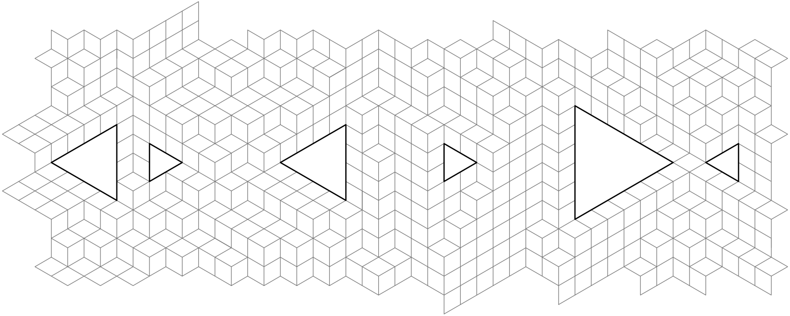
Holey matrimony: marrying two approaches to the dimer problem.



Tomack Gilmore
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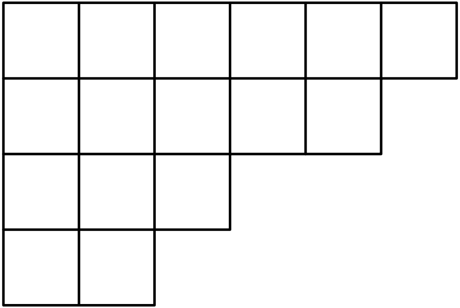
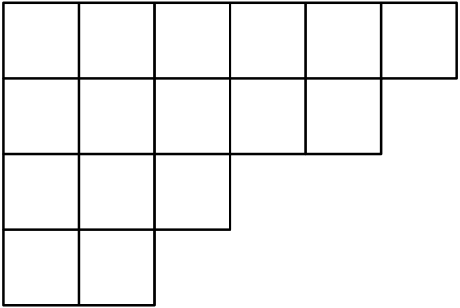




A *partition* of an integer m is a left and top justified array of m boxes where the lengths of the rows are weakly decreasing from top to bottom.

6	6	5	4	2	2
5	5	4	3	1	
5	3	3			
2	1				

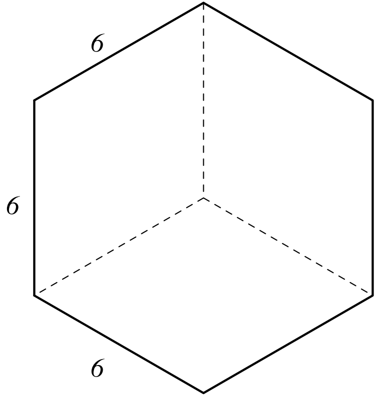
6	6	5	4	2	2
5	5	4	3	1	
5	3	3			
2	1				

A *plane partition* is a partition together with a filling of the boxes with integers that are weakly decreasing along rows and down columns.

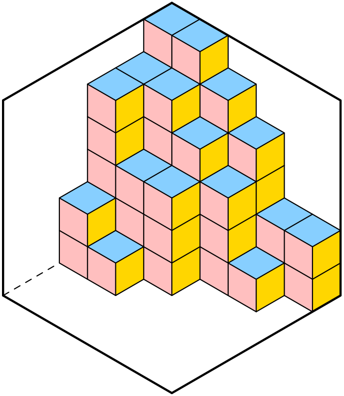
6	6	5	4	2	2
5	5	4	3	1	
5	3	3			
2	1				

An $a \times b \times c$ *boxed plane partition* is a plane partition with row and column length at most a, b respectively, and entries bounded above by c.

