Interactions of holes in two dimensional dimer systems

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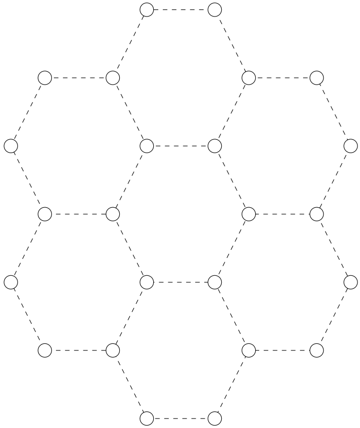
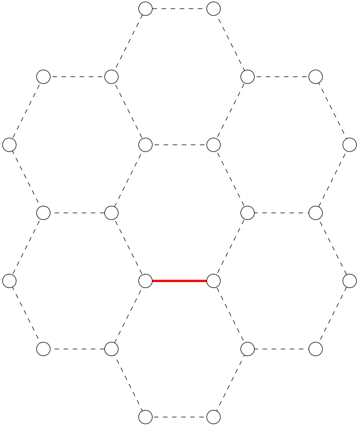
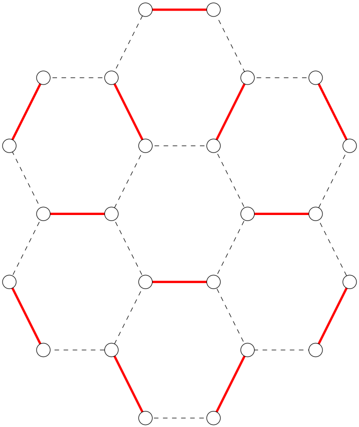


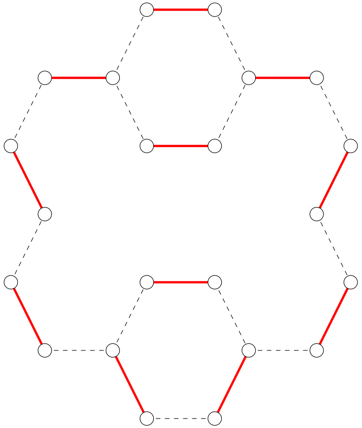
Let L be a subset of the hexagonal lattice.



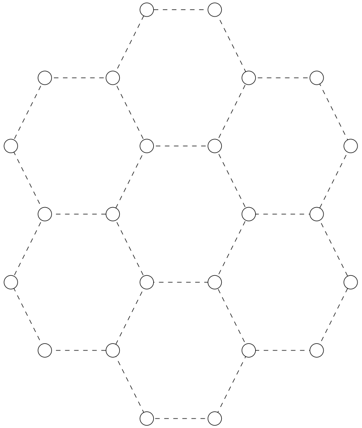
A *dimer* on L is a pair of adjacent vertices, joined by precisely one edge.

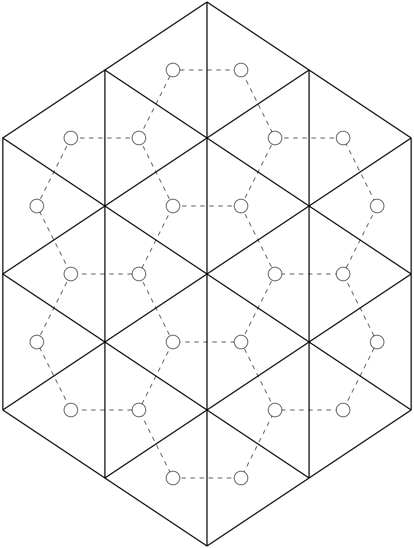


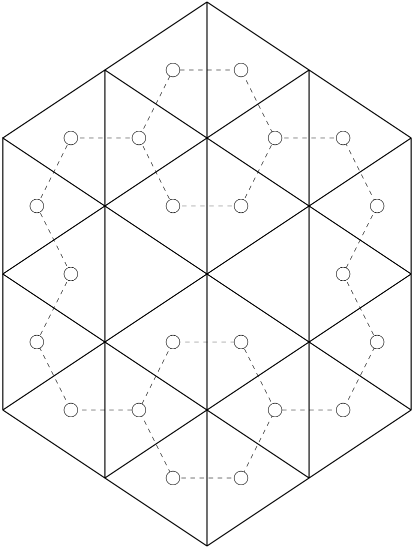
A *dimer covering* on L is a set of dimers that cover every vertex of L exactly once.

