Interactions of holes in two dimensional dimer systems

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Let $L$ be a subset of the hexagonal lattice.
A *dimer* on $L$ is a pair of adjacent vertices, joined by precisely one edge.
A dimer covering on $L$ is a set of dimers that cover every vertex of $L$ exactly once.
A dimer covering on $L$ containing two holes.
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A hexagon, $H_{n,m}$. 
A set of triangles, $T$, contained within $H_{n,m}$. 
The holey hexagon $H_{n,m} \setminus T$. 
**Definition**

Given a hexagon $H_{n,n}$ and a set of triangles $T$, the *interaction* (or *correlation function*) of the holes is defined to be

$$\omega(T) = \lim_{n \to \infty} \frac{M(H_{n,n} \setminus T)}{M(H_{n,n})},$$

where $M(R)$ denotes the total number of rhombus tilings of the region $R$.

**Conjecture (M. Ciucu, 2008)**

The asymptotic interaction of a set of holes $T$ within a sea of dimers is governed (up to a multiplicative constant) by Coulomb’s law for two dimensional electrostatics.
Coulomb’s Law

The magnitude of the electrostatic force $F$ between two point charges ($q_1$ and $q_2$), each with a signed magnitude, is given by

$$|F| = k_e \frac{|q_1 q_2|}{d^2},$$

where $k_e$ is Coulomb’s constant.
If $T$ denotes the above pair of triangles then according to Ciucu’s conjecture

$$\omega(T) \sim C \cdot \frac{1}{d^2}.$$ 

**Theorem (TG)**

The interaction, $\omega(T)$, between two inward pointing triangular holes of side length two within a sea of dimers is asymptotically

$$\left( \frac{\sqrt{3}}{2\pi d} \right)^2.$$
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Matchings Factorisation Theorem (M. Ciucu)

$$M(H_{n,n}) = 2^l \cdot M(H_{n,n}) \cdot M(w(H_{n,n}))$$
$T = \{\uparrow_r, \downarrow_l\}$
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Matchings Factorisation Theorem (M. Ciucu)

\[ M(H_{n,n} \setminus T) = 2^l \cdot M(H_{n,n}^{-} \setminus T) \cdot M_{\text{w}}(H_{n,n}^{+} \setminus T) \]
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Matchings Factorisation Theorem (M. Ciucu)

\[ M(H_{n,n} \setminus T) = 2^l \cdot M(H_{n,n} \setminus T) \cdot M_{w}(H_{n,n} \setminus T). \]
**Interactions of holes in two dimensional dimer systems**

\[ H_{n,n}^+ \setminus T \]

\[ H_{n,n}^- \setminus T \]

\[
\text{Matchings Factorisation Theorem (M. Ciucu)}
\]

\[
M(H_{n,n}^+ \setminus T) = 2^l \cdot M(H_{n,n}^- \setminus T) \cdot M_{\text{weak}}(H_{n,n}^+ \setminus T).
\]
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\[ H_{n,n}^+ \setminus T \]

\[ n/2 \]

\[ H_{n,n}^- \setminus T \]

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Matchings Factorisation Theorem (M. Ciucu)

\[ M(H_{n,n} \setminus T) = 2^l \cdot M(H_{n,n}^- \setminus T) \cdot M_w(H_{n,n}^+ \setminus T). \]
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\[ M(H_{n,n}^\perp \setminus T) = \mathcal{P}(V \rightarrow W), \]

where \( \mathcal{P}(V \rightarrow W) \) denotes the set of non-intersecting paths starting at a set of points \( V \) and ending at a set of points \( W \) where

\[
V = \{(i, 1-i) : 1 \leq i \leq \frac{n}{2}\} \cup \{(1 + \frac{n+l}{2}, \frac{n+l}{2})\},
\]

\[
W = \{(n+j, n+1-j) : 1 \leq j \leq \frac{n}{2}\} \cup \{(1 + \frac{n+r}{2}, \frac{n+r}{2})\}
\]

and such that no path crosses the line \( y = x \).
**Theorem (Lindström-Gessel-Viennot)**

The number of non-intersecting paths that begin at $V$ and end at $W$ is given by $|\det(G)|$, where the matrix $G = (g_{i,j})_{1 \leq i,j \leq n/2+1}$ has $(i,j)$-entry $g_{i,j} = \mathcal{P}(V_i \rightarrow W_j)$.