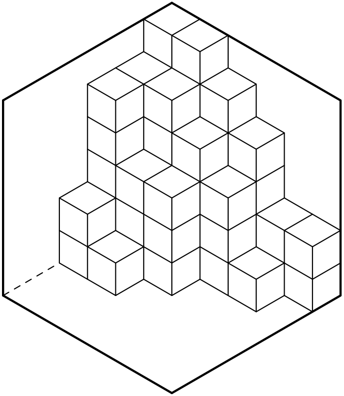
6	6	5	4	2	2
5	5	4	3	1	
5	3	3			
2	1				



6	6	5	4	2	2
5	5	4	3	1	
5	3	3			
2	1				



6	6	5	4	2	2
5	5	4	3	1	
5	3	3			
2	1				



6	6	5	4	2	2
5	5	4	3	1	
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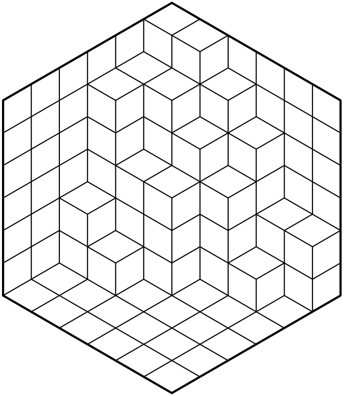
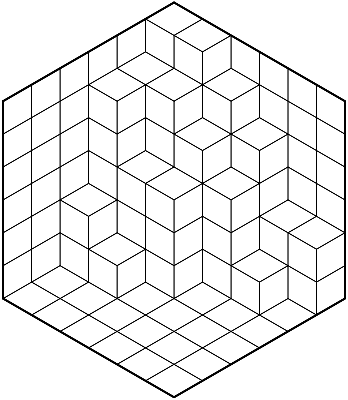
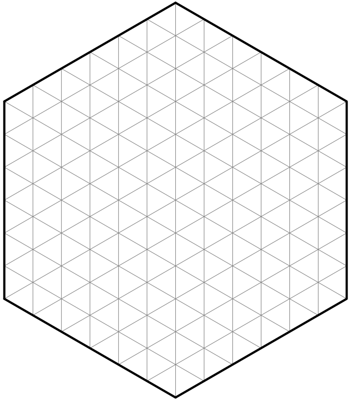
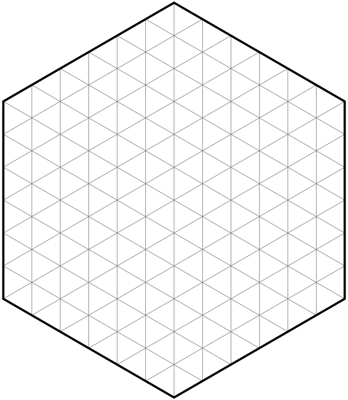






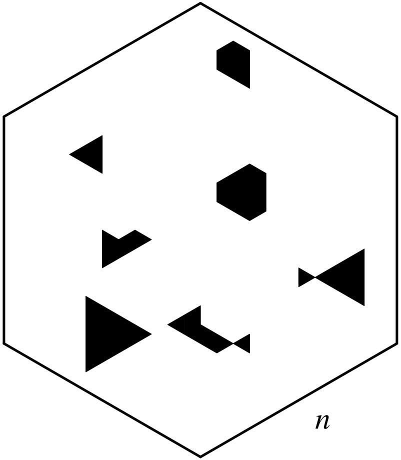
$\# n \times n \times n$ boxed plane partitions $= \#$ rhombus tilings of a regular hexagon of side length n.



MacMahon

The number of rhombus tilings of a regular hexagon of side length n is

$$\dagger(n) = \prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{n}\frac{(i+j+k-1)}{(i+j+k-2)}$$



A hexagon containing a set of holes H is referred to as a *holey hexagon*. The number of rhombus tilings of a holey hexagon is denoted $\dagger(n, H)$.