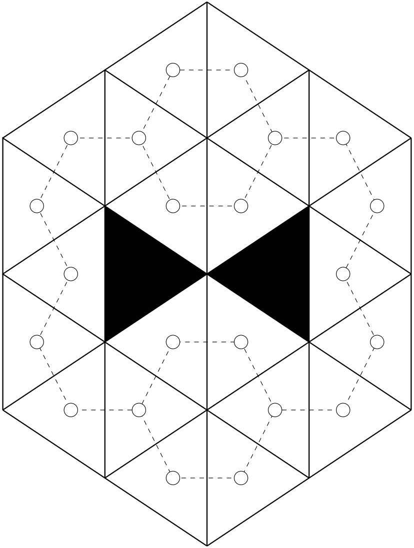


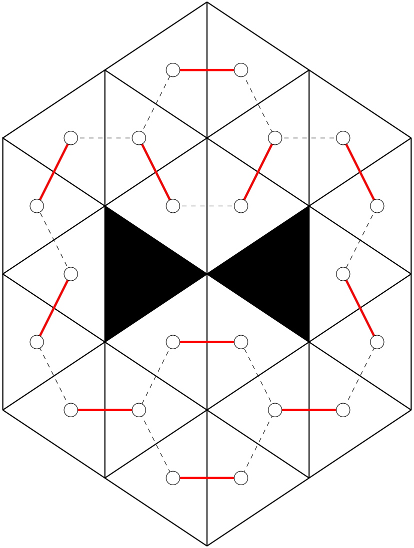
A dimer covering on L containing two holes.

