By (67), in that case we can thus write $A_\infty = \mathbf{U} g$, with $\mathbf{U}$ a uniform random variable on (0,1) independent from g. On the other hand, if $g > 1$, then A_∞ is uniformly distributed on $(0, g)$, conditionally on being smaller than 1 (by (70)). Thus A_∞ is uniformly distributed on (0,1) in that case. To finish the proof of Theorem 4, it suffices to make a computation: for $0 < u < 1$,

$$\begin{aligned} \mathbb{P}(A_\infty < u) &= \frac{1}{2}\mathbb{P}(A_\infty < u | g > 1) + \frac{1}{2}\mathbb{P}(A_\infty < u | g < 1) \\ &= \frac{1}{2} u + \frac{1}{2}\left(\mathbb{P}(g < u | g < 1) + \int_{u<y<1} \mathbb{P}(g \in dy | g < 1) \frac{u}{y}\right) \\ &= \frac{1}{2} u + \frac{1}{2}(u^{1/2} + u(u^{-1/2} - 1)) = u^{1/2} = \mathbb{P}(\mathbf{U}^2 < u). \end{aligned}$$

Hence the proof is finished. □

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