A parking problem

Enumeration

Asymptotics

Conclusion

Counting Defective Parking Functions

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Erwin Schrödinger Institute, May 30, 2008
1. A parking problem (defective parking functions)

2. Enumeration (generating functions)

3. Asymptotics (limit distributions)

4. Conclusion and outlook
1 A parking problem (defective parking functions)

2 Enumeration (generating functions)

3 Asymptotics (limit distributions)

4 Conclusion and outlook
A parking problem

Consider \( n \) parking spaces in a one-way street.
A parking problem

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- Drivers of \( m \) cars choose their preferred parking space
A parking problem

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Drivers of \( m \) cars choose their preferred parking space

- If the space is free, the driver parks the car there
- If the space is occupied, the driver parks in the next free space
- If no free space is available, the driver leaves
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\( n^m \) sequences of choices.
A parking problem

Consider $n$ parking spaces in a one-way street.

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If all drivers park successfully, the sequence is called a parking function.
A parking problem

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\( n^m \) sequences of choices

If all drivers park successfully, the sequence is called a parking function

If \( k \) drivers fail to park, the sequence is called a defective parking function of degree \( k \)
- $n = 2$ parking spaces, $m = 2$ cars:
- $n = 2$ parking spaces, $m = 2$ cars:
  - $k = 0$: 11, 12, 21
  - $k = 1$: 22
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\( n = 2 \) parking spaces, \( m = 2 \) cars:
- \( k = 0 \): 11, 12, 21
- \( k = 1 \): 22

\( n = 3 \) parking spaces, \( m = 3 \) cars:
A parking problem

Enumeration  Asymptotics  Conclusion

$n = 2$ parking spaces, $m = 2$ cars:

- $k = 0$: 11, 12, 21
- $k = 1$: 22

$n = 3$ parking spaces, $m = 3$ cars:

- $k = 0$: 111, 112, 121, 211, 113, 131, 311, 122,
  212, 221, 123, 132, 213, 231, 312, 321
**n = 2 parking spaces, m = 2 cars:**

- $k = 0$: 11, 12, 21
- $k = 1$: 22

**n = 3 parking spaces, m = 3 cars:**

- $k = 0$: 111, 112, 121, 211, 113, 131, 311, 122, 212, 221, 123, 132, 213, 231, 312, 321
- $k = 1$: 133, 313, 331, 222, 223, 232, 322, 233, 323, 332
n = 2 parking spaces, m = 2 cars:

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k = 0: 111, 112, 121, 211, 113, 131, 311, 122, 212, 221, 123, 132, 213, 231, 312, 321
k = 1: 133, 313, 331, 222, 223, 232, 322, 233, 323, 332
k = 2: 333
- $n = 2$ parking spaces, $m = 2$ cars:
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  - $k = 1$: 133, 313, 331, 222, 223, 232, 322, 233, 323, 332
  - $k = 2$: 333

**Theorem**

*Every permutation of a defective parking function of degree $k$ is also a defective parking function of degree $k$.***
The counting problem

Enumerate the number

\[ \text{cp}(n, m, k) \]

of assignments of \( m \) drivers to \( n \) spaces such that exactly \( k \) drivers leave
The counting problem

Enumerate the number

\[ cp(n, m, k) \]

of assignments of \( m \) drivers to \( n \) spaces such that exactly \( k \) drivers leave.

The probabilistic question

What is the probability

\[ p_{n,m}(k) = \frac{1}{n^m} cp(n, m, k) \]

that for a randomly chosen assignment exactly \( k \) drivers leave? In particular, are there interesting limiting distributions?
Reference for this work:

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Related work by Alois Panholzer, TU Wien:

- “Limiting distribution results for a discrete parking problem”, GOCPS 2008 (talk presented on March 5, 2008)
- “On a discrete parking problem”, AofA 2008 (talk presented on April 17, 2008)
Many equivalent or related formulations, for example

- Hashing with linear probing
  
  [Konheim and Weiss, 1966]
  [Flajolet, Poblete and Viola, 1998]

- Drop-push model for percolation
  
  [Majumdar and Dean, 2002]
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- Hashing with linear probing
  
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- Drop-push model for percolation
  
  [Majumdar and Dean, 2002]

Connections to other combinatorial objects:

- labelled trees, major functions, acyclic functions, Prüfer code, non-crossing partitions, hyperplane arrangements, priority queues, Tutte polynomial of graphs, inversion in trees
A parking problem (defective parking functions)

