3. Arbitrage Pricing Model

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Single factor model

- Assumptions: Asset returns are correlated ONLY because they all respond to one common risk factor, which is usually the market return.

\[ r_{jt} = \alpha_j + \beta_j r_{mt} + \varepsilon_{jt}, \quad COV(\varepsilon_i, \varepsilon_j) = 0 \quad \text{if} \; i \neq j \]

- Macro events affect returns of all assets
- Micro events affect only individual assets
- After controlling for the common risk factor, there is no correlations between returns of different assets.
- It follows that  
  \[ COV(r_i, r_j) = \beta_i \beta_j \sigma^2(r_m) \]
Portfolio variance

- Total variance = Systematic risk + idiosyncratic risk

\[
\sigma^2(r_j) = \beta_j^2 \sigma^2(r_m) + \sigma^2(\varepsilon_j)
\]
\[
\sigma^2(r_p) = \beta_p^2 \sigma^2(r_m) + \sigma^2(\varepsilon_p)
\]
\[
= \left(\sum_{i=1}^{n} w_i \beta_i\right)^2 \sigma^2(r_m) + \sum_{i=1}^{n} w_i^2 \sigma^2(\varepsilon_i)
\]
\[
\rightarrow \left(\sum_{i=1}^{n} w_i \beta_i\right)^2 \sigma^2(r_m) \text{ as } n \rightarrow +\infty
\]

- Factor structure greatly simplifies the calculation of portfolio variance.
- This is the main motivation for Sharpe (1963) to introduce the factor model.
Number of parameters

- To estimate the variance of a portfolio with N assets, originally we would need to estimate:
  - N variances
  - N*(N-1)/2 covariances
- With the factor structure, we only need to estimate:
  - N idiosyncratic variances
  - N betas
  - 1 factor variance
Multifactor models

- Security returns are driven by multiple common risk factors

\[ r_{jt} = \alpha_j + \beta_{1j}F_{1t} + \beta_{2j}F_{2t} + \ldots + \beta_{Lj}F_{Lt} + \varepsilon_{jt} \]

- If the common risk factors are uncorrelated with each other, then

\[
\sigma^2(r_p) = \left( \sum_{i=1}^{n} w_i \beta_{1i} \right)^2 \sigma^2(F_1) + \left( \sum_{i=1}^{n} w_i \beta_{2i} \right)^2 \sigma^2(F_2) + \ldots
\]

\[ + \left( \sum_{i=1}^{n} w_i \beta_{Li} \right)^2 \sigma^2(F_L) + \sum_{i=1}^{n} w_i^2 \sigma^2(\varepsilon_i) \]
Multifactor model: example

- Example: 2 factors

<table>
<thead>
<tr>
<th></th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>Residual variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Security I</td>
<td>0.6</td>
<td>0.2</td>
<td>0.05</td>
</tr>
<tr>
<td>Security II</td>
<td>0.9</td>
<td>0.1</td>
<td>0.02</td>
</tr>
</tbody>
</table>

- Assume that \( \sigma_{F_1}^2 = 0.12 \) and \( \sigma_{F_2}^2 = 0.10 \) and \( \text{COV}(F_1, F_2) = 0 \). Compute the variance of the following portfolio \( w_1 = w_2 = 0.5 \)
Constructing zero-beta portfolio

- Example:

<table>
<thead>
<tr>
<th>Security</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-0.4</td>
<td>1.75</td>
</tr>
<tr>
<td>B</td>
<td>1.6</td>
<td>-0.75</td>
</tr>
<tr>
<td>C</td>
<td>0.667</td>
<td>-0.25</td>
</tr>
</tbody>
</table>

- How can you construct a zero-beta portfolio?
Rewriting the factor models

- For simplicity, define $f_i$ as the unexpected component of $F_i$,

\[ f_i \equiv F_i - E(F_i) \]