Interaction

The *interaction* between the set of holes H is given by

$$\omega_n(H) = \frac{\dagger(n, H)}{\ddagger(n)}.$$

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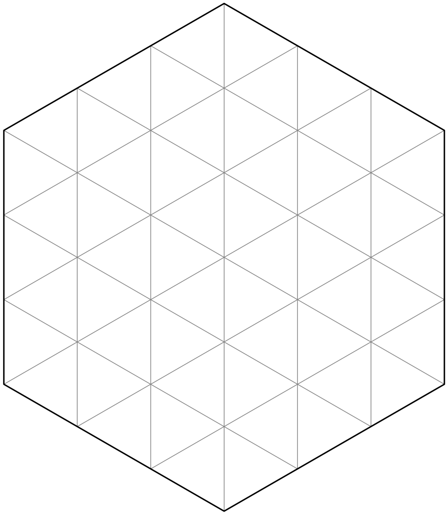
$$\omega_n(H) = \frac{\dagger(n,H)}{\rotatebox[origin=c]{180}{$\dagger$}(n)}.$$

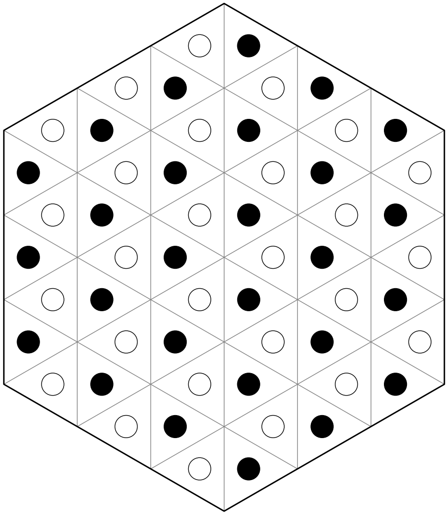
Ciucu's conjecture, '08

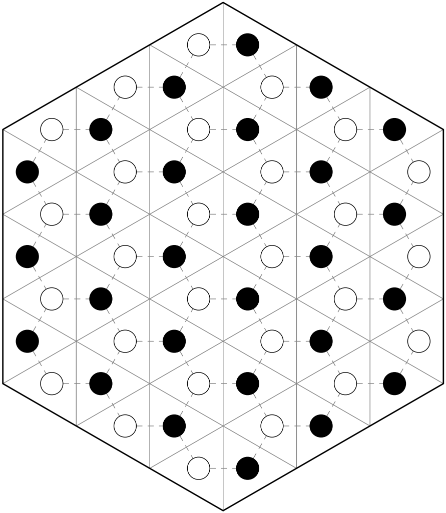
As $n \to \infty$ and the distance between the holes in H grows large,

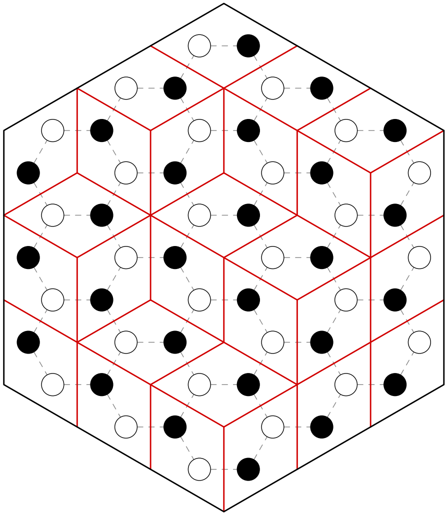
$$\omega_n(H) \sim \prod_{h \in H} C_h \prod_{1 \leq i < j \leq |H|} d(h_i, h_j)^{\frac{1}{2} q(h_i) q(h_j)},$$

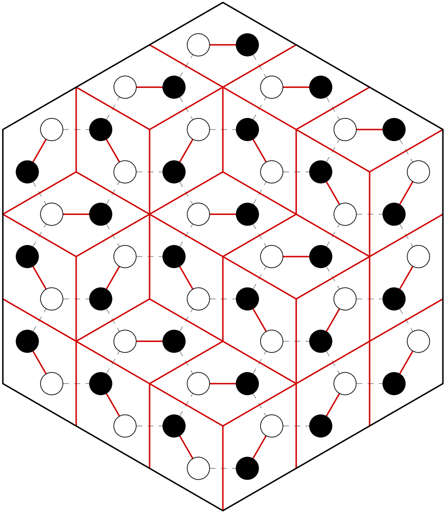
where the charge of a hole, $q(h)$, is the difference between the left and right pointing unit triangles that comprise it.

