The Context of Tunings

Thirds and Septimal Intervals in Ancient Greek Music

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ZUSAMMENFASSUNG

Aus der griechischen Antike sind von mehreren Autoren Intervallangaben für verschiedene ‚Stimmungen' überliefert. Sämtliche Systeme vom Ende des fünften vorchristlichen bis zum zweiten nachchristlichen Jh. werden hier im jeweiligen Kontext mathematisch-philosophischer Theorie und zeitgenössischer Musikpraxis untersucht. Dies ermöglicht abzuschätzen, welche spezifischen Eigenheiten tatsächliche Instrumentenstimmungen widerspiegeln und welche mathematischer Idealisierung entspringen. Besonderes Augenmerk liegt dabei auf reinen Terzen und Septimalintervallen und ihrer potentiellen Rolle in den verschiedenen Epochen antiker Musik.

1. THIRDS AND SEPTIMAL INTERVALS

Most people are familiar with thirds: there is a major and a minor one, and in this paper we are going to use the unqualified term third for these two intervals. Our focus will be on pure, i.e. resonant[1], thirds, represented by pitch ratios of 5:4 and 6:5 respectively. Impure thirds, on the other hand, are a side product of any pentatonic or diatonic tuning, and therefore of little interest to us as such. As their ratios suggest, pure thirds are characterized by the prime number five (while the lower prime numbers two and three are associated with the most resonant intervals: the octave and the fifth and fourth, respectively).

Septimal intervals are those whose mathematical description as ratios of pitches involves the number seven. In this paper, mainly the two simple, 'superparticular'[2] variants come into focus:

- the 'septimal third', 7:6, which is somewhat smaller than the minor third, and
- the 'septimal tone', 8:7, which is a bit larger than the large whole tone.

These intervals, which are largely absent from Western music, play a substantial role in the mathematical tunings given by Archytas in the early 4th century BC, and by Ptolemy in the 2nd century AD. On the other hand, in the five hundred years between these two outstanding philosophers not one of the music theorists we know of made use of septimal intervals, although they present us with a lot of thirds.

Furthermore, septimal intervals and resonant (minor and major) thirds are almost mutually exclusive, at least on stringed instruments of the lyre type, such as Ptolemy has in mind. A look at the tuning tables which are found in Ptolemy's *Harmonics* reveals the full consequences of the septimal approach: in the six cithara tuning patterns he gives, each comprising one octave, we encounter not one major or minor third, but no fewer than nineteen septimal thirds or tones (Table 1)[3].

This is all the more remarkable because in ancient lyre playing often two or more strings

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1 The term 'resonance' is to be understood as denoting the objective physical basis of 'consonance'; cf. Franklin 2005, 12–13.

2 Superparticular (or 'epimoric') ratios are those of the form $(n+1)/n$. Since they are found to correspond, together with multiple ratios of the form $k \cdot n$, to pitch relations perceived as concordant, mathematical relations of this type are of primary importance in Greek musical theory of Pythagorean hue.

3 According to Ptolemy, in certain cases the voice used a slightly different tuning from the instrument. Taking these deviations into account, some resonant thirds are possible between the voice and the instrument, mainly at the expense of the resonant fifths and fourths. The figures are then as follows: *lýdia* sung: *3:2*: 2; *4:3*: 2; *5:4*: 1; *6:5*: 2; *7:6*: 0; *8:7*: 1; *iastiaiólia* sung: *3:2*: 1; *4:3*: 1; *5:4*: 1; *6:5*: 2; *7:6*: 0; *8:7*: 1; Total in all tunings: *3:2:* 13; *4:3*: 18; *5:4*: 3; *6:5*: 3; *7:6*: 8; *8:7*: 10. That makes: fifths and fourths: 31; resonant thirds: 6; septimal intervals: 18.

	fifth	fourth	major third	minor third	sept. third	sept. tone
Tuning name	**3:2**	**4:3**	**5:4**	**6:5**	**7:6**	**8:7**
lýdia	3	4	–	–	–	1
trópoi	2	3	–	–	2	2
hypértropa	4	3	–	–	2	2
iastiaiólia	3	4	–	–	1	1
trítai	2	5	–	–	2	2
parypátai	2	4	–	–	2	2
	16	**23**	**0**	**0**	**9**	**10**
	39		0		19	

Table 1 Pure intervals in Ptolemy's *cithara* tunings.

were sounded together[4]. The well-known 'negative' playing technique of sweeping the plectrum over the whole range of strings with the right hand, while muting those which are not to be heard with the fingers of the left, for instance[5], a technique which painters seem to have considered the most typical one, makes sense only if more than one note is played simultaneously. One might assume that this kind of music making led to tunings comprising a maximum of maximally concordant intervals. Such tunings, however, must include major and minor thirds, and even a larger number of fifths and fourths than Ptolemy's. Nevertheless, Ptolemy provides tests, with the help of which, he claims, his musically trained contemporaries were able to assess the exact correspondence between his figures and the tunings of citharodic practice. His scientific method even earned Ptolemy the faith of more critical scholars, who dismissed much of the earlier accounts as purely mathematical constructions.

Before we pursue the history of Greek tuning mathematics further, let us consider possible musical meanings of septimal intervals. Most naturally these arise in the context of trumpet playing. When the series of overtones is produced on a natural horn, the seventh harmonic produces a septimal third with the sixth, and a septimal tone with the eighth. Trumpets, however, play no distinctly 'musical' role in Classical antiquity or in the ancient Near East, serving rather as signal instruments in military or cultic context. Therefore we can hardly attribute the origin of septimal intervals on the lyre to the trumpet, which would presume a transfer of musical structure from a marginal instrument to one of central cultural focus.

Yet overtones also occur as partials, where the ear takes them as a clue to the fundamental frequency of a given note (to such an extent that the actually perceived basic frequency need not even be present in the oscillatory pattern). This built-in mechanism of the human auditory system can be put to additional use in musical cultures for defining a tonal hierarchy: by picking out partials of the tonic, or their octave counterparts, as (melodic or harmonic) notes, the status of the tonic can be brought out or reinforced. Thus, in Western music the third partial serves as the 'dominant', and the fifth and sixth build the 'major chord' above the tonic (which is repeated at octave intervals by the second, fourth and eighth partials); cf. Diagram 1.

A similar mechanism can be derived from the seventh partial when combined with the sixth. Together with the eight partial, these pitches form a typical sequence of septimal third and septimal tone, which points to its highest note as a tonic, insofar this note constitutes an octave counterpart of the 'basic frequency' of the 6–7–8 series. If arbitrary sequences of these notes are played, one gets the distinct feeling that the septimal intervals create a tension which is subsequently relieved only by returning to the tonic (in the guise of its eighth overtone)[6]. This is a possible musical function of the septimal intervals, if arranged in the right order below a melodic focal note. To be sure, nothing of

4 For heterophony in ancient Greek music, see West 1992, 103f.; 205–207; Barker 1995; Hagel 2004, 374–379. For lyre playing, see especially Ptol., *Harm.* 2.12, p. 67.6–11 Düring.

5 Cf. Roberts 1980; West 1992, 64–70; Lawson 2005.

6 This was demonstrated at the conference, using computer-generated pitches. It is important to note that this effect is not due to Western preconceptions, but arises *in spite of* our hearing conventions. In Western music, the septimal intervals play a role in the septimal chord (with the note in question tuned too high on most instruments), which is built on the basic note of the harmonic series it constitutes part of, and consequently resolves to a chord based on another note, namely that a fifth below ($C^7 \rightarrow F$). Still, the fundamental note is also present in the tonic chord in the second hierarchical position (f – a – c̲).