- By definition, $E(f_i)=0$
- Rewrite the factor models as

\[ r_{jt} = E(r_j) + \beta_{1j} f_{1t} + \beta_{2j} f_{2t} + \ldots + \beta_{Lj} f_{Lt} + \varepsilon_{jt} \]

- Asset return = expected return + response to unexpected factor realization + idiosyncratic return
APT

- Ross (1976) argued that to prevent arbitrage, we must have

\[ E(r_j) = r_f + \sum_{l=1}^{L} \beta_{jl} \lambda_l \]

where \( \lambda_l \) is the risk premium of factor \( l \), i.e., the expected excess return of an asset with \( \beta_i = 1 \) and \( \beta_i = 0 \) for all \( i \neq l \).

=> Expected returns increase linearly with the loadings on common risk factors
A simple derivation of APT

- For simplicity, consider just the single factor model without idiosyncratic risk
  \[ r_j = E(r_j) + \beta_j f \]

- The return of a portfolio with two assets is
  \[ r_p = w r_i + (1 - w) r_j = [w E(r_i) + (1 - w) E(r_j)] + [w \beta_i + (1 - w) \beta_j] f \]

- If we set \( w = -\beta_j / (\beta_i - \beta_j) \)
  then this portfolio is risk free, therefore its expected return must be \( r_f \).

- Otherwise there will be arbitrage opportunity
A simple derivation of APT

- It follows that

\[ \frac{\beta_j}{\beta_i - \beta_j} E(r_i) + (1 + \frac{\beta_j}{\beta_i - \beta_j})E(r_j) = r_f \]

\[ \Rightarrow \beta_i[E(r_j) - r_f] = \beta_j[E(r_i) - r_f] \]

\[ \Rightarrow \frac{E(r_i) - r_f}{\beta_i} = \frac{E(r_j) - r_f}{\beta_j} = \lambda \]

- The derivation is similar for the case of multiple factors without idiosyncratic risk
- When they are idiosyncratic risk, APT hold only approximately.
- Key argument: If APT does not hold, then arbitrageurs can create a well-diversified zero beta portfolios with an expected return different from the risk free rate.
- This would make arbitrage possible!
Arbitrage in expectations

Figure B  Expected Return and Exposure

Expected Return, E

25%
20%
15%
15%

Factor Three Sensitivity, b(3)

1 2

Figure C  Actual Returns: Stack B vs. Portfolio P

Return, R

25%
20%
15%

Factor Three Return, f(3)

Bond A

Asset Management  Youchang Wu
Arbitrage in expectations (2)

**Figure 9.1** Infeasible relationship between \( E(r_j) \) and \( \beta_{1,j} \) in a one-factor model.
# APT Example

<table>
<thead>
<tr>
<th>Security</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>Expected return</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.0</td>
<td>1.6</td>
<td>16.2</td>
</tr>
<tr>
<td>B</td>
<td>3.0</td>
<td>2.8</td>
<td>21.6</td>
</tr>
<tr>
<td>$r_f$</td>
<td>0.0</td>
<td>0.0</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Suppose APT holds

- What is the risk premium for factor 1?
- What is the risk premium for factor 2?
What are the risk factors

- APT does not specify what the factors stand for
- The factor structure is ambiguous.
- If securities’ returns can be written using the $k$-factor vector $f$ and the $n \times k$ beta matrix $B$, they can also be written using a factor vector $f^*$ and a beta vector $B^*$ as long as

$$f^* = L^{-1}f \quad \text{and} \quad B^* = BL,$$

where $L$ is a non-singular $k \times k$ matrix.
Chen, Roll and Ross (1986) find the following macroeconomic factors are relevant:

- Unanticipated changes in industrial production
- Unanticipated changes in inflation
- Unanticipated changes in risk premiums (as measured by the return difference between low grade and high grade bonds)
- Unanticipated changes in the term structure of interest rates
Why are those factors chosen

- Stocks prices in a discounted cash flow framework

\[ P = \frac{E[C]}{r} \]

