Complexity of Semi-Stable and Stage Semantics in Argumentation Frameworks

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Outline

1. Argumentation in AI
2. Abstract Argumentation
3. Complexity of Stage / Semi-Stable Semantics
4. Fixed-Parameter-Tractability
5. Conclusion
Argumentation in AI

- Very general idea: representation of an argument
- Different views: modeling the process, verifying the correctness, analyzing the conflicts, etc.
- Thus, representation of arguments came in many different flavors
Argumentation in AI

- Very general idea: representation of an argument
- Different views: modeling the process, verifying the correctness, analyzing the conflicts, ... etc.
- Thus, representation of arguments came in many different flavors

Abstract Argumentation

- Arguments are “atomic”
- Argumentation frameworks (AFs) formalize relations (rebuttals) between arguments
- Semantics gives an abstract handle to solve the inherent conflicts between statements by selecting acceptable subsets
An argumentation framework (AF) is a pair $(A, R)$ where
- $A$ is a set of arguments
- $R \subseteq A \times A$ is a relation representing “attacks” (“defeats”)
Argumentation Frameworks

An argumentation framework (AF) is a pair \((A, R)\) where
- \(A\) is a set of arguments
- \(R \subseteq A \times A\) is a relation representing “attacks” (“defeats”)

Example

\[AF=\{(a,b,c,d,e),\{(a,b),(c,b),(c,d),(d,c),(d,e),(e,e)\}\}\]
Conflict-free Extension

Given an AF \((A, R)\).
A set \(S \subseteq A\) is conflict-free in \(F\), if, for each \(a, b \in S\), \((a, b) \notin R\).
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Example

\[
\begin{align*}
\text{cf} (F) &= \{ \{a, c\}, \}
\end{align*}
\]
2. Abstract Argumentation

Conflict-free Extension

Given an AF \((A, R)\).
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Example

\[ cf(F) = \{\{a, c\}, \{a, d\}, \} \]
Conflict-free Extension

**Conflict-Free Extension**

Given an AF \((A, R)\).
A set \(S \subseteq A\) is **conflict-free** in \(F\), if, for each \(a, b \in S\), \((a, b) \notin R\).

**Example**

\[ cf(F) = \{\{a, c\}, \{a, d\}, \{b, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \emptyset\} \]
2. Abstract Argumentation

Admissible Extension

Given an AF \((A, R)\). A set \(S \subseteq A\) is admissible in \(F\), if

- \(S\) is conflict-free in \(F\)
- each \(a \in S\) is defended by \(S\) in \(F\),
  - \(a \in A\) is defended by \(S\) in \(F\), if for each \(b \in A\) with \((b, a) \in R\), there exists a \(c \in S\), such that \((c, b) \in R\).
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$$adm(F) = \{\{a, c\}\},$$
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Given an AF \((A, R)\). A set \(S \subseteq A\) is **admissible** in \(F\), if

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Example

\[
adm(F) = \{\{a, c\}, \{a, d\}\},
\]

[Diagram showing the relationships between elements a, b, c, d, and e with arrows indicating dependencies.]
Admissible Extension

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Example

\[ \text{adm}(F) = \{ \{a, c\}, \{a, d\}, \{b, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \emptyset \} \]
Stable Extensions

Stable Extension

Given an AF \((A, R)\). A set \(S \subseteq A\) is \textit{stable} in \(F\), if

- \(S\) is conflict-free in \(F\)
- for each \(a \in A \setminus S\), there exists a \(b \in S\), such that \((b, a) \in R\).
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Complexity of Semi-Stable and Stage Semantics
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**Example**

\[
\begin{align*}
\text{stable}(F) &= \{\{a, c\}, \{a, d\}, \{b, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \emptyset, \}\n\end{align*}
\]
Some AFs have no stable extension:

- The argumentation framework (AF) shows a cycle among arguments a, b, and c, with directed links indicating the attack relations.

For $S \subseteq A$, we define $S^+ = S \cup \{a : \exists b \in S : (b, a) \in R\}$ minimizing $A \setminus S^+ \iff$ maximizing $S^+$.

If $S$ is a stable extension then $S^+ = A$. 

Complexity of Semi-Stable and Stage Semantics
Stable Extensions

Some AFs have no stable extension:

![Diagram of AFs](attachment:af_diagram.png)

Idea: Using extensions minimizing the unattacked arguments in $A \setminus S$. 
Stable Extensions

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Idea: Using extensions minimizing the unattacked arguments in \( A \setminus S \).