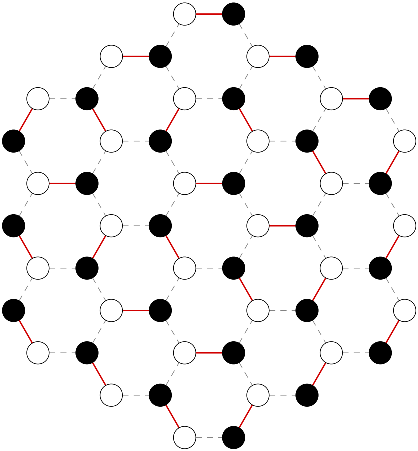












A *dimer covering* of a *hexagonal subgraph* of the hexagonal lattice.

Kasteleyn's method

A bipartite graph G together with edge weights taken from some commutative ring is considered *flat* if:

- every face that consists of $4k$ edges contains an odd number of edges with negative weight;
- every face that consists of $4k+2$ edges contains an even number of edges of even weight.

If G_Z is the bipartite graph G with a flat weighting Z then the number of (weighted) dimer coverings of G is given by $|\det(D)|$, where $D = (D_{i,j})_{b_i, w_j \in G_Z}$ is the matrix with entries given by $D_{i,j} = e(b_i, w_j)$ (that is, the weight of the edge between b_i and w_j).

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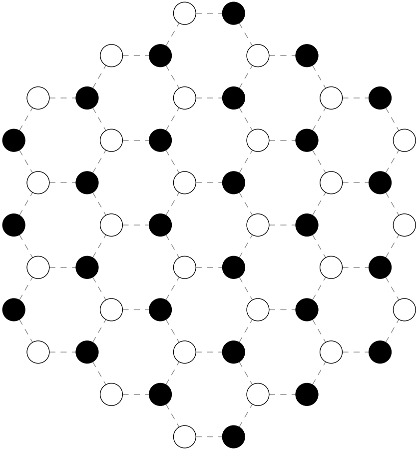
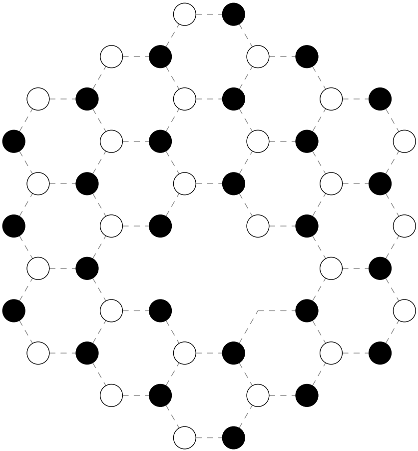
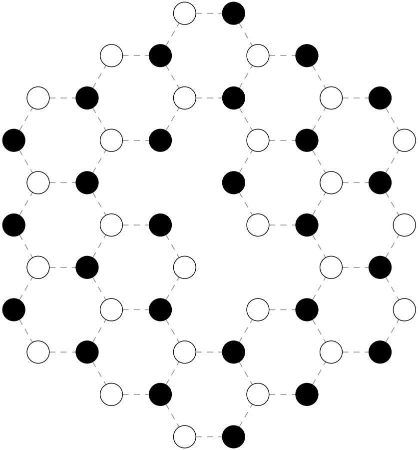
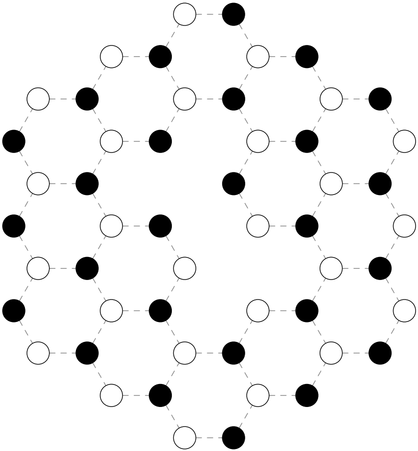
The number of dimer coverings of a hexagonal subgraph G of the hexagonal lattice, denoted $M(G)$, is the determinant of its *bi-adjacency matrix*.