**Proposition**

$$M(H_{n,n}^- \setminus T) = |\det(G)|$$

where $G = (g_{i,j})_{1 \leq i,j \leq n/2+1}$ is the $(n/2 + 1) \times (n/2 + 1)$ matrix with $(i,j)$-entries given by

$$g_{i,j} = \begin{cases} 
\binom{2n}{n+j-i} - \binom{2n}{n+j-1+i}, & 1 \leq i,j \leq n/2 \\
\binom{n-l}{n/2-l/2+j-1} - \binom{n-l}{n/2-l/2-j}, & i = n/2 + 1, 1 \leq j \leq n/2 \\
\binom{n+r}{n/2+r/2+1-i} - \binom{n+r}{n/2+r/2-i}, & j = n/2 + 1, 1 \leq i \leq n/2 \\
\binom{r-l}{r/2-l/2} - \binom{r-l}{r/2-l/2-1}, & i = j = n/2 + 1.
\end{cases}$$
Theorem (TG)

The positive determinant of the matrix $G$, which counts rhombus tilings of $H_{n,n}^- \setminus T$, is given by

$$\begin{vmatrix} \left( \frac{r-l}{r/2-l/2} \right) - \left( \frac{r-l}{r/2-l/2-1} \right) - \sum_{s=1}^{n/2} B_{n,l}(s) \cdot D_{n,r}(s) \end{vmatrix}$$

$$\times \left( \frac{n/2}{\prod_{i=1}^{n/2} (2i-1)! (2i+2n-2)!} \right) \left( \frac{(2i-1)! (2i+3n-3)!}{(2i+n-2)! (2i+n-1)!} \right),$$

where

$$B_{n,l}(j) = \frac{(-1)^{j+1} (j+n-2)! (2j+n-1)! (n-l)! (j+l/2+n-2)!}{2(j-1)! (2j+2n-3)! (n-l)! (j+1/2+n-2)! (j-l+1/2+n-2)!},$$

$$D_{n,r}(i) = \frac{(-1)^{i+1} (2i)! (i+n-1)! (n+r)! (i+1/2+n-r-2)!}{2(i)! (2i+n-2)! (n-r-2)! (i+1/2+n-r)! (i+1/2+n+r)!}.$$
Rate

Rate is a Mathematica package written by Christian Krattenthaler to guess closed form expressions for generic terms in a sequence. It is available at http://www.mat.univie.ac.at/~kratt/rate/rate.html

Example

In[1]:= <<rate2.m
In[2]:= Rate[1,2,6,20,70]
   -1 + i0
   ----- 2 (-1 + 2 i1)
Out[2]= { | | -------------}
      | | i1
      i1=1

It can be shown that this product is equal to $\binom{2n}{n}$. 
Proposition

\[ G = L \cdot U, \]

where \( L = (l_{i,j})_{1 \leq i,j \leq m+1} \) is the matrix given by

\[
l_{i,j} = \begin{cases} 
A_n(i, j), & 1 \leq j \leq i \leq m \\
B_n,i(j), & i = m + 1, j \in \{1, \ldots, m\} \\
1, & i = j = m + 1, \\
0 & \text{otherwise},
\end{cases}
\]

and \( U = (u_{i,j})_{1 \leq i,j \leq m+1} \) is given by

\[
u_{i,j} = \begin{cases} 
C_n(i, j), & 1 \leq i \leq j \leq m, \\
D_{n,i}(i), & j = m + 1, i \in \{1, \ldots, m\}, \\
\left( \frac{r-l}{r/2-l/2} \right) - \left( \frac{r-l}{r/2-l/2-1} \right) - \sum_{s=1}^{n/2} B_n,i(s) \cdot D_{n,r}(s), & i = j = m + 1.
\end{cases}
\]
Doodle of Proof

In order to prove this proposition it must be shown that

\[
\min(i,j) \sum_{s=1}^{\min(i,j)} l_{i,s} \cdot u_{s,j} = g_{i,j},
\]

for all \(1 \leq i, j \leq m + 1\). For example, it must be shown that

\[
\min(i,j) \sum_{s=1}^{\min(i,j)} A_n(i,s) \cdot C_n(s,j) = \binom{2n}{n+j-i} - \binom{2n}{n+j-1+i},
\]

where

\[
A_n(i,s) = \frac{(2i-1)!n!(i+s-2)!(n+2s-1)!}{(2i-2)!(2s-1)!(i-s)!(-i+n+s)!(i+n+s-1)!},
\]

\[
C_n(s,j) = \frac{(2j-1)!n!(j+s-2)!(2n+2s-2)!}{(2j-2)!(j-s)!(n+2s-2)!(-j+n+s)!(j+n+s-1)!}.
\]
Zeilberger’s (fast) Algorithm

- Generates recurrences for sums of hypergeometric terms
- An excellent Mathematica implementation is available from RISC:

Example

\textbf{In}[1]:= \texttt{<<fastZeil.m;}
\textbf{In}[2]:= \texttt{Zb[Binomial[m,j]Binomial[n-m,k-j],\{j,0,k\},k,1]}

If ‘k’ is a natural number, then:
\textbf{Out}[2]= \{(-k + n) \text{SUM}[k] + (-1 - k) \text{SUM}[1 + k] == 0\}
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If ‘k’ is a natural number, then:
Out[2]= {(-k + n) SUM[k] + (-1 - k) SUM[1 + k] == 0}

It can be easily checked that \((n - k)\binom{n}{k} - (k + 1)\binom{n}{k+1} = 0\), thus verifying the Chu-Vandermonde Identity.
Zeilberger’s algorithm is used to prove that $G = L \cdot U$.