Enumeration (generating functions)

Asymptotics (limit distributions)

Conclusion and outlook
### A parking problem

- **Enumeration**
- **Asymptotics**
- **Conclusion**

<table>
<thead>
<tr>
<th>cp((n, n, k))</th>
<th>(k = 0)</th>
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\( cp(n, n, k) \): the number of car parking assignments of \( n \) cars to \( n \) spaces such that \( k \) cars are not parked.
Theorem

For $m \leq n$,

$$cp(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$
Theorem

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$$\text{cp}(n, m, 0) = (n + 1 - m)(n + 1)^{m-1}$$

Proof (adapted from Pollak ($m = n$), 1974).

- Consider a circular car park with $m$ cars and $n + 1$ spaces.
Theorem

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- There are $(n + 1)^m$ choices of sequences.
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- Consider a circular car park with $m$ cars and $n + 1$ spaces.
- There are $(n + 1)^m$ choices of sequences.
- A sequence will be a parking function for the original problem if and only if space $n + 1$ is empty.
Theorem

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- Consider a circular car park with $m$ cars and $n + 1$ spaces.
- There are $(n + 1)^m$ choices of sequences.
- A sequence will be a parking function for the original problem if and only if space $n + 1$ is empty.
- This will happen with a probability given by the fraction of empty spaces $(n + 1 - m)/(n + 1)$. 
Definition

For \( r, s, k \in \mathbb{N}_0 \) let \( a(r, s, k) \) denote the number of choices for which \( r \) spaces remain empty, \( s \) spaces are occupied in the end and \( k \) people drive home.
Definition

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- \( n = s + r \) parking spaces, \( m = s + k \) drivers
- \( \text{cp}(n, m, k) = a(n - m + k, m - k, k) \)
Lemma (A recursion)

For $r, s, k \in \mathbb{N}_0$, the number of assignments of $s + k$ drivers to $s + r$ spaces such that $r$ spaces remain empty, $s$ spaces are occupied and $k$ drivers leave is recursively defined by

$$a(r, s, k) = \begin{cases} \end{cases}$$
Lemma (A recursion)

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$$a(r, s, k) = \begin{cases} 
1 & \text{if } r = s = k = 0, \\
\end{cases}$$
Lemma (A recursion)

For \( r, s, k \in \mathbb{N}_0 \), the number of assignments of \( s + k \) drivers to \( s + r \) spaces such that \( r \) spaces remain empty, \( s \) spaces are occupied and \( k \) drivers leave is recursively defined by

\[
a(r, s, k) = \begin{cases} 
1 & \text{if } r = s = k = 0, \\
\sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r, s - 1, i) & \text{if } k > 0,
\end{cases}
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\sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r - 1, s, 0) + \\
a(r - 1, s, 0) & \text{if } k = 0 \text{ and } \\
\sum_{i=0}^{k+1} \binom{s+k}{k+1-i} \cdot a(r, s - 1, i) & (r > 0 \text{ or } s > 0). 
\end{cases}$$
Lemma (A functional equation)

Let $A$ be the formal power series in the three variables $u$, $v$, and $t$ defined by

$$A(u, v, t) := \sum_{r,s,k \geq 0} a(r, s, k) \cdot u^r \frac{v^s t^k}{(s+k)!}.$$  

Then $A(u, v, t)$ is the unique solution of

$$0 = \left(\frac{v}{t} e^t - 1\right) \cdot A(u, v, t) + (u - \frac{v}{t}) \cdot A(u, v, 0) + 1$$

in the ring of formal power series in $u$, $v$ and $t$. 
Lemma (The explicit generating function)

The generating function for the car parking problem is given by

\[ A(u, v, t) = \frac{1}{1 - \frac{v}{t} e^t} + \frac{u - \frac{v}{t}}{1 - \frac{v}{t} e^t} \cdot \frac{e^{T(v)}}{1 - u e^{T(v)}}, \]

where \( T(v) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} \cdot v^i \) is the tree function \( (T(v) = v e^{T(v)}). \)
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Proof (using the Kernel Method).
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  \[ 0 = K(v, t) \cdot A(u, v, t) + (u - \frac{v}{t}) \cdot A(u, v, 0) + 1 \]

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The generating function for the car parking problem is given by