A set of even holes

$E :=$ a set of vertices such that $G_Z \setminus E$ is also flat weighted.

Proposition

Let G be an hexagonal subgraph, A_G its bi-adjacency matrix, and E a set of even holes. Then

$$M(G \setminus E) = |\det(A_G)_E|,$$

where $(A_G)_E$ is the submatrix obtained by deleting the rows and columns from A_G corresponding to the black and white (resp.) vertices in E.

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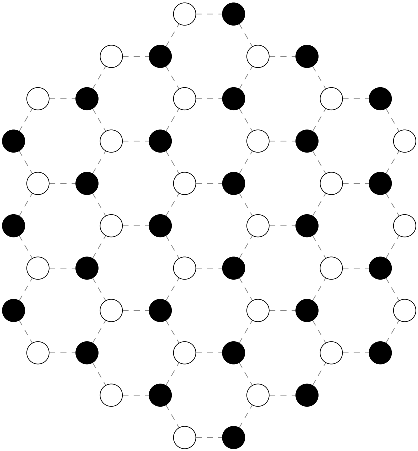
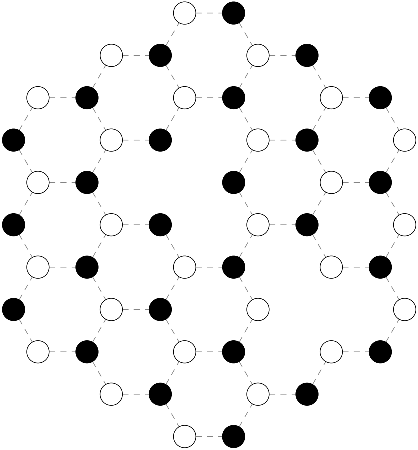
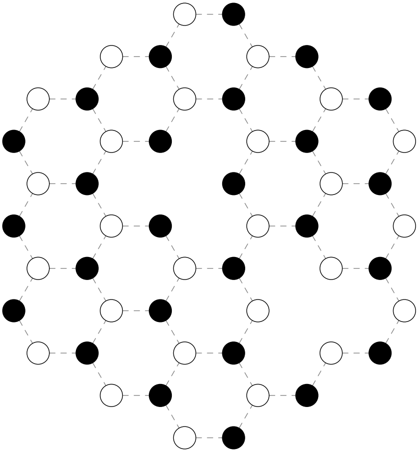
Kenyon '97

$$\det(A_G)_E = \det A_G \cdot \det((A_G)^{-1})_{E^*},$$

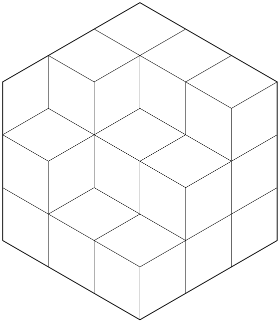
where $((A_G)^{-1})_{E^*}$ is the submatrix indexed by the rows and columns corresponding to the white and black vertices (resp.) of E.

