Propagation of minimality in the supercooled Stefan problem

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Abstract

Supercooled Stefan problems describe the evolution of the boundary between the solid and liquid phases of a substance, where the liquid is assumed to be cooled below its freezing point. Following the methodology of Delarue, Nadtochiy and Shkolnikov, we construct solutions to the one-phase one-dimensional supercooled Stefan problem through a certain McKean–Vlasov equation, which allows to define global solutions even in the presence of blow-ups. Solutions to the McKean–Vlasov equation arise as mean-field limits of particle systems interacting through hitting times, which is important for systemic risk modeling. Our main contributions are: (i) we prove a general tightness theorem for the Skorokhod M_1-topology which applies to processes that can be decomposed into a continuous and a monotone part. (ii) We prove propagation of chaos for a perturbed version of the particle system for general initial conditions. (iii) We prove a conjecture of Delarue, Nadtochiy and Shkolnikov, relating the solution concepts of so-called minimal and physical solutions, showing that minimal solutions of the McKean–Vlasov equation are physical whenever the initial condition is integrable.

Keywords: supercooled Stefan problem, McKean–Vlasov equations, singular interactions, propagation of chaos, systemic risk
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Contents

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1 Introduction

1.1 The classical supercooled Stefan problem

Stefan problems are models for the evolution of the interface between two phases of a substance undergoing a phase transition. Historically, the study of these problems goes back to the eponymous physicist Stefan (1891) studying the growth of ice, and to Lamé and Clapeyron (1831), studying the formation of the earth's crust. The classical one-dimensional supercooled Stefan problem is a simple model for the freezing of a supercooled liquid on the semi-infinite strip $[0, \infty)$. It can be formulated using the set of equations

$$\partial_t u = \frac{1}{2}\partial_{xx} u, \qquad \alpha\Lambda_t < x < \infty, \quad t > 0, \tag{1.1a}$$

$$u(t, \alpha\Lambda_t) = 0, \qquad t > 0, \tag{1.1b}$$

$$\partial_x u(t, \alpha\Lambda_t) = -2\alpha\dot{\Lambda}_t, \qquad t > 0, \tag{1.1c}$$

$$u(0, x) = -\alpha f(x), \qquad x > 0. \tag{1.1d}$$

Here, we interpret t, x and $u = u(t, x)$ as time, position and temperature, respectively. The freezing point is at $u = 0$, and the freezing front (the interface between solid and liquid) at time t is located at position $x = \alpha\Lambda_t$. We assume that $\alpha > 0$ is a known constant and that f is a known probability density function. Equation (1.1d) then implies that the liquid is initially below or at its freezing point - which is precisely the reason why we refer to the problem as *supercooled*. Equation (1.1a) describes the heat transport in the fluid phase. Condition (1.1b) asserts that the phase change is isothermal, and condition (1.1c) is a so-called *Stefan condition*, which balances the discontinuity in heat flux across the freezing front with the release of latent heat during freezing. The set of equations (1.1) describes a *one-phase* model, which means that we do not account for heat transport in the solid phase, which amounts to assuming that the temperature in the solid is constant and equal to $u = 0$. Solving the supercooled Stefan problem then amounts to finding functions u and Λ such that (1.1) is satisfied.

It is well-known that for certain initial conditions, problem (1.1) exhibits blow-ups in finite time (see Sherman (1970)). The classical remedy for this issue is a modification of the boundary condition, see Dewynne (1992) for a survey. We follow the approach introduced in Delarue et al. (2019), which allows us to globally define solutions to (1.1) in the presence of blow-ups by virtue of a probabilistic reformulation.

1.2 Probabilistic reformulation

Following Delarue et al. (2019), we consider the probabilistic reformulation of the supercooled Stefan problem (1.1), given by the following McKean–Vlasov equation

$$\begin{cases} X_t = X_{0-} + B_t - \alpha\Lambda_t \\ \tau = \inf\{t \geqslant 0 : X_t \leqslant 0\} \\ \Lambda_t = \mathbb{P}\left(\tau \leqslant t\right), \end{cases} \tag{1.2}$$

where the initial condition X_{0-} is supported in $[0, \infty)$, and B is an independent Brownian motion. (The assumption about the support of X_{0-} is not needed, but natural for the applications we have in mind). A solution to this equation is a triple (X, τ, Λ), where $\Lambda : [0, \infty) \to [0, 1]$ is a deterministic, non-decreasing càdlàg function such that (1.2) holds for any Brownian motion B, and X and τ are the resulting process and stopping time, respectively.

We heuristically motivate the connection between (1.1) and (1.2) by means of a formal calculation. Suppose that (X, τ, Λ) solves (1.2) for some continuously differentiable loss function Λ, let $\varphi \in C^2$ be a test function with $\varphi(0) = 0$ and let X_{0-} admit the density f. Applying Itô's formula and taking expectations yields

$$\mathbb{E}\left[\varphi(X_t)\mathbb{1}_{[\tau>t]}\right] = \mathbb{E}\left[\varphi(X_{0-})\right] + \frac{1}{2}\int_0^t \mathbb{E}\left[\varphi''(X_s)\mathbb{1}_{[\tau>s]}\right]\,\mathrm{d}s - \alpha\int_0^t \mathbb{E}\left[\varphi'(X_s)\mathbb{1}_{[\tau>s]}\right]\,\mathrm{d}\Lambda_s.$$

Denoting the subdensity of $X_t\mathbb{1}_{[\tau>t]}$ by $p(t, \cdot)$, integrating by parts yields

$$\partial_t p = \frac{1}{2}\partial_{xx}p + \alpha\dot{\Lambda}_t\partial_x p, \quad p(0, \cdot) = f, \quad p(\cdot, 0) = 0.$$

Taking the derivative with respect to time of the equation $\Lambda_t = 1 - \int_0^\infty p(t, x)\,\mathrm{d}x$, we see that

$$\dot{\Lambda}_t = -\int_0^\infty \partial_t p(t, x)\,\mathrm{d}x = -\int_0^\infty \frac{1}{2}\partial_{xx}p(t, x)\,\mathrm{d}x - \int_0^\infty \alpha\dot{\Lambda}\partial_x p(t, x)\,\mathrm{d}x = \frac{1}{2}\partial_x p(t, 0).$$

Therefore, setting $u(t,x) := -\alpha \cdot p(t, x - \alpha\Lambda_t)$, we (formally) obtain a solution to (1.1). As (Delarue et al., 2019, Theorem 1.1) shows, this argument can be made rigorous for so-called *physical solutions* to the McKean–Vlasov problem (1.2) in between jump times.

Apart from the physical significance of this equation, systemic risk and neuro-science applications have recently sparked considerable interest in the McKean–Vlasov problem (1.2) and variants thereof, see e.g. Delarue et al. (2015a); Nadtochiy and Shkolnikov (2019); Hambly et al. (2019); Delarue et al. (2019); Ledger and Søjmark (2020). The existence question could be clarified in Delarue et al. (2015a), where (for a slightly different equation) it is shown that global solutions to (1.2) can be obtained as limit points of the following particle system

$$\begin{cases} X_t^{i,N} = X_{0-}^i + B_t^i - \alpha L_t^N \\ \tau_{i,N} = \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\} \\ L_t^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \mathbb{1}_{[\tau_{i,N} \leqslant t]}, \end{cases} \tag{1.3}$$

with $N \in \mathbb{N}$ and $i = 1, \ldots, N$. Here, the initial conditions X_{0-}^i are iid and distributed as X_{0-} in (1.2), and B^i are independent Brownian motions independent of $(X_{0-}^i)_{i=1}^N$. A solution to this equation is a triple (X^N, τ_N, L^N), where L^N is a $[0,1]$-valued, non-decreasing, càdlàg process such that (1.2) holds for all Brownian motions $(B^i)_{i=1}^N$, and X^N and τ_N are the resulting N-dimensional processes and stopping times, respectively.

1.3 Connections to systemic risk

From the particle system (1.3) the connection to systemic risk in finance becomes apparent. Indeed, consider N banks and assume that $X^{i,N}$ stands for the evolution of bank i's asset value (or rather the distance-to-default, see the introduction of Hambly et al. (2019) regarding possible economic interpretations). The dynamics described by (1.3) then correspond to a perfectly homogeneous lending network, where every bank lends a quantity of α/N to every other bank in the system. Whenever one $X^{i,N}$ hits 0, then – due to the exposure to all other banks – a contagion effect occurs. It results in a loss of α/N for all other banks, which in turn can cause further defaults leading to default cascades. Such default cascades can trigger so-called *systemic events,* whose definition, introduced in Nadtochiy and Shkolnikov (2019), is based on the limiting McKean–Vlasov equation (1.2). Indeed, for a solution (X, τ, Λ) to (1.2), the function Λ can be interpreted as the aggregate loss of a typical bank in a large banking system caused by defaults from other banks. Mathematically, a *systemic event* is then defined as a jump-discontinuity of Λ, meaning that a considerable fraction of banks defaults. For more on the relevance of models such as (1.2) to the topic of systemic risk, we refer the reader to Hambly and Søjmark (2019).

A natural question at this point is under which conditions jump-discontinuities of Λ occur. Not surprisingly, the smoothness of Λ depends on the tuple (X_{0-}, α). A remarkably short argument (Hambly et al., 2019, Theorem 1.1) shows that for any solution (X, τ, Λ) to the McKean–Vlasov problem (1.2), the function Λ must be discontinuous whenever $2\mathbb{E}[X_{0-}] < \alpha$. Conversely, in the weak feedback regime (if the feedback parameter α is "small" relative to the sup-norm of the density of the initial condition), global uniqueness and continuity of Λ is proved in Ledger and Søjmark (2020). A similar result is obtained in Delarue et al. (2015b) for deterministic initial conditions.

1.4 Solution concepts for the McKean–Vlasov problem

The preceding definition of a systemic event only makes sense when a suitable *propagation of chaos-result* for the sequence of empirical measures that corresponds to the particle system (1.3) can be established. By the results of Delarue et al. (2015a), this holds true provided that there is uniqueness among the limit points. But this is exactly (one of) the crucial and still unresolved issue(s): it is open whether uniqueness of limit points of the particle system and uniqueness of solutions to (1.2) hold within the class of so-called *physical solutions* in general. Following Delarue et al. (2015a), we call a solution (X, τ, Λ) to (1.2) *physical*, if

$$\Delta\Lambda_t = \inf\{x > 0 \colon \mathbb{P}\left(\tau \geqslant t, X_{t-} \in [0, \alpha x]\right) < x\}, \quad t \geqslant 0. \tag{1.4}$$

The meaning of this condition may appear rather opaque at first sight. Proposition 6.3 reveals that for each instant in time, (1.4) amounts to choosing the smallest possible jump size of Λ that allows for a càdlàg solution of (1.2). In particular, solutions whose loss function Λ is continuous are physical.

There is an analogue of the notion of physical solution for the particle system; we provide more details in Section 3.2. In the particle system, restricting to physical solutions is economically and physically meaningful, and allows to preclude economically elusive solutions as constructed in Example 3.2 and to conclude uniqueness of (1.3) (for every fixed N) among the set of physical solutions. Moreover, it turns out that the physical solution coincides with another meaningful type of solution, namely the *minimal solution* (see Lemma 3.5).

In the current paper we shall focus on this solution concept and illustrate that this is in several respects the right way to look at the problem. We call a solution $(\underline{X}^N, \underline{\tau}_N, \underline{L}^N)$ to (1.3) *minimal* if, almost surely, for any other solution (X^N, τ_N, L^N) to (1.3) coupled to the same Brownian motions $(B^i)_{i=1}^N$, we have

$$\underline{L}_t^N \leqslant L_t^N, \quad t \geqslant 0. \tag{1.5}$$

In an analogous manner, we call a solution $(\underline{X}, \underline{\tau}, \underline{\Lambda})$ to the McKean–Vlasov problem (1.2) *minimal*, if for every solution (X, τ, Λ) to (1.2) we have

$$\underline{\Lambda}_t \leqslant \Lambda_t, \quad t \geqslant 0. \tag{1.6}$$

Note that this condition is deterministic as $\underline{\Lambda}$ and Λ are necessarily deterministic.

Significant progress regarding the question of uniqueness of physical solutions has been made recently, going beyond the weak feedback regime. In Delarue et al. (2019), uniqueness of the physical solution to the McKean–Vlasov problem (1.2) is established for initial conditions with a bounded and non-oscillatory density for any $\alpha > 0$. However, uniqueness within the class of physical solutions for general X_{0-} and the question whether the (unique) minimal solution coincides with (one of) the physical solution(s), are open problems.

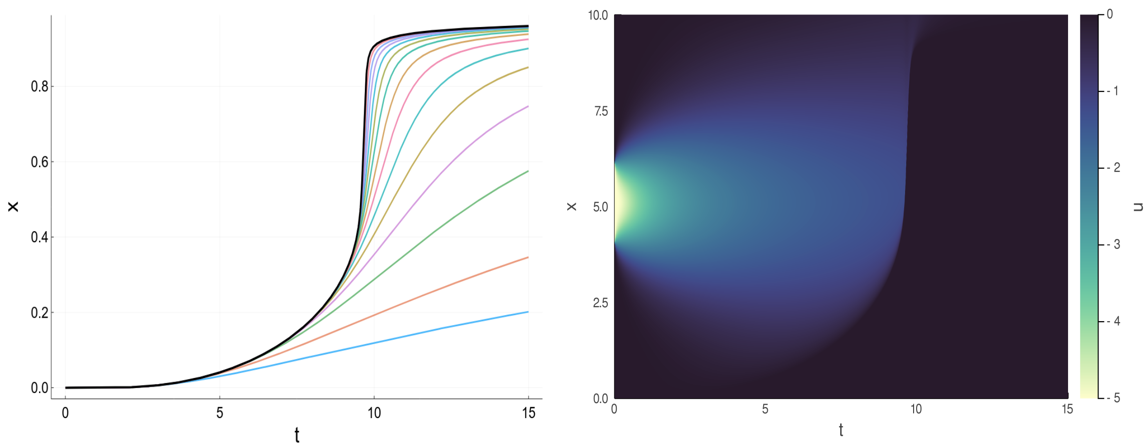

Figure 1: The left-hand side shows the iterates $\Gamma^{(k)}[0]$. The right-hand side shows the solution $u(t,x)$ to the supercooled Stefan problem. The parameters were $\alpha = 10$, and X_{0-} was chosen to be uniformly distributed on (4,6).

1.5 Main results

One of the main contributions of the current paper is to answer the latter question raised by Delarue et al. (2019) with "yes": *the minimal solution is a physical solution* (see Theorem 6.5). The key for proving this is a propagation of chaos-result for a perturbed particle system. Indeed, consider (1.3) with an initial condition that is slightly larger, namely

$$X_{0-}^i + \alpha N^{-\gamma},$$

for $\gamma \in (0, \frac{1}{2})$. Then we can show that the empirical distribution of the minimal (and thus physical) solution to the perturbed particle system converges in probability to the law of the minimal solution to (1.2). This is what we call *perturbed propagation of minimality* (see Theorem 6.2).

In the case where the physical solution happens to be unique, the minimal and physical solution coincide and *true propagation of minimality* holds true (see Theorem 6.6). By the latter we mean propagation of chaos of the unperturbed particle system (1.3) to the minimal solution to (1.2). This follows from the known fact that any limit point (of the laws) of the empirical distribution of the minimal (=physical) solution to (1.3) corresponds to the law of a physical solution of (1.2). By uniqueness of the latter and the fact that the minimal solution is physical, we can thus conclude. The general case, in particular for deterministic initial conditions, is still open.

In this paper we also make a step towards answering this question. Indeed, for any fixed $\alpha > 0$ and for Lebesgue-almost every $x > 0$, we can prove true propagation of minimality and uniqueness of the limit points of the particle system with initial condition $X_{0-}^i - x$ (see Theorem 6.9). This does not necessarily imply uniqueness of the physical solution to the McKean–Vlasov equation but shows again that *the minimal solution is the right solution concept.*

Our results also open up the way to *new numerical methods.* For existing algorithms we refer to Kaushansky and Reisinger (2018) and Lipton et al. (2019), where numerical schemes based on the particle system are proposed. Knowing that the minimal solution to

(1.2) is physical, we can directly work with the McKean–Vlasov equation and representations of the minimal solution. Indeed we show in Proposition 2.3 that $\underline{\Lambda}$ can be computed via

$$\lim_{k\to\infty} \Gamma^{(k)}[0] = \underline{\Lambda}, \tag{1.7}$$

where $\Gamma^{(k)}$ denotes the k^{th} iteration of the operator Γ, which acts on càdlàg functions ℓ and is defined according to

$$\begin{cases} X_t^\ell = X_{0-} + B_t - \alpha\ell_t \\ \tau^\ell = \inf\{t \geqslant 0 : X_t^\ell \leqslant 0\} \\ \Gamma[\ell]_t = \mathbb{P}\left(\tau^\ell \leqslant t\right). \end{cases} \tag{1.8}$$

Note here that $(X^\ell, \tau^\ell, \ell)$ solves (1.2) if and only if ℓ is a fixed-point of Γ and (1.7) is nothing else than a fixed-point iteration starting with 0, the smallest possible value. Computing $\Gamma[\ell]$ amounts to solving the first passage problem for the function $X_{0-} - \alpha\ell$. Numerical methods for that, e.g. based on the so-called Master equation (see Peskir and Shiryaev (2006), Theorem 14.3 and the discussion on p.230), can thus be applied iteratively. Figure 1 illustrates an implementation of this method. This also provides a numerical method to directly assess the first time of a systemic event, defined via

$$t_{\text{sys}} = \inf\{t \geqslant 0 \,|\, \underline{\Lambda}_t \neq \underline{\Lambda}_{t-}\}.$$

Finally, note that our results on the physical nature of the minimal solution and the propagation of minimality show that systemic events should correspond to jump discontinuities of the minimal solution, which is intuitive from an economic point of view.