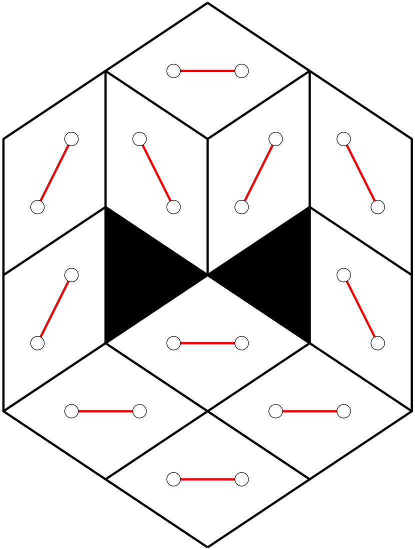


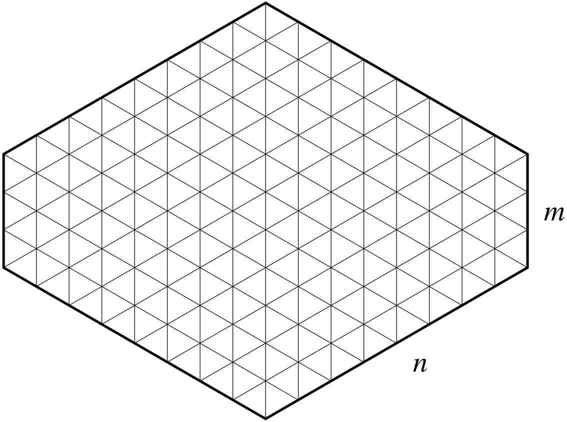




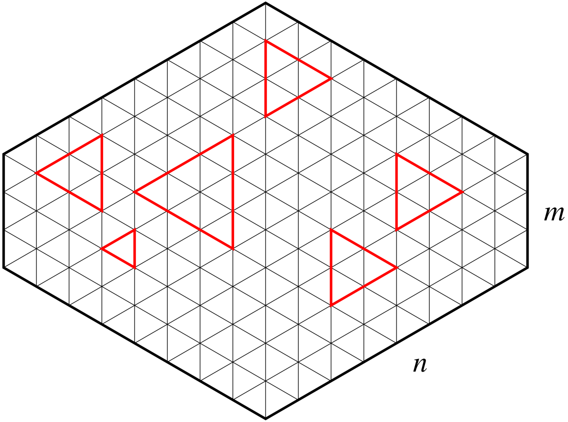




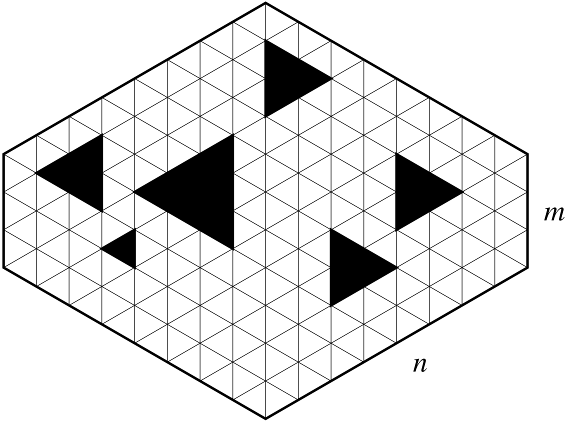




A hexagon, $H_{n,m}$.



A set of triangles, T, contained within $H_{n,m}$.



The holey hexagon $H_{n,m} \setminus T$.

Definition

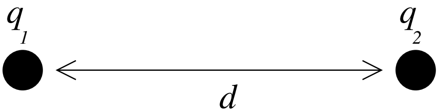
Given a hexagon $H_{n,n}$ and a set of triangles T, the *interaction* (or *correlation function*) of the holes is defined to be

$$\omega(T) = \lim_{n\to\infty} \frac{M(H_{n,n} \setminus T)}{M(H_{n,n})},$$

where $M(R)$ denotes the total number of rhombus tilings of the region R.

Conjecture (M. Ciucu, 2008)

The asymptotic interaction of a set of holes T within a sea of dimers is governed (up to a multiplicative constant) by Coulomb's law for two dimensional electrostatics.

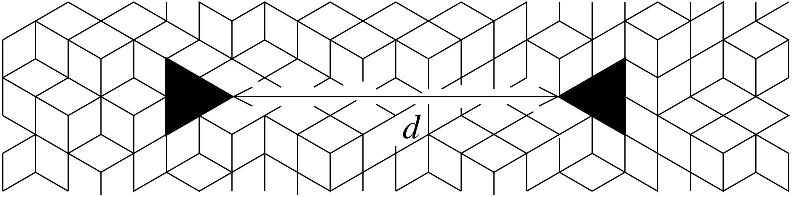


Coulomb's Law

The magnitude of the electrostatic force F between two point charges (q_1 and q_2), each with a signed magnitude, is given by

$$|F| = k_e \frac{|q_1 q_2|}{d^2},$$

where k_e is Coulomb's constant.



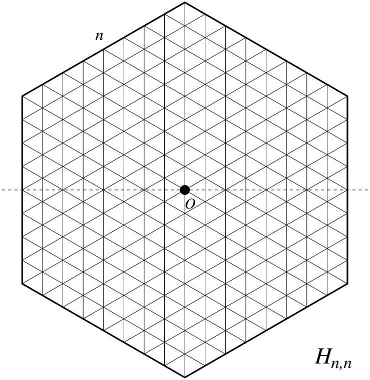
If T denotes the above pair of triangles then according to Ciucu's conjecture

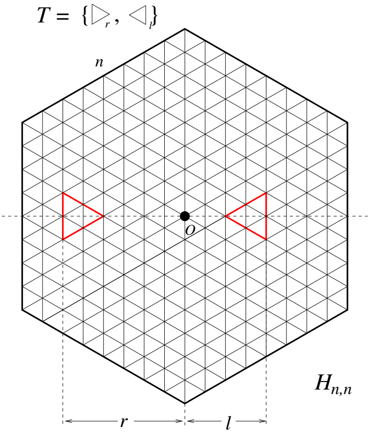
$$\omega(T) \sim C \cdot \frac{1}{d^2}.$$