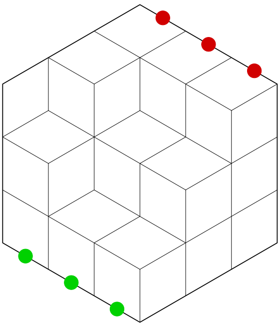


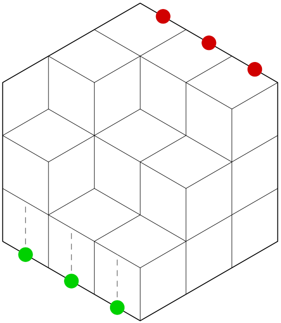


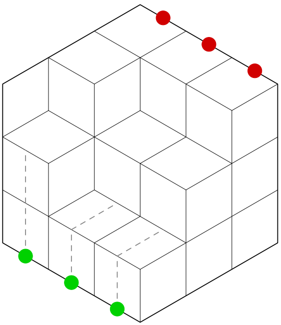


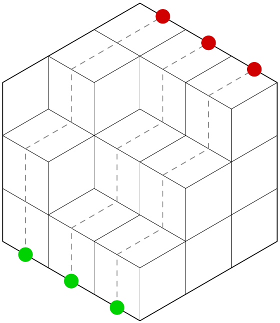
$|\det(A_G)_{\{b_j,w_i\}}|$ counts the number of *signed perfect matchings* of $G \setminus \{b_j, w_i\}$.

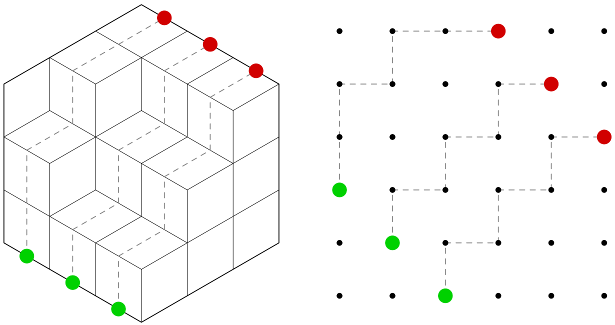


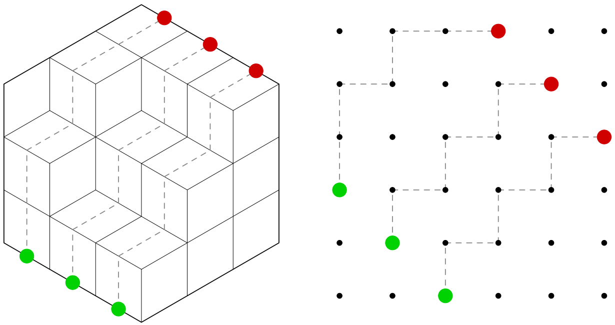






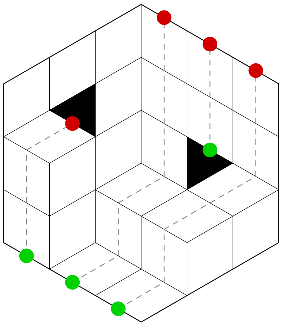


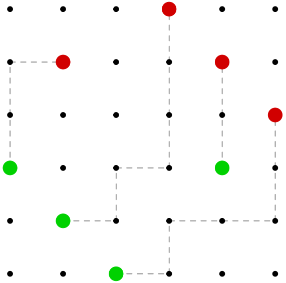


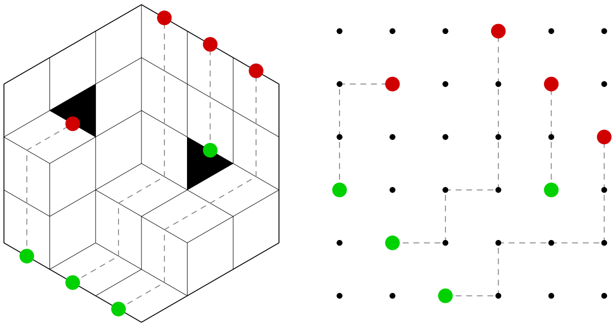


Lindström, Gessel-Viennot

Suppose S and F are *d-compatible*. The number of non-intersecting lattice paths consisting of north and east unit steps beginning at S and ending at F is given by $|\det P|$, where $P = (P_{i,j})_{1 \leq i,j \leq |S|}$ has entries given by the number of paths from the point S_i to the point F_j.