- For \( S \subseteq A \) we define \( S^+ = S \cup \{a : \exists b \in S : (b, a) \in R\} \)
- minimizing \( A \setminus S^+ \) ⇔ maximizing \( S^+ \)
- If \( S \) is a stable extension then \( S^+ = A \)
Stage/Semi-Stable Extension

Given an AF \((A, R)\). A set \(S \subseteq A\) is **stage** (resp. **semi-stable**) in \(F\), if

- \(S\) is conflict-free (resp admissible) in \(F\)
- for each \(S' \subseteq A\), if \(S'\) conflict-free (admissible) then \(S^+ \not\subset S'^+\)
Stage/Semi-Stable Extension

Given an AF \((A, R)\). A set \(S \subseteq A\) is \textit{stage} (resp. \textit{semi-stable}) in \(F\), if

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Example

\[
\begin{align*}
\text{cf}(F) &= \{\emptyset, \{a\}, \{b\}, \{c\}\} \\
\text{adm}(F) &= \{\emptyset\} \\
\text{stage}(F) &= \{\{a\}, \{b\}, \{c\}\} \\
\text{semi}(F) &= \{\emptyset\}
\end{align*}
\]
Decision Problems on AFs

Let be $\sigma$ a semantic for AFs then we are interested in the following problems:

- **Credulous Acceptance** ($\text{Cred}_\sigma$): Given AF $F = (A, R)$ and $a \in A$; is $a$ contained in at least one $\sigma$-extension of $F$?

- **Skeptical Acceptance** ($\text{Skept}_\sigma$): Given AF $F = (A, R)$ and $a \in A$; is $a$ contained in every $\sigma$-extension of $F$?

Theorem ([Dunne and Caminada(2008)])

$\text{Cred}_\sigma$ and $\text{Skept}_\sigma$ are $\text{P}$-NP-hard.

$\text{Cred}_\sigma$ is $\Sigma_p^p$-easy.

$\text{Skept}_\sigma$ is $\Pi_p^p$-easy.
Decision Problems on AFs

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**Theorem ( [Dunne and Caminada(2008)] )**

$\text{Cred}_{\text{semi}}$ and $\text{Skept}_{\text{semi}}$ are $\text{P}^\text{NP}_{\|}$-hard.  
$\text{Cred}_{\text{semi}}$ is $\Sigma^p_2$-easy.  
$\text{Skept}_{\text{semi}}$ is $\Pi^p_2$-easy.
Complexity of stage / semi-stable semantics

Theorem ( [Dvořák and Woltran(2009)] )

Cred for stage / semi-stable semantics is $\Sigma^p_2$-complete.
Skept for stage / semi-stable semantics is $\Pi^p_2$-complete.
Complexity of stage / semi-stable semantics

Theorem ( [Dvořák and Woltran(2009)] )

Cred for stage / semi-stable semantics is $\Sigma^p_2$-complete.
Skept for stage / semi-stable semantics is $\Pi^p_2$-complete.

Proof membership.

Credulous Acceptance of $a \in A$
- Guess a set $S$ such that $a \in S$.
- Verify that $S$ is conflict-free (admissible)
- Verify that $S$ is $\subseteq^+$-maximal (in co-NP)
  - Guess a set $S'$ such that $S^+ \subset S'^+$
  - Test if $S'$ is conflict-free (admissible)
3. Complexity of Stage / Semi-Stable Semantics

Complexity of stage / semi-stable semantics

Theorem ( [Dvořák and Woltran(2009)] )

Cred for stage / semi-stable semantics is $\Sigma^p_2$-complete.
Skept for stage / semi-stable semantics is $\Pi^p_2$-complete.

Proof membership.

co-Skeptical Acceptance of $a \in A$

- Guess a set $S$ such that $a \notin S$.
- Verify that $S$ is conflict-free (admissible)
- Verify that $S$ is $\subseteq^+$-maximal (in co-NP)
  - Guess a set $S'$ such that $S^+ \subset S'^+$
  - Test if $S'$ is conflict-free (admissible)
To prove the hardness we reduce the $\Pi^p_2$-hard problem $\text{QSAT}^{\forall}_2$ to Skept.

**Definition (QSAT$^{\forall}_2$)**

Given: A quantified boolean formula in CNF: $\Phi = \forall Y \exists Z \Psi(Y, Z)$.
Question: Is $\Phi$ true?