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  with the kernel \( K(v, t) = \frac{v}{t}e^t - 1 \).
- Setting the kernel equal to zero gives \( t = T(v) \) and
  \[ A(u, v, 0) = \frac{e^{T(v)}}{1 - ue^{T(v)}} . \]
After some further work (Lagrange inversion, resummation of quadruple sums) ...
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**Theorem**

*The number of car parking assignments of \( m \) cars on \( n \) spaces such that at least \( k \) cars do not find a parking space is given by*

\[
S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n-m+k) \cdot (n-m+k+i)^{i-1} \cdot (m-k-i)^{m-i}.
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- \( S(n, m, k) = \sum_{j=k}^{m} \text{cp}(n, m, j) \)
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\]

- \( S(n, m, k) = \sum_{j=k}^{m} \text{cp}(n, m, j) \)
- The car parking numbers \( \text{cp}(n, m, k) \) are given by
  \[
  \text{cp}(n, m, k) = S(n, m, k) - S(n, m, k + 1)
  \]
Notice that

\[ S(n, m, k) = \sum_{i=0}^{m-k} \binom{m}{i} \cdot (n - m + k) \cdot (n - m + k + i)^{i-1} \cdot (m - k - i)^{m-i} \]

are partial sums occurring in

**Lemma (Abel’s Binomial Identity, 1826)**

*For all* \( a, b \in \mathbb{R}, m \in \mathbb{N}_0, *\)

\[ \sum_{i=0}^{m} \binom{m}{i} \cdot a \cdot (a + i)^{i-1} \cdot (b - i)^{m-i} = (a + b)^m \]
Notice that

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- In fact, \( S(n, m, 0) = n^m \) proves Abel’s identity for \( a = n - m \) and \( b = m \).
1. A parking problem (defective parking functions)

2. Enumeration (generating functions)

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4. Conclusion and outlook
Consider the distribution of the probability of a parking function with defect $k$,

$$p_{n,m}(k) = \frac{1}{n^m} \text{cp}(n, m, k),$$

for $n$ (parking spaces) or $m$ (cars) large.
Consider the distribution of the probability of a parking function with defect $k$,

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for $n$ (parking spaces) or $m$ (cars) large.

Different regimes:

- $m \ll n$: cars park with probability 1
- $m \sim \lambda n$, $\lambda < 1$: discrete limit law
  - ($\sqrt{m} \ll m - n \ll m$: exponential distribution)
  - $m - n \sim \lambda \sqrt{m}$: linear-exponential distribution
  - ($\sqrt{m} \ll n - m \ll m$: exponential distribution)
- $m \sim \lambda n$, $\lambda > 1$: discrete limit law
- $m \gg n$: cars leave with probability 1
Define

\[ P_{n,m}(k) = \frac{1}{n^m} S(n, m, k) = \sum_{j=k}^{m} p_{n,m}(j) \]

- \( m \ll n: \ P_{n,m}(k) \to 0 \)
Define

\[ P_{n,m}(k) = \frac{1}{n^m} S(n, m, k) = \sum_{j=k}^{m} p_{n,m}(j) \]

- \( m \ll n \): \( P_{n,m}(k) \to 0 \)
- \( m \sim \lambda n, \lambda < 1 \): discrete limit law

\[ P_{n,m}(k) \to (1 - \lambda) \sum_{l=0}^{k} (-1)^{k-l} \frac{(l + 1)^{k-l}}{(k - l)!} \lambda^{k-l} e^{\lambda(l+1)} \]
Define

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- \( m \sim \lambda n, \lambda > 1 \): discrete limit law

\[ P_{n,m}(n - m + k) \to k \sum_{l=0}^{\infty} \frac{(l + k)^{l-1}}{l!} \lambda^{l} e^{-\lambda(l+k)} \]
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\[ P_{n,m}(k) = \frac{1}{n^m} S(n, m, k) = \sum_{j=k}^{m} p_{n,m}(j) \]

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- \( m \sim \lambda n, \lambda > 1 \): discrete limit law

\[ P_{n,m}(n - m + k) \to k \sum_{l=0}^{\infty} \frac{(l + k)^{l-1}}{l!} \lambda^l e^{-\lambda(l+k)} \]

- \( m \gg n \): \( P_{n,m}(k) \to 1 \)
The interesting scaling regime \( m - n \sim \lambda \sqrt{m} \)
The interesting scaling regime $m - n \sim \lambda \sqrt{m}$