Since $l_{i,i} = 1$ for $i \in \{1, \ldots, m+1\}$, this decomposition is unique.

The formula follows from considering the product of the diagonal entries of $U$, that is,

$$\det(G) = \prod_{s=1}^{m+1} u_{s,s}.$$ 

Precisely the same approach may be used to determine the formula that counts weighted tilings of the region $H_{n,n}^+ \setminus T$.

Combining the two enumeration results by way of Ciucu’s Factorisation Theorem, one obtains an enumerative formula for the total number of tilings of $H_{n,n} \setminus T$. 
Theorem (TG)

The total number of tilings of $H_{n,n} \setminus T$ is

\[
\left| \begin{pmatrix} r - l \\ r/2 - l/2 \end{pmatrix} - \begin{pmatrix} r - l \\ r/2 - l/2 - 1 \end{pmatrix} - \sum_{s=1}^{n/2} B_{n,l}(s) \cdot D_{n,r}(s) \right| \\
\times \left| \begin{pmatrix} r - l \\ r/2 - l/2 \end{pmatrix} + \begin{pmatrix} r - l \\ r/2 - l/2 - 1 \end{pmatrix} - \sum_{s=1}^{n/2} B'_{n,l}(s) \cdot D'_{n,r}(s) \right| \\
\times M(H_{n,n}),
\]

with $B_{n,l}(s)$ and $D_{n,r}(s)$ as before and

\[
B'_{n,l}(s) = \frac{(-1)^{s+1}(-l+n+1)!(n+s-1)!(n+2s-1)!\left(\frac{l}{2} + \frac{n}{2} + s - 2\right)!}{(s-1)!\left(\frac{n}{2} - \frac{l}{2}\right)!\left(\frac{l}{2} + \frac{n}{2} - 1\right)!\left(2n+2s-1\right)!\left(-\frac{l}{2} + \frac{n}{2} + s\right)!}
\]

\[
D'_{n,r}(s) = \frac{(-1)^{s+1}(2s-2)!(n+r+1)!(n+s-1)!\left(\frac{n}{2} - \frac{r}{2} + s - 2\right)!}{(s-1)!\left(\frac{n}{2} - \frac{r}{2} - 1\right)!\left(\frac{n}{2} + \frac{r}{2}\right)!\left(n+2s-2\right)!\left(\frac{n}{2} + \frac{r}{2} + s\right)!}.
\]
Interaction

According to the definition of the correlation function, the interaction between the holes in $H_{n,n \setminus T}$, denoted $\omega_H(r,l)$, is given by

$$\lim_{n \to \infty} \left| \begin{array}{c} \binom{r-l}{r/2 - l/2} - \binom{r-l}{r/2 - l/2 - 1} - \sum_{s=1}^{n/2} B_{n,l}(s) \cdot D_{n,r}(s) \\ \binom{r-l}{r/2 - l/2} + \binom{r-l}{r/2 - l/2 - 1} - \sum_{s=1}^{n/2} B'_{n,l}(s) \cdot D'_{n,r}(s) \end{array} \right|$$
Interaction

According to the definition of the correlation function, the interaction between the holes in $H_{n,n} \setminus T$, denoted $\omega_H(r,l)$, is given by

$$\lim_{n \to \infty} \left| \begin{pmatrix} r - l \\ r/2 - l/2 + 1 \end{pmatrix} - \begin{pmatrix} r - l \\ r/2 - l/2 - 1 \end{pmatrix} - \sum_{s=1}^{n/2} B_{n,l}(s) \cdot D_{n,r}(s) \right|$$

$$\times \left| \begin{pmatrix} r - l \\ r/2 - l/2 - 1 \end{pmatrix} + \begin{pmatrix} r - l \\ r/2 - l/2 + 1 \end{pmatrix} - \sum_{s=1}^{n/2} B'_{n,l}(s) \cdot D'_{n,r}(s) \right|$$

HYP

HYP is a do-it-yourself Mathematica package, also written by Christian Krattenthaler, that allows one to manipulate and transform hypergeometric series. It is available here: http://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html
Interaction

The finite sums consisting of hypergeometric terms in $\omega_H(r, l)$ may be written as limits of hypergeometric series, for example

$$\sum_{s=1}^{n/2} B_{n,l}(s) \cdot D_{n,r}(s) = \lim_{\epsilon \to 0} \left( \sum_{s=1}^{\infty} B_{n,l}(s) \cdot D_{n,r}(s) \frac{(-n/2)_s}{(-n/2 + \epsilon)_s} \right).$$

Manipulating these hypergeometric series using the HYP package, it may be shown that

$$\omega_H(r, l) \sim \frac{3}{4\pi^2d(r, l)^2},$$

where $d(r, l)$ is the Euclidean distance between the triangular holes.
Further Results

Using similar methods it is possible to show that the interaction between a right pointing triangular hole and a free boundary that borders a sea of lozenges on the right is asymptotically

\[ \frac{3}{4\pi r'} . \]
Thank you.