In the following let us summarize the main contributions of the current article.

- Global continuity of the operator Γ defined in (1.8) (Proposition 2.1), which as a consequence yields representations of the minimal solution to the McKean–Vlasov problem (1.2) as fixed-point iterations (Proposition 2.3). We also obtain similar results for the particle system (Lemma 3.1).
- A general tightness result in the M_1 topology for stochastic processes that can be decomposed into a continuous and a non-decreasing càdlàg process (Theorem 4.4).
- (Perturbed) Propagation of minimality (Theorem 6.2 and Theorem 6.6) via the trilogy of arguments consisting of tightness with respect to Skorokhod's M_1 topology (Corollary 4.7), convergence of olutions (Proposition 5.8) and identification of the limit (Section 6), which involves an argument proving that limit points of minimal solutions to the perturbed particle system are minimal solutions to the McKean–Vlasov equation (Proposition 6.1).
- The physical nature of the minimal solution and hence global existence of physical solutions whenever the initial condition is integrable (Theorem 6.5).
- Uniqueness of the limit points of the particle system for almost all deterministic initial conditions (Theorem 6.9).

The remainder of the article is structured as follows. In Section 1.6 we summarize frequently used notation. In Section 2, we show continuity of the operator Γ and construct the minimal solution through a fixed-point iteration. In Section 3, we discuss the notions of physical and minimal solutions for the particle system and show that they are equivalent. In Section 4 we show tightness of the empirical measures associated the particle system, in Section 5 we show that limit points of such empirical measures correspond to solutions of the McKean–Vlasov problem in a certain sense. Finally, in Section 6, we prove propagation of minimality under various perturbations and deduce that the minimal solution of the McKean–Vlasov problem is physical whenever the initial condition is integrable.

1.6 Frequently used notation

For a solution (X, τ, Λ) to the McKean–Vlasov problem (1.2) we refer to X as the solution process and, with a slight abuse of notation, to Λ as the solution. Moreover, we write (X, Λ) for (X, τ, Λ) whenever the specification of τ is not needed. The same terminology is used for solutions of the particle system (1.3).

α	feedback/network connectivity parameter in $(0, \infty)$ appearing in the particle system (1.3) and the McKean–Vlasov problem (1.2).
B, B^i	Brownian motion (superscript i indicates the i-th Brownian motion in the particle system).
$\mathcal{B}(E)$	Borel σ-field on the space E.
$C_b(E)$	space of continuous and bounded functions from the Polish space E to $\mathbb{R}$.
$C([0,T])$	space of continuous paths $w\colon [0,T] \to \mathbb{R}$ endowed with the uniform topology.
$C([0,\infty))$	space of continuous paths $w\colon [0,\infty) \to \mathbb{R}$ endowed with the topology of compact convergence, which is defined through uniform convergence on compact subsets of $[0,\infty)$.
$D([T_0,T])$	space of càdlàg paths $x : [T_0,T] \to \mathbb{R}$ which are left-continuous at T furnished with the M_1-topology, see (A.1).
$D([T_0,\infty))$	space of càdlàg paths $x : [T_0,\infty) \to \mathbb{R}$ furnished with the M_1-topology, see Definition A.6.
δ_x	Dirac measure concentrated on x.
d_L	Lévy-metric metrizing weak convergence of probability measures on $[0,\infty)$, see Definition 4.1.
E	Polish space.
$\bar{E}$	The space $C([0,\infty)) \times M$ furnished with the product topology induced by the topology of compact convergence on $C([0,\infty))$ and the Lévy-metric on M, see Theorem 4.2.
Γ	fixed-point operator for the McKean–Vlasov problem introduced in (1.8). $\Gamma[\ell]$ corresponds to the solution to the first-passage time problem with barrier function ℓ.
$\Gamma^{(k)}$	k-th iterate of Γ.

Γ_N	fixed-point operator for the particle system introduced in (3.1). If ℓ is deterministic, $\Gamma_N[\ell]$ corresponds to the empirical distribution function of the solution to the first-passage time problem with barrier function ℓ.
$\Gamma_N^{(k)}$	k-th iterate of Γ_N.
Λ	solution to the McKean–Vlasov problem (1.2).
$\underline{\Lambda}$	minimal solution to the McKean–Vlasov problem (1.2) as defined in (1.6).
$\underline{\Lambda}(x)$	minimal solution to the McKean–Vlasov problem (6.11)
L^N	solution to the N-particle system (1.3).
$\underline{L}^N$	minimal solution to the N-particle system (1.3) as defined in (1.5).
λ_t	path functional; $\lambda_t(x)$ is equal to 1 if $\inf_{s\leqslant t} x_s \leqslant 0$ and equal to 0 otherwise (see Def. 5.1).
M	space of cumulative distribution functions of probability measures on $[0,\infty]$ endowed with the Lévy-metric, see (2.2).
μ	(random) probability measure on $D([-1,\infty))$.
$\underline{\mu}$	law of the minimal solution on $\mathcal{P}(D([-1,\infty))$.
$\langle\mu, f\rangle$	$\int f(x)\ \mathrm{d}\mu(x)$
μ_N	empirical measure corresponding to the minimal solution to the N-particle system
N	number of particles
ν_{0-}	law of the initial condition X_{0-}
$\underline{\nu}_{t-}^N$	for a Borel set $A \in \mathcal{B}(\mathbb{R})$, the quantity $N\underline{\nu}_{t-}^N(A)$ is the number of particles (corresponding to the minimal solution) surviving until time t which have values in A at time $t-$, see Definition 3.3
$\underline{\nu}_{t-}$	Subprobability measure as given in Definition B.1.
ω	elementary event.
Ω	probability space.
$\mathcal{P}(E)$	space of probability measures on the Polish space E endowed with the topology of weak convergence.
$\tau, (\tau^N)$	(vector of) stopping time(s) corresponding to some solution of the McKean–Vlasov problem (the particle system).
τ_0	path functional, $\tau_0(x)$ denotes the first hitting time of 0 of the path $x \in D([-1,\infty))$, see Definition 5.1.
$X, (X^N)$	solution process corresponding to a solution to the McKean–Vlasov problem (N-particle system).
$\underline{X}, (\underline{X}^N)$	solution process corresponding to the minimal solution to the McKean–Vlasov problem (N-particle system).
ξ_N	empirical measure on $\bar{E}$, see Corollary 4.7.
ξ	(random) probability measure on $\bar{E}$.

2 The Minimal Solution of the McKean–Vlasov Problem

As noted in the introduction, it is helpful to decouple system (1.2) and to write it as a fixed-point problem. Considering the operator Γ as introduced in (1.8), it is straightforward to see that Γ is monotone in the sense that

$$\ell^1_t \leqslant \ell^2_t, \quad t \geqslant 0 \qquad \Longrightarrow \qquad \Gamma[\ell^1]_t \leqslant \Gamma[\ell^2]_t, \quad t \geqslant 0. \tag{2.1}$$

We are now interested in finding a space on which the operator Γ stabilizes. Let $\overline{\mathbb{R}}$ denote the one-point compactification of $\mathbb{R}$. As a consequence of $\overline{\mathbb{R}}$ being a compact metric space and $[0,\infty]$ being a closed subset thereof, the space of probability measures on $[0,\infty]$, denoted as $\mathcal{P}([0,\infty])$, endowed with the topology of weak convergence of probability measures is a compact Polish space (for more detail see e.g. Klenke (2013))[1]. Set

$$M := \{\ell \colon \overline{\mathbb{R}} \to [0,1] \mid \ell \text{ càdlàg and increasing}, \;\; \ell_{0-} = 0, \;\ell_\infty = 1\}, \tag{2.2}$$

then we may identify the elements of M with distribution functions of measures in $\mathcal{P}([0,\infty])$ via the map $\ell \mapsto \mu_\ell$, where we set

$$\mu_\ell([0,t]) := \ell_t, \quad t \geqslant 0.$$

Convergence in M is then equivalent to weak convergence of probability measures in $\mathcal{P}([0,\infty])$. In particular, we have that $\ell^n \to \ell$ in M if and only if $\mu_{\ell^n} \to \mu_\ell$ weakly in $\mathcal{P}([0,\infty])$, if and only if $\ell^n_t \to \ell_t$ for all $t \in [0,\infty]$ that are continuity points of ℓ. Equipped with this topology, M is a compact Polish space. We now show that Γ is a continuous operator on M.

Proposition 2.1. *The operator $\Gamma : M \to M$ is continuous.*

Proof. Let $\ell^n \to \ell$ in M. We have to show that $\Gamma[\ell^n]_t \to \Gamma[\ell]_t$ for all $t \in [0,\infty]$ that are continuity points of $\Gamma[\ell]$ as described above. As Brownian motion hits every level with probability 1 in finite time, the stopping time τ^ℓ is almost surely finite and hence $\Gamma[\ell]$ does not have an atom at infinity. This implies

$$\Gamma[\ell^n]_\infty = \Gamma[\ell]_\infty = 1,$$

so the case $t = \infty$ is clear. By the Portmanteau theorem, we have for any $s \geqslant 0$

$$\limsup_{n\to\infty} \ell^n_s = \limsup_{n\to\infty} \mu_{\ell^n}([0,s]) \leqslant \mu_\ell([0,s]) = \ell_s. \tag{2.3}$$

Note that for any $t \geqslant 0$, the reverse Fatou lemma together with (2.3) imply

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}\left(\tau^{\ell^n} \leqslant t\right) &= \limsup_{n\to\infty} \mathbb{P}\left(\exists s \in [0,t] : X_{0-} + B_s \leqslant \alpha \ell^n_s\right) \\ &\leqslant \mathbb{P}\Big(\exists s \in [0,t] : X_{0-} + B_s \leqslant \limsup_{n\to\infty} \alpha \ell^n_s\Big) \\ &\leqslant \mathbb{P}\left(\exists s \in [0,t] : X_{0-} + B_s \leqslant \alpha \ell_s\right) \\ &= \mathbb{P}\left(\tau^\ell \leqslant t\right).\end{aligned}$$

Or, equivalently,

$$\limsup_{n\to\infty} \Gamma[\ell^n]_t \leqslant \Gamma[\ell]_t. \tag{2.4}$$

[1]Note that here we use the language of probabilists, this mode of convergence corresponds to the weak*-convergence of measures in the language of functional analysis.

Let $t = 0$. If zero is not a point of continuity for $\Gamma[\ell]$, there is nothing to prove. If zero is a point of continuity, then $\Gamma[\ell]_0 = 0$. Inequality (2.4) shows that $\Gamma[\ell^n]_0$ goes to 0 as n goes to infinity.

Now let $t > 0$ be a point of continuity for $\Gamma[\ell]$. We claim that

$$\lim_{n\to\infty} (\Gamma[\ell]_t - \Gamma[\ell^n]_t)^+ = 0. \tag{2.5}$$

For the following estimates, we follow the proof of Proposition 3.1 in Hambly et al. (2019). We may write

$$(\Gamma[\ell]_t - \Gamma[\ell^n]_t)^+ \leqslant \mathbb{P}\left(\tau^{\ell^n} > t, \tau^\ell \leqslant t\right) = \int_{[0,t]} \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) d\Gamma[\ell]_s.$$

We now split up the integrand in its continuous and jump part, writing $\Gamma[\ell]_s^c$ for the continuous part, which we estimate in the following.

$$\begin{aligned}
&\int_0^t \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) d\Gamma[\ell]_s^c \\
&= \int_0^t \mathbb{P}\left(X_u^{\ell^n} > 0, u \in [0,t] \mid \tau^\ell = s\right) d\Gamma[\ell]_s^c \\
&\leqslant \int_0^t \mathbb{P}\left(X_s^{\ell^n} + B_u - B_s - \alpha(\ell_u^n - \ell_s^n) > 0, u \in [s,t] \mid \tau^\ell = s\right) d\Gamma[\ell]_s^c \\
&\leqslant \int_0^t \mathbb{P}\left(X_s^{\ell^n} + B_u - B_s > 0, u \in [s,t] \mid \tau^\ell = s\right) d\Gamma[\ell]_s^c,
\end{aligned}$$

where in the second to last inequality, the requirement $X_u^{\ell^n} > 0$ for $u \in [0,t]$ was relaxed to $X_u^{\ell^n} > 0$ for $u \in [s,t]$, and in the last inequality, we used that $\alpha > 0$ and ℓ^n is increasing. On the event $[\tau^\ell = s]$, we have $X_s^\ell \leqslant 0$, so

$$X_s^{\ell^n} \leqslant X_s^{\ell^n} - X_s^\ell = \alpha(\ell_s - \ell_s^n).$$

We conclude, using the reflection principle,

$$\begin{aligned}
&\int_0^t \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) d\Gamma[\ell]_s^c \\
&\leqslant \int_0^t \mathbb{P}\left(B_u - B_s > -\alpha(\ell_s - \ell_s^n), u \in [s,t] \mid \tau^\ell = s\right) d\Gamma[\ell]_s^c \\
&= \int_0^t \mathbb{P}\left(\inf_{u\in[s,t]} \{B_u - B_s\} > -\alpha(\ell_s - \ell_s^n)\right) d\Gamma[\ell]_s^c \\
&= \int_0^t \left(2\Phi\left(\alpha\frac{\ell_s - \ell_s^n}{\sqrt{t-s}}\right) - 1\right) d\Gamma[\ell]_s^c,
\end{aligned} \tag{2.6}$$

where Φ denotes the cumulative distribution function of a standard normal random variable. Because the set of discontinuity times of ℓ is at most countable, the integrand in (2.6) converges $\Gamma[\ell]^c$-almost everywhere to 0. Consequently, the integral in (2.6) vanishes as $n \to \infty$ by the dominated convergence theorem.

Regarding now the integral with respect to the jump part of $\Gamma[\ell]$, we obtain

$$
\begin{aligned}
&\int_0^t \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) d(\Gamma[\ell] - \Gamma[\ell]^c)_s \\
&= \sum_{s \leqslant t} \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) \Delta\Gamma[\ell]_s \\
&= \sum_{s < t} \mathbb{P}\left(\tau^{\ell^n} > t \mid \tau^\ell = s\right) \mathbb{P}\left(\tau^\ell = s\right) \\
&= \sum_{s < t} \mathbb{P}\left(\tau^{\ell^n} > t, \tau^\ell = s\right), \qquad (2.7)
\end{aligned}
$$

where we may take the sum over $s < t$ because t was assumed to be a continuity point of $\Gamma[\ell]$. For any $s > 0$, we have by the Portmanteau theorem

$$
\liminf_{n \to \infty} \ell^n_{s-} = \liminf_{n \to \infty} \mu_{\ell^n}([0, s)) \geqslant \mu_\ell([0, s)) = \ell_{s-}. \qquad (2.8)
$$

For $s < t$, we find

$$
\mathbb{P}\left(\tau^{\ell^n} > t, \tau^\ell = s\right) \leqslant \mathbb{P}\left(\forall \varepsilon \in (0, t - s) : X_{0-} + B_{s+\varepsilon} > \alpha \ell^n_{s+\varepsilon-}, \tau^\ell = s\right),
$$

now taking the limes superior and using (2.8) yields, again by the reverse Fatou lemma,

$$
\begin{aligned}
\limsup_{n \to \infty} \mathbb{P}\left(\tau^{\ell^n} > t, \tau^\ell = s\right) &\leqslant \mathbb{P}\left(\forall \varepsilon \in (0, t - s) : X_{0-} + B_{s+\varepsilon} \geqslant \alpha \ell_{s+\varepsilon-}, \tau^\ell = s\right) \\
&\leqslant \mathbb{P}\left(X_{0-} + B_s \geqslant \alpha \ell_s, \tau^\ell = s\right) \\
&\leqslant \mathbb{P}\left(X_{0-} + B_s = \alpha \ell_s\right) \\
&= 0.
\end{aligned}
$$

Again, by the dominated convergence theorem we find that the sum in (2.7) converges to zero as n goes to infinity. Together with (2.6) this yields claim (2.5), which, combined with (2.4), proves $\lim_{n \to \infty} \Gamma[\ell^n]_t = \Gamma[\ell]_t$. □

Remark 2.2. By definition, $\Gamma[\ell]$ is the solution to the first-passage time problem with lower function $X_{0-} - \alpha \ell$. As such, we may view Proposition 2.1 as a statement on the stability of first-passage time problems.