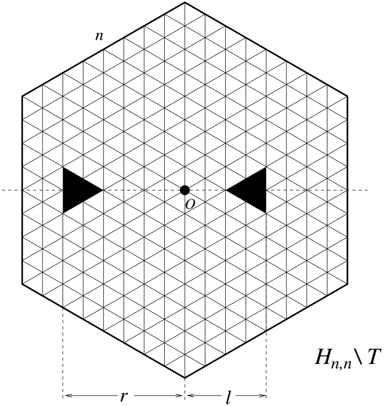
Theorem (TG)

The interaction, $\omega(T)$, between two inward pointing triangular holes of side length two within a sea of dimers is asymptotically

$$\left(\frac{\sqrt{3}}{2\pi d}\right)^2.$$


n
O
$H_{n,n}$


T = {▷r , ◁l}
n
O
Hn,n
r
l


n
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$H_{n,n} \setminus T$
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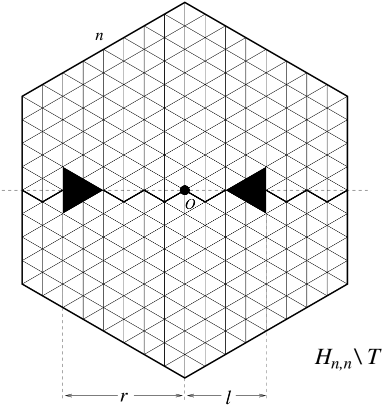
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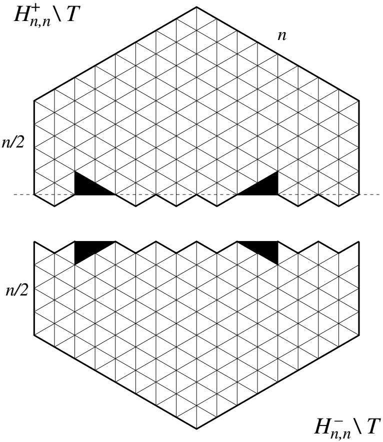
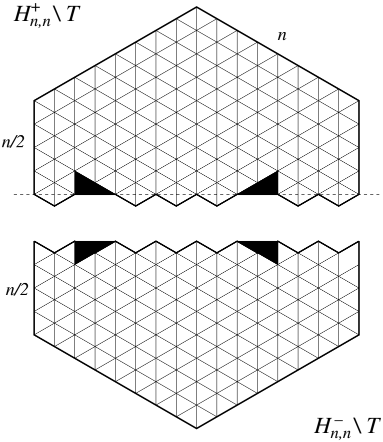
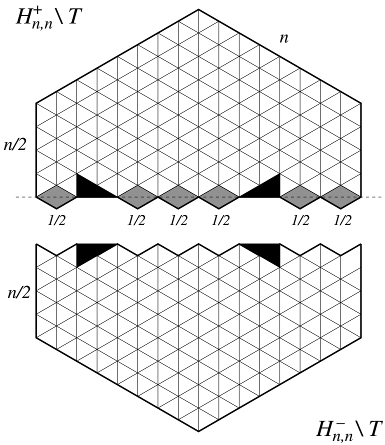

$H^+_{n,n} \setminus T$
n
n/2
n/2
$H^-_{n,n} \setminus T$



Matchings Factorisation Theorem (M. Ciucu)

$$M(H_{n,n} \setminus T) = 2^{l} \cdot M(H^{-}_{n,n} \setminus T) \cdot M_{w}(H^{+}_{n,n} \setminus T).$$



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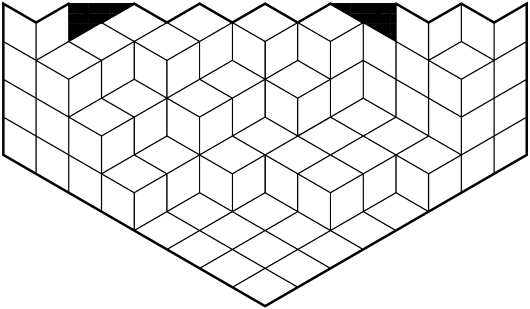