- \( E[C] \) = expected dividends
- \( r \) = discount rate

- Relevant factors are those that influence either the discount rate or the expected dividends:
  - Discount rate:
    - Change in risk premium
    - Changes in Term Structure
  - Dividends:
    - Changes in inflation rate
    - Changes in industrial production
Construction of the factors

- **Industrial production growth:**
  
  \[ MP(t) = \ln IP(t) - \ln IP(t-1) \]

- **Inflation:**
  
  - Unexpected inflation rate
    
    \[ UI(t) = I(t) - E[I(t)|t-1] \]
  
  - Changes in expected inflation rate
    
    \[ DEI(t) = E[I(t+1)|t] - E[I(t)|t-1] \]
Construction of the factors (2)

- Risk premium:
  \[ UPR(t) = "Baa and under" \text{ bond return} - LGB(t) \]
  note that UPR(t) is negatively related to unexpected changes in aggregate risk aversion
- Term Structure:
  \[ UTS(t) = LGB(t) - TB(t - 1) \]
- Additionally examined variables:
  - EWNY(t): return on equally weighted NYSE index
  - VWNY(t): return on value-weighted NYSE index
Empirical Results

- Model equations:
  \[ R = a + \beta_{MP} MP + \beta_{DEI} DEI + \beta_{UI} UI + \beta_{UPR} UPR + \beta_{UTS} UTS + e \]
  \[ R = \alpha + b_{MP} \beta_{MP} + b_{DEI} \beta_{DEI} + b_{UI} \beta_{UI} + b_{UPR} \beta_{UPR} + b_{UTS} \beta_{UTS} + \varepsilon \]

- Results (see the table):
  - \(b_{MP} > 0\)
  - \(b_{DEI} < 0\)
  - \(b_{UI} < 0\)
  - \(b_{UPR} > 0\)
  - \(b_{UTS} < 0\)
### Results (Panel C of table 4)

<table>
<thead>
<tr>
<th></th>
<th>EWNY</th>
<th>MP</th>
<th>DEI</th>
<th>UI</th>
<th>UPR</th>
<th>UTS</th>
<th>Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1958–84</td>
<td>5.021</td>
<td>14.009</td>
<td>-.128</td>
<td>-.848</td>
<td>8.130</td>
<td>-5.017</td>
<td>6.409</td>
</tr>
<tr>
<td></td>
<td>(1.218)</td>
<td>(3.774)</td>
<td>(-1.666)</td>
<td>(-2.541)</td>
<td>(2.855)</td>
<td>(-1.576)</td>
<td>(1.848)</td>
</tr>
<tr>
<td>1958–67</td>
<td>6.575</td>
<td>14.936</td>
<td>-.005</td>
<td>-.279</td>
<td>5.747</td>
<td>-.146</td>
<td>7.349</td>
</tr>
<tr>
<td></td>
<td>(1.199)</td>
<td>(2.336)</td>
<td>(-.060)</td>
<td>(-.558)</td>
<td>(2.070)</td>
<td>(-.067)</td>
<td>(1.591)</td>
</tr>
<tr>
<td></td>
<td>(.283)</td>
<td>(2.715)</td>
<td>(-3.039)</td>
<td>(-3.366)</td>
<td>(2.758)</td>
<td>(-2.015)</td>
<td>(.558)</td>
</tr>
<tr>
<td></td>
<td>(.906)</td>
<td>(1.253)</td>
<td>(-.529)</td>
<td>(-.847)</td>
<td>(.663)</td>
<td>(-.520)</td>
<td>(1.245)</td>
</tr>
</tbody>
</table>
Results (Panel D of table 4)