With Proposition 2.1 in hand, we can apply the Schauder-Tychonoff fixed-point theorem to prove the existence of global solutions to (1.2). A similar strategy is pursued in Nadtochiy and Shkolnikov (2020), where more general networks are considered and a version of the Schauder-Tychonoff theorem is proved for the Skorokhod M_1-topology. Using the monotonicity of the operator Γ, one can do even better: one can iteratively construct the minimal solution to the McKean–Vlasov equation, an idea due to (Delarue et al., 2019). Figure 1 illustrates this iteration.

Proposition 2.3. *For any initial condition X_{0-} and $\alpha > 0$, there is a minimal solution to* (1.2)*, which we denote by $(\underline{X}, \underline{\Lambda})$. It holds that*

$$
\lim_{k \to \infty} \Gamma^{(k)}[0] = \underline{\Lambda},
$$

in M, where $\Gamma^{(k)}$ denotes the k-th iterate of the operator Γ as defined in (1.8).

Proof. By definition, we have that $0 \leqslant \Gamma[0]$. Using the monotonicity of Γ, this implies that

$$\Gamma[0] \leqslant \Gamma[\Gamma[0]] = \Gamma^{(2)}[0]$$

and a straightforward induction shows that $\Gamma^{(k)}[0] \leqslant \Gamma^{(k+1)}[0]$, for each $k \in \mathbb{N}$. The sequence $(\Gamma^{(k)}[0]_t)_{k\in\mathbb{N}}$ is therefore increasing and bounded by 1 for every $t \geqslant 0$, which implies that we may define $\tilde{\Lambda}$ to be its pointwise limit

$$\tilde{\Lambda}_t := \lim_{k\to\infty} \Gamma^{(k)}[0]_t, \quad t \geqslant 0.$$

Clearly, $\tilde{\Lambda}$ is increasing with $\tilde{\Lambda}_{0-} = 0$ and $\tilde{\Lambda}_\infty = 1$, so its càdlàg modification $\underline{\Lambda}_t := \tilde{\Lambda}_{t+}$ lies in M. By construction, we have that $\lim_{k\to\infty} \Gamma^{(k)}[0] = \underline{\Lambda}$ in M. By continuity of Γ on M (Proposition 2.1), we obtain

$$\Gamma[\underline{\Lambda}] = \Gamma[\lim_{k\to\infty} \Gamma^{(k)}[0]] = \lim_{k\to\infty} \Gamma^{(k+1)}[0] = \underline{\Lambda},$$

so $\underline{\Lambda}$ solves (1.2). Now suppose that Λ is another solution to (1.2). By definition, it holds that $\Lambda \geqslant 0$, and using the monotonicity of Γ this leads to

$$\Gamma[0] \leqslant \Gamma[\Lambda] = \Lambda.$$

A straightforward induction shows that $\Gamma^{(k)}[0] \leqslant \Lambda$, for each $k \in \mathbb{N}$. If t is a continuity point of $\underline{\Lambda}$, this implies

$$\underline{\Lambda}_t = \lim_{k\to\infty} \Gamma^{(k)}[0]_t \leqslant \Lambda_t, \tag{2.9}$$

which by right-continuity implies $\underline{\Lambda} \leqslant \Lambda$. As Λ was an arbitrary solution to (1.2), this proves that $\underline{\Lambda}$ is in fact the minimal solution. □

The next example illustrates how the solution to the McKean–Vlasov problem (1.2) can fail to be unique.

Example 2.4 (Several solutions to the McKean–Vlasov problem)**.** Let $X_{0-} \equiv 1$ and set $\alpha = 1$. Then, $\overline{\Lambda} \equiv 1$ solves the McKean–Vlasov problem (1.2). Later on, we will prove (see Theorem 6.5) that the minimal solution is physical if the initial condition is integrable, which implies by definition that

$$\underline{\Lambda}_0 = \underline{\Lambda}_0 - \underline{\Lambda}_{0-} = \inf\{x \geqslant 0 : \mathbb{P}\left(X_{0-} \in [0, x]\right) < x\} = 0.$$

Therefore, clearly we have $\underline{\Lambda} \neq \overline{\Lambda}$, and hence nonuniqueness of the McKean–Vlasov problem (1.2).

3 Solutions of the particle system

3.1 Minimal solutions

Although it may seem intuitively obvious, we have not shown yet that there is a minimal solution to the particle system (1.3). A natural question at this point is whether the same construction outlined in Lemma 2.3 works, and a moment's reflection shows that it does. Define the operator Γ_N via

$$\begin{cases} X_t^{i,N}[L] = X_{0-}^i + B_t^i - \alpha L_t \\ \tau_{i,N}[L] = \inf\{t \geqslant 0 : X_t^{i,N}[L] \leqslant 0\} \\ \Gamma_N[L]_t = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N}[L]\leqslant t]}, \end{cases} \tag{3.1}$$

where L is some càdlàg process. In analogy to (2.1), Γ_N is monotone in the sense that

$$L_t^1 \leqslant L_t^2, \quad t \geqslant 0 \quad \Longrightarrow \quad \Gamma[L^1]_t \leqslant \Gamma[L^2]_t, \quad t \geqslant 0.$$

Making use of this montonicity, we readily see by straightforward induction arguments that

$$\Gamma_N^{(k)}[0] \leqslant L^N, \quad \Gamma_N^{(k)}[0] \leqslant \Gamma_N^{(k+1)}[0], \quad k \in \mathbb{N}, \tag{3.2}$$

holds almost surely, where L^N is any solution to the particle system and $\Gamma_N^{(k)}$ denotes the k-th iterate of Γ_N. We did not show any suitable continuity properties of Γ_N to conclude in the same way as in Lemma 2.3, but as we prove in the next lemma, the iteration $(\Gamma_N^{(k)}[0])_{k\in\mathbb{N}}$ is constant after at most N steps.

Lemma 3.1. *For $N \in \mathbb{N}$, let Γ_N be defined as in* (3.1)*. Then $\underline{L}^N := \Gamma_N^{(N)}[0]$ is the minimal solution to the particle system* (1.3)

and the error bound

$$\|\Gamma_N^{(k)}[0] - \underline{L}^N\|_\infty \leqslant \frac{(N-k)^+}{N} \tag{3.3}$$

holds almost surely.

Proof. For $k \in \mathbb{N}$, define the stopping times

$$\sigma_k := \inf\left\{t \geqslant 0 \colon \Gamma_N^{(k)}[0]_t \geqslant k/N\right\}.$$

First off, we show by induction on k that

$$\Gamma_N^{(k-1)}[0]_t = \Gamma_N^{(k)}[0]_t, \quad t < \sigma_k, \tag{3.4}$$

for each $k \in \mathbb{N}$. For the base case, let $k = 1$ (we define $\Gamma_N^{(0)}$ to be the identity operator). Observe that the first time that $\Gamma_N[0]$ jumps coincides with the first time any of the Brownian motions $(X_{0-}^i + B^i)_{i\in\mathbb{N}}$ hit 0, and $\Gamma_N[0]$ is equal to 0 before that time. This means that we have $0 = \Gamma_N^{(0)}[0]_t = \Gamma_N^{(1)}[0]_t$ for each $t < \sigma_1$. For the inductive step, assume the claim holds for all natural numbers up to k. Applying Γ_N to both sides of (3.4), we obtain

$$\Gamma_N^{(k)}[0]_t = \Gamma_N^{(k+1)}[0]_t, \quad t < \sigma_k. \tag{3.5}$$

We distinguish two cases: In the first case, suppose that $\Gamma_N^{(k)}[0]_{\sigma_k} > \frac{k}{N}$. Due to (3.2), we then must have $\Gamma_N^{(k+1)}[0]_{\sigma_k} \geqslant \frac{k+1}{N}$ and hence $\sigma_k = \sigma_{k+1}$, completing the inductive step. In the second case, we have $\Gamma_N^{(k)}[0]_{\sigma_k} = \frac{k}{N}$, and using (3.2) we find for $t \in (\sigma_k, \sigma_{k+1})$

$$\frac{k}{N} = \Gamma_N^{(k)}[0]_{\sigma_k} \leqslant \Gamma_N^{(k)}[0]_t \leqslant \Gamma_N^{(k+1)}[0]_t < \frac{k+1}{N}$$

which shows that $\Gamma_N^{(k)}[0]$ and $\Gamma_N^{(k+1)}[0]$ agree on all of $[0, \sigma_{k+1})$, completing the inductive step.

Having established (3.4), we show that $\underline{L}^N = \Gamma_N^{(N)}[0]$ solves the particle system (1.3). For $k \in \mathbb{N}$, repeatedly applying Γ_N to (3.4), we find

$$\Gamma_N^{(k)}[0]_t = \Gamma_N^{(k+1)}[0]_t = \ldots = \underline{L}_t^N = \Gamma_N[\underline{L}^N]_t, \quad t < \sigma_k. \tag{3.6}$$

Choosing $k = N$, we obtain $\underline{L}^N = \Gamma_N[\underline{L}^N]$ on $[0, \sigma_N)$. Recall that by (3.2) it holds that $\underline{L}^N \leqslant \Gamma_N[\underline{L}^N]$, so we see that $1 = \underline{L}_{\sigma_N}^N \leqslant \Gamma_N[\underline{L}^N]_{\sigma_N} \leqslant 1$ and therefore $\underline{L}^N = \Gamma_N[\underline{L}^N]$. By (3.2), it follows that $\underline{L}^N$ is indeed the minimal solution.

The error bound (3.3) is now straightforward: By (3.6), $\Gamma_N^{(k)}[0]$ agrees with $\underline{L}^N$ for $t < \sigma_k$ and is increasing in t, and therefore

$$\sup_{t \geqslant 0} |\underline{L}_t^N - \Gamma_N^{(k)}[0]_t| = \sup_{t \geqslant \sigma_k} |\underline{L}_t^N - \Gamma_N^{(k)}[0]_t| \leqslant 1 - \Gamma_N^{(k)}[0]_{\sigma_k} \leqslant \left(1 - \frac{k}{N}\right),$$

for each $k \leqslant N$. □

As mentioned in the introduction, in the weak feedback regime the solution to the McKean–Vlasov problem (1.2) is unique. As we see in the next example, no such condition can guarantee uniqueness of solutions for the particle system.

Example 3.2 (Several solutions to the particle system)**.** We show that solutions to system (1.3) are not necessarily pathwise unique. The example given here is adapted from Delarue et al. (2015a) to fit our framework. Let $N = 3$, $t > 0$, and suppose no jump occurred before time t. Assume that

$$X_{t-}^1 = 0, \quad X_{t-}^2 \in (\alpha/3, 2\alpha/3), \quad \text{and} \quad X_{t-}^3 \in (\alpha/3, 2\alpha/3).$$

By definition, for $i = 1, 2, 3$ we have

$$X_t^i = X_{t-}^i - \alpha L_t. \tag{3.7}$$

One solution is then given by letting only X^1 default at time t, which implies $L_t = 1/3$, and is consistent with X^2 and X^3 not defaulting. Another solution is given by letting all three banks default at time t, which implies $L_t = 1$ and is consistent with equations (3.7), leading to the collapse of the system.

3.2 Physical solutions

Definition 3.3. If X^N is a solution process to the particle system (1.3), define the random subprobability measure

$$\nu_{t-}^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_{t-}^{i,N}} \mathbb{1}_{[\tau^{i,N}\geqslant t]}$$

for $t \geqslant 0$.

We describe the concept of physical solution $(\hat{X}^N, \hat{L}^N)$ to the particle system, introduced by Delarue et al. (2015a). Before stating it rigorously, we briefly explain the intuition behind the concept. Clearly, the system (1.3) only jumps at a time t when there is a particle j which has not defaulted until t such that $\hat{X}_{t-}^{j,N} = 0$. Supposing that is the case, if letting $\Delta\hat{L}_t^N = 1/N$ results in there being a particle $i \neq j$ (which survived until time t) such that $\hat{X}_{t-}^{i,N} - \alpha\Delta\hat{L}_t^N \leqslant 0$, then clearly the value of $\Delta\hat{L}_t^N$ must be at least $2/N$ in order to obtain a solution. Continuing in this manner, we resolve the cascade by demanding that $\Delta\hat{L}_t^N$ equals the smallest k/N such that the kick the system experiences by taking $\Delta\hat{L}_t^N = k/N$ results in no more than k particles defaulting at time t.This condition is easily converted into the following definition[2].

Definition 3.4. With the notation of Definition 3.3, we call a solution $(\hat{X}^N, \hat{L}^N)$ of (1.3) *physical*, if we have

$$\Delta\hat{L}_t^N = \inf\left\{\frac{k}{N} \geqslant 0 : \ k \in \mathbb{N},\ \nu_{t-}^N\left(\left[0, \alpha\frac{k}{N}\right]\right) \leqslant \frac{k}{N}\right\} \tag{3.8}$$

for all $t \geqslant 0$.

Note that $N\nu_{t-}^N([0, \alpha\frac{k}{N}])$ is the number of particles surviving up to time t which would not survive a kick of $\alpha\frac{k}{N}$. Therefore we see that (3.8) corresponds to choosing the smallest possible jump size at any $t \geqslant 0$. We recognize formula (3.8) as the discrete analogue of the physical jump condition. Note that a similar

Lemma 3.5. *The physical solution to* (1.3) *is equal to the minimal solution to* (1.3).

Proof. Note that the physical solution $(\hat{X}^N, \hat{L}^N)$ to (1.3) is pathwise unique, as it is unique between jump times and (3.8) uniquely specifies the size of the jump at any given time. Since the minimal solution $(\underline{X}^N, \underline{L}^N)$ is unique by definition, it is sufficient to show that the physical solution is minimal.

Let σ be the first time the physical solution $(\hat{X}^N, \hat{L}^N)$ jumps, then for all $t < \sigma$ we have $\hat{L}_t^N = \underline{L}_t^N = 0$, because if the first jump of the minimal solution would happen before σ this would contradict the minimality of $(\underline{X}^N, \underline{L}^N)$. As physical solutions have minimal jumps, we find $\hat{L}_\sigma^N \leqslant \underline{L}_\sigma^N$, which due to the minimality of $\underline{L}^N$ implies $\hat{L}_\sigma^N = \underline{L}_\sigma^N$. Repeating this argument for each jump of the physical solution proves the claim. □

[2] A similar convention was adopted in Dembo and Tsai (2019), where another type of particle system approximating the supercooled Stefan problem is studied.

4 Tightness

As in the previous works on problem at hand, we will deal with the Skorokhod M_1-topology in this paper. We explain this choice and collect some fundamental results regarding the M_1-topology in Section A in the appendix.

Definition 4.1. Let $w^n, w \in C([0,\infty))$. We say that w^n converges to w with respect to the topology of compact convergence if $w^n \to w$ in $C([0,T])$ for every $T > 0$. We also define d_L to be the Lévy-metric on M, that is if $\ell^1, \ell^2 \in M$ we have

$$d_L(\ell^1, \ell^2) = \inf\{\varepsilon > 0 : \ell^1_{t+\varepsilon} + \varepsilon \geqslant \ell^2_t \geqslant \ell^1_{t-\varepsilon} - \varepsilon, \text{ for all } t \geqslant 0\}.$$

It is well-known that the Lévy-metric metrizes weak convergence. In order to avoid having to work directly with the unwieldy M_1-metric, the following theorem comes in handy. Recall that $D([T_0,\infty))$ denotes the space of càdlàg paths from $[T_0,\infty)$ to $\mathbb{R}$ furnished with the M_1-topology (see Definition A.6 for more detail).

Theorem 4.2. *Define the space* $\bar{E}$ *as*

$$\bar{E} := C([0,\infty)) \times M,$$

where M *is defined as in* (2.2). *Endowed with the product topology induced by compact convergence on* $C([0,\infty))$ *and the Lévy-metric on* M*, the space* $\bar{E}$ *is Polish. For* $w \in C([0,\infty))$ *and* $\ell \in M$*, define*

$$\hat{w}_t := \begin{cases} w_0 & t \in [-1,0) \\ w_t & t \in [0,\infty) \end{cases} \qquad \check{\ell}_t = \begin{cases} 0 & t \in [-1,0) \\ \ell_t & t \in [0,\infty). \end{cases}$$

Then, for any $\alpha \in \mathbb{R}$*, the embedding* $\iota_\alpha \colon \bar{E} \to D([-1,\infty))$ *defined via*

$$\iota_\alpha(w,\ell) = \hat{w} - \alpha\check{\ell}$$

is continuous.

Proof. As the product of Polish spaces is again Polish, by construction $\bar{E}$ is a Polish space. We show the continuity of ι_α. Let $(w^n, \ell^n) \to (w,\ell)$ in $\bar{E}$. Let $T > 0$ be a continuity point of ℓ. By assumption, we have that $w^n \to w$ in $C([0,T])$, which implies that $\widehat{w^n} \to \hat{w}$ in $C([-1,T])$ and thus in $D([-1,T])$. Again, by assumption, ℓ^n_t converges to ℓ_t for all $t > 0$ that are continuity points of ℓ, from which we deduce (using Lemma A.3) that $-\alpha\check{\ell^n} \to -\alpha\check{\ell}$ in $D([-1,T])$. By Lemma A.5, it follows that

$$\lim_{n\to\infty} \iota_\alpha(w^n,\ell^n) = \lim_{n\to\infty}(\widehat{w^n} - \alpha\check{\ell^n}) = \hat{w} - \alpha\check{\ell} = \iota_\alpha(w,\ell)$$

in $D([-1,T])$. The conclusion now follows from Lemma A.7. □

Remark 4.3. It is necessary to extend the domain artificially to the left to obtain a continuous embedding of $\bar{E}$ into the càdlàg functions equipped with the M_1-topology as in Theorem 4.2. The reason lies in the requirements of Lemma A.3, more precisely the requirement of pointwise convergence in the left interval endpoint. The extension procedure might now look like a cheap trick to circumvent having to show pointwise convergence in 0 (which it is), but indeed, one easily constructs examples of sequences $(\ell^n)_{n\in\mathbb{N}}$ that converge in M, but do not converge pointwise in 0 against their càdlàg limit, such as $\ell^n_t := \mathbb{1}_{[1/n \leqslant t]}$.

Theorem 4.4. *Let $(X^N)_{N\in\mathbb{N}}$ be a sequence of stochastic processes with paths in $D([0,\infty))$. Suppose that X^N almost surely admits a decomposition*

$$X^N = Z^N - \alpha_N L^N \tag{4.1}$$

where α_N is a (possibly random) real number, Z^N is the continuous part of X^N, and $L^N \in M$. Suppose that $(Z^N)_{N\in\mathbb{N}}$ is tight on $C([0,T])$ for each $T > 0$ and that $(\alpha_N)_{N\in\mathbb{N}}$ is tight on $\mathbb{R}$. Set

$$\hat{X}^N_t := \begin{cases} Z^N_0, & t \in [-1,0), \\ X^N_t, & t \in [0,\infty). \end{cases} \tag{4.2}$$

Then, the random variables $((Z^N, L^N))_{N\in\mathbb{N}}$ are tight on $\bar{E}$ and random variables $(\hat{X}^N)_{N\in\mathbb{N}}$ are tight on $D([-1,\infty))$.

Proof. Let $\varepsilon > 0$ and let $r > 0$ be such that $\mathbb{P}(|\alpha_N| > r) < \varepsilon/2$ or each $N \in \mathbb{N}$. Choose a sequence of positive numbers $(T_k)_{k\in\mathbb{N}}$ such that $T_k \nearrow \infty$. Because $(Z^N)_{N\in\mathbb{N}}$ is tight on $C([0,T_k])$, we may pick $K^k_\varepsilon \subseteq C([0,T_k])$ compact, such that

$$\mathbb{P}\left(Z^N \notin K^k_\varepsilon\right) < \varepsilon 2^{-(k+1)}, \quad N \in \mathbb{N}.$$

Set $K_\varepsilon := \{w \in C([0,\infty)) : w \in K^k_\varepsilon,\ k \in \mathbb{N}\}$. We show that K_ε is compact in the topology of compact convergence. The topological space

$$K^\infty_\varepsilon := \prod_{k=1}^\infty K^k_\varepsilon$$

endowed with the product topology of uniform convergence is compact by Tychonoff's theorem. Let $(w^n)_{n\in\mathbb{N}}$ be a sequence in K_ε, then $(w^n, w^n, \dots)_{n\in\mathbb{N}}$ is a sequence in K^∞_ε and thus we may assume that it converges (passing to a subsequence if necessary). This allows us to define $w_t := \lim_{n\to\infty} w^n_t$ for $t \geqslant 0$, and by construction we must have

$$(w^n, w^n, \dots) \to (w, w, \dots) \text{ in } K^\infty_\varepsilon,$$

which implies $w \in K_\varepsilon$ and $w^n \to w$ in the topology of compact convergence. We obtain that

$$\mathbb{P}\left((Z^N, L^N) \notin K_\varepsilon \times M\right) = \mathbb{P}\left(Z^N \notin K_\varepsilon\right) \leqslant \sum_{k=1}^\infty \mathbb{P}\left(Z^N \notin K^k_\varepsilon\right) \leqslant \varepsilon/2,$$

and as M is compact, this shows that $((Z^N, L^N))_{N\in\mathbb{N}}$ is tight on $\bar{E}$. Theorem 4.2 now shows that

$$K := (\hat{K}_\varepsilon - r\check{M}) \cup (\hat{K}_\varepsilon + r\check{M})$$

is compact in $D([-1,\infty))$. We obtain, uniformly in $N \in \mathbb{N}$,

$$\mathbb{P}\left(\hat{X}^N \notin K\right) \leqslant \mathbb{P}\left(Z^N \notin K_\varepsilon\right) + \mathbb{P}\left(|\alpha_N| > r\right) < \varepsilon.$$

So $(\hat{X}^N)_{N\in\mathbb{N}}$ is tight on $D([-1,\infty))$. □

When we are concerned with a process admitting a decomposition of the form (4.1) in $D([0,\infty))$, from now on we will not distinguish between the process and its extension as given in (4.2) to $D([-1,\infty))$ in our notation.

To prove tightness of empirical measures, we make use of a famous result of Sznitman. Set $X^N := (X^{1,N}, \dots, X^{N,N})$.

Definition 4.5. X^N is N-exchangeable, if

$$\mathrm{law}(X^N) = \mathrm{law}((X^{\sigma(1),N}, X^{\sigma(2),N}, \dots, X^{\sigma(N),N})),$$

for any permutation σ of $\{1, \dots, N\}$.

Proposition 4.6. *Let E be a Polish space and let X^N be N-exchangeable on E^N for every $N \in \mathbb{N}$, then the $\mathcal{P}(E)$-valued random variables $(\mu_N)_{N\in\mathbb{N}}$ given by*

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^{i,N}}$$

are tight if and only if $(X^{1,N})_{N\in\mathbb{N}}$ is tight on E.

Proof. See (Sznitman, 1991, Proposition 2.2). □

Corollary 4.7. *Suppose that X^N satisfies the dynamics*

$$X_t^{i,N} = X_{0-}^{i,N} + B_t^i - \alpha L_t^N$$

where $\alpha > 0$, $(X_{0-}^N)_{N\in\mathbb{N}}$ are N-exchangeable random vectors, $(B_i)_{i\in\mathbb{N}}$ are independent Brownian motions, and $L^N \in M$. If $(X_{0-}^{1,N})_{N\in\mathbb{N}}$ is tight on $\mathbb{R}$, then the empirical measures

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^{i,N}} \qquad \text{and} \qquad \xi_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_{0-}^{i,N} + B^i, L^N)} \tag{4.3}$$

are tight on $\mathcal{P}(D([-1,\infty)))$ and $\mathcal{P}(\bar{E})$, respectively.

Proof. Fix $\varepsilon > 0$ and let $T > 0$. As $C([0,T])$ is a Polish space, there is a compact set $K \subseteq C([0,T])$ such that $\mathbb{P}(B^1 \in K) > 1 - \varepsilon/2$. By assumption, there is a compact set $K_{0-} \subseteq \mathbb{R}$ such that $\mathbb{P}(X_{0-}^{1,N} \in K_{0-}) > 1 - \varepsilon/2$, for each $N \in \mathbb{N}$. As $K_{0-} + K$ is compact in $C([0,T])$, and

$$\mathbb{P}\left(X_{0-}^{1,N} + B^1 \notin K_{0-} + K\right) \leqslant \varepsilon, \quad N \in \mathbb{N},$$

we find that $(X_{0-}^{1,N} + B^1)_{N\in\mathbb{N}}$ is tight on $C([0,T])$ for every $T > 0$. The claim now follows from Theorem 4.4 and Proposition 4.6. □

We now state a technical result for future reference.

Corollary 4.8. *Let $(\mu_N)_{N\in\mathbb{N}}$ be given as in Corollary 4.7. Then there are $\mathcal{P}(\bar{E})$-valued random variables ξ, ξ_N such that, after passing to subsequences if necessary, $\mathrm{law}(\mu_N) = \mathrm{law}(\iota_\alpha(\xi_N))$, $\xi_N \to \xi$, and $\iota_\alpha(\xi_N) \to \iota_\alpha(\xi)$ almost surely. Moreover,*

$$\mathrm{law}(\xi_N) = \mathrm{law}\left(\frac{1}{N} \sum_{i=1}^{N} \delta_{(X_{0-}^{i,N} + B^i, L^N)}\right). \tag{4.4}$$

Proof. Set ξ_N as in Corollary 4.7 and note that $\mu_N = \iota_\alpha(\xi_N)$ for every $N \in \mathbb{N}$. After passing to subsequences if necessary, we may assume that $\mathrm{law}(\xi_N) \to \mathrm{law}(\xi)$ by virtue of Corollary 4.7 for some random variable ξ. By the Skorokhod representation theorem, we may assume as well that $\lim_{N\to\infty} \xi_N = \xi$ holds in $\mathcal{P}(\bar{E})$ for a representation sequence. Since ι_α is continuous by Theorem 4.2, this implies $\lim_{N\to\infty} \iota_\alpha(\xi_N) = \iota_\alpha(\xi)$ almost surely. □

5 Convergence of solutions

Consider the setting described in Corollary 4.7 where

$$X_t^{i,N} = X_{0-}^{i,N} + B_t^i - \alpha L_t^N$$

for $\alpha > 0$, some N-exchangeable random vectors $(X_{0-}^N)_{N\in\mathbb{N}}$, some independent Brownian motions $(B_i)_{i\in\mathbb{N}}$, and some $L^N \in M$ adapted to the filtration $(\mathcal{F}_t^N)_{t\geqslant 0}$ generated by $(X_{0-}^N, B^1, \ldots, B^N)$. Assume that $(X_{0-}^N)_{N\in\mathbb{N}}$ is independent of $(B_i)_{i\in\mathbb{N}}$ as well as $(X_{0-}^{1,N})_{N\in\mathbb{N}}$ is tight on $\mathbb{R}$. Setting again

$$\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}} \tag{5.1}$$

we can then make use of Corollary 4.8 to deduce that there are $\mathcal{P}(\bar{E})$-valued random variables ξ, ξ_N such that ξ_N satisfies (4.4) and, after passing to subsequences if necessary,

$$\mathrm{law}(\mu_N) = \mathrm{law}(\iota_\alpha(\xi_N)), \quad \xi_N \to \xi, \quad \text{and} \quad \iota_\alpha(\xi_N) \to \iota_\alpha(\xi) \tag{5.2}$$

almost surely. The goal of this section is now to study the properties of the random measure ξ. This analysis will lead to the conclusion that solutions to the particle systems converge to solutions of the McKean–Vlasov problem (1.2).

The proofs in this section are adaptations of arguments originally presented in Delarue et al. (2015a) to our setting.

Definition 5.1. For $t \in \mathbb{R}$ and $x \in D([-1,\infty))$, define the path functionals

$$\tau_0(x) := \inf\{s \geqslant 0 : x_s \leqslant 0\} \quad \text{and} \quad \lambda_t(x) := \mathbb{1}_{[\tau_0(x)\leqslant t]}.$$

Considering a sequence of constant positive functions converging to zero shows that λ_t is not continuous on $D([-1,\infty))$ with the M_1-topology. However, it turns out that λ_t is continuous at paths that satisfy a certain crossing property (going back to Delarue et al. (2015a)), which ensures that the path actually dips below the x-axis at the first hitting time of 0. Before we show continuity of λ_t assuming this crossing property, we prove a technical lemma.

Lemma 5.2. *Let $x^n, x \in \iota_\alpha(\bar{E})$ and suppose that $x^n \to x$ in $D([-1,\infty))$. Then, if $t \geqslant 0$ is any continuity point of x, it holds that*

$$\lim_{n\to\infty} \inf_{0\leqslant s\leqslant t} x_s^n = \inf_{0\leqslant s\leqslant t} x_s.$$

Proof. The result follows by Lemma A.9 and the fact that for each $y \in \iota_\alpha(\bar{E})$ and $t \geqslant 0$ we have $\inf_{-1\leqslant s\leqslant t} y_s = \inf_{0\leqslant s\leqslant t} y_s$. □

Remark 5.3. The same example given in Remark 4.3 (now interpreted as a sequence of paths in $D([-1,\infty))$) shows that Lemma 5.2 is no longer true in general if we drop the requirement that $x^n, x \in \iota_\alpha(\bar{E})$.

Lemma 5.4. *Let $x^n, x \in \iota_\alpha(\bar{E})$ and suppose that $x^n \to x$ in $D([-1,\infty))$ where x satisfies the crossing property*

$$\inf_{0\leqslant s\leqslant h}(x_{\tau_0+s} - x_{\tau_0}) < 0, \quad h > 0, \tag{5.3}$$

where we write τ_0 as a shorthand for $\tau_0(x)$. Then, it follows that

$$\lim_{n\to\infty} \lambda_t(x^n) = \lambda_t(x),$$

holds for all $t \in [0,\infty)$ in a co-countable set.

Proof. Let D be a co-countable set such that every point in D is a continuity point of x. For all $t \in D$ such that $\inf_{0\leqslant s\leqslant t} x_s \neq 0$ Lemma 5.2 yields

$$\lim_{n\to\infty} \lambda_t(x^n) = \lim_{n\to\infty} \mathbb{1}_{\left[\inf_{s\in[0,t]} x^n_s \leqslant 0\right]} = \mathbb{1}_{\left[\inf_{s\in[0,t]} x_s \leqslant 0\right]} = \lambda_t(x).$$

Now suppose $t \in D$ satisfies $\inf_{s\in[0,t]} x_s = 0$. Then $t \geqslant \tau_0(x)$ by definition. Observe that if $t > \tau_0(x)$, by (5.3) and the fact that $x_{\tau_0} \leqslant 0$ we have that $\inf_{0\leqslant s\leqslant t} x_s < 0$, leading to a contradiction. It follows that the only point in D where convergence can fail is τ_0, so $D\backslash\{\tau_0\}$ is a co-countable set on which we have the desired convergence. □

The following lemma lifts the continuity of the loss functional shown in the previous lemma to probability measures and is the analogue of Proposition 5.8 in Delarue et al. (2015a). We follow the proof almost verbatim.

Lemma 5.5. *Assume that $(\xi^n)_{n\in\mathbb{N}}$ is a convergent sequence of probability measures on $\bar{E}$ with limit ξ. Define $\mu^n := \iota_\alpha(\xi^n)$ and $\mu := \iota_\alpha(\xi)$. If μ-almost every path satisfies the crossing property* (5.3)*, i.e., we have*

$$\mu\Big(\big\{x \in D([-1,\infty)) : \inf_{0\leqslant s\leqslant h}(x_{\tau_0+s} - x_{\tau_0}) = 0\big\}\Big) = 0, \quad h > 0, \tag{5.4}$$

then $\lim_{n\to\infty}\langle \mu^n, \lambda\rangle = \langle \mu, \lambda\rangle$ holds in M.

Proof. By Theorem 4.2, we have $\mu^n \to \mu$ in $\mathcal{P}(D([-1,\infty)))$. For brevity, denote $\langle \mu^n, \lambda_t\rangle$ by ℓ^n_t. Note that for all $n \in \mathbb{N}$ it holds that $\ell^n \in M$, and therefore, by compactness of M we may assume that $\ell^n_t \to \ell_t$ for some $\ell \in M$ at all points $t \geqslant 0$ where ℓ is continuous. We devote the rest of the proof to proving that $\ell = \langle \mu, \lambda\rangle$.

Let $T > 0$ and let f be a bounded and measurable function from $[0,T]$ to $\mathbb{R}$. The dominated convergence theorem implies

$$\lim_{n\to\infty} \int_0^T \ell^n_t f_t \,\mathrm{d}t = \int_0^T \ell_t f_t \,\mathrm{d}t.$$

By the Skorokhod representation theorem, we may write

$$\int_0^T \ell^n_t f_t \,\mathrm{d}t = \mathbb{E}\left[\int_0^T \lambda_t(Y^n) f_t \,\mathrm{d}t\right],$$

where $(Y^n)_{n\in\mathbb{N}}$ is a sequence of processes distributed according to $(\mu^n)_{n\in\mathbb{N}}$ and converging almost surely in $D([-1,\infty))$ towards some process Y with law μ. By assumption, Y^n and Y almost surely satsify the assumptions of Lemma 5.4, therefore, by the dominated convergence theorem, it holds almost surely that

$$\lim_{n\to\infty}\int_0^T \lambda_t(Y^n)f_t\ \mathrm{d}t = \int_0^T \lambda_t(Y)f_t\ \mathrm{d}t.$$

Using the dominated convergence theorem yet another time, we obtain

$$\int_0^T \ell_t f_t\ \mathrm{d}t = \lim_{n\to\infty}\mathbb{E}\left[\int_0^T \lambda_t(Y^n)f_t\ \mathrm{d}t\right] = \mathbb{E}\left[\int_0^T \lambda_t(Y)f_t\ \mathrm{d}t\right] = \int_0^T \langle\mu,\lambda_t\rangle f_t\ \mathrm{d}t,$$

where the last inequality follows from the fact that the law of Y is μ. Using right-continuity, this proves that $\ell_t = \langle\mu,\lambda_t\rangle$ for all $t\in[0,T]$. As $T>0$ was arbitrary the claim follows. □

Using a martingale convergence argument, we prove that the dynamics of limit points of the particle system are essentially given by Brownian motion in the next lemma.

Lemma 5.6. *For almost every realization ω, if* $\mathrm{law}((W,L)) = \xi(\omega)$, *then $W - W_0$ is a Brownian motion with respect to the filtration generated by (W,L). In particular, $W-W_0$ is independent of W_0.*

Proof. We employ Lévy's characterization of Brownian motion. Note the (general) fact that if M^1, M^2 are independent, continuous $(\mathcal{F}_t^N)_{t\geqslant 0}$-martingales, then the product M^1M^2 is an $(\mathcal{F}_t^N)_{t\geqslant 0}$-martingale as well and we have

$$\mathbb{E}\left[(M_t^1 - M_s^1)(M_t^2 - M_s^2)\mid \mathcal{F}_s^N\right] = 0. \tag{5.5}$$

Now let $g_i \in C_b(\mathbb{R}^2)$ for $i = 1,\dots,n$ and let $0\leqslant s_1 < \cdots < s_n \leqslant s < t$, and set $G(w,\ell) := \prod_{i=1}^n g(w_{s_i},\ell_{s_i})$. Writing $B_{t,s}^i$ for $B_t^i - B_s^i$ and $W^{i,N}$ for $X_{0-}^{i,N} + B_t^i$, we calculate

$$\begin{aligned}\mathbb{E}\Bigg[\bigg(\int_{\bar E}(w_t - w_s)G(w,\ell)\ d\xi_N(w,\ell)\bigg)^2\Bigg] &= \mathbb{E}\Bigg[\bigg(\frac{1}{N}\sum_{i=1}^N B_{t,s}^i G(W^{i,N},L^N)\bigg)^2\Bigg]\\ = \frac{1}{N}\mathbb{E}\left[(B_{t,s}^1 G(W^{1,N},L^N))^2\right] &+ \frac{N^2-N}{N^2}\mathbb{E}\left[B_{t,s}^1 B_{t,s}^2 G(W^{1,N},L^N)G(W^{2,N},L^N)\right].\end{aligned}$$

The cross-term can be rewritten as $\mathbb{E}\left[\mathbb{E}\left[B_{t,s}^1B_{t,s}^2\mid\mathcal{F}_s^N\right]G(W^{1,N},L^N)G(W^{2,N},L^N)\right]$, which vanishes by (5.5). It follows by uniform integrability that

$$\begin{aligned}\mathbb{E}\Bigg[\bigg(\int_{\bar E}(w_t - w_s)G(w,\ell)\ d\xi(w,\ell)\bigg)^2\Bigg] &= \lim_{N\to\infty}\mathbb{E}\left[\left(\int_{\bar E}(w_t-w_s)G(w,\ell)\ d\xi_N(w,\ell)\right)^2\right]\\ &= \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[(B_{t,s}^1G(W^{1,N},L^N))^2\right] = 0.\end{aligned}$$

Using (right-)continuity and the fact that the Borel σ-algebra on $\bar E$ is generated by the evaluation mappings, we conclude that for almost every realization ω, if $\mathrm{law}((W,L)) =$

$\xi(\omega)$, then $W - W_0$ is a martingale with respect to the filtration generated by (W, L). Repeating this reasoning with $M^i_{t,s} := (B^i_t)^2 - (B^i_s)^2 - (t-s)$ in place of $B^i_{t,s}$, we find that

$$\mathbb{E}\left[\left(\int_{\bar{E}} [(w_t - w_0)^2 - (w_s - w_0)^2 - (t-s)]G(w,l)\, d\xi(w,\ell)\right)^2\right] = 0,$$

which shows that $(W_t - W_0)^2 - t$ is a martingale with respect to the filtration generated by (W, L). By Lévy's characterization of Brownian motion the claim follows. □

The next lemma shows that the limiting measures of particle systems such as (1.3) satisfy the crossing property, rendering the loss functional continuous. Results of this kind have been obtained in (Delarue et al., 2015a, Lemma 5.9) and (Ledger and Søjmark, 2018, Lemma 3.13). We follow a slightly different route here.

Lemma 5.7. *Suppose that* $\mathrm{law}(\mu_N) \to \mathrm{law}(\mu)$ *for some random variable* μ. *Then* μ *satisfies the crossing property* (5.4) *almost surely.*

Proof. Recall that by (5.2) we have $\mathrm{law}(\mu) = \mathrm{law}(\iota_\alpha(\xi))$.

Fixing $h > 0$ we can then compute

$$\begin{aligned}
&\mathbb{E}\big[\mu\big(\big\{x \in D([-1,\infty))\colon \inf_{0\leqslant s\leqslant h}(x_{\tau_0+s} - x_{\tau_0}) = 0\big\}\big)\big] \\
&\quad = \mathbb{E}\big[\xi\big(\big\{(w,\ell) \in \bar{E}\colon \inf_{0\leqslant s\leqslant h}[(w_{\tau_0+s} - w_{\tau_0}) - \alpha(\ell_{\tau_0+s} - \ell_{\tau_0})] = 0\big\}\big)\big] \\
&\quad \leqslant \mathbb{E}\big[\xi\big(\big\{(w,\ell) \in \bar{E}\colon \inf_{0\leqslant s\leqslant h}(w_{\tau_0+s} - w_{\tau_0}) = 0\big\}\big)\big],
\end{aligned}$$

where the inequality is due to the fact that ℓ is increasing. Observe that for almost every realization ω, if $\mathrm{law}((W,L)) = \xi(\omega)$, then $\tau_0 = \tau_0(\iota_\alpha(W,L))$ is a stopping time with respect to the filtration generated by (W,L). Since $W - W_0$ is a Brownian motion with respect to the same filtration by Lemma 5.6, the strong Markov property yields

$$\mathbb{E}\big[\xi\big(\big\{(w,\ell) \in \bar{E}\colon \inf_{0\leqslant s\leqslant h}(w_{\tau_0+s} - w_{\tau_0}) = 0\big\}\big)\big] = \mathbb{P}\big(\inf_{0\leqslant s\leqslant h} B_s = 0\big) = 0.$$

We have shown that

$$\mu\big(\big\{x \in D([-1,\infty)) : \inf_{0\leqslant s\leqslant h}(x_{\tau_0+s} - x_{\tau_0}) = 0\big\}\big) = 0$$

holds almost surely. Repeating this reasoning for $h = h_n$ for a sequence $(h_n)_{n\in\mathbb{N}}$ such that $h_n > 0$ and $\lim_{n\to\infty} h_n = 0$ yields the result. □

The next result shows that weak limits of laws pertaining to the particle system (1.3) correspond to laws of solution processes to the McKean–Vlasov problem (1.2).

Proposition 5.8. *For* $N \in \mathbb{N}$, *let* (X^N, L^N) *be a solution to the particle system*

$$\begin{cases} X^{i,N}_t = X^{i,N}_{0-} + B^i_t - \alpha L^N_t \\ \tau_{i,N} = \inf\{t \geqslant 0 : X^{i,N}_t \leqslant 0\} \\ L^N_t = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N}\leqslant t]}, \end{cases} \tag{5.6}$$

and let $(\mu_N)_{N\in\mathbb{N}}$ *denote the corresponding empirical measures* (5.1). *Suppose that for some random variable* μ *and some measure* $\nu_{0-} \in \mathcal{P}(\mathbb{R})$ *we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}_{0-}} = \nu_{0-},$$

and $\mathrm{law}(\mu_N) \to \mathrm{law}(\mu)$ *along some subsequence. Then* μ *coincides almost surely with the law of a solution process to the McKean–Vlasov problem* (1.2) *for* $\mathrm{law}(X_{0-}) = \nu_{0-}$.

Proof. Without loss of generality we may assume that $\mu_N = \iota_\alpha(\xi_N)$ (and thus (4.3)) and set $\mu = \iota_\alpha(\xi)$. Observe that the map $t \mapsto \mathbb{E}\left[\langle \mu, \lambda_t\rangle\right]$ is increasing, and therefore has at most countably many discontinuities, and the same holds for the map $t \mapsto \mathbb{E}\left[\int \ell_t \, d\xi(w,\ell)\right]$. Let J be the set of discontinuities of these maps and fix $t \notin J$. Then $\langle \mu, \lambda_{t-}\rangle = \langle \mu, \lambda_t\rangle$ almost surely. By Lemma 5.7 we can apply Lemma 5.5 to obtain that

$$\lim_{N\to\infty} \langle \mu_N, \lambda_t\rangle = \langle \mu, \lambda_t\rangle \tag{5.7}$$

holds almost surely. In the next three steps we prove that μ coincides almost surely with the law of a solution process to (1.2).

<u>Step 1:</u> We show that for almost every realization ω, if $\mathrm{law}((W,L)) = \xi(\omega)$, then $L \equiv \langle \mu(\omega), \lambda\rangle$ almost surely. Note that we have

$$\mathbb{E}\left[\int_{\bar{E}} \big||\ell_t - \langle \mu, \lambda_t\rangle| - |\ell_t - \langle \mu_N, \lambda_t\rangle|\big| \; \mathrm{d}\xi_N(w,\ell)\right] \leqslant \mathbb{E}\left[|\langle \mu, \lambda_t\rangle - \langle \mu_N, \lambda_t\rangle|\right]$$

which vanishes as $N \to \infty$ by (5.7) and the dominated convergence theorem. By choice of $t \notin J$, the map $\ell \to \ell_t$ is continuous from M to $\mathbb{R}$ for ξ-almost every ℓ, and it follows that

$$\begin{aligned}\mathbb{E}\left[\int_{\bar{E}} |\ell_t - \langle \mu, \lambda_t\rangle| \; \mathrm{d}\xi(w,\ell)\right] &= \lim_{N\to\infty} \mathbb{E}\left[\int_{\bar{E}} |\ell_t - \langle \mu, \lambda_t\rangle| \; \mathrm{d}\xi_N(w,\ell)\right] \\ &\leqslant \lim_{N\to\infty} \mathbb{E}\left[\int_{\bar{E}} |\ell_t - \langle \mu_N, \lambda_t\rangle| \; \mathrm{d}\xi_N(w,\ell)\right].\end{aligned}$$

Since $\langle \mu_N, \lambda\rangle = \ell$ holds for ξ_N-almost every $\ell \in M$, almost surely, we can conclude that this expression is equal to 0. The claim follows by letting t range through a countable dense subset of $[0,\infty)\backslash J$ and using right-continuity.

<u>Step 2:</u> We show that for almost every realization ω, if $\mathrm{law}((W,L)) = \xi(\omega)$, then $\mathrm{law}(W_0) = \nu_{0-}$. To that end, let that the evaluation map $\overline{\pi}_0$ defined on $\bar{E}$ be given by $(w,\ell) \mapsto w_0$. Since $\overline{\pi}_0$ is a continuous map, letting $\overline{\pi}_0(\xi)$ denote the pushforward of ξ by $\overline{\pi}_0$, the continuous mapping theorem shows

$$\overline{\pi}_0(\xi) = \lim_{N\to\infty} \overline{\pi}_0(\xi_N) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}_{0-}} = \nu_{0-}.$$

Step 3: We conclude by showing that for almost every realization ω, if $\mathrm{law}(X) = \mu(\omega)$, then $\mathrm{law}(X_{0-}) = \nu_{0-}$ and $X - X_{0-} + \langle \mu(\omega), \lambda \rangle$ is a Brownian motion independent of X_{0-}. Since $\iota_{-\alpha}(\xi) = \mu$, by Step 1 we know that if $\mathrm{law}((W, L)) = \xi(\omega)$

$$\mathrm{law}((W_0, W - W_0)) = \mathrm{law}((X_{0-}, X - X_{0-} + \langle \mu(\omega), \lambda \rangle)).$$

The claim now follows by Step 2 and Lemma 5.6. □

6 Identification of the Limit

Having proved convergence of the particle system to solutions of the McKean–Vlasov problem, we are now interested in characterizing the limit points of minimal and physical solutions of the particle system. We shed some light on this intricate topic by either perturbing the initial condition of the particle system (Section 6.1) or of the McKean–Vlasov problem (Section 6.2).

6.1 The perturbed particle system

The next result is a key technical result for the theorems in this section.

Proposition 6.1. *Fix $\alpha > 0$, $\gamma \in (0, 1/2)$, and let $\underline{\tilde{L}}^N$ denote the perturbed minimal solution to the particle system, which for $N \in \mathbb{N}$ is the minimal solution to the perturbed particle system*

$$\begin{cases} X_t^{i,N} &= X_{0-}^i + \alpha N^{-\gamma} + B_t^i - \alpha L_t^N \\ \tau_{i,N} &= \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\} \\ L_t^N &= \dfrac{1}{N} \displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N} \leqslant t]}. \end{cases} \tag{6.1}$$

If $\underline{\Lambda}$ is the minimal solution to the McKean–Vlasov problem (1.2), *then*

$$\limsup_{N \to \infty} \mathbb{E}\left[\underline{\tilde{L}}_t^N\right] \leqslant \underline{\Lambda}_t, \tag{6.2}$$

holds for every $t > 0$.

Proof. The key idea of this proof is to compare both $\underline{\Lambda}$ and $(\underline{\tilde{L}}^N)_{N \in \mathbb{N}}$ to a sequence of optimizers of suitable optimization problems. To that end, let $\varepsilon > 0$ be such that $\beta := \gamma + \varepsilon < 1/2$. Set $\mathcal{R}$ to be the set of all random variables $\mu : \Omega \to \mathcal{P}(E)$. Fix $t > 0$ and define the cost functional

$$c_N(\mu) := \langle \mu, \lambda_t \rangle + N^\beta \|\Gamma_N[\langle \mu, \lambda \rangle] - \langle \mu, \lambda \rangle\|_\infty,$$

where Γ_N is the operator introduced in (3.1) and $\|\cdot\|_\infty$ is the uniform norm on $[0, \infty)$. For $t \geqslant 0$, let V_t^N be the optimal value

$$V_t^N := \inf_{\mu \in \mathcal{R}} \mathbb{E}\left[c_N(\mu)\right]. \tag{6.3}$$

The definition (6.2) depends on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. However, our comparison argument works for any choice of $(\Omega, \mathcal{F}, \mathbb{P})$ as long as $\mu_N \in \mathcal{R}$ for all $N \in \mathbb{N}$

and for any $\mu \in \mathcal{R}$, the map $\omega \mapsto c_N(\mu(\omega))$ is measurable. As (6.2) does not depend on the choice of probability space, it is sufficient to carry out the argument for a specific choice of $(\Omega, \mathcal{F}, \mathbb{P})$. For the construction of a probability space satisfying the required properties see Example C.1 in Section C in the appendix.

Step 1: We show that asymptotically, $\underline{\Lambda}$ is an upper bound for V_t^N. Taking μ to be constant and equal to the law of the minimal solution process of (1.2), that is $\mu(\omega) = \underline{\mu}$, we find

$$V_t^N \leqslant \underline{\Lambda}_t + N^\beta \mathbb{E}\left[\|\Gamma_N[\underline{\Lambda}] - \underline{\Lambda}\|_\infty\right].$$

Note that by definition, $\Gamma_N[\underline{\Lambda}]$ is the N-th empirical distribution function associated to the iid sequence $\tau^1[\underline{\Lambda}], \tau^2[\underline{\Lambda}], \dots$, where

$$\tau^i[\underline{\Lambda}] := \inf\{t \geqslant 0 : X_{0-}^i + B_t^i - \alpha\underline{\Lambda}_t \leqslant 0\}, \quad i \in \mathbb{N},$$

and that $\mathbb{P}\left(\tau^i[\underline{\Lambda}] \leqslant t\right) = \underline{\Lambda}_t$ for each t. By the Dvoretzky-Kiefer-Wolfowitz inequality (cf Corollary 1 in Massart (1990)), it holds for $x > 0$ that

$$\mathbb{P}\left(\sqrt{N}\|\Gamma_N[\underline{\Lambda}] - \underline{\Lambda}\|_\infty > x\right) \leqslant 2\exp(-2x^2).$$

It therefore follows that

$$\begin{aligned}
\mathbb{E}\left[N^\beta\|\Gamma_N[\underline{\Lambda}] - \underline{\Lambda}\|_\infty\right] &= \int_0^\infty \mathbb{P}\left(N^\beta\|\Gamma_N[\underline{\Lambda}] - \underline{\Lambda}\|_\infty > x\right)\,\mathrm{d}x \\
&= \int_0^\infty \mathbb{P}\left(N^{1/2}\|\Gamma_N[\underline{\Lambda}] - \underline{\Lambda}\|_\infty > N^{1/2-\beta}x\right)\,\mathrm{d}x \\
&\leqslant 2\int_0^\infty \exp(-2N^{1-2\beta}x^2)\,\mathrm{d}x \\
&= \sqrt{\frac{\pi}{2}}N^{\beta-1/2}.
\end{aligned}$$

As $\beta < 1/2$, this quantity vanishes as $N \to \infty$, which proves

$$\limsup_{N\to\infty} V_t^N \leqslant \underline{\Lambda}_t.$$

Step 2: We show that asymptotically, $\limsup_{n\to\infty}\mathbb{E}[\underline{\tilde{L}}_t^N] \leqslant \limsup_{n\to\infty} V_t^N$. Consider a sequence of $\mathcal{P}(E)$-valued random variables $(\zeta_N)_{N\in\mathbb{N}}$ such that

$$\mathbb{E}\left[c_N(\zeta_N)\right] - V_t^N \leqslant \frac{1}{N}, \quad N \in \mathbb{N}.$$

Plugging the empirical measure associated to the solution of the N-particle system into the objective function in (6.3) yields $V_t^N \leqslant 1$ and consequently

$$\mathbb{E}\left[N^\beta\|\Gamma_N[\langle\zeta_N, \lambda\rangle] - \langle\zeta_N, \lambda\rangle\|_\infty\right] \leqslant 1 + \frac{1}{N}, \quad N \in \mathbb{N}. \tag{6.4}$$

Let $A_N := [\|\Gamma_N[\langle\zeta_N, \lambda\rangle] - \langle\zeta_N, \lambda\rangle\|_\infty \leqslant N^{-\gamma}]$, then the Markov inequality together with (6.4) yield

$$\mathbb{P}\left(A_N^c\right) \leqslant N^{\gamma-\beta}(1 + \frac{1}{N}) \leqslant 2N^{-\varepsilon}. \tag{6.5}$$

For the last step, in analogy to (3.1), define the operator $\tilde{\Gamma}_N$ via

$$\begin{cases} \tilde{X}^{i,N}_t[\ell] := X^i_{0-} + \alpha N^{-\gamma} + B^i_t - \alpha \ell_t \\ \tilde{\tau}_{i,N}[\ell] := \inf\{t \geqslant 0 : \tilde{X}^{i,N}_t[\ell] \leqslant 0\} \\ \tilde{\Gamma}_N[\ell]_t := \dfrac{1}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{[\tilde{\tau}_{i,N}[\ell] \leqslant t]}. \end{cases}$$

On the set A_N, we have by definition

$$\langle \zeta_N, \lambda_t \rangle \geqslant \Gamma_N[\langle \zeta_N, \lambda \rangle]_t - N^{-\gamma}, \quad t \geqslant 0. \tag{6.6}$$

As $\langle \zeta_N, \lambda \rangle$ is nonnegative and Γ_N is monotone, this implies for all $t \geqslant 0$

$$\langle \zeta_N, \lambda_t \rangle \geqslant \Gamma_N[-N^{-\gamma}]_t - N^{-\gamma} = \tilde{\Gamma}_N[0]_t - N^{-\gamma}$$

Using the monotonicity of Γ_N and applying (6.6) again, this leads to

$$\langle \zeta_N, \lambda_t \rangle \geqslant \Gamma_N[\tilde{\Gamma}_N[0] - N^{-\gamma}]_t - N^{-\gamma} = \tilde{\Gamma}^{(2)}_N[0]_t - N^{-\gamma}$$

for all $t \geqslant 0$. A straightforward induction shows that

$$\langle \zeta_N, \lambda_t \rangle \geqslant \tilde{\Gamma}^{(k)}_N[0]_t - N^{-\gamma}, \quad t \geqslant 0$$

holds for all $k \in \mathbb{N}$ on the event A_N. Lemma 3.1 shows that $\tilde{\Gamma}^{(N)}_N[0] = \underline{\tilde{L}}^N$ and so, finally we obtain that on A_N we have

$$\langle \zeta_N, \lambda \rangle \geqslant \underline{\tilde{L}}^N - N^{-\gamma}. \tag{6.7}$$

As $\underline{\tilde{L}}^N$ is bounded by 1, we find

$$0 \leqslant \mathbb{E}\big[\underline{\tilde{L}}^N\big] - \mathbb{E}\big[\underline{\tilde{L}}^N \mathbb{1}_{A_N}\big] = \mathbb{E}\big[\underline{\tilde{L}}^N \mathbb{1}_{A_N^c}\big] \leqslant \mathbb{P}\left(A_N^c\right) \tag{6.8}$$

which goes to zero as $N \to \infty$ by (6.5). Conditions (6.8) and (6.7) yields

$$\begin{aligned} \limsup_{N\to\infty} \mathbb{E}\big[\underline{\tilde{L}}^N_t\big] &= \limsup_{N\to\infty} \mathbb{E}\big[\underline{\tilde{L}}^N_t \mathbb{1}_{A_N}\big] \leqslant \limsup_{N\to\infty} \mathbb{E}\big[\langle \zeta_N, \lambda_t \rangle \mathbb{1}_{A_N}\big] \\ &\leqslant \limsup_{N\to\infty} \mathbb{E}\left[c_N(\zeta_N)\right] = \limsup_{N\to\infty} V^N_t. \end{aligned}$$

Combining Step 1 and Step 2 then yields $\limsup_{N\to\infty} \mathbb{E}\big[\underline{\tilde{L}}^N_t\big] \leqslant \limsup_{N\to\infty} V^N_t \leqslant \underline{\Lambda}_t$. □

By virtue of Proposition 6.1, we are now in a position to prove a propagation of chaos result for the perturbed particle system. We obtain this result without imposing any restrictions on the distribution of the initial condition X_{0-}.

Theorem 6.2 (Propagation of chaos, perturbed)**.** *Fix $\gamma \in (0, 1/2)$ and define $(\underline{\tilde{X}}^N, \underline{\tilde{L}}^N)$ to be the minimal solution to the perturbed particle system* (6.1)*. Let $\underline{\mu}$ be the law of the minimal solution process $\underline{X}$ to the McKean–Vlasov problem* (1.2)*. Then, it holds that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \delta_{\underline{\tilde{X}}^{i,N}} = \underline{\mu}$$

in probability on $\mathcal{P}(D([-1,\infty))$. Furthermore, the sequence of loss functions $(\underline{\tilde{L}}^N)_{N\in\mathbb{N}}$ converges to $\underline{\Lambda}$ in probability with respect to the Lévy-metric, i.e., for every $\varepsilon > 0$ it holds that

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\underline{\tilde{L}}^N, \underline{\Lambda}) > \varepsilon\right) = 0,$$

where d_L denotes the Lévy-metric.

Proof. We follow the classical sequence of arguments, showing tightness of the empirical measures first, then their convergence to the law of a solution's process to the McKean–Vlasov problem, and finally we check that the corresponding solution is minimal.

Step 1: Since the sequence $(X^1_{0-} + \alpha N^{-\gamma})_{N\in\mathbb{N}}$ is tight, tightness of the empirical measures follows from Corollary 4.7.

Step 2: Let $\underline{\tilde{\mu}}_N$ be the empirical measures associated to the minimal solution to (6.1). Then by Step 1 there is a random variable μ such that, after passing to a subsequence if necessary, $\text{law}(\underline{\tilde{\mu}}_N) \to \text{law}(\mu)$. Moreover, because of Varadarajan's theorem (cf (Dudley, 2018, Theorem 11.4.1)) and the fact that uniformly continuous functions are convergence determining (cf. Proposition 3.4.4 in Ethier and Kurtz (2009)) we can establish that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_{0-}+\alpha N^{-\gamma}} = \nu_{0-}$$

holds almost surely in $\mathcal{P}(\mathbb{R})$. Proposition 5.8 then shows that μ coincides almost surely with the law of a solution process to the McKean–Vlasov problem (1.2).

Step 3: We now show that $\mu = \underline{\mu}$ almost surely, i.e. that μ coincides with law of the minimal solution's process to (1.2). Let $J \subseteq [0,\infty)$ be the countable set of discontinuities of the increasing map $t \mapsto \mathbb{E}\left[\langle \mu, \lambda_t \rangle\right]$ and fix $t \notin J$. Then, arguing as in the proof of Proposition 5.8, we obtain that $\lim_{N\to\infty}\langle \underline{\tilde{\mu}}_N, \lambda_t\rangle = \langle \mu, \lambda_t\rangle$ almost surely. To be precise the convergence holds for a subsequence of a representative sequence of $(\underline{\tilde{\mu}}_N)_{n\in\mathbb{N}}$ but we can assume without loss of generality that it coincides with the original one. Letting D be a countable, dense subset of $[0,\infty)$ with $D \cap J = \varnothing$, we then have that

$$\lim_{N\to\infty} \langle \underline{\tilde{\mu}}_N, \lambda_t \rangle = \langle \mu, \lambda_t \rangle, \quad t \in D \tag{6.9}$$

holds almost surely. By Step 2 and the definition of minimal solution, we have $\langle \mu, \lambda \rangle \geqslant \underline{\Lambda}$ almost surely. The dominated convergence theorem and Proposition 6.1 imply

$$\mathbb{E}\left[\langle \mu, \lambda_t \rangle\right] = \lim_{N\to\infty} \mathbb{E}\left[\langle \underline{\tilde{\mu}}_N, \lambda_t \rangle\right] = \lim_{N\to\infty} \mathbb{E}\big[\underline{\tilde{L}}^N_t\big] \leqslant \underline{\Lambda}_t, \quad t \in D.$$

We conclude that $\mathbb{P}\left(\underline{\Lambda}_t = \langle \mu, \lambda_t \rangle,\ t \in D\right) = 1$ and therefore we have $\underline{\Lambda} = \langle \mu, \lambda \rangle$ almost surely by right-continuity. It follows that $\mu = \underline{\mu}$ almost surely.

Step 4: By (6.9) and right continuity we know that $\text{law}(\underline{\tilde{L}}^N) \to \text{law}(\langle \mu, \lambda \rangle)$. Since $\text{law}(\langle \mu, \lambda \rangle) = \delta_{\underline{\Lambda}}$ by Step 3, we can conclude that $\underline{\tilde{L}}^N \to \underline{\Lambda}$ in probability on M. □

Similarly to the situation in the particle system, it turns out that physical solutions to the McKean–Vlasov problem have minimal jumps, which is the content of the following proposition.

Proposition 6.3. *Suppose* (X, τ, Λ) *solves* (1.2). *Then it holds that*

$$\Delta\Lambda_t \geqslant \inf\{x > 0 : \mathbb{P}(\tau \geqslant t, X_{t-} \in [0, \alpha x]) < x\}$$

for any $t \geqslant 0$.

Proof. See Proposition 1.2 in Hambly et al. (2019). □

It has been established in previous works that weak limits of physical solutions of the particle system (1.3) correspond to physical solutions to the McKean–Vlasov problem (1.2). Recall that we call a solution (X, Λ) to (1.2) physical, if it satisfies the physical jump condition (1.4).

Theorem 6.4 (Physical solutions converge to physical solutions)**.** *Let* $(\hat{X}^N, \hat{L}^N)$ *be a physical solution to the particle system* (5.6), *where* $X_{0-}^{i,N} = X_{0-}^i + a_N$ *for some deterministic sequence* $(a_N)_{N\in\mathbb{N}}$ *converging to* 0 *and some iid sequence* $(X_{0-}^i)_{i\in\mathbb{N}}$ *such that* $\mathbb{E}[X_{0-}^i] < \infty$. *Then, if for some* $\hat{\Lambda} \in M$ *we almost surely have* $\hat{L}^N \to \hat{\Lambda}$ *along some subsequence in* M, *it follows that* $\hat{\Lambda}$ *is a physical solution to the McKean–Vlasov problem* (1.2).

Proof. See Section B in the appendix. □

The next theorem gives a positive answer to a conjecture of Delarue, Nadtochiy and Shkolnikov (see (Delarue et al., 2019, Section 5.2)).

Theorem 6.5. *Suppose that* $\mathbb{E}[X_{0-}] < \infty$. *Then, the minimal solution* $(\underline{X}, \underline{\Lambda})$ *to the McKean–Vlasov problem* (1.2) *is physical.*

Proof. Let $(\tilde{\underline{X}}^N, \tilde{\underline{L}}^N)$ be a minimal solutions to the perturbed particle system (6.1). By Theorem 6.2 we know that $\tilde{\underline{L}}^N \to \underline{\Lambda}$ in probability on M. We also know by Lemma 3.5 that $\tilde{\underline{L}}^N$ is a physical solution to the perturbed particle system. After passing to a subsequence if necessary, we may assume that $\tilde{\underline{L}}^N \to \underline{\Lambda}$ almost surely on M. Theorem 6.4 then shows that $\underline{\Lambda}$ is a physical solution to the McKean–Vlasov problem (1.2). □

Theorem 6.6 (Propagation of minimality)**.** *Suppose that* $\mathbb{E}[X_{0-}] < \infty$ *and that the physical solution to the McKean–Vlasov problem* (1.2) *is unique. Denote by* $(\underline{X}^N, \underline{L}^N)$ *the minimal solution to the particle system* (1.3). *Let* $\underline{\mu}$ *be the law of the minimal solution process* $\underline{X}$ *to the McKean–Vlasov problem* (1.2). *Then, it holds that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\underline{X}^{i,N}} = \underline{\mu} \tag{6.10}$$

in probability on $\mathcal{P}(D([-1, \infty)))$. *Furthermore, the sequence of loss functions* $(\underline{L}^N)_{N\in\mathbb{N}}$ *converges to* $\underline{\Lambda}$ *in probability with respect to the Lévy-metric, i.e., for every* $\varepsilon > 0$ *it holds that*

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\underline{L}^N, \underline{\Lambda}) > \varepsilon\right) = 0,$$

where d_L *denotes the Lévy-metric.*

Proof. We argue again in the classical way. Tightness of the empirical measures (6.10) follows from Corollary 4.7 and their convergence to the law of a solution process of the McKean–Vlasov problem (1.2) follows by Proposition 5.8. We know from Lemma 3.5 that minimal solutions of the particle system are physical, which allows us to apply Theorem 6.4, telling us that the limit must be the unique physical solution to the McKean–Vlasov problem. By Theorem 6.5, the physical solution must be equal to the minimal solution. □

Remark 6.7. By (Delarue et al., 2019, Theorem 1.4), if the initial condition X_{0-} admits a bounded Lebesgue density that changes monotonicity finitely often on compact subsets of $[0, \infty)$, the physical solution to the McKean–Vlasov problem (1.2) is unique.

6.2 The perturbed McKean–Vlasov problem

A modification of the proof of Proposition 6.1 allows us to "shift" the perturbation from the particle system to the McKean–Vlasov problem.

Proposition 6.8. *Fix $\alpha > 0$, $\gamma \in (0, 1/2)$, and for $x \in \mathbb{R}$ let $\underline{\Lambda}(x)$ denote the minimal solution of the perturbed McKean–Vlasov problem*

$$\begin{cases} X_t = X_{0-} - x + B_t - \alpha\Lambda_t \\ \tau = \inf\{t \geqslant 0 : X_t \leqslant 0\} \\ \Lambda_t = \mathbb{P}\left(\tau \leqslant t\right). \end{cases} \tag{6.11}$$

If $\underline{L}^N$ is the minimal solution to the particle system (1.3)*, then*

$$\limsup_{N\to\infty} \mathbb{E}\left[\underline{L}_t^N\right] \leqslant \limsup_{N\to\infty} \underline{\Lambda}_t(\alpha N^{-\gamma}), \tag{6.12}$$

for every $t > 0$.

Proof. The proof is analogous to the proof of Proposition 6.1, only that now we consider the sequence of cost functionals

$$c_N(\mu) := \langle\mu, \lambda_t\rangle + N^{\beta}\|\Gamma_N[\langle\mu, \lambda\rangle + N^{-\gamma}] - \langle\mu, \lambda\rangle\|_\infty$$

where $\beta = \gamma + \varepsilon$. □

Shifting the perturbation to the McKean–Vlasov problem allows us to show that propagation of minimality holds true for Lebesgue almost every (fixed) additive perturbation of the initial condition.

Theorem 6.9 (Propagation of chaos, almost everywhere)**.** *For $x \in \mathbb{R}$, let $(\underline{X}^N(x), \underline{L}^N(x))$ be the minimal solution to the particle system*

$$\begin{cases} X_t^{i,N} = X_{0-}^i - x + B_t^i - \alpha L_t^N \\ \tau_{i,N} = \inf\{t \geqslant 0 : X_t^{i,N} \leqslant 0\} \\ L_t^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \mathbb{1}_{[\tau_{i,N}\leqslant t]}, \end{cases}$$

Furthermore, let $\underline{\mu}(x)$ be the law of the minimal solution's process $\underline{X}(x)$ to the perturbed McKean–Vlasov problem (6.11)*. Then, there is a co-countable set $D \subseteq \mathbb{R}$ such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\underline{X}^{i,N}(x)} = \underline{\mu}(x)$$

in probability on $\mathcal{P}(D([-1,\infty))$ for $x \in D$. Furthermore, the sequence of loss functions $(\underline{L}^N(x))_{N\in\mathbb{N}}$ converges to $\underline{\Lambda}(x)$ in probability with respect to the Lévy-metric for every $x \in D$, i.e., for every $\varepsilon > 0$ it holds that

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\underline{L}^N(x), \underline{\Lambda}(x)) > \varepsilon\right) = 0,$$

where d_L denotes the Lévy-metric.

Proof. Let D_0 be a countable dense subset of $[0,\infty)$. Fix $t \in D_0$ and note that the map $x \mapsto \underline{\Lambda}_t(x)$ is increasing and therefore has at most countably many discontinuities. Let the set of all such discontinuities be denoted by J_t. Then, $J := \bigcup_{t\in D_0} J_t$ is countable, and we set $D := [0,\infty)\backslash J$. Then, for all $x \in D$ it follows from Proposition 6.8 that we have

$$\limsup_{N\to\infty} \mathbb{E}\left[\underline{L}_t^N(x)\right] \leqslant \limsup_{N\to\infty} \underline{\Lambda}_t(x + \alpha N^{-\gamma}) = \underline{\Lambda}_t(x), \quad t \in D_0.$$

The remainder of the proof is analogous to the proof of Theorem 6.2. □

Following the proof of Theorem 6.9 we can see that if we would have stability of the minimal solution to the McKean–Vlasov problem under additive perturbations of the initial condition, we would obtain propagation of minimality as in Theorem 6.6 without having to assume that the physical solution to the McKean–Vlasov problem is unique. We conjecture that such a stability result holds true.

Conjecture 6.10. The map $x \mapsto \underline{\Lambda}(x)$ is continuous from $\mathbb{R}$ to M.

Appendices

A On the M_1- and J_1-topologies

A.1 Why the J_1-topology is ill-suited to the problem

Of the four topologies initally proposed by Skorokhod, the J_1-topology is the most popular, and often simply referred to as "the Skorokhod topology". However, for the present purpose, the J_1-topology seems to be simply too strong - in particular, local accumulations of small jumps can obstruct convergence in the J_1-topology. We illustrate this point with an example, for which we need the following theorem.

Theorem A.1. *Let ℓ^n, ℓ be increasing càdlàg functions on $[0, \infty)$. Then, $\ell^n \to \ell$ as $n \to \infty$ in the J_1-topology if and only if there is a dense subset $D \subseteq [0, \infty)$ consisting of continuity points of ℓ such that for all $t \in D$*

(i) $\ell_t^n \to \ell_t$

(ii) $\sum_{s \leqslant t} |\Delta \ell_s^n|^2 \to \sum_{s \leqslant t} |\Delta \ell_s|^2$.

Proof. See Theorem VI.2.15 in Jacod and Shiryaev (2013) □

Example A.2. For $n \in \mathbb{N}$, let ℓ^n be an increasing step function such that ℓ^n has n jumps of size $1/n$ in the interval $[1/2 - 1/n, 1/2 + 1/n]$ and is constant otherwise. Interpreting ℓ_t^n as the proportion of banks that defaults up to time t, this would mean that for all $n \in \mathbb{N}$, all of the banks in the system default after time $1/2 - 1/n$ and before time $1/2 + 1/n$. Certainly, as n goes to infinity, we would expect the limit to be the function $\ell_t = \mathbb{1}_{[1/2,\infty]}(t)$, which corresponds to all the banks defaulting at time $1/2$. Assuming that ℓ^n converges to some function g in the J_1-topology, condition (i) in Theorem A.1 yields that $g(t) = \mathbb{1}_{[1/2,\infty]}(t)$. However, as $\sum_{s \leqslant t} |\Delta \ell_s^n|^2 = \frac{1}{n}$, condition (ii) of Theorem A.1 yields that g must be continuous, a contradiction.

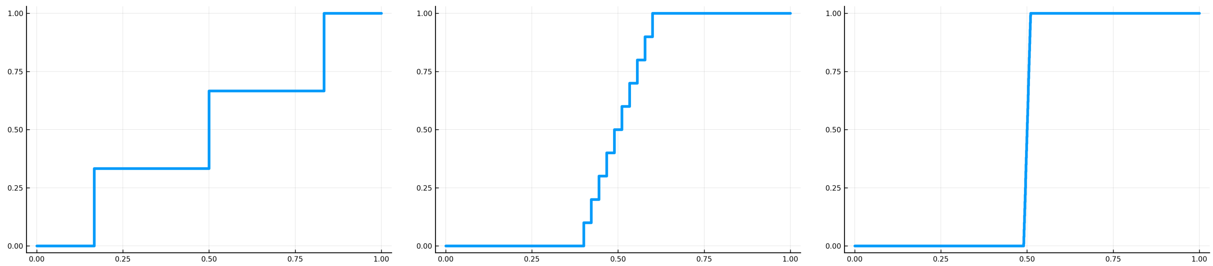

Figure 2: The functions ℓ^n as defined in Example A.2 for $n = 3, 10, 100$.

Roughly speaking, the J_1-topology allows for some flexibility in the location of jumps in convergent sequences, while requiring that the size of the jumps in the approximating sequence remains close to the size of the jumps in the limit in a certain sense. With this intuition in mind, it is not surprising that the space of continuous functions is closed in the space of càdlàg functions endowed with the J_1-topology. In contrast, continuous functions are dense in the space of càdlàg functions endowed with the M_1-topology, and the sequence given in Example A.2 is convergent in the M_1-topology.

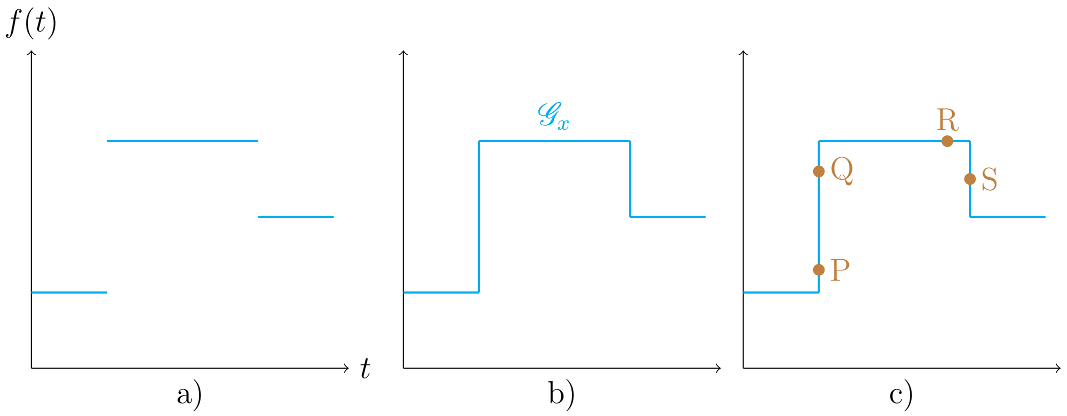

Figure 3: In a), the graph of a piecewise constant path x is plotted, b) shows the completion of its graph $\mathscr{G}_x$. In c), four points on the completed graph are marked to illustrate the order on the graph: For $A, B \in \mathscr{G}_x$, we have $A \preceq B$ iff A is reached before B when $\mathscr{G}_x$ is traced out from left to right. In the situation depicted in c), we find $P \prec Q \prec R \prec S$.

A.2 Some properties of the M_1-topology

The M_1-topology is strictly weaker than the J_1-topology (Example A.2 serves as an example of a sequence which converges in M_1 but not in J_1 in view of Lemma A.3). This is not necessarily a weakness: whenever we want to show tightness, a weaker topology is favorable because the conditions for compactness are less strict. We will see that the M_1-topology is particularly well-suited to deal with monotone functions in this context.

We mention some fundamental properties of the M_1-topology in the following. For a path $x \in D([T_0, T])$, the space of càdlàg paths defined on $[T_0, T]$ taking values in $\mathbb{R}$ that are left-continuous at T, we define the completed graph

$$\mathscr{G}_x := \{(t, y) \in [T_0, T] \times \mathbb{R} : y \in [x_{t-}, x_t]\},$$

where $[x_{t-}, x_t]$ is the non-ordered segment between x_{t-} and x_t (this takes into account that x_{t-} might be larger than x_t). The set $\mathscr{G}_x$ can be ordered in the following way: for $(t_1, y_1), (t_2, y_2) \in \mathscr{G}_x$, we say that $(t_1, y_1) \preceq (t_2, y_2)$ if either $t_1 < t_2$, or $t_1 = t_2$ and $|x_{t_1-} - y_1| \leqslant |x_{t_2-} - y_2|$. This order can be conceptualized more easily in the following way. The completion $\mathscr{G}_x$ can be imagined as a path in 2-dimensional space, where we complete the graph of x by connecting the discontinuities with straight lines going through the points (t, x_{t-}) and (t, x_t). We can imagine a particle traveling along $\mathscr{G}_x$ from left to right; for $A, B \in \mathscr{G}_x$, we have $A \preceq B$ iff the particle reaches A before it reaches B. This is illustrated in Figure A.2.

We then define a *parametric representation* of $\mathscr{G}_x$ to be a continuous function (r, u) that maps $[T_0, T]$ onto $\mathscr{G}_x$ such that $t \mapsto (r(t), u(t))$ is increasing with respect to $(\mathscr{G}_x, \preceq)$. Denote the set of all parametric representations of x as R_x.

For $x^1, x^2 \in D([T_0, T])$, we define the M_1-metric as

$$d_T^{M_1}(x^1, x^2) = \inf_{\substack{(r^j, u^j) \in R_{x^j} \\ j=1,2}} \max(\|r^1 - r^2\|_\infty, \|u^1 - u^2\|_\infty), \tag{A.1}$$

where $\|.\|_\infty$ is the supremum norm on $C([T_0, T])$. If we want to relate this to the picture with particles described earlier, we let the particles travel along the respective completed graphs and are allowed some freedom in choosing the velocities of the particles (albeit, due to the requirement that the parametric representations are increasing with respect to $(\mathscr{G}_x, \preceq)$, the velocities can never be negative). Then, two functions are close to each other in the M_1-topology, if there are velocity profiles for the particles such that the vertical and horizontal distance between the particles remains uniformly small.

We provide some fundamental results regarding the M_1-topology which play a crucial role in many proofs of this paper.

The space $D([T_0, T])$ endowed with the M_1-topology is a Polish space, even though the metric defined in (A.1) is incomplete. It is generally not very pleasant to work directly with the metric defined above, and we will seek to avoid doing so whenever possible. The following theorem will prove to be expedient in this endeavor, as it allows us to relate convergence in the M_1-topology to pointwise convergence when every path in the sequence is monotone. Let M be the space defined in (2.2).

Lemma A.3. *Let $(\ell^n)_{n\in\mathbb{N}}, \ell \subset M$. Then $\ell^n \to \ell$ in the M_1-topology if and only if $\ell^n_t \to \ell_t$ for each t in a subset of full Lebesgue measure of $[T_0, T]$ that includes T_0 and T.*

Proof. See (Whitt, 2002, Theorem 12.5.1). □

Convergence in the uniform norm implies convergence in the M_1-topology. The next theorem shows that if the limit path is continuous, the converse holds as well.

Lemma A.4. *Suppose that $x^n \to x$ in $D([T_0, T])$ equipped with the M_1-topology. Then we have locally uniform convergence at all continuity point of x. In particular, for all points t at which x is continuous it holds that $x^n_t \to x_t$.*

Proof. See (Whitt, 2002, Theorem 12.4.1). □

Similarly to the J_1-topology, the M_1-topology does not turn $D([T_0, T])$ into a topological vector space. In particular, addition is not continuous in general. However, the following result holds.

Lemma A.5. *Assume that $x^n \to x$ and $y^n \to y$ in $D([T_0, T])$ equipped with the M_1-topology. If x and y have no common jumps of opposite sign, that is*

$$\Delta x_t \cdot \Delta y_t \geqslant 0, \quad t \in [T_0, T],$$

then $x^n + y^n \to x + y$ in $D([T_0, T])$.

Proof. See (Whitt, 2002, Theorem 12.7.3). □

If $(\ell_n)_{n\in\mathbb{N}}$ is a sequence of distribution functions such that $\ell_n \to \ell$ in M, Lemma A.3 tells us that $\ell_n \to \ell$ in the M_1-topology if and only if ℓ^n converges pointwise to ℓ in the interval endpoints. To remove this restriction of convergence in the interval endpoints, we will consider processes on all of $[0, \infty)$, and (somewhat artifically) extend the domain to $[-1, \infty)$, where we let the processes stay constant for all negative times. Following Whitt (2002), we need to define the M_1-metric on noncompact domains.

Definition A.6. We say that $x^n \to x$ in $D([T_0,\infty))$ if x^n converges to x in $D([T_0,T_k])$ for each T_k in some sequence $(T_k)_{k\in\mathbb{N}}$ with $T_k \to \infty$, where the sequence $(T_k)_{k\in\mathbb{N}}$ may depend on x. For $t>0$, let $\hat{d}_t$ be a metric that makes $D([T_0,t])$ complete, then $D([T_0,\infty))$ is metrized by

$$d^{M1}_{[T_0,\infty]}(x^1,x^2) := \int_{T_0}^{\infty} e^{-t}(\hat{d}_t(x^1,x^2)\wedge 1)\ \mathrm{d}t.$$

Equipped with this metric, $D([T_0,\infty))$ is a Polish space by construction. We have the following equivalence.

Lemma A.7. *For $x^n, x \in D([T_0,\infty))$, convergence of x^n against x with respect to $d^{M_1}_{[T_0,\infty]}$ is equivalent to convergence of x^n against x with respect to $\hat{d}_t$ for all $t>T_0$ where x is continuous.*

Proof. See (Whitt, 2002, Theorem 12.9.3). □

Lemma A.8. *The Borel σ-field of $D([T_0,\infty))$ endowed with the M_1-topology is generated by the evaluation mappings.*

Proof. By (Jacod and Shiryaev, 2013, Theorem 1.14c), the claim holds for the Borel σ-field on $D([T_0,\infty))$ generated by the J_1-topology. By definition, the J_1-topology is stronger than the M_1-topology, and as any two comparable Lusin spaces have the same Borel sets (Schwartz, 1973, p.101), the claim follows. □

Lemma A.9. *Assume that $x^n \to x$ in $D([T_0,\infty))$. Then if $t\in[T_0,\infty)$ is a continuity point of x, it holds that $\lim_{n\to\infty}\inf_{s\in[T_0,t]} x^n_s = \inf_{s\in[T_0,t]} x_s$.*

Proof. Fix a continuity point t of x and choose $T>t$ as a continuity point of x as well. Then by Lemma A.7 we have $x^n \to x$ in $D([T_0,T])$. By Theorem 13.4.1 in Whitt (2002), the map $x \mapsto \inf_{s\in[T_0,\cdot]} x_s$ is continuous from $D([T_0,T])$ to $D([T_0,T])$, and therefore the claim follows from Lemma A.4. □

B Proofs regarding physical solutions

We start by introducing some useful notation.

Definition B.1. If (X,τ,Λ) is a solution to the McKean–Vlasov problem (1.2), we denote by ν_{t-} the marginal subprobability distribution at time $t-$ of the particles surviving up to time t, i.e.,

$$\nu_{t-}(A) := \mathbb{P}\left(\tau \geqslant t, X_{t-}\in A\right), \quad t\geqslant 0$$

and we denote the measure corresponding to the minimal solution as $\underline{\nu}$.

The next technical lemma is a key result when it comes to showing that physical solutions of the particle system converge to physical solutions of the McKean–Vlasov problem. Roughly speaking, it says that there is a very small chance of observing a macroscopic proportion of particles spiking at least twice in a small interval and can be seen as an extension of (3.8) to small intervals. This result in its original form is due to (Delarue et al., 2015a, Proposition 5.3). We follow the proof given in (Ledger and Søjmark, 2018, Lemma 3.10) here.

Lemma B.2. *Suppose that* $\mathbb{E}[X_{0-}] < \infty$ *and fix* $T > 0$. *Let* $(\hat{X}^N, \hat{L}^N)$ *be a physical solution to the particle system* (1.3) *and let* ν^N_{t-} *be the corresponding subprobabilty measure as defined in Definition 3.3. Then there is a constant* $C > 0$ *such that for every (sufficiently small)* $\varepsilon > 0$

$$\mathbb{P}\left(\nu^N_{t-}([0, \alpha z + C\varepsilon^{1/3}]) \geqslant z \quad \forall z \leqslant \hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-} - C\varepsilon^{1/3}\right) \geqslant 1 - C\varepsilon^{1/3}, \quad t < T,$$

whenever $N \geqslant \varepsilon^{-1/3}$.

Proof of Lemma B.2. Note that $N(\hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-})$ equals the number of particles defaulting in the interval $[t, t+\varepsilon]$. By definition, as $\hat{L}^N$ is a physical solution, if t_0 is any jump time in $[t, t+\varepsilon]$, we must have

$$\nu^N_{t_0-}\left(\left[0, \alpha\frac{k}{N}\right]\right) \geqslant \frac{k}{N} \quad \text{for} \quad k = 0, 1, \ldots, N\Delta\hat{L}^N_{t_0}.$$

Let $t_1, \ldots, t_m$ be the jump times of $\hat{L}^N$ in $[t, t+\varepsilon]$. If $k \leqslant N(\hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-})$, then there are numbers $k_1, \ldots, k_m$ such that $k_i \leqslant N\Delta\hat{L}^N_{t_i}$ and $k_1 + \cdots + k_m = k$, which then shows that

$$\sum_{i=1}^m \nu^N_{t_i-}\left(\left[0, \alpha\frac{k_i}{N}\right]\right) \geqslant \frac{1}{N}\sum_{i=1}^m k_i = \frac{k}{N}.$$

Now by definition, the number on the left-hand side of the above inequality is dominated by $\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_1^{i,k}}$ where

$$E_1^{i,k} := \big\{\hat{X}^{i,N}_{t-} - \alpha\frac{k}{N} - \varepsilon(1 + \sup_{s\leqslant t+\varepsilon}|\hat{X}^{i,N}_s|) - \sup_{h\leqslant\varepsilon}|B^i_{t+h} - B^i_t| \leqslant 0,\ \tau^{i,N} \geqslant t\big\}.$$

We have obtained that for $k = 0, 1, \ldots, N(\hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-})$

$$\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_1^{i,k}} \geqslant \frac{k}{N}. \tag{B.1}$$

Now fix $z \in \mathbb{R}$ such that $z \leqslant \hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-} - 2\varepsilon^{1/3}$ and set $k_0 := \lfloor z + 2\varepsilon^{1/3}\rfloor \leqslant N(\hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-})$. This choice implies that $z \geqslant \frac{k_0}{N} - 2\varepsilon^{1/3}$ as well as $\frac{k_0}{N} \geqslant z + 2\varepsilon^{1/3} - \frac{1}{N}$ by definition, and we also have that (B.1) holds for $k = k_0$. Define

$$E_2^i := \big\{\varepsilon(1 + \sup_{s\leqslant t+\varepsilon}|\hat{X}^{i,N}_s|) + \sup_{h\leqslant\varepsilon}|B^i_{t+h} - B^i_t| \geqslant \varepsilon^{1/3}\big\}.$$

On the event $E_1^{i,k_0} \cap (E_2^i)^c$, we have $\hat{X}^{i,N}_{t-} - \alpha\frac{k_0}{N} \leqslant \varepsilon^{1/3}$, and hence

$$\hat{X}^{i,N}_{t-} - \alpha z \leqslant \hat{X}^{i,N}_{t-} - \alpha\left(\frac{k_0}{N} - 2\varepsilon^{1/3}\right) \leqslant (1 + 2\alpha)\varepsilon^{1/3}.$$

It follows that

$$\nu^N_{t-}([0, \alpha z + (1+2\alpha)\varepsilon^{1/3}]) \geqslant \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_1^{i,k_0}}\mathbb{1}_{(E_2^i)^c}.$$

Finally, letting $E := \{\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_2^i} \leqslant \varepsilon^{1/3}\}$, we find that on the event E we have

$$\nu_{t-}^N([0, \alpha z + (1+2\alpha)\varepsilon^{1/3}]) \geqslant \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_1^{i,k_0}} - \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_2^i} \geqslant \frac{k_0}{N} - \varepsilon^{1/3}$$
$$\geqslant (z + 2\varepsilon^{1/3} - 1/N) - \varepsilon^{1/3} = z + \varepsilon^{1/3} - 1/N.$$

Taking $N \geqslant \varepsilon^{-1/3}$ implies $\varepsilon^{1/3} - 1/N \geqslant 0$, so we conclude that

$$\nu_{t-}^N([0, \alpha z + (1+2\alpha)\varepsilon^{1/3}]) \geqslant z \quad \text{on} \quad E,$$

for any $z \leqslant \hat{L}_{t+\varepsilon}^N - \hat{L}_{t-}^N - 2\varepsilon^{1/3}$. It remains to prove that there is a $C > 0$ such that $\mathbb{P}(E^c) \leqslant C\varepsilon^{1/3}$. To do that, note that $\mathbb{E}\big[\sup_{h\leqslant\varepsilon} |B_{t+h}^i - B_t^i|\big] \leqslant \sqrt{\pi/2}\varepsilon$ and

$$\mathbb{E}\big[1 + \sup_{s\leqslant T} |\hat{X}_s^{i,N}|\big] \leqslant 1 + \mathbb{E}[X_{0-}] + \mathbb{E}\big[\sup_{s\leqslant T}|B_s|\big] + \alpha \leqslant 1 + \alpha + \mathbb{E}[X_{0-}] + \sqrt{\pi/2}T.$$

Now letting $c \geqslant 1 + \alpha + \mathbb{E}[X_{0-}] + \sqrt{\frac{\pi}{2}}(1+T)$ we find, by applying the Markov inequality twice

$$\mathbb{P}(E^c) = \mathbb{P}\Big(\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{E_2^i} > \varepsilon^{1/3}\Big) \leqslant \varepsilon^{-1/3}\frac{1}{N}\sum_{i=1}^N \mathbb{P}(E_2^i)$$
$$\leqslant \varepsilon^{1/3}(\mathbb{E}\big[1 + \sup_{s\leqslant T}|\hat{X}_s^{i,N}|\big] + \mathbb{E}\big[\sup_{h\leqslant\varepsilon}|B_{t+h}^i - B_t^i|\big]) \leqslant 2c\varepsilon^{1/3}.$$

Now $C := \max(2c, (1+2\alpha))$ satisfies the requirements of the lemma. □

Proof of Theorem 6.4. Let $\hat{\mu}_N$ be the empirical measure corresponding to $\hat{X}^N$. From Corollary 4.8 we know that there are random variables $\hat{\xi}, \hat{\xi}_N$ such that, after passing to subsequences if necessary, $\text{law}(\hat{\mu}_N) = \text{law}(\iota_\alpha(\hat{\xi}_N))$, $\hat{\xi}_N \to \hat{\xi}$, and $\iota_\alpha(\hat{\xi}_N) \to \iota_\alpha(\hat{\xi})$ almost surely. Without loss of generality we may assume that $\hat{\mu}_N = \iota_\alpha(\hat{\xi}_N)$ and set $\hat{\mu} = \iota_\alpha(\hat{\xi})$. Note that this implies $\hat{\Lambda} = \langle \hat{\mu}, \lambda\rangle$ and thus, by Proposition 5.8, that $\hat{\Lambda}$ solves the McKean–Vlasov problem (1.2). In the following three steps we prove that $\hat{\Lambda}$ is physical by verifying Condition (1.4).

Step 1: We show convergence of the laws of the subprobability measures $\hat{\nu}_{t-}^N$. For $t \geqslant 0$, let $\pi_{t-}(x)$ denote the map $x \mapsto x_{t-}$ and define the transformation $S\colon D([-1,\infty)) \to \mathbb{R}$ through $S_{t-}(x) = \pi_{t-}(x)(1 - \lambda_{t-}(x))$. Note that for $f \in C_b(\mathbb{R})$ and $\mu \in \mathcal{P}(D([-1,\infty)))$ it holds that

$$\langle S_{t-}(\mu), f\rangle = \int f(x_{t-}\mathbb{1}_{[\tau_0(x)\geqslant t]})\, \mathrm{d}\mu(x) = \int f(x_{t-})\mathbb{1}_{[\tau_0(x)\geqslant t]}\, \mathrm{d}\mu(x) + f(0)\langle \mu, \lambda_{t-}\rangle.$$

It follows that

$$\langle \hat{\nu}_{t-}^N, f\rangle = \langle S_{t-}(\hat{\mu}_N), f\rangle - f(0)\hat{L}_t^N, \quad \langle \hat{\nu}_{t-}, f\rangle = \langle S_{t-}(\hat{\mu}), f\rangle - f(0)\hat{\Lambda}_t.$$

Let J be the set of discontinuity points of $\hat{\Lambda}$. We claim that $S_{t-} \circ \iota_\alpha$ is continuous at $\hat{\xi}$-almost every $(w, \ell) \in \bar{E}$ whenever $t \notin J$. To see this, consider that π_{t-} is continuous at all paths $x \in D([-1,\infty))$ such that t is a continuity point of x by Lemma A.4. Secondly, suppose that $x = \iota_\alpha(w,\ell)$ satisfies the crossing property (5.3). Then, applying Lemma 5.5 for $\xi = \delta_{(w,\ell)}$ and for a sequence $(\xi^n)_{n\in\mathbb{N}}$ converging to ξ with $\xi^n = \delta_{(w^n,\ell^n)}$, it follows that $\lambda_t \circ \iota_\alpha$ is continuous at (w,ℓ) except possibly for $t = \tau_0(x)$. The same holds for $\lambda_{t-} \circ \iota_\alpha$. We have

$$\hat{\xi}(\{(w,\ell) \in \bar{E} : \tau_0(\iota_\alpha(w,\ell)) = t\}) = \hat{\mu}(\{x \in D([-1,\infty)) \colon \tau_0(x) = t\}) = \mathbb{P}\left(\hat{\tau} = t\right)$$

and by definition, $\mathbb{P}\left(\hat{\tau} = t\right) > 0$ if and only if $t \in J$. Since $x = \iota_\alpha(w,\ell)$ satisfies (5.3) for $\hat{\xi}$-almost every $(w,\ell) \in \bar{E}$, we have thus proved the aforementioned continuity property of $S_{t-} \circ \iota_\alpha$. It follows from the Portmanteau theorem that for every $t \notin J$

$$\lim_{N\to\infty} S_{t-}(\hat{\mu}_N) = \lim_{N\to\infty} (S_{t-} \circ \iota_\alpha)(\hat{\xi}_N) = (S_{t-} \circ \iota_\alpha)(\hat{\xi}) = S_{t-}(\hat{\mu}),$$

almost surely on $\mathcal{P}(\mathbb{R})$. Therefore we obtain, for $t \notin J$,

$$\lim_{N\to\infty} \hat{\nu}^N_{t-} = \hat{\nu}_{t-},$$

almost surely on $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the space of subprobability measures on $\mathbb{R}$ endowed with the topology of weak convergence.

Step 2: Fix $T > 0$ and a sufficiently small $\varepsilon > 0$. We take the limit as $N \to \infty$ of

$$\mathbb{P}\left(\hat{\nu}^N_{t-}([0, \alpha z + C\varepsilon^{1/3}]) \geqslant z \quad \forall z \leqslant \hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-} - C\varepsilon^{1/3}\right) \geqslant 1 - C\varepsilon^{1/3}, \qquad t < T. \quad \text{(B.2)}$$

The above equation holds due to Lemma B.2, where we make use of the assumption $\mathbb{E}\left[X_{0-}\right] < \infty$. Fix $t, t+\varepsilon \in [0,T)\backslash J$ and $z \in \mathbb{R}$ such that $z < \hat{\Lambda}_{t+\varepsilon} - \hat{\Lambda}_{t-} - C\varepsilon^{1/3}$. Introduce the events

$$A^N_z := \{z \leqslant \hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-} - C\varepsilon^{1/3}\}.$$

By assumption, we know that $\hat{L}^N_{t+\varepsilon} - \hat{L}^N_{t-} \to \hat{\Lambda}_{t+\varepsilon} - \hat{\Lambda}_{t-}$, and therefore it holds that $\lim_{N\to\infty} \mathbb{P}\left(A^N_z\right) = 1$. Recalling Step 1 and (B.2), on applying the Portmanteau theorem and the reverse Fatou lemma we find

$$\begin{aligned}\hat{\nu}_{t-}[0, \alpha z + C\varepsilon^{1/3}] &\geqslant \mathbb{E}\big[\limsup_{N\to\infty} \hat{\nu}^N_{t-}[0, \alpha z + C\varepsilon^{1/3}]\big] \geqslant \limsup_{N\to\infty} \mathbb{E}\left[\hat{\nu}^N_{t-}[0, \alpha z + C\varepsilon^{1/3}]\right] \\ &\geqslant \limsup_{N\to\infty} \mathbb{E}\left[\hat{\nu}^N_{t-}[0, \alpha z + C\varepsilon^{1/3}]\mathbb{1}_{A^N_z}\right] \geqslant z(1 - C\varepsilon^{1/3}).\end{aligned}$$

We have established

$$\hat{\nu}_{t-}[0, \alpha z + C\varepsilon^{1/3}] \geqslant z(1 - C\varepsilon^{1/3}), \quad z < \hat{\Lambda}_{t+\varepsilon} - \hat{\Lambda}_{t-} - C\varepsilon^{1/3}, \quad t, t+\varepsilon \in [0,T)\backslash J.$$

Step 3: We take the limit as $\varepsilon \to 0$. To that end, consider that for $f \in C_b(\mathbb{R})$, by the dominated convergence theorem, the map

$$t \mapsto \int f(x_{t-}) \mathbb{1}_{[\tau_0(x) \geqslant t]} \, \mathrm{d}\hat{\mu}(x) = \langle \hat{\nu}_{t-}, f \rangle$$

is left-continuous, and hence we have $\lim_{s\nearrow t}\hat{\nu}_{s-} = \hat{\nu}_{t-}$ in $\mathcal{S}(\mathbb{R})$. Now fix $t \in [0,T)$ and let t_n, ε_n be such that $t_n, t_n + \varepsilon_n \notin J$ with $t_n < t < t_n + \varepsilon_n$, $t_n \nearrow t$ and $\varepsilon_n \searrow 0$. Let $0 \leqslant z < \Delta\hat{\Lambda}_t$, then as $\hat{\Lambda}$ is càdlàg , for all sufficiently large n we have $z < \hat{\Lambda}_{t_n+\varepsilon_n} - \hat{\Lambda}_{t_n-} - C\varepsilon_n^{1/3}$. Then, for any $\delta > 0$, by the Portmanteau theorem and Step 2, we obtain

$$\begin{aligned}\hat{\nu}_{t-}[0, \alpha z + \delta] &\geqslant \limsup_{n\to\infty} \hat{\nu}_{t_n-}[0, \alpha z + \delta] \geqslant \limsup_{n\to\infty} \hat{\nu}_{t_n-}[0, \alpha z + C\varepsilon_n^{1/3}] \\ &\geqslant \limsup_{n\to\infty} z(1 - C\varepsilon_n^{1/3}) = z.\end{aligned}$$

Letting $\delta \to 0$, as $T > 0$ was arbitrary we finally see that $\hat{\nu}_{t-}[0, \alpha z] \geqslant z$ for each $z < \Delta\hat{\Lambda}_t$ and $t \geqslant 0$, which implies

$$\Delta\hat{\Lambda}_t \leqslant \inf\{z > 0 \colon \hat{\nu}_{t-}[0, \alpha z] < z\}.$$

The reverse inequality follows from Proposition 6.3. □

C Supplements

Example C.1. Define

$$\Omega := C([0,\infty))^{\mathbb{N}}$$

and let ω be the the canonical process on $C([0,\infty))^{\mathbb{N}}$. We take $\mathcal{F}$ to be the product σ-field induced by the Borel σ-field on $C([0,\infty))$. Let $\mathbb{P} = \bigotimes_{n\in\mathbb{N}} \eta$ for a probability measure η on $C([0,\infty))$, such that the law of ω^i under η coincides with the law of $X_{0-} + B$ for a Brownian motion B and a random variable X_{0-}. We write $E := D([-1,\infty))$ in the following.

Let $\mathcal{R}$ be the space of all random variables $\mu\colon \Omega \to \mathcal{P}(E)$ and let $\mu \in \mathcal{R}$. As the Borel σ-field in $\mathcal{P}(E)$ is generated by the mappings $\mu \mapsto \mu(A)$ where A is a Borel set in E (cf. (Carmona and Delarue, 2018, Proposition 5.7)), any such map is measurable. In addition, Lemma A.9 shows that the map $x \mapsto \inf_{s\leqslant\cdot} x_s$ is continuous on E, and as the evaluation mappings $\pi_t := x \mapsto x_t$ are measurable, it follows that the set

$$A_t := \bigl\{x \in E \colon \pi_t\bigl(\inf_{s\leqslant\cdot} x_s\bigr) \in (-\infty, 0]\bigr\}$$

is a Borel set in E. We find that $\langle\mu, \lambda_t\rangle = \mu(A_t)$, so we obtain that $\omega \mapsto \langle\mu(\omega), \lambda_t\rangle$ is measurable for every $t \geqslant 0$. As the Borel σ-field on E is generated by the evaluation mappings, it follows that $\omega \mapsto \langle\mu(\omega), \lambda\rangle$ is measurable as a map into E.

By Lemma A.5, for $N \in \mathbb{N}$ the map

$$\begin{aligned}\Psi^N \colon E \times C([0,\infty))^{\mathbb{N}} &\to \mathcal{P}(E) \\ (x, \omega) &\mapsto \frac{1}{N}\sum_{i=1}^{N} \delta_{\omega^i - \alpha x}\end{aligned}$$

is continuous. It follows that

$$\Gamma_N[\langle\mu, \lambda\rangle](\omega) = \langle\Psi^N(\langle\mu(\omega), \lambda\rangle, \omega), \lambda\rangle = \Psi^N(\langle\mu(\omega), \lambda\rangle, \omega)(A_t)$$

and is thus measurable. This shows that the map $\omega \mapsto c_N(\mu(\omega))$ is measurable for every $\mu \in \mathcal{R}$. Defining

$$\mu_N^{(1)} := \Psi^N(0, \omega), \quad \mu_N^{(k)} := \Psi^N(\langle\mu_N^{(k-1)}(\omega), \lambda\rangle, \omega), \quad k \in \mathbb{N},$$

by Lemma 3.1 we find that $\mu_N = \mu_N^{(N)} \in \mathcal{R}$.

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