Optimal Insurance Coverage

A DM, whose wealth is threatened by a potential damage (loss) has to make the decision whether or not or rather to what extent to insure his wealth against that potential loss, i.e. what part of the risk he deals with should he optimally transfer on a different subject, the so called insurer [if there is one at all].

In other words the DM simply seeks to find the optimal insurance coverage, which will maximize the value of his value function, representing his individual preferences!

Note that this problem is in fact very similar to the portfolio optimization problem! Eventually, it may be considered as a special case of portfolio optimization!
Consider a DM who is endowed with initial wealth denoted by $w_0$.

Eventually, with probability $p$, the DM’s wealth may be reduced by a loss denoted by $L$. i.e. with probability, $1-p$, no loss will occur.

The DM can insure himself, **fully** or **partially**, against the potential loss, i.e. he can choose the amount of **insurance coverage**, denoted by $\beta$, which is defined as the weight of the **indemnification payment** (the compensation for the loss which the DM receives from the insurer) relative to the **realized loss**.

The respective **insurance premium** is denoted by $\pi$. Initially, we assume $\pi$ to be fair, i.e. it equals the expected value of the indemnification payment.
In order to remain rather general, we assume the DM to follow the EU criterion. To be precise, the DM seeks to maximize expected utility of his final wealth, by choosing a particular level of insurance coverage, i.e. a particular value of $\beta$.

Formally, the objective function is represented by the expected utility of DM’s final wealth, and the only decision variable is $\beta$. Accordingly, the DM seeks to find the optimal level of insurance coverage $\beta^*$.

Furthermore, it is reasonable to assume the following boundaries for $\beta$:

$$0 \leq \beta \leq 1$$
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The potential loss can be expressed by the following lottery:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$p(x_i)$</th>
</tr>
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<tbody>
<tr>
<td>L</td>
<td>p</td>
</tr>
<tr>
<td>0</td>
<td>1-p</td>
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</tbody>
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Final wealth in the respective two states of nature, denoted by $Z_L$ and $Z_N$, can be expressed as:

$$Z_L = w_f|_{\bar{x}=L} = w_0 - p\beta L - L + \beta L$$

$$Z_N = w_f|_{\bar{x}=0} = w_0 - p\beta L$$
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Since there are only two states of nature and hence only two possible outcomes of final wealth, it is possible [and also very convenient] to depict the whole feasible set [in dependence of varying the value of $\beta$] in a 2D-diagram.

On the x-axis we will graph the state $Z_N$, i.e. the amount of final wealth in case that $x = 0$.

On the y-axis we will graph the state $Z_L$, i.e. the amount of final wealth in case that $x = L$. 
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Note that line AB represents the feasible set of the underlying optimization problem.

Whereas point A represents the case $\beta = 1$ and point B the case $\beta = 0$.

It can simply be shown that varying the value of $\beta$ the resulting points $[Z_N, Z_L]$ form exactly the AB line. Mathematically line AB is expressed by equation AB of the following form:

$$Z_L = -\frac{1-p}{p}Z_N + \frac{w_0 - pL}{p}$$
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**Proof:** We know that for every value $\beta = c$ the following holds:

\[
Z_N(\beta = c) = w_0 - pcL
\]
\[
Z_L(\beta = c) = w_0 - pcL - L + cL
\]

Plugging the expressions for $Z_N$ and $Z_L$ for an arbitrary value $\beta = c$ in the equation AB the latter will be satisfied, i.e. any value pair $[Z_N, Z_L]$ obtained by varying the value of $\beta$ lies exactly on line AB; and for any point on AB a corresponding value of $\beta$ can be found.

Hence line (equation) AB represents the feasible set of the underlying optimization problem!
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As already mentioned the DM follows the **EU criterion** and hence he seeks to maximize the expected utility of his final wealth:

\[ \max_{\beta} \{ \mathbb{E}[U(\tilde{w}_f)] = \mathbb{E}[U(w_0 - \beta \mathbb{E}(\tilde{x}) - \tilde{x} + \beta \tilde{x})] \} \]

The corresponding necessary conditions for the underlying optimization problem are the following:

\[ \frac{\partial \mathbb{E}(U)}{\partial \beta} = \mathbb{E}[U'(\tilde{w}_f)(\tilde{x} - \mathbb{E}(\tilde{x}))] = 0 \]

\[ \frac{\partial^2 \mathbb{E}(U)}{\partial \beta^2} = \mathbb{E}[U''(\tilde{w}_f)(\tilde{x} - \mathbb{E}(\tilde{x}))^2] < 0 \]
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Obviously, the second condition will be satisfied if and only if the DM is risk averse, i.e. if $U'' < 0$.