**Theorem**

Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then the limiting probability that in a random assignment of $n + \lfloor y \sqrt{n} \rfloor$ drivers to $n$ spaces at least $\lfloor x \sqrt{n} \rfloor$ drivers fail to park is

$$\lim_{n \to \infty} P_{n, n + \lfloor y \sqrt{n} \rfloor}(\lfloor x \sqrt{n} \rfloor) = \begin{cases} e^{-2x(x-y)} & \text{if } x > y, \\ 1 & \text{otherwise.} \end{cases}$$
The interesting scaling regime $m - n \sim \lambda \sqrt{m}$

**Theorem**

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\exp^{-2x(x-y)} & \text{if } x > y, \\
1 & \text{otherwise}.
\end{cases}
$$

Therefore we have approximately

$$
\frac{cp(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}
$$
Proof.

Approximate the expression for $S(n, m, k)$ by an integral

$$\lim_{n \to \infty} P_{n,n+\lfloor y\sqrt{n} \rfloor}(\lfloor x\sqrt{n} \rfloor) = \int_0^1 \frac{x-y}{\sqrt{2\pi \alpha^3(1-\alpha)}} \cdot \exp \left( - \frac{(x - (1 - \alpha)y)^2}{2\alpha(1 - \alpha)} \right) d\alpha.$$
Proof.

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Under the substitution $\alpha = \frac{u(x - y)}{x + u(x - y)}$ this integral simplifies to

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \sqrt{\frac{x(x - y)}{u^3}} \cdot \exp \left( -x \cdot (x - y) \cdot \frac{(1 + u)^2}{2u} \right) du$$

$$= \exp(-2x(x - y)).$$
Proof.

Approximate the expression for $S(n, m, k)$ by an integral

$$\lim_{n \to \infty} P_{n,n+\lfloor y\sqrt{n} \rfloor} (\lfloor x\sqrt{n} \rfloor)$$

$$= \int_{0}^{1} \frac{x - y}{\sqrt{2\pi}\alpha^3(1 - \alpha)} \cdot \exp \left( -\frac{(x - (1 - \alpha)y)^2}{2\alpha(1 - \alpha)} \right) d\alpha.$$  

Under the substitution $\alpha = \frac{u(x-y)}{x+u(x-y)}$ this integral simplifies to

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$$= \exp(-2x(x - y)).$$

NB: neither Maple nor Mathematica can do the above integral!
Less cars than parking spaces ($n > m$): 

$$\rho(100, 90, k)$$

$$\frac{cp(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$
Equal number of cars and parking spaces ($n = m$):

$$\frac{cp(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}$$
More cars than parking spaces \((n < m)\):

\[
cp(n, m, k) \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}
\]
The approximation is surprisingly accurate:

\[
\frac{\text{cp}(n, m, k)}{n^m} \approx \frac{2}{n} \cdot (2k - m + n) \cdot e^{-2k(k-m+n)/n}
\]
What is the probability that the car park is full?
What is the probability that the car park is full?

**Theorem**

Let $\lambda \in \mathbb{R}^+$. Then the limiting probability that in a random assignment of $\lfloor \lambda n \rfloor$ drivers to $n$ spaces all spaces are occupied is

$$
\lim_{n \to \infty} \frac{cp(n, \lfloor \lambda n \rfloor, \lfloor \lambda n \rfloor - n)}{n^{\lfloor \lambda n \rfloor}} = \begin{cases} 
0 & \text{if } \lambda \leq 1, \\
1 - \frac{1}{\lambda} \cdot T(\lambda e^{-\lambda}) & \text{if } \lambda > 1.
\end{cases}
$$
$n = 10, 20, \infty$:

\[
\frac{\text{cp}(n, m, m - n)}{n^m} \bigg|_{m = \lfloor \lambda n \rfloor} \approx 1 - \frac{1}{\lambda} T(\lambda e^{-\lambda})
\]


1. A parking problem (defective parking functions)

2. Enumeration (generating functions)

3. Asymptotics (limit distributions)

4. Conclusion and outlook
Summary:

- Introduced interesting and apparently new parking problem
- Solved the counting problem (Kernel method!)
- Discussed limiting probability distributions
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- Introduced interesting and apparently new parking problem
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Outlook:
- So far only “weak limit laws” - can refine analysis
- Extension to several generalised parking problems in the literature possible
The End