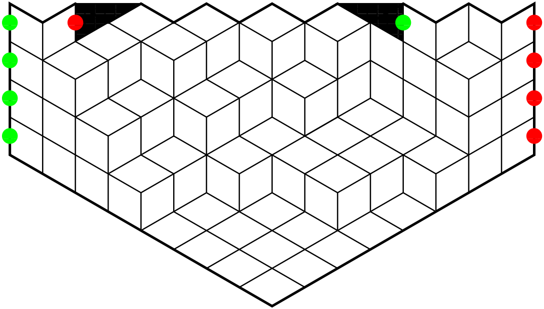

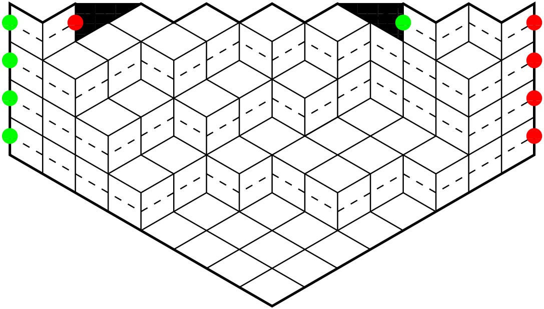

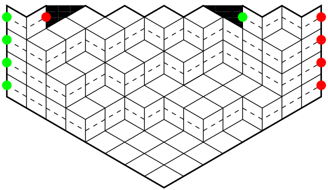

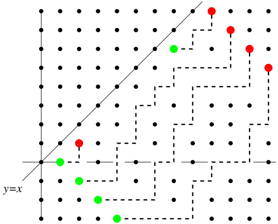

$$M(H^{-}_{n,n} \setminus T) = \mathscr{P}(V \to W),$$

where $\mathscr{P}(V \to W)$ denotes the set of non-intersecting paths starting at a set of points V and ending at a set of points W where

$$V = \{(i, 1-i) : 1 \leq i \leq \tfrac{n}{2}\} \cup \{(1 + \tfrac{n+l}{2}, \tfrac{n+l}{2})\},$$
$$W = \{(n+j, n+1-j) : 1 \leq j \leq \tfrac{n}{2}\} \cup \{(1 + \tfrac{n+r}{2}, \tfrac{n+r}{2})\}$$

and such that no path crosses the line $y = x$.

Theorem (Lindström-Gessel-Viennot)

The number of non-intersecting paths that begin at V and end at W is given by $|\det(G)|$, where the matrix $G = (g_{i,j})_{1 \le i,j \le n/2+1}$ has (i,j)-entry $g_{i,j} = \mathscr{P}(V_i \to W_j)$.

Proposition

$$M(H^{-}_{n,n} \setminus T) = |\det(G)|$$

where $G = (g_{i,j})_{1 \le i,j \le n/2+1}$ is the $(n/2+1) \times (n/2+1)$ matrix with (i,j)-entries given by

$$g_{i,j} = \begin{cases} \binom{2n}{n+j-i} - \binom{2n}{n+j-1+i}, & 1 \le i,j \le n/2 \\ \binom{n-l}{n/2-l/2+j-1} - \binom{n-l}{n/2-l/2-j}, & i = n/2+1, 1 \le j \le n/2 \\ \binom{n+r}{n/2+r/2+1-i} - \binom{n+r}{n/2+r/2-i}, & j = n/2+1, 1 \le i \le n/2 \\ \binom{r-l}{r/2-l/2} - \binom{r-l}{r/2-l/2-1}, & i = j = n/2+1. \end{cases}$$

Theorem (TG)

The positive determinant of the matrix G, which counts rhombus tilings of $H^{-}_{n,n} \setminus T$, is given by

$$\left| \binom{r-l}{r/2-l/2} - \binom{r-l}{r/2-l/2-1} - \sum_{s=1}^{n/2} B_{n,l}(s) \cdot D_{n,r}(s) \right| \times \left(\prod_{i=1}^{n/2} \frac{(2i-1)!(2i+2n-2)!}{(2i+n-2)!(2i+n-1)!} \right),$$

where

$$B_{n,l}(j) = \frac{(-1)^{j+1}(j+n-2)!(2j+n-1)!(n-l)!\left(j+\frac{l}{2}+\frac{n}{2}-2\right)!}{2(j-1)!(2j+2n-3)!\left(\frac{n}{2}-\frac{l}{2}\right)!\left(\frac{l}{2}+\frac{n}{2}-1\right)!\left(j-\frac{l}{2}+\frac{n}{2}\right)!},$$