The determinant of the matrix $P = (P_{i,j})_{1 \le i,j \le |S|+1}$ whose entries are given by the number of paths from $s_i \in S \cup \{r\}$ to $f_j \in F \cup \{l\}$ is equal to $\pm$ the number of *families of signed non-intersecting paths* that begin at $S \cup \{r\}$ and end at $F \cup \{l\}$.

Cook & Nagel, '15

$$|\det(A_G)_{b_i,w_j}| = |\det P|$$

where P is the lattice path matrix with entries given by paths from $S \cup \{r\}$ to $F \cup \{l\}$, where r and l are the unit triangles corresponding to b_i and w_j.

Cook & Nagel, '15

$$|\det(A_G)_{b_i,w_j}| = |\det P|$$

where P is the lattice path matrix with entries given by paths from $S \cup \{r\}$ to $F \cup \{l\}$, where r and l are the unit triangles corresponding to b_i and w_j.

$$\det P = \dagger(n) \cdot \left(\binom{(w_j)_x + (w_j)_y - (b_i)_x - (b_i)_y}{(w_j)_x - (b_i)_x} - \sum_{t=1}^{n} \left[\left(\sum_{u=0}^{t-1} \frac{(-1)^{t-u+1}(n+t-1)!(n+u)!(-(b_i)_x-(b_i)_y+n)!(n+t-u-2)!}{(n-1)!u!\left(-(b_i)_y+u+\frac{1}{2}\right)!(2n+t-1)!(t-u-1)!\left(-(b_i)_x+n-u-\frac{1}{2}\right)!} \right) \cdot \left(\sum_{v=0}^{t-1} \frac{(t-1)!(-1)^{t-v+1}(n+v)!(n+t-v-2)!(n+(w_j)_x+(w_j)_y)!}{(n-1)!v!(n+t-1)!(t-v-1)!\left(v+(w_j)_x+\frac{1}{2}\right)!\left(n-v+(w_j)_y-\frac{1}{2}\right)!} \right) \right] \right)$$

Theorem, TG '16

For a regular holey hexagon containing a set of even holes H,

$$\begin{aligned} \dagger(n,H) &= |\det(A_G)_H| \\ &= |\det A_G \cdot \det((A_G)^{-1})_{H^*}| \\ &= |\ddagger(n) \cdot \det K|, \end{aligned}$$

where $K = (K_{i,j})_{b,w\in H}$ has entries given by

$$\binom{(w_j)_x + (w_j)_y - (b_i)_x - (b_i)_y}{(w_j)_x - (b_i)_x} -$$

$$\sum_{t=1}^{n} \left[\left(\sum_{u=0}^{t-1} \frac{(-1)^{t-u+1}(n+t-1)!(n+u)!(-(b_i)_x-(b_i)_y+n)!(n+t-u-2)!}{(n-1)!u!\left(-(b_i)_y+u+\frac{1}{2}\right)!(2n+t-1)!(t-u-1)!\left(-(b_i)_x+n-u-\frac{1}{2}\right)!} \right) \cdot \left(\sum_{v=0}^{t-1} \frac{(t-1)!(-1)^{t-v+1}(n+v)!(n+t-v-2)!(n+(w_j)_x+(w_j)_y)!}{(n-1)!v!(n+t-1)!(t-v-1)!\left(v+(w_j)_x+\frac{1}{2}\right)!\left(n-v+(w_j)_y-\frac{1}{2}\right)!} \right) \right]$$

Corollary

The interaction between the set of holes H is given by

$$\omega_n(H) = |\det K|.$$

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Open problem

What are the entries of K as $n \to \infty$? What are the asymptotics if the distances between the holes grows large?