<table>
<thead>
<tr>
<th></th>
<th>VWNY</th>
<th>MP</th>
<th>DEI</th>
<th>UI</th>
<th>UPR</th>
<th>UTS</th>
<th>Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1958–84</td>
<td>-2.403</td>
<td>11.756</td>
<td>-.123</td>
<td>-.795</td>
<td>8.274</td>
<td>-5.905</td>
<td>10.713</td>
</tr>
<tr>
<td></td>
<td>(-.633)</td>
<td>(3.054)</td>
<td>(-1.600)</td>
<td>(-2.376)</td>
<td>(2.972)</td>
<td>(-1.879)</td>
<td>(2.755)</td>
</tr>
<tr>
<td>1958–67</td>
<td>1.359</td>
<td>12.394</td>
<td>.005</td>
<td>-.209</td>
<td>5.204</td>
<td>-.086</td>
<td>9.527</td>
</tr>
<tr>
<td></td>
<td>(.277)</td>
<td>(1.789)</td>
<td>(.064)</td>
<td>(-.415)</td>
<td>(1.815)</td>
<td>(-.040)</td>
<td>(1.984)</td>
</tr>
<tr>
<td></td>
<td>(-.717)</td>
<td>(2.038)</td>
<td>(-3.237)</td>
<td>(-3.106)</td>
<td>(2.955)</td>
<td>(-2.299)</td>
<td>(1.167)</td>
</tr>
<tr>
<td>1978–84</td>
<td>-3.683</td>
<td>8.402</td>
<td>-.116</td>
<td>-.739</td>
<td>61.056</td>
<td>-5.928</td>
<td>15.452</td>
</tr>
<tr>
<td></td>
<td>(-.491)</td>
<td>(1.432)</td>
<td>(-.458)</td>
<td>(-.869)</td>
<td>(.782)</td>
<td>(-.644)</td>
<td>(1.867)</td>
</tr>
</tbody>
</table>
Interpretation

- $b_{MP}>0$: stocks less sensitive to production risk are more valuable (therefore has a lower return)
- $b_{UPR}>0$: stocks less sensitive to the return of low-grade bond returns (relative to that of high-grade bonds) are more valuable
- $b_{UI}<0, b_{DEI}<0$: stocks doing well when inflation is unexpectedly high are more valuable
- $b_{UTS}<0$: stocks doing well when long-term real rate is unexpectedly low are more valuable.
Further remarks

- Market return as a factor:
  - The macroeconomic factors don’t lose their significance, even if market return is included as a factor.
  - The influence of market return is not significant
  - CAPM would suggest the opposite
  - Therefore the results support APT against CAPM
APT and portfolio management

- The central focus of portfolio strategy is the choices of an appropriate pattern of factor sensitivities
- This choice should depend on the economic characteristics of the investors
Example

- Two factors: inflation and industrial production.
- Is zero exposure to both factors desirable?
  - No hedge against inflation
  - No risk premium
Analyzing portfolio strategies

- Analysis of the investors’ economic situation
  - What is the investor’s expenditure structure?
  - What is the investor’s income structure?
  - E.g. Institutional investors: Expenditures are less influenced by food prices; but their travelling expenses are likely to be higher than those of an average investor.

- Compare with the market portfolio
Strategic portfolio decisions

- If an institution does not need to hedge so much against inflation due to low sensitivity to food price
  \[ \Rightarrow \text{choose a low } \beta_i \text{ (relative to the average).} \]
- If an institution does not need to hedge so much against production fluctuations
  \[ \Rightarrow \text{choose a high } \beta_{IP}. \]
- By bearing such additional systematic risks the institution can earn a higher expected return
Tactical portfolio decisions

- If the firm’s idiosyncratic risk depends on energy cost
  => heavily invest in the stocks of energy sector.
- If the firm is unconcerned about inflation in agricultural price
  => skew portfolio holdings out of this sector
- Common mistake: pension funds investing in the stock of the sponsoring firm
APT and CAPM

- APT allows for multiple risk factors but requires zero correlation among idiosyncratic returns.
- CAPM considers only one factor but does not require zero correlation among idiosyncratic returns.
- CAPM is derived under the assumption that investors hold mean-variance efficient portfolios, while APT is derived under the assumption of no arbitrated in expectations.
- CAPM recommends investors to hold the market portfolio, while APT suggests investors to optimize exposures to risk factors according to their own profiles.