$$D_{n,r}(i) = \frac{(-1)^{i+1}(2i)!(i+n-1)!(n+r)!\left(i+\frac{n}{2}-\frac{r}{2}-2\right)!}{2(i!)(2i+n-2)!\left(\frac{n}{2}-\frac{r}{2}-1\right)!\left(\frac{n}{2}+\frac{r}{2}\right)!\left(i+\frac{n}{2}+\frac{r}{2}\right)!}.$$

Rate

`Rate` is a Mathematica package written by Christian Krattenthaler to guess closed form expressions for generic terms in a sequence. It is available at
`http://www.mat.univie.ac.at/~kratt/rate/rate.html`

Example

```
In[1]:= <<rate2.m
In[2]:= Rate[1,2,6,20,70]
           -1 + i0
            ----- 2 (-1 + 2 i1)
Out[2]= {  | |   -------------}
           | |        i1
           i1=1
```

It can be shown that this product is equal to $\binom{2n}{n}$.

Proposition

$$G = L \cdot U,$$

where $L = (l_{i,j})_{1 \le i,j \le m+1}$ is the matrix given by

$$l_{i,j} = \begin{cases} A_n(i,j), & 1 \le j \le i \le m \\ B_{n,l}(j), & i = m+1, j \in \{1,\ldots,m\} \\ 1, & i = j = m+1, \\ 0 & otherwise, \end{cases}$$

and $U = (u_{i,j})_{1 \le i,j \le m+1}$ is given by

$$u_{i,j} = \begin{cases} C_n(i,j), & 1 \le i \le j \le m, \\ D_{n,r}(i), & j = m+1, i \in \{1,\ldots,m\}, \\ \binom{r-l}{r/2-l/2} - \binom{r-l}{r/2-l/2-1} \\ \quad - \sum_{s=1}^{n/2} B_{n,l}(s) \cdot D_{n,r}(s), & i = j = m+1. \end{cases}$$

Doodle of Proof

In order to prove this proposition it must be shown that

$$\sum_{s=1}^{\min(i,j)} l_{i,s} \cdot u_{s,j} = g_{i,j},$$

for all $1 \leq i, j \leq m+1$. For example, it must be shown that

$$\sum_{s=1}^{\min(i,j)} A_n(i,s) \cdot C_n(s,j) = \binom{2n}{n+j-i} - \binom{2n}{n+j-1+i},$$

where

$$A_n(i,s) = \frac{(2i-1)!n!(i+s-2)!(n+2s-1)!}{(2i-2)!(2s-1)!(i-s)!(-i+n+s)!(i+n+s-1)!},$$

$$C_n(s,j) = \frac{(2j-1)!n!(j+s-2)!(2n+2s-2)!}{(2j-2)!(j-s)!(n+2s-2)!(-j+n+s)!(j+n+s-1)!}.$$

Zeilberger's (fast) Algorithm

- Generates recurrences for sums of hypergeometric terms
- An excellent Mathematica implementation is available from RISC:
  `http://www.risc.jku.at/research/combinat/software/ergosum/RISC/fastZeil.html`

Example

```
In[1]:= <<fastZeil.m;
In[2]:= Zb[Binomial[m,j]Binomial[n-m,k-j],{j,0,k},k,1]
If ‘k’ is a natural number, then:
Out[2]= {(-k + n) SUM[k] + (-1 - k) SUM[1 + k] == 0}
```

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```

It can be easily checked that $(n-k)\binom{n}{k} - (k+1)\binom{n}{k+1} = 0$, thus verifying the Chu-Vandermonde Identity.

- Zeilberger's algorithm is used to prove that $G = L \cdot U$.
- Since $l_{i,i} = 1$ for $i \in \{1, \ldots, m+1\}$, this decomposition is unique.
- The formula follows from considering the product of the diagonal entries of U, that is,

$$\det(G) = \prod_{s=1}^{m+1} u_{s,s}.$$

- Precisely the same approach may be used to determine the formula that counts weighted tilings of the region $H^+_{n,n} \setminus T$.
- Combining the two enumeration results by way of Ciucu's Factorisation Theorem, one obtains an enumerative formula for the total number of tilings of $H_{n,n} \setminus T$.