Example:

$$\forall y_1, y_2 \exists z_3, z_4 (y_1 \lor y_2 \lor z_3) \land (\neg y_2 \lor \neg z_3 \lor \neg z_4) \land (\neg y_1 \lor \neg y_2 \lor z_4)$$
To prove the hardness we reduce the \( \Pi_2^p \)-hard problem \( \text{QSAT}^{\forall}_2 \) to Skept.

**Definition (\( \text{QSAT}^{\forall}_2 \))**

Given: A quantified boolean formula in CNF: \( \Phi = \forall Y \exists Z \, \Psi(Y, Z) \).

Question: Is \( \Phi \) true?

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\]

In our reduction

- we map each formula to \( \Phi \) to an AF \( F_\Phi \) and an argument \( t \in F_\Phi \)
- such that \( \Phi \) is true iff \( t \) is skeptically accepted in \( F_\Phi \).
3. Complexity of Stage / Semi-Stable Semantics

Reduction (informal)

We first demonstrate our reduction on an example QBF:

$$\forall y_1, y_2 \exists z_3, z_4 \ (y_1 \lor y_2 \lor z_3) \land (\neg y_2 \lor \neg z_3 \lor \neg z_4) \land (\neg y_1 \lor \neg y_2 \lor z_4)$$

The resulting framework $F_\Phi$:
Reduction (formal)

Given a $QBF^2_\forall$ formula $\Phi = \forall Y \exists Z \bigwedge_{c \in C} c$, we define $F_\Phi = (A, R)$, where

$$A = \{t, \bar{t}, b\} \cup C \cup Y \cup \bar{Y} \cup Y' \cup \bar{Y}' \cup Z \cup \bar{Z}$$

$$R = \{\langle c, t \rangle \mid c \in C\} \cup \{\langle x, \bar{x} \rangle, \langle \bar{x}, x \rangle \mid x \in Y \cup Z\} \cup \{\langle y, y' \rangle, \langle \bar{y}, \bar{y}' \rangle, \langle y', y' \rangle, \langle \bar{y}', \bar{y}' \rangle \mid y \in Y\} \cup \{\langle l, c \rangle \mid \text{literal } l \text{ occurs in } c \in C\} \cup \{\langle t, \bar{t} \rangle, \langle \bar{t}, t \rangle, \langle t, b \rangle, \langle b, b \rangle\}.$$
Lemma

For every stage (resp. semi-stable) extension $S$ of an AF $F_{\Phi} = (A, R)$:

1. $b \notin S$, as well as $y' \notin S$ and $\bar{y}' \notin S$ for each $y \in Y$.
2. $x \notin S \iff \bar{x} \in S$ for each $x \in \{t\} \cup Y \cup Z$. 

Proof.

ad 1) clear, since all this arguments attack themselves

ad 2) Obviously $\{x, \bar{x}\} \subseteq S$ cannot hold ($S$ is conflict-free).

Let us assume there exists an $x$, such that $\{x, \bar{x}\} \cap S = \emptyset$.

If $x = t$ then $T = S \cup \{\bar{t}\}$ is conflict-free and we have $S^+ \subseteq T^+$.

Further $T$ is admissible if $S$ is.

If $x \in Y \cup Z$ then we define $T = (S \setminus \{c \in C | \langle \bar{x}, c \rangle \in R\}) \cup \{\bar{x}\}$.

Once more we have that $T$ is conflict-free and that $T$ is admissible if $S$ is.

For the removed arguments $c \in C$, we have $c \in T^+$.

The only argument attacked by such $c$ is $t$, but $t \in T^+$, since we can already assume $\{t, \bar{t}\} \cap S \neq \emptyset$.

Therefore we have $S^+ \subseteq T^+$. 

Lemma

For every stage (resp. semi-stable) extension \( S \) of an AF \( F_\Phi = (A, R) \):

1. \( b \not\in S \), as well as \( y' \not\in S \) and \( \bar{y}' \not\in S \) for each \( y \in Y \).
2. \( x \not\in S \iff \bar{x} \in S \) for each \( x \in \{t\} \cup Y \cup Z \).

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Lemma

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Let us assume there exists an $x$, such that $\{x, \bar{x}\} \cap S = \emptyset$.
If $x = t$ then $T = S \cup \{\bar{t}\}$ is conflict-free and we have $S^+ \subset T^+$. Further $T$ is admissible if $S$ is. \\If $x \in Y \cup Z$ then we define $T = (S \setminus \{c \in C \mid \langle \bar{x}, c \rangle \in R\}) \cup \{\bar{x}\}$. Once more we have that $T$ is conflict-free and that $T$ is admissible if $S$ is. For the removed arguments $c \in C$, we have $c \in T^+$. The only argument attacked by such $c$ is $t$, but $t \in T^+$, since we can already assume $\{t, \bar{t}\} \cap S \neq \emptyset$. Therefore we have $S^+ \subset T^+$. \\
Lemma

Let \( Y^* = Y \cup \bar{Y} \cup Y' \cup \bar{Y}' \) and \( S, T \) be conflict-free sets then:

1. \( S \cap Y^* \subseteq T \cap Y^* \iff (S \cap Y^*)^+ \subseteq (T \cap Y^*)^+ \)
2. \( S \cap Y^* = T \cap Y^* \iff (S \cap Y^*)^+ = (T \cap Y^*)^+ \)
3. Complexity of Stage / Semi-Stable Semantics

Lemma

Let $Y^* = Y \cup \bar{Y} \cup Y' \cup \bar{Y}'$ and $S, T$ be conflict-free sets then:

1. $S \cap Y^* \subseteq T \cap Y^*$ iff $(S \cap Y^*)^+ \subseteq (T \cap Y^*)^+$
2. $S \cap Y^* = T \cap Y^*$ iff $(S \cap Y^*)^+ = (T \cap Y^*)^+$

Proof.

We first prove (1):

$\Rightarrow$: First, assume $S \cap Y^* \subseteq T \cap Y^*$.

By the monotonicity of $(.)^+$ we get $(S \cap Y^*)^+ \subseteq (T \cap Y^*)^+$. ✓

$\Leftarrow$: Assume now $(S \cap Y^*)^+ \subseteq (T \cap Y^*)^+$ and let $l \in S \cap Y^*$. ($l$ is either of form $y$ or $\bar{y}$)

As $l \in S \cap Y^*$ we have $l, \bar{l}, l' \in (S \cap Y^*)^+$ and thus $l, \bar{l}, l' \in (T \cap Y^*)^+$.

But then, $l \in T \cap Y^*$ follows from $l' \in (T \cap Y^*)^+$. ✓
Lemma

Let \( Y^* = Y \cup \bar{Y} \cup Y' \cup \bar{Y}' \) and \( S, T \) be conflict-free sets then:

1. \( S \cap Y^* \subseteq T \cap Y^* \) iff \((S \cap Y^*)^+ \subseteq (T \cap Y^*)^+\)
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Proof.

We first prove (1):

\(\Rightarrow\): First, assume \( S \cap Y^* \subseteq T \cap Y^* \).

By the monotonicity of \((.)^+\) we get \((S \cap Y^*)^+ \subseteq (T \cap Y^*)^+\). ✓

\(\Leftarrow\): Assume now \((S \cap Y^*)^+ \subseteq (T \cap Y^*)^+\) and let \( l \in S \cap Y^* \). (\( l \) is either of form \( y \) or \( \bar{y} \))

As \( l \in S \cap Y^* \) we have \( l, \bar{l}, l' \in (S \cap Y^*)^+ \) and thus \( l, \bar{l}, l' \in (T \cap Y^*)^+ \).

But then, \( l \in T \cap Y^* \) follows from \( l' \in (T \cap Y^*)^+ \). ✓

By symmetry (2) follows.
Lemma

If $\Phi$ is true, then $t$ is contained in every stage and in every semi-stable extension of $F_\Phi$. 
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If $\Phi$ is true, then $t$ is contained in every stage and in every semi-stable extension of $F_\Phi$.

Proof.
Suppose $\Phi = \forall Y \exists Z C$ is true and let $S$ be a stage or a semi-stable extension of such that $t \notin S$. Let $I_Y = Y \cap S$. Since $\Phi$ is true we know there exists an $I_Z \subseteq Z$, such that for each $c \in C$ holds:

$$
(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}) \cap c \neq \emptyset.
$$
Lemma

If $\Phi$ is true, then $t$ is contained in every stage and in every semi-stable extension of $F_\Phi$.

Proof.

Suppose $\Phi = \forall Y \exists Z C$ is true and let $S$ be a stage or a semi-stable extension of such that $t \notin S$. Let $I_Y = Y \cap S$. Since $\Phi$ is true we know there exists an $I_Z \subseteq Z$, such that for each $c \in C$ holds:

$$(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)} \} \cap c \neq \emptyset.$$  

Consider now the set

$$T = I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)} \} \cup \{t\}.$$  

$T$ is admissible and $T^+ = A \setminus I_Y'$. As $S \cap I_Y' = \emptyset$ and $b \notin S^+$ this implies $S^+ \subseteq T^+ \notin$. 