In order to derive the exact solution for $\beta^*$ based on the first order condition one must know the exact (particular) form of the utility function. However some important properties of the optimal solution can be well analyzed also in general!

Since it is a 2-dimensional problem it is quite convenient to derive the solution graphically. For this purpose it is necessary to draw the isolines of the objective function. The graphical solution will also offer us some additional intuition of the underlying problem!

What about the analytical solution?
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The objective function can be rewritten in the following way:

\[ E[U(w_f)] = pU(Z_L) + (1-p)U(Z_N) \]

Using this information the slope of \( Z_L \) in dependence of \( Z_N \) for a fixed expected utility \( \bar{U} \), in other words the slope of the corresponding isoline at \( E(U) = \bar{U} \), can simply be derived as:

\[
\frac{\partial Z_L}{\partial Z_N} \bigg|_{E[U(w_f)]=\bar{U}} = -\frac{\frac{\partial E(U)}{\partial Z_N}}{\frac{\partial E(U)}{\partial Z_L}} = -\frac{1-p}{p} \frac{U'(Z_N)}{U'(Z_L)}
\]
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The optimum must be represented by the **tangency point** of the isolines on line AB. In other words the **slope** of the corresponding **isoline** passing through the optimal point must exactly at that point be **equal** to the slope of **line AB**:

\[
\frac{1-p}{p} \frac{U'(Z_N)}{U'(Z_L)} = \frac{1-p}{p}
\]

This will only be the case if

\[
U'(Z_N) = U'(Z_L)
\]

If U is monotonic the optimum is represented by the point located on line AB where \(Z_N = Z_L\) which is in fact **point A**, i.e. \(\beta^* = 1\).
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This basically means that a *risk averse* DM will *optimally* buy *full insurance coverage* if the *insurance premium* is *fair*!

This is in fact not really surprising, since in case of a fair insurance premium the *expected final wealth* stays *constant* with each additional unit of insurance coverage, while *risk decreases*.

Thus the obtained result $\beta^* = 1$ is fully reasonable and intuitive for a risk averse DM.

But let us now ask the following question: *What will happen [to the optimal insurance coverage] if the insurance premium is not fair?*
Let us now assume that the insurer requires additionally to the fair insurance premium an administration fee $\lambda$, expressed as a weight in terms of the fair insurance premium $\pi$.

Accordingly the final wealth can then be expressed in the following way:

$$\tilde{w}_f = w_0 - (1 + \lambda) \beta E(\tilde{x}) - \tilde{x} + \beta \tilde{x}$$

In the two possible states $w_f$ will take the following respective values:

$$Z_N = w_f \bigg|_{x=0} = w_0 - (1 + \lambda) p \beta L$$

$$Z_L = w_f \bigg|_{x=L} = w_0 - (1 + \lambda) p \beta L - L + \beta L$$
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[Unfair Premium]

In this case line AB (A: \( \beta = 1 \); B: \( \beta = 0 \)) will have the following slope:

\[
k = -\frac{1-(1+\lambda)p}{(1+\lambda)p} \iff k = 1 - \frac{1}{(1+\lambda)p}
\]

It is obvious that:

\[
\frac{\partial k}{\partial \lambda} > 0
\]

Thus the following must be true:

\[
k(\lambda > 0) > k(\lambda = 0)
\]
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[Unfair Premium]
Optimal Insurance Coverage
[Unfair Premium]

Accordingly the tangency condition in case \( \lambda > 0 \) says:

\[
- \frac{1-p}{p} U'(Z_N) = - \frac{1-(1+\lambda)p}{(1+\lambda)p} > - \frac{1-p}{p}
\]

This will only be the case if:

\[ U'(Z_N) < U'(Z_L) \]

Thus if \( \lambda > 0 \) the optimum of a risk averse DM (\( U'' < 0 \)) must satisfy \( Z_N > Z_L \), i.e. \( \beta^* < 1 \).

As long as \( \lambda \) does not exceed a particular upper bound it will also hold that \( \beta^* > 0 \).
Optimal Insurance Coverage
[Unfair Premium]

This means that a risk averse DM will optimally not buy full insurance coverage if the insurance premium is not fair i.e. if the insurer requires any additional fees; which in reality is always the case!

If the amount of additional fees is not too high a risk averse DM will in optimum always buy partial insurance coverage $0 < \beta^* < 1$.

Challenging questions:

How would the optimal decision of a risk seeking DM look like?
How would the optimal decision of a risk neutral DM look like?