Theorem (TG)

The total number of tilings of $H_{n,n} \setminus T$ *is*

$$\left| \binom{r-l}{r/2-l/2} - \binom{r-l}{r/2-l/2-1} - \sum_{s=1}^{n/2} B_{n,l}(s) \cdot D_{n,r}(s) \right|$$
$$\times \left| \binom{r-l}{r/2-l/2} + \binom{r-l}{r/2-l/2-1} - \sum_{s=1}^{n/2} B'_{n,l}(s) \cdot D'_{n,r}(s) \right|$$
$$\times M(H_{n,n}),$$

with $B_{n,l}(s)$ *and* $D_{n,r}(s)$ *as before and*

$$B'_{n,l}(s) = \frac{(-1)^{s+1}(-l+n+1)!(n+s-1)!(n+2s-1)!\left(\frac{l}{2}+\frac{n}{2}+s-2\right)!}{(s-1)!\left(\frac{n}{2}-\frac{l}{2}\right)!\left(\frac{l}{2}+\frac{n}{2}-1\right)!(2n+2s-1)!\left(-\frac{l}{2}+\frac{n}{2}+s\right)!}$$
$$D'_{n,r}(s) = \frac{(-1)^{s+1}(2s-2)!(n+r+1)!(n+s-1)!\left(\frac{n}{2}-\frac{r}{2}+s-2\right)!}{(s-1)!\left(\frac{n}{2}-\frac{r}{2}-1\right)!\left(\frac{n}{2}+\frac{r}{2}\right)!(n+2s-2)!\left(\frac{n}{2}+\frac{r}{2}+s\right)!}.$$

Interaction

According to the definition of the correlation function, the interaction between the holes in $H_{n,n} \setminus T$, denoted $\omega_H(r,l)$, is given by

$$\lim_{n\to\infty} \left| \binom{r-l}{r/2-l/2} - \binom{r-l}{r/2-l/2-1} - \sum_{s=1}^{n/2} B_{n,l}(s)\cdot D_{n,r}(s) \right| \times \left| \binom{r-l}{r/2-l/2} + \binom{r-l}{r/2-l/2-1} - \sum_{s=1}^{n/2} B'_{n,l}(s)\cdot D'_{n,r}(s) \right|$$

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HYP

`HYP` is a do-it-yourself Mathematica package, also written by Christian Krattenthaler, that allows one to manipulate and transform hypergeometric series. It is available here:
`http://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html`

Interaction

The finite sums consisting of hypergeometric terms in $\omega_H(r,l)$ may be written as limits of hypergeometric series, for example

$$\sum_{s=1}^{n/2} B_{n,l}(s)\cdot D_{n,r}(s) = \lim_{\epsilon\to 0}\left(\sum_{s=1}^{\infty} B_{n,l}(s)\cdot D_{n,r}(s)\frac{(-n/2)_s}{(-n/2+\epsilon)_s}\right).$$

Manipulating these hypergeometric series using the `HYP` package, it may be shown that

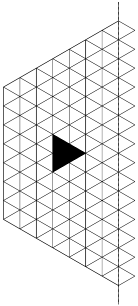
$$\omega_H(r,l) \sim \frac{3}{4\pi^2 \mathrm{d}(r,l)^2},$$

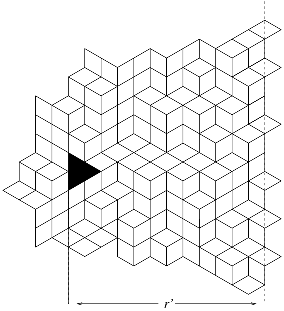
where $\mathrm{d}(r,l)$ is the Euclidean distance between the triangular holes.

Further Results

Using similar methods it is possible to show that the interaction between a right pointing triangular hole and a free boundary that borders a sea of lozenges on the right is asymptotically

$$\frac{3}{4\pi r'}.$$





Thank you.