Hardness - Skeptical Acceptance Semi-Stable

**Theorem**

\[ \text{Skept}_{\text{semi}} \text{ is } \Pi^p_2 \text{-hard.} \]
Hardness - Skeptical Acceptance Semi-Stable

Theorem

Skept_{semi} is $\Pi_2^p$-hard.

We have to show that $t$ is contained in all semi-stable extensions of $F\Phi$ iff $\Phi$ is true. (The if direction is already captured by the last lemma)
Hardness - Skeptical Acceptance Semi-Stable

**Theorem**

\( \text{Skept}_{\text{semi}} \) is \( \Pi^p_2 \)-hard.

We have to show that \( t \) is contained in all semi-stable extensions of \( F_\Phi \) iff \( \Phi \) is true. (The if direction is already captured by the last lemma)

**Proof.**

We prove the only-if direction by showing that if \( \Phi \) is false, then there exists a semi-stable extension \( S \) of \( F_\Phi \) such that \( t \not\in S \).

In case \( \Phi \) is false, there exists an \( I_Y \subseteq Y \), such that for each \( I_Z \subseteq Z \), there exists a \( c \in C \), such that

\[
(I_Y \cup I_Z \cup \{ \bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z) \}) \cap c = \emptyset. \tag{1}
\]

Consider now a maximal (wrt. \( \leq^+ \)) admissible (in \( F_\Phi \)) set \( S \), such that \( I_Y \subseteq S \). \( S \) then has to be a semi-stable extension.
proof (ctd).

Consider now a maximal (wrt. \( \leq^+ \)) admissible (in \( F_\Phi \)) set \( S \), such that \( I_Y \subseteq S \). \( S \) then has to be a semi-stable extension.

It remains to show \( t \not\in S \). We prove this by contradiction and assume \( t \in S \).

As \( S \) is admissible, \( S \) defends \( t \) and therefore it defeats all \( c \in C \).

Further as all attacks against \( C \) come from \( Y \cup \tilde{Y} \cup Z \cup \tilde{Z} \), the set

\[
U = (I_Y \cup (S \cap (Z \cup \tilde{Z}))) \cup \{ \tilde{y} \mid y \in Y \setminus I_Y \}
\]

defeats all \( c \in C \).

As we know that for each \( z \in Z \), either \( z \) or \( \tilde{z} \) is contained in \( S \). We get an equivalent characterization for \( U \) by

\[
U = (I_Y \cup I_Z \cup \{ \tilde{x} \mid x \in (Y \cup Z) \setminus (I_Y \cap I_Z) \}) \text{ with } I_Z = S \cap Z.
\]

Thus, for all \( c \in C \),

\[
(I_Y \cup I_Z \cup \{ \tilde{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z) \}) \cap c \neq \emptyset,
\]

which contradicts assumption (1). \( \square \)
Hardness - Skeptical Acceptance under Stage Semantics

**Theorem**

$\text{Skept}_{\text{stage}}$ is $\Pi^p_2$-hard.
Hardness - Skeptical Acceptance under Stage Semantics

Theorem

\[\text{Skept}_{\text{stage}} \text{ is } \Pi^p_2\text{-hard.}\]

Proof.

Similar to the proof of the previous theorem. For details see [Dvořák and Woltran(2009)].
Hardness - Credulous Acceptance

Theorem

Credulous acceptance for stage or semi-stable semantics is $\Sigma^p_2$-hard.
Hardness - Credulous Acceptance

Theorem

*Credulous acceptance for stage or semi-stable semantics is $\Sigma_2^P$-hard.*

Proof.

We have shown that a $QBF_\forall^2$ formula $\Phi$ is true iff $t$ is contained in each semi-stable extension of $F_\Phi$. This is equivalent to $\bar{t}$ is not contained in any semi-stable extension of $F_\Phi$. Thus the co-credulous acceptance is also $\Pi_2^P$-hard.
Fixed-Parameter-tractability

Stage and Semi-stable Extensions can be specified in MSOL:

\[
U \subseteq^+_R V = \forall x \left( (x \in U \lor \exists y (y \in U \land \langle y, x \rangle \in R)) \rightarrow \right.
\]
\[
\left( x \in V \lor \exists y (y \in V \land \langle y, x \rangle \in R) \right)
\]

\[
U \subseteq^+_R V = U \subseteq^+_R V \land \neg (V \subseteq^+_R U)
\]

\[
cf_R(U) = \forall x, y \left( \langle x, y \rangle \in R \rightarrow (\neg x \in U \lor \neg y \in U) \right)
\]

\[
adm_R(U) = cf_R(U) \land \forall x, y \left( \langle x, y \rangle \in R \land y \in U \rightarrow \right.
\]
\[
\exists z (z \in U \land \langle z, x \rangle \in R) \right)
\]

\[
semi_{(A,R)}(U) = adm_R(U) \land \neg \exists V (V \subseteq A \land adm_R(V) \land U \subseteq^+_R V)
\]

\[
stage_{(A,R)}(U) = cf_R(U) \land \neg \exists V (V \subseteq A \land cf_R(V) \land U \subseteq^+_R V)
\]
Fixed-Parameter-Tractability

Stage and Semi-stable Extensions can be specified in MSOL:

\[
\begin{align*}
U \subseteq^+ R V &= \forall x \left( (x \in U \lor \exists y (y \in U \land \langle y, x \rangle \in R)) \rightarrow (x \in V \lor \exists y (y \in V \land \langle y, x \rangle \in R)) \right) \\
U \subset^+ R V &= U \subseteq^+ R V \land \neg (V \subseteq^+ R U) \\
\text{cf}_R(U) &= \forall x, y (\langle x, y \rangle \in R \rightarrow (\neg x \in U \lor \neg y \in U)) \\
\text{adm}_R(U) &= \text{cf}_R(U) \land \forall x, y (\langle x, y \rangle \in R \land y \in U) \rightarrow \exists z (z \in U \land \langle z, x \rangle \in R) \\
\text{semi}_{(A,R)}(U) &= \text{adm}_R(U) \land \neg \exists V (V \subseteq A \land \text{adm}_R(V) \land U \subset^+ R V) \\
\text{stage}_{(A,R)}(U) &= \text{cf}_R(U) \land \neg \exists V (V \subseteq A \land \text{cf}_R(V) \land U \subset^+ R V)
\end{align*}
\]

By Courcelles theorem the problems $\text{Cred}_{\text{semi}}$, $\text{Skept}_{\text{semi}}$, $\text{Cred}_{\text{stage}}$, $\text{Skept}_{\text{stage}}$ are fixed parameter tractable wrt tree-width of $\text{AF}$. 
Definition (cycle rank)

An acyclic graph has \( cr(G) = 0 \).
If \( G \) is strongly connected then \( cr(G) = 1 + \min_{v \in V_G} cr(G \setminus v) \).
Otherwise, \( cr(G) \) is the maximum cycle rank among all strongly connected components of \( G \).
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Theorem

The problems Skept$_{semi}$, Skept$_{stage}$ (resp. Cred$_{semi}$, Cred$_{stage}$) remain $\Pi^p_2$-hard (resp. $\Sigma^p_2$-hard), even if restricted to AFs which have a cycle-rank of 1.
Fixed-Parameter-tractability

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**Theorem**

The problems $\text{Skept}_{\text{semi}}, \text{Skept}_{\text{stage}}$ (resp. $\text{Cred}_{\text{semi}}, \text{Cred}_{\text{stage}}$) remain $\Pi^p_2$-hard (resp. $\Sigma^p_2$-hard), even if restricted to AFs which have a cycle-rank of 1.

**Proof.**

Every framework of the form $F_\Phi$ has cycle-rank 1 and therefore we have an reduction from $QBF^2_\forall$ formulas to an AF with cycle-rank 1.
Main Results:

- We answered two questions about the complexity of semi-stable semantics raised by Dunne and Caminada (2008).
  - Cred_{semi} is $\Sigma_2^p$-complete / Skept_{semi} is $\Pi_2^p$-complete
- We extended this results to stage semantics:
  - Cred_{stage} is $\Sigma_2^p$-complete / Skept_{stage} is $\Pi_2^p$-complete
- But these problems are tractable on AFs of bounded tree-width.
Main Results:

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- But these problems are tractable on AFs of bounded tree-width.

Future Work:

- Finding tractable algorithms for AFs of bounded tree-width.
- Identify further tractable fragments.
Paul E. Dunne and Martin Caminada.  
Computational complexity of semi-stable semantics in abstract argumentation frameworks.  

Wolfgang Dvořák and Stefan Woltran.  
Technical note: Complexity of stage semantics in argumentation frameworks.  