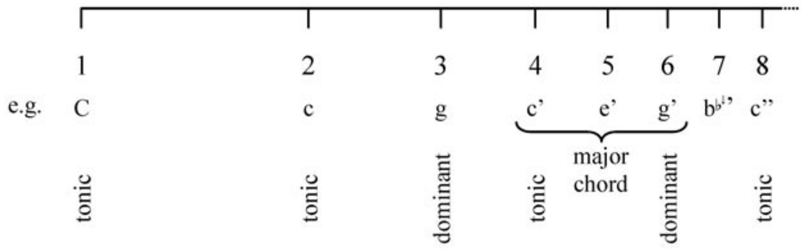

Diagram 1 The overtone series as defining the tonic.

this kind is ever said in ancient sources; but at any rate, the question of tonal hierarchies is hardly ever addressed there.

Another possible application of septimal intervals concerns the division of the fourth. Just as a major and a minor third add up to a fifth ($5/4 \times 6/5 = 3/2$), the fourth can be divided into a septimal third and a septimal tone ($7/6 \times 8/7 = 4/3$). This is the simplest division of the fourth into two superparticular intervals, simplest both in the sense that the smallest numbers are involved, and that the two parts are as similar in size as possible. But what might be its significance in practice? Within Greek scales, the fourth was regularly divided into three intervals, so that an even melodic bisection per se was irrelevant. Only if it was for some reason desirable to play the two bounding notes of a fourth together with one of the inner notes, does the septimal division make sense: it produces, for the fourth, what a major or minor chord is for the fifth. Even so, the sound is not very resonant; and again, there is not the slightest evidence for such a convention, or for any kind of three-note harmony in the ancient world[7].

2. ANCIENT TUNING AND FINE TUNING

But what do we know of the intervals of ancient Greek music, apart from what Pythagoreanizing theorists tell us? By the 4th century BC, there was a confusing variety of scalar 'shades', differing only by microtonal amounts. Earlier, certain theorists had tried to establish the quartertone as a general measure[8], but Aristoxenus finds it necessary to introduce even finer divisions[9]. On top of this, he freely admits that one can only give examples of musically useful divisions of the pitch continuum, while there is a principally infinite number of possibilities – although these must adhere to a fixed set of rules. Nevertheless, even the weirdest scales in practical use were analyzed against a heptatonic standard: for the ancients, it seemed self-evident that an octave 'naturally' comprises seven notes, however these are arranged. And secondly, the framework of these scales, as well as modulations between them, followed a pattern of fifths and fourths. It is one of the most important insights of the last years that these two constants, underlying heptatony and the pattern of fifths and fourths, derive from an age-old tradition of lyre-playing, which we find first attested in a Babylonian text from the early 2nd millennium BC, going back to Sumerian sources[10]. Within this tradition, stringed instruments were tuned in alternating fifths and fourths, until none of the steps in the resulting scale was larger than a tone; and the strings were arranged in order of pitch[11]. Thus, fifths and fourths were doubtlessly, and in perfect accordance with the textual evidence, of primary importance in Greek music. Still, after a general tuning was set up using these maximally resonant intervals, some notes could be altered to achieve specific fine-tunings[12]. Here we would again expect, above all, resonant thirds. The primary 'Pythagorean'[13] diatonic produces dissonant major and minor thirds. These can subsequently be changed to resonant thirds by minimal changes of string

7 By "three-note harmony", I do not refer to three-note sounds in which a functional two-note interval comes to be played on three strings by means of octave doubling. Cf. Hagel 2005b on ancient Near Eastern 'harmonic', i.e. 'dichordal', music.

8 Barker 1982; West 1992, 225; Hagel 2000, 181-182.

9 Cf. e.g. West 1992, 168-169.

10 Franklin 2002; Franklin 2002a. While Franklin argues for the introduction of Near Eastern music into Greece in the Orientalizing Period, I have considered possible evidence for an unbroken line of Greek heptatony from at least the second half of the 2nd millennium BC on (Hagel 2005a).

11 Interestingly enough, both these defining features are not true for the last surviving lyre cultures in Africa, which employ pentatonic tuning (achieved by alternating fifths and fourths until no gap larger than a minor third remains) in re-entrant string arrangement.

12 This is not to imply that the instrumentalists did always tune by pure fifths and fourth and only then fine-tune. Although this may have been convenient when newly stringing a lyre, I have found it much more convenient in daily use to tune resonant thirds directly.

13 The tuning in pure fourths and fifths came to be termed as 'Pythagorean' mostly because it appears in the fragments of the Pythagorean Philolaus, and was adopted by Plato in the *Timaeus*. The term is, however, misleading: by the time of Plato, the 'Pythagorean' Archytas had already proposed his mathematically refined tunings.

tension, while maintaining the general character of the scale as a sequence of tones and semitones. The extant fragments of Hurrian music from the second half of the 2nd millennium BC show that precisely such a procedure was followed at least in parts of the Near Eastern tradition[14]. Septimal intervals, on the other hand, call for considerably larger modifications; but Greek music of the Classical era was without doubt almost fully emancipated from strictly diatonic/heptatonic forms and had adopted even much more marked deviations from a 'Pythagorean' tuning standard. So far, nothing speaks a priori for, but also not much against septimal intervals in ancient Greek music.

As regards Greek fine tuning, it is now generally accepted that in the enharmonic genus one of the dissonant 'Pythagorean' thirds was often 'sweetened' to resonance by slightly raising the lower note (cf. Diagram 2)[15]. This is a plausible reconstruction, although Aristoxenus discusses the phenomenon not under the aspect of resonance, but talks about a tendency of decreasing the large melodic gap of two whole tones[16]. Yet Archytas' mathematical description, which predates Aristoxenus' account by several decades, gives the figures for a resonant third at the scalar position in question, so that it is more than tempting to connect the notion of 'sweetness' with this now much more concordant interval. Accordingly, there is strong evidence for the employment of resonant thirds beside fifths and fourths not long after 500 BC. Aristoxenus leaves no doubt that the 'sweeter' tuning has become standard by his time. Whether earlier music really favoured the harsh ditone, as he claims as "obvious to those acquainted with the first and second ancient styles", can be doubted; but there must have been a basis for his argument, and it is unlikely that the performances he alludes to were the result of a falsely archaizing movement, which had replaced former resonant intervals by discords.

Important for our study is that Archytas' mathematically pure third is supported by a witness as independent as we could wish: Aristoxenus not only displays emphatic disinterest in musical ratios[17], he even expresses his dislike for the 'modern' kind of tuning. Accordingly, scholars were not unwilling to take the rest of Archytas' figures, including the septimal intervals, as oriented towards musical practice, as well.

Now Archytas was the first to publish a meaningful mathematical description not only of the diatonic genus with its tones and semitones, but also of the chromatic and the enharmonic, which contain pairs of semitones and quartertones, respectively. Moreover, his diatonic differs significantly from the standard version, as we know it from Philolaus and Plato, which reproduces the tuning in alternating fifths and fourths, leading to an awkward semitone of 256:243, known as the *leîmma*. Archytas strove for mathematical elegance as expressed by the use of superparticular ratios and the application of his newly developed theory of mathematical means.

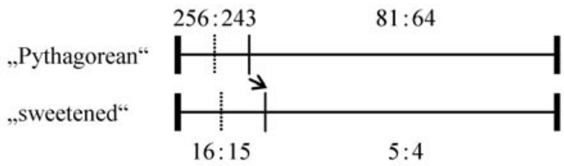

Diagram 2 'Sweetening' of the enharmonic tetrachord.

3. METHODOLOGY

Before we discuss the possible musical significance of Archytas' figures, we ought to be very precise about our methodology in dealing with mathematical tunings of the Pythagorean tradition. Firstly, we must acknowledge that all ancient writers who set out for a new mathematical description of music believed, at least to a certain extent, that the beauty of music was ultimately due to the beauty of the numerical structures which it incorporated, so that sensuous pleasure could be translated into aesthetical values perceived by reason alone. As a consequence, it was not doubted that the consonances were identical with numerical ratios between the lowest numbers, although no experiment could have been devised to prove that a fifth, for instance, corresponds to a string length ratio of 3:2 rather than, say, 3001:2001. Only Aristoxenus, perhaps the only innovative author to care for music for its own sake, acknowledged this epistemological problem, conceding that consonance might even occupy a certain space in the continuum of intervals, instead of just a point, as the definition by ratios implies[18]. Correspondingly, we cannot expect the authors to give accurate results of experiments – even if they can be assumed to have carried out any at all –, whenever a mathematically more satisfying description was

14 Hagel 2005b.

15 Winnington-Ingram 1932, 200; cf. Barker 1989, 50; Barker 2000, 122; Franklin 2005, 26–28.

16 Aristox., *Harm.* 1.23, 29.14–30.8 Da Rios.

17 Aristox., *Harm.* 2.32, 41.19–42.3 Da Rios.

18 Aristox., *Harm.* 2.55, 68.10–12 Da Rios. Aristoxenus was forced to part with the traditional view in order to establish a tonal space that allowed for every kind of modulation. As this implied using the full circle of fifths, he had to postulate fifths (and fourths) of equal temperament, which indeed cannot be distinguished from pure fifths by the experimental means of antiquity (cf. Hagel 2000,17–20).

at hand that did not violate the musical facts all too much. To put it the other way round, we are a priori justified to take the numerical values as mere approximations, judging the possible aberration individually for each author.

On the other hand, we can expect clues to the underlying musical practice wherever we encounter definite lack of mathematical beauty: here the author was obviously forced to forsake ideological principles in order to keep up with reality. Yet even in such cases, we are by no means justified in taking the ratios literally, i.e. as the interval sizes actually used. There were other elements which had to be taken into account, and which were much more conspicuous than interval sizes. In the first place, we have to reckon with identification between functionally different notes. Modulations between different genera or keys, quite frequent in late Classical and Hellenistic music, often required that the same string on an instrument was used, in different contexts, for functionally different notes – which were sometimes even notated by different signs. Their theoretical pitches, however, differed little, if at all, and in any case the differences were so minute that they did not justify the introduction of an extra string – which is a valuable resource on an instrument of the lyre type[19]. As long as frequent modulation was part of the musical culture, theoretical writers could hardly overlook such equations between notes, and we will come across examples where considerations of that kind clearly played a role. We must also keep in mind that for each author the musical conventions of his period must be considered.

Another kind of information that was much more readily accessible to ancient authors than actual interval sizes is the tuning procedure by which a given scale was set up on a stringed instrument. Consequently, we can expect that attention is sometimes given to intervals important in tuning. Only where this is the case there is also a good chance that the figures reflect the actual intervals of the scale: an exact tuning can be achieved only by means of resonance, which in turn is indeed adequately described by ratios of small numbers[20].

A last clue concerns only Ptolemy, the only author in question whose work we can read (almost) completely. As he discusses his methodical principles extensively, we can detect where he fails to apply these principles. Such cases, if not accounted for otherwise, may again reflect musical practice.

Finally, we are well advised to apply some kind of Ockham's razor. Wherever we can explain the figures given by a certain author by such reasons as given above – knowledge about tuning procedures, identification of strings, mathematical principles –, we are not justified in assuming that experiments had been carried out to further assess the validity of the construction, unless we are expressly told that this was the case.

4. ARCHYTAS, I

Let us now turn to Archytas' account. His ratios for the division of the fourth in all three genera are set out in Diagram 3. Since it has long been seen that Archytas constructed his system with respect to the whole tone that often resides below tetrachords when these are put together into scales (cf. e.g. the Classical Dorian as shown in Diagram 4)[21], this 'disjunctive' tone is also printed in the diagram (modern note names serve only for orientation and do not indicate absolute pitch).

The most striking characteristic of Archytas' division is that he gives the same lowest interval for all three genera, so that the lower of the two 'movable' notes coincide. This has been associated with the fact that these notes are also written with an identical sign in the ancient Greek musical notation, the invention of which is usually placed in the fifth century. It seemed therefore that only Archytas has preserved a most interesting detail of late Classical or early post-Classical music[22].

5. PHILOLAUS

But this can hardly be true. Not only do we find all later theorists unanimous in attributing different pitches to the notes in question, but this is obviously also true for the only earlier source, Philolaus' work, fragments of which survive in later treatises[23]. Philolaus' account is all the more significant because he gives ratios only for the diatonic, while approaching the intervals of the other two genera by improper additive calculations. Thus, his picture of enharmonic and chromatic intervals emerges quite plainly. Actually, Philolaus tries to connect the Pythagorean approach, which requires multiplication but had been elaborated only for the diatonic genus, with the simple way of adding intervals together, as inferred by the human ear and enshrined linguistically in expressions such

19 In a very similar way, functionally different notes, such as d sharp and e flat, are equated on many modern instruments.

20 For an investigation of resonance behind Greek theoretical tunings cf. Franklin 2005.

21 Cf. Tannery 1995, 78 with n. 1; 111; Winnington-Ingram 1932, 206–207; Winnington-Ingram 1936, 25–28; West 1992, 221; Hagel 2000, 89–93; Franklin 2005, 29 and passim.

22 Cf. e.g. Tannery 1915, 110.

23 Boeth., *Inst. mus.* 3.5, 276f Friedlein = Philol. A26 D.-K.

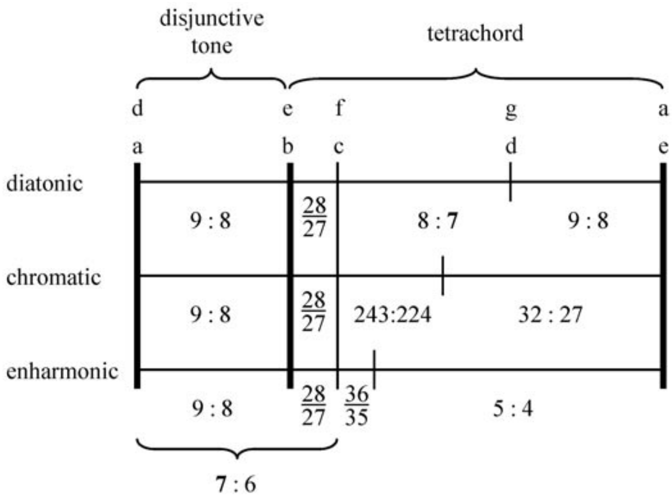

Diagram 3 Archytas' ratios

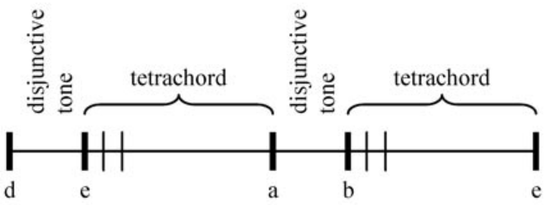

Diagram 4 The Classical Dorian scale according to Aristides Quintilianus.

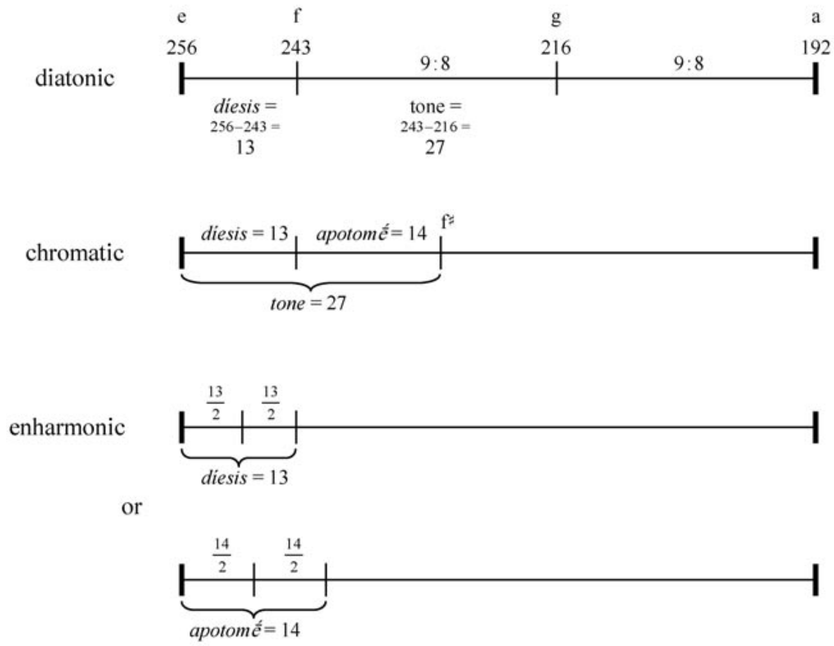

Diagram 5 Philolaus' tetrachord tunings.

as 'ditone', 'semitone', 'double octave', and the like[24]. Not having logarithmic tables at his disposal, Philolaus could not devise a mathematical way of doing so; still, unlike his mathematics, his results are not so completely off the rails, especially if we consider that they could not easily be tested experimentally: no incremental change of any physical parameter (such as length, tension, or whatever) in equal units can result in a series of equal intervals.

Philolaus' reconstructed tetrachord schemes are presented in Diagram 5[25]. His starting point was the diatonic ratios, with the highest number indicating the lowest pitch, as would go along with experiments with string lengths. In the usual Greek manner, he finds the set of smallest whole numbers that incorporate the necessary ratios. For the diatonic tetrachord, these are 256 : 243 : 216 :

24 In Greek, no respective term is attested before Philolaos; but this is no wonder, given the overall lack of technical texts before the fourth century BC. The general line of thought was doubtlessly exploited not only by practising musicians but also by the *harmonikoí*, whose achievements prefigure Aristoxenus' work in some respects (cf. Barker 1978).

25 For the reconstruction see Burkert 1962, 376, 372–377 (Burkert 1972, 394–399); West 1992, 167-168; 235-236. Huffman 1993, 364–374, doubts the authenticity of the fragments and seeks their origin in the early Academy. His argument, however, is conclusive only if it is assumed that

192. From these numbers, he derives two figures, which he subsequently uses as additive measures, by subtracting the respective boundaries: the *leîmma*, which he calls *díesis*, becomes 256–243=13, while the tone is 243–216=27. This seems peculiar, especially as there is another tone of equal size that would yield the different number of 216–192=24. Yet Philolaus obviously choose his figures because they made some additional sense. Above all, 13 is less than half of 27, just as the *díesis* is smaller than half of a tone. And the relations prove to be not far from the truth. If we assume 27 as a logarithmic measure corresponding to a tone, just as we are accustomed to define the tone as 204 cents, the *díesis*, which is actually 90 cents, would amount to 11.95 'Philolaean units', a value reasonably close to his figure of 13 – less than the twentieth part of a tone off.

For the rest of Philolaus' assumptions compare Diagram 5, where all the intervals are drawn to scale. Philolaus determined the size of the 'larger semitone', which he calls by the name of *apotomḗ*, 'offcut' (namely, from the tone, by subtracting the *díesis*) as the difference between a tone and a *díesis*, 27–13=14. Apparently this *apotomḗ* served as his higher chromatic interval, so that both chromatic semitones added up to a tone[26]. The difference between the larger and the smaller semitone, 14–13=1, was called *kómma* and attributed some significance as the "unit"[27]. Interestingly, Philolaus goes on to divide both the *díesis* and the *kómma* into two equal halves. This can only be understood in the context of the enharmonic, and especially of an enharmonic of equal quartertones. Moreover, an enharmonic with two half-*diéseis* (called *schísmata*) does not account for the splitting of the *kómma*. Obviously Philolaus recognized another form of the enharmonic, in which two slightly larger 'quartertones' added up to the *apotomḗ*. This might have been his only form, although it is at least as probable that he gave both variants. In any case, Philolaus conceived of the two small enharmonic intervals as exactly equal. The splitting of the established unit, on the other hand, must have been regarded as a major drawback, especially in a Pythagorean context. Such an assumption would hardly have been made unless under the force of evidence from musical practice. Thus, we must conclude that the equality of the 'quartertones' was commonplace at Philolaus' times.

It is of utmost importance for our inquiry that Philolaus' views are in almost perfect accordance with Aristoxenus, though perhaps even nearer to musical practice. Although Aristoxenus recognizes several shades of each genus in the chapters devoted to this topic[28], in the rest of his work he talks only about those which were also to become standard in later treatises and handbooks:

diatonic: semitone + tone + tone
chromatic: semitone + semitone + 1½ tones
enharmonic: quartertone + quartertone + ditone

These are exactly the divisions Philolaus already had in mind. If they are directly related to tuning practice, it becomes clear how the minute differences between the two writers emerge from their different basic principles:

A simple diatonic was created by alternating fifths and fourths. All ancient authorities agree that the tone is defined as the difference between these two intervals. Philolaus, calculating with ratios, finds that after subtracting two tones (9:8) from the fourth (4:3), the remainder (256:243) is smaller than half a tone. Aristoxenus, on the other hand, is implicitly working with tempered fifths and fourths in an octave comprising exactly six tones, so that his semitone must indeed be half a tone. Consequently, Aristoxenus needed not bother about distinguishing between two different 'semitones', *díesis* and *apotomḗ*.

In the chromatic, it is agreed that a whole tone is tuned above the lowest note of the tetrachord, presumably once more by a sequence of fifth and fourth. This 'chromatic note', obviously established before 400 BC, remained stable at least until the time of Ptolemy, who still knows it under the name of *khrōmatikḗ*, and at the same relative pitch[29]. Moreover, a chromatic tetrachord

pre-Platonic number mysticism was not concerned with the powers of 3 at all. Moreover, Huffman neglects the music theoretical side, which places the fragments rather before than after Archytas. In any case, they are pre-Aristoxenian, and the exact dating is not crucial for our general argument.

26 This traditional tuning, which can reasonably be called the 'Pythagorean chromatic' is preserved by Gaudentius (16, 344.17–24 Jan), along with the 'Pythagorean diatonic' (cf. also Theo Smyrn. 91f. Hiller; Anecdota Studemund, 5–7; Barbera 1977, 306; West 1992, 168 n. 32; Mathiesen 1999, 504–505). Gaudentius, however, presented the 'correct' ratios instead of Philolaus' fanciful numbers. It is significant that Gaudentius does not complement his tables with an account of the enharmonic. While Philolaus' chromatic could easily be converted to ratios, this is impossible for the enharmonic with its equal division of intervals established by ratios.

27 Boethius, *Inst. mus.* 3.5, 277.4–18 Friedlein.

28 In his systematical treatment, Aristoxenus insists on there being only one proper enharmonic genus (Aristox., *Harm.* 2.50, 63.1f. Da Rios), although he plainly admits that this type of enharmonic is actually dying out, so that even the existence of the truly quartertonic enharmonic came to be denied (1.23, 29.12f. Da Rios).

29 Ptol., *Harm.* 2.1, 43.10 Düring (read χρωματικήν: Düring 1934, 18). It is important that Ptolemy uses this name for referencing the note not in the framework of his system, but in relation to contemporary musical practice; the passage cannot be understood merely out of the *Harmonics*. Actually, the chromatic connotation of *khrōmatikḗ* severely compromises Ptolemy's overall system of genera, and he never discusses the chromaticism of this part of the scale.

emerged as a by-product of modulation from a disjunct to a conjunct tetrachord of any genus[30]; in this case, the sum of the semitones was defined as a whole tone by the scalar structure (cf. Diagram 6). The second lowest note is identical with its diatonic counterpart, which leads Philolaus to his *apotomḗ*, while Aristoxenus gets two equal semitones.

The enharmonic, finally, uses and bisects the semitone. The idea behind both Philolaus' and Aristoxenus' account is obviously to tune an interval known from the other genera, before bisecting it into two equal halves. Interestingly, Philolaus' postulated *apotomḗ* variant corresponds closely to the 'modern' variant of the enharmonic that we have already discussed as probably including a pure major third. As Aristoxenus leaves no doubts that this type of enharmonic was used practically to the exclusion of the ditonic variant by his times, it is no wonder that Philolaus already felt the need to account for it, especially if he was less narrow-minded on this point than Aristoxenus. And indeed, a calculation of Philolaus' values yields perfect accordance with the pure-third theory[31]. If we subtract the *apotomḗ* (114 cents) from the fourth (498 cents), the remaining interval of 384 cents is practically identical with, and certainly indistinguishable from, a major third (386 cents). Can this be coincidence? If not, how would Philolaus have been able to find this relation? Certainly not by the use of the monochord, since this would have required the calculation of ratios. If we start from units that were available from musical practice, we can write the connection Philolaus has established as

> tone – *díesis* = fourth – resonant third = larger enharmonic pyknón

Surprisingly, to verify this equation it needs no more than a seven-stringed lyre as was readily available in any average upper-class household. For the test, five strings will suffice; the seven strings are necessary merely for setting up the 'Pythagorean' diatonic tuning from which we start. What we need is the typical 'disjunct' tuning, with the tetrachord in question at the lower end, followed by a 'disjunctive' whole tone. We know from another fragment of Philolaus that he regarded such a tuning, which incorporated the basic division of the octave as fourth–tone–fourth, as standard[32]. Starting from such a typically tuned lyre, not more than two steps will be necessary (cf. Diagram 7):

1) Tune the second highest note of the tetrachord down to a pitch one whole tone above its lowest note. This is easily done by means of the higher note of the disjunctive tone, which stands a fourth above the required pitch. In ancient terms:
   Tune *likhanós* down to a fourth below *trítē/paramésē*[33].
2) Tune the higher note of the disjunctive tone down to a pitch one fourth above the second lowest note of the tetrachord. Once more, in ancient terms:
   Tune *trítē/paramésē* down to a fourth above *parypátē*.

As a result, it could be shown that the interval between *likhanós* and *trítē*, the third and fifth strings from the lower end, was identical to the resonant third used in the 'modern' form of enharmonic. This demonstration must have been all the more striking if the second string was played, too, as it completed an enharmonic tetrachord with undivided 'semitone' – a structure that Aristoxenus regarded as the older form of the enharmonic; his evolutionary model rested on evidence from old tunes and is nowadays generally accepted[34].

Did Philolaus actually detect this relation? The two necessary retuning procedures are by no means uncharacteristic. The first leads us merely from diatonic to chromatic, and must have been carried out innumerable times, especially as we were able to deduce from the agreement between Philolaus and Aristoxenus that the result was the standard form of chromatic tuning. But the second step was no less typical. It represents the 'Pythagorean' way of establishing a 'semitone' above the central string, *mésē*. And such a semitone was part of the second ancient standard tuning, the 'conjunct' one, in which the lower tetrachord was immediately followed by a higher one[35]. Consequently, what we have established as the final step of a procedure was in all probability a common lyre tuning in itself: the chromatic variant of the conjunct scale.

At the least, we can state that all prerequisites for the construction of the 'pure third' that prevailed in enharmonic music were part of the tradi-

30 Cf. Hagel 2000, 25 with Fig. 4; 87–89.

31 Cf. West 1992, 168.

32 Philolaus fr. 6a = Nicomachus, *Harm.* 9, 252.17–253.3 Jan.

33 The higher boundary note of the disjunctive tone is called *trítē* by Philolaos, who obviously worked with a 'gapped' lyre, which lacked one scalar degree in the upper region (cf. Philolaus as in note 31 above; Burkert 1962, 369–372; Burkert 1972, 391–394; West 1992, 176), but *paramésē* in later standard terminology.

34 Cf. Ps.-Plutarch, De musica 1135a; 1137b; Winnington-Ingram 1928; West 1981; West 1992, 163-164 (assuming a pentatonic precursor of all genera); Franklin 2002 (plausibly arguing with Aristoxenus for a diatonic origin).

35 Cf. West 1992, 219-220. Note that both notes generated by this string are called *trítē*, 'third [string note]', if we apply Philolaus' terminology to the disjunct structure.

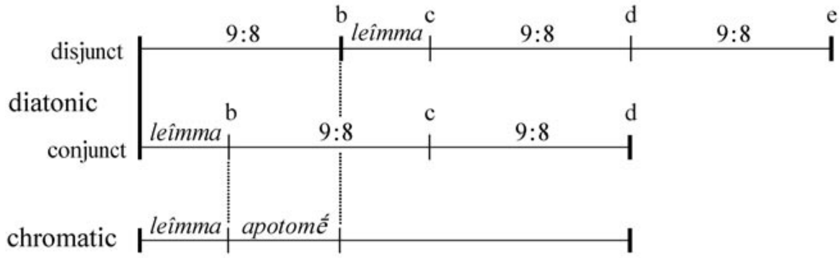

Diagram 6 The generation of the ‘Pythagorean’ chromatic by *synēmménon* modulation.

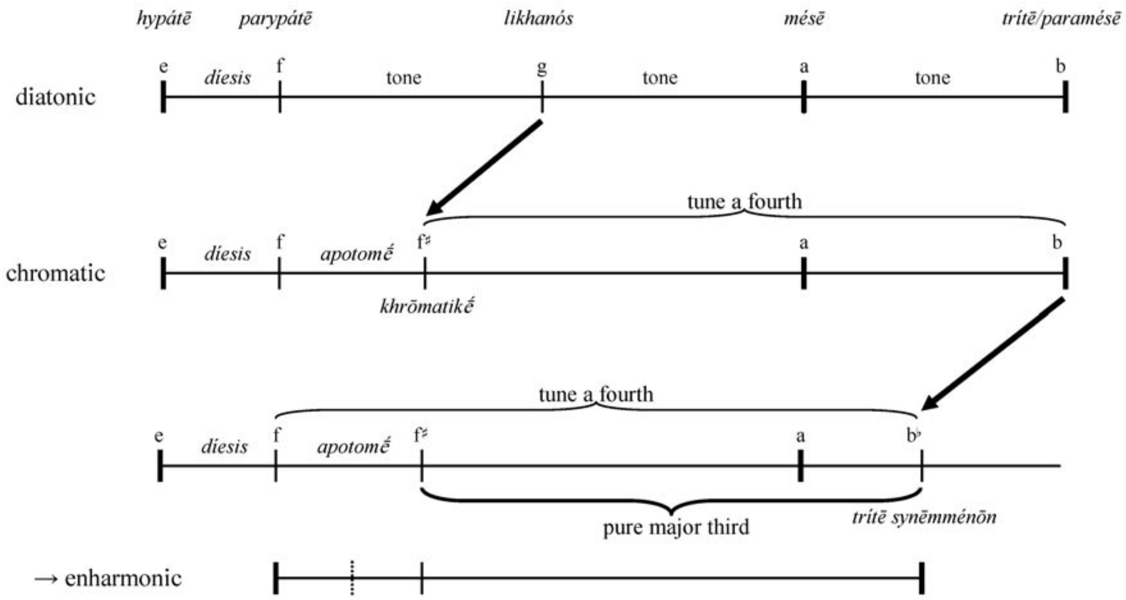

Diagram 7 The tuning procedure behind Philolaus’ divisions.

tional standard repertory of lyre tuning and retuning. Philolaus may have been the first to think it over and devise names and figures for all the intervals involved, perhaps also the first to prove that the upper ‘semitone’ of the chromatic was equal to the sum of the two quartertones in the modern form of enharmonic (as indeed it was for all practical purposes). But it is unlikely that he was the first to detect the resonant third, which had perhaps been exploited in music-making for quite a long time.

Philolaus’ system, as we have seen, makes perfect sense in the context of lyre tunings, to which he himself refers, and is also consistent with Aristoxenus. There can be no doubt that we are here in the position to determine the characteristics of standard lyre tunings of the late fifth and much of the fourth century. These tunings are still based mainly on a procedure of alternating fifths and fourths. Nevertheless, resonant thirds are also present, either as the side-product of the deviations produced by the circle of fifths, or explicitly tuned instead of the ‘Pythagorean’ ditone in one form of enharmonic.

Additional evidence comes from the views of the pre-Aristoxenian but non-Pythagorean theorists whom we find referred to as “the so-called *harmonikoí*”, although they probably constitute no uniform group[36]. There we find a quartertone grid employed as the uniform measure of all tonal structures. This necessarily implies an enharmonic with two equal quartertones and a chromatic with two equal semitones, and strongly suggests a canonical diatonic tetrachord formed of a semitone and two tones: the standard forms of the genera until late antiquity, which must have been derived directly from musical practice.

6. ARCHYTAS, II

Against this rather uniform background the oddities of Archytas’ figures stand out even more clearly. Still, his account has much in common with the works of the other theorists, and it is best to start

36 Cf. Barker 1978.

with the agreed on points. Firstly, the highest interval in the diatonic is a tone. Only thus can there be modulation between a disjunct and a conjunct tetrachord, where the second highest diatonic note must be interchangeable with a tetrachord boundary note. Such modulations were so common that they were accounted for in the standard 'Unmodulating System' of Greek musical theory, its name notwithstanding[37].

Secondly, the two lower chromatic intervals add up to a tone. Here Archytas clearly dares not deviate from musical practice, the size of this interval being conspicuous to everyone, as it was established through the consonances of a fifth and a fourth. As a consequence, Archytas' higher chromatic interval is determined as 32:27 (the fourth minus a tone), the interval below as 243:224 (the common lowest interval of 28:27 subtracted from a 9:8 tone)[38], which constitutes the major aesthetical flaw of his system and pins down the note common to both intervals as the culprit. If more evidence were needed for the chromatic tone, the lack of elegance in this detail would suffice to prove it.

Finally, Archytas correctly determines the resonant third of the preeminent enharmonic style as 5:4. Perhaps he had started by calculating the accurate values for Philolaus' schemes. If so, he got a ratio of 8192:6561 for the interval in question, and may have realized that this tuning is acceptable merely because these figures happen to be a good approximation of 5:4 (in decimal notation, it gives 1.2486 instead of 1.25). Thus, he would have been able to determine the 'correct' value for the enharmonic pure third without any experiment.

Only in the lowest intervals, for which he uniformly defines a ratio of 28:27, does Archytas disagree with his contemporaries. In fact, they did not offer him anything useful. The 'Pythagorean' tuning gave the traditional result of 256:243, which he could have used for the diatonic and chromatic. Yet, apart from the numbers themselves being rather horrible, he would have come up with an upper chromatic 'semitone' of 2187:2048 (the ratio for Philolaus' *apotomḗ*), which is definitely worse than the 243:224 he ultimately adopted. The *harmonikoí* approach, on the other hand, as well as both variants of Philolaus' enharmonic, required exact bisection of an interval, which Archytas himself had proven to be impossible (if one works with ratios)[39]. Thus, he had to develop a totally new approach.

As regards the identification of the three *parypátai*, Archytas may have been encouraged by the practice of ancient notation, which, as mentioned above, used identical signs for the notes in question. Yet in ancient notation there was never a one-to-one relation between signs and pitches. Above all, the chromatic and enharmonic *likhanoí*, which were most clearly of different pitch, were notated with identical signs, as well[40]. Thus, the notation cannot serve as an external confirmation of Archytas' *parypátai*.

7. ARISTOXENUS

But there is also a passage from Aristoxenus, which has been quoted in support of Archytas, and taken as the ultimate proof that "alike in the 4th century BC and the 2nd century AD the Greeks used a diatonic scale containing septimal tones"[41]. It concerns not the general identification of the *parypátai*, but the diatonic and chromatic tetrachords. For these, Aristoxenus gives possible variants as sequences of ⅓+1⅙+1 tones and ⅓+⅔+1½ tones, respectively[42]. Although these are expressed not as ratios but as parts of a tone, they emerge as practically identical in size with Archytas' tetrachord divisions. On the other hand, one must not forget that Aristoxenus explicitly excludes an enharmonic like that of Archytas[43], and insists that the enharmonic *parypátē* was always different from any chromatic or diatonic one[44].

And indeed the context in Aristoxenus' work does not allow taking the cited tetrachord divisions as widely used in musical practice. Aristoxenus describes and names quite a number of common tuning shades, and if those were among them, it is hard to see what would have prevented him from including them, as well. It is especially significant that he does not even provide a name for

[37] Cf. Hagel 2000, 37 with n. 58.

[38] Cf. Barbera 1977, 297. As Tannery 1915, 71 n. 1, points out, Archytas could easily have incorporated a chromatic tetrachord with superparticular ratios into his system, namely 28:27 – 15:14 – 6:5.

[39] Cf. Boethius, *Inst. mus.* 3.11, 285f. Friedlein.

[40] Originally, the sign triplets seem to have stood for an enharmonic *pyknón*, then also for a chromatic *pyknón*, whence it was consistent to notate the diatonic *parypátē* with the sign used for the other two *parypátai*, all the more since it was identical in pitch with the chromatic one. In any case, the alternative of using the sign designating the enharmonic *likhanós* for the diatonic *parypátē* would have caused much more confusion, especially when the chromatic became dominant in the fourth century BC.

[41] Winnington-Ingram 1932, 202-203.

[42] Aristox., *Harm.* 1.27, 35.3–7; 2.51, 65.4–20 Da Rios.

[43] Cf. Winnington-Ingram 1932, 197 n. 2.

[44] Aristox., *Harm.* 1.26, 34.14–17 Da Rios: παρυπάτης δὲ δύο εἰσὶ τόποι· ὁ μὲν κοινὸς τοῦ τε διατόνου καὶ τοῦ χρώματος, ὁ δ' ἕτερος ἴδιος τῆς ἁρμονίας· κοινωνεῖ γὰρ δύο γένη τῶν παρυπατῶν. ἐναρμόνιος μὲν οὖν ἐστι παρυπάτη πᾶσα ἡ βαρυτέρα τῆς βαρυτάτης χρωματικῆς […].

them. A closer inspection of Aristoxenus' argument reveals their true meaning.

What we read as Aristoxenus' '*Harmonics*' contains the remains of more than one work. Accordingly, we find two complete discussions of tuning shades, with emphasis on slightly different points. Nevertheless both contain the same set of one enharmonic, three chromatic and two diatonic divisions, with consistent nomenclature. Yet of the two additional nameless shades in question only one is given in each passage, and in both cases they do not form part of the shades list, but are introduced to prove general assumptions. These are reproduced most easily if we label the three intervals within the tetrachord as A, B, and C (with ascending pitch). Aristoxenus holds that

1) always A≤B, while
2) there is no such rule for the relation between B and C[45].

The first statement cannot be proven conclusively in the strict sense of the word, since it is not possible to test all of the in principle infinite divisions that either apply to or contradict the rule. All that can be done is to give examples for 'correct' and 'wrong' tunings, which appeal to the ear as melodic (*emmelḗs*) or as out of tuning (*anármostos*), respectively. For the second principle, it suffices to give one example for each relation in question (B<C, B=C, B>C).

For our investigation, the accepted divisions are of primary interest. Examples for most cases are already provided by the standard tunings, where we find

1) A<B (diatonic)
   A=B (enharmonic, chromatic)
2) B<C (enharmonic, chromatic, soft diatonic)
   B=C (diatonic)

As is easily seen, a positive example is lacking only for B>C. Actually, there is only one possible combination of pitches that were already defined in Aristoxenus' foregoing discussion that meets the requirements: that of the lowest non-enharmonic *parypátē* with the highest possible *likhanós*, which results in the division of $\frac{1}{3}+1\frac{1}{6}+1$ tones, which is indeed specified in the first passage. It is crucial to understand that this is the only available example which does not complicate the matter further by defining new points of orientation within the pitch continuum. Therefore, these figures (which are, by the way, implied, not explicitly given) arise necessarily from Aristoxenus' principles and his standard shades. Since there is no alternative, they can by no means serve as a confirmation of Archytas' ratios. This becomes entirely clear from the second, more elaborated passage. Here Aristoxenus indeed makes mention of the pitch continuum, alluding to the in principle infinite divisions with B>C. Although the argument is in principle identical, nothing like Archytas' division emerges, only a combination of 'the highest diatonic *likhanós* with any *parypátē* which is lower than that at the semitone"[46].

The other seemingly Archytean tuning, the chromatic of $\frac{1}{3}+\frac{2}{3}+1\frac{1}{2}$ tones, is introduced only in the more elaborate passage, as an example that cases of A<B are to be found not only in the diatonic. Here there would have been a theoretical alternative of $\frac{1}{3}+\frac{5}{12}+1\frac{3}{4}$ (choosing the 'hemiolic' instead of the 'soft' chromatic *likhanós*). But apart from the problems in using less commensurable numbers, the difference between the intervals in question would then amount to merely a twelfth of a tone, which does not make it a good example for an unequal division at all. Naturally, Aristoxenus has chosen the other possibility, which makes one interval twice as large as the other, so that the question of melodic acceptability of an unequal chromatic division can be judged easily.

It emerges that Aristoxenus is certainly no witness to Archytean shades being frequently employed in musical practice. We should take his account on face value: Archytas' chromatic and diatonic are compatible with the demands of melody, and similar divisions may have been in use. But the standard tunings were still those postulated for Philolaus, enriched by a number of secondary forms. One should even consider the possibility that Aristoxenus allowed for the Archytas-like divisions (wherever possible) not so much with regard to musical practice, but out of reverence for his great colleague and fellow countryman, a deferential biography of whom is among Aristoxenus' lost works[47]. Even so, he merely acknowledges that Archytas' diatonic and chromatic are not against the rules of harmony. In any case, an enharmonic with its lower intervals similar to those Archytas gives had to be plainly ruled out.

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45 Aristox., *Harm.* 1.27, 34.19–35.3 Da Rios: τῶν δὲ διαστημάτων τὸ μὲν ὑπάτης καὶ παρυπάτης τῷ παρυπάτης καὶ λιχανοῦ ἤτοι ἴσον μελῳδεῖται ἢ ἔλαττον, τὸ δὲ παρυπάτης καὶ λιχανοῦ τῷ λιχανοῦ καὶ μέσης καὶ ἴσον καὶ ἄνισον ἀμφοτέρως. Cf. 2.52, 65.2–4;15f Da Rios.

46 Aristox., *Harm.* 2.51, 65.18–20 Da Rios: …ὅταν (τις) λιχανῷ μὲν τῇ συντονωτάτῃ τῶν διατόνων, παρυπάτῃ δὲ τῶν βαρυτέρων τινὶ τῆς ἡμιτονιαίας χρήσηται.

47 Cf. esp. Aristox. fr. 48.

8. ARCHYTAS, III

All in all, at least part of Archytas' identical *parypátai* must be wrong[48], which casts serious doubts on the reliability of their position in all three genera. Is it probable at all that Archytas determined the *parypátē* of one genus correctly, perhaps by means of some experimental device, and then wrongly generalized its position? Apart from the note in question, we could easily explain all of Archytas' choices without assuming him to have carried out any experiment[48]. Need we make such an assumption for that which of all intervals seems to be at odds with musical practice, as it can be deduced from other roughly contemporaneous writers? Furthermore, it is not even clear that Archytas focussed on stringed instruments, as Philolaus apparently did, and as later writers have done, most prominently Ptolemy. To the contrary, there is evidence that Archytas was involved in the theoretical discussion that went hand in hand with the evolution of a new type of aulos[50]. In this case, a rather careless treatment of the *parypátai* of musical practice appears in new light. These, whether lying a quartertone or a semitone above the basic note of the tetrachord, were originally played by partially opening one of the reed instrument's finger holes[51]; hence the ambiguous term *díesis*, 'letting-through', which is applied to the semitone by Philolaus, but comes later to designate the quartertone. Such a fingering will, of course, produce no unequivocal pitch, as can be used for measurements and tests[52]. If Archytas' main concern was auletic music, his focus audience aulos players, he probably needed not fear reproaches regarding the position of *parypátai*.

In any case, if we are able to find a plausible derivation of the septimal interval in Archytas' tunings other than from musical practice, it should be preferred on methodological grounds. And indeed there is a possible explanation that draws exclusively on mathematical considerations. But to detect it, it is necessary to retranslate the figures into a form as Archytas may have used them. Today, we are accustomed to write ancient tunings as a set of ratios, just as we have done so far. This however, is just for our convenience. In ancient times, one usually determined the set of smallest numbers which incorporated all relations in question. Ptolemy, however, gives both. As we have already presented the ratios for Archytas' tunings[53], here are Ptolemy's numbers[54]:

	enharmonic	chromatic	diatonic
mésē	1512	1512	1512
likhanós	1890	1792	1701
parypátē	1944	1944	1944
hypátē	2016	2016	2016

Whatever Ptolemy found in Archytas' now lost work, he has converted it to his own standard format: numbers chosen so that comparisons between the different genera are readily available, with higher numbers indicating lower pitch, so that they can easily be converted to measurements on the experimental device. Archytas, in all probability, gave his figures the other way round, higher numbers indicating higher pitch, in accordance with his physics[55]. If the interrelations of all three genera are considered, we obtain the following numbers:

	enharmonic	chromatic	diatonic
mésē	1440	1440	1440
likhanós	1152	1215	1280
parypátē	1120	1120	1120
hypátē	1080	1080	1080

Still, it need not be assumed that Archytas compiled such a comprehensive rendition. The next step is to reduce each of the genera to its smallest numbers:

	enharmonic	chromatic	diatonic
mésē	180	288	36
likhanós	144	243	32
parypátē	140	224	28
hypátē	135	216	27

Here, while neither the enharmonic nor the chromatic seem of any numerological significance, the diatonic – which always remained in the focus of Pythagorean music theorists – stands out in containing only small numbers. Actually, these numbers are as small as is possible at all in the diatonic. This can easily be proven, on the base of merely two assumptions:

1) The fourth corresponds to a ratio of 4:3 (which is commonplace).
2) The highest interval of the diatonic is a 9:8 tone (as necessary for *synēmménon* modulation and assumed by all theorists until well into the Roman era).

48 It has been suspected that the idiosyncrasies of Archytas' account reflect local tuning practice (Barker 2000, 122); but by the fourth century, a constant exchange of music and musicians must have generated a rather uniform musical high culture.

49 The existence of experimental instruments for deducing and testing ratios before 300 BC is disputed; cf. van der Waerden 1943, 177.

50 Cf. Hagel 2005a, 79–80.

51 Cf. West 1992, 235 n. 42.

52 Cf. Aristox., *Harm.* 2.41–43, 52.9–54.4; Plato, *Phlb.* 56a.

53 See above, p. 286, Diagram 3.

54 Ptol., *Harm.* 1.13, p. 31 Düring.

55 Cf. Archytas fr. B 1 D.-K.; van der Waerden 1943, 173–175; Hagel 2005a, 79.

From this minimal common ground, the following structure emerges:

	fourth	tone		
mésē	4	9	→	36
likhanós		8	→	32
parypátē				?
hypátē	3		→	27

In other words, no diatonic tetrachord with a highest number below 36 meets the basic requirements. If Archytas searched for small numbers for their own sake, his diatonic is sufficiently explained: the missing position can only be supplied by the number 28; an alternative 29 has not only the great mathematical disadvantage of not generating superparticular ratios, but also deviates even more from the traditional *díesis* and, on top of this, cannot function as chromatic or enharmonic *parypátē*.

If the disjunctive tone below the tetrachord is taken into consideration, as is usually postulated for Archytas' divisions, another point comes into view:

	fourth			
mésē	4	9	→	36
likhanós		8	→	32
parypátē		(7)	→	28
hypátē	3		→	27
hyperypátē		6	_	24

In relation to the 9:8 tone at the top of the tetrachord, the disjunctive tone takes the value of 6. Was it not obvious to complete the series 6:7:8:9, if the missing 7 was by any means acceptable as a *parypátē*? All the more so, because otherwise 7 would have remained the only number below ten which does not take part in the divisions of the tetrachord, and this in spite of its considerable numerological significance[56]. In this way, Archytas managed to assign each number its place within musical harmony. The first ones, up to four, granted the basic structure built from fifths and fourths, while the higher numbers defined the 'movable' notes: the enharmonic *likhanós* as well as *likhanós* and *parypátē* in the diatonic genus:

	fixed notes		movable notes	
	fifth	fourth	enharmonic	diatonic
mésē	3	4	5	9
likhanós			4	8
parypátē				7
hypátē		3		
hyperypátē	2			6

If the note names of the upper tetrachord are used, the number one can be included, as well, and the full symmetry of the system up to the number five becomes visible[57]:

	fixed notes			movable notes	
	octave	fifth	fourth	enharmonic	diatonic
nḗtē	2	3	4	5	9
paranḗtē				4	8
trítē					7
paramésē			3		
mésē		2			6
likhanós					
parypátē					
hypátē	1				

The chromatic could not be fitted well into this scheme; but its name implied anyway that it was just a 'colouring' subordinate to the diatonic, the 'first and most important'[58] genus, and to the enharmonic, most highly esteemed in Classical music[59]. From Ptolemy's words it transpires how Archytas determined the chromatic *likhanós* in relation to the diatonic and by reference to the traditional tuning in fifths and fourths[60]. It seems that the non-superparticular ratios of 32:27 and 243:224 were not mentioned by Archytas at all; not even Ptolemy gives them in this form. The identification of the three *parypátai* probably resulted from a happy coincidence: if the (originally diatonic) interval of 28:27 was used are the lower enharmonic 'quartertone', the upper one emerged as 36:35, a superparticular ratio. Archytas may have regarded the ratios of 28:27 and 36:35 as close enough in size to stand for the alleged (but never proved) 'equal' quartertones (actually, their difference amounts to 14 cents, which is indeed negligible for intervals that are not established by resonance). On the other hand, the identification of chromatic and diatonic *parypátē* was traditional – and as we have seen, Archytas did not bother much about chromatic ratios.

Thus it becomes clear how Archytas could establish his divisions, relying only on traditional relations, but searching for a better representation of the diatonic and for a first true mathematical account of the enharmonic, all built on small numbers and superparticular ratios. He did not base his figures on experiments, nor was he oriented

56 For its place in fourth-century speculation, cf. e.g. Aristot., *Met.* 1093a.

57 In view of such a symmetry, one ought to doubt the reliability of Archytas' enharmonic pure third, were it not for the additional evidence from Aristoxenus and Philolaus.

58 πρῶτον καὶ πρεσβύτατον: Aristox., *Harm.* 1.19, 24.20 Da Rios.

59 Cf. now the original interpretation of the term 'chromatic' by Franklin 2005, 23–38 (which can, however, not be applied to Philolaus or Archytas).

60 Ptol. *Harm.* 1.13, 31.2–6.

towards musical practice any more than Philolaus; his advancements in tetrachordal mathematics were of a purely mathematical kind[61]. The correct question in assessing Archytas' value as a witness to early tunings is not so much, what relations he encoded, but rather, how far he dared to deviate from musical practice.

As regards our general question, we have established two points with reasonable certainty: while the theory of resonant thirds in fourth-century tunings holds, we had to discard the alleged septimal intervals as a mathematical fiction. The evidence from the following centuries will confirm this picture.

9. ERATOSTHENES

At the height of Hellenistic music, Eratosthenes of Cyrene devised his set of tunings, which would, just as those of Archytas, have been lost, were it not for Ptolemy. Eratosthenes' ratios and their respective intervallic distances are set out in Diagram 8. It appears that Eratosthenes was not very impressed by Archytas' divisions, and although he continues to search for harmonic relations in superparticular ratios, he makes no use of septimal intervals, entirely disregarding the prime number seven. All his figures are readily explained in the canonical tradition reflected by Philolaus and Aristoxenus, though mathematically accommodated to a musical environment that had changed significantly.

Most surprisingly, Eratosthenes' diatonic adheres to the strictly 'Pythagorean' tuning standard. The *leîmma* of 256:243 is his nastiest interval, and was probably a concession as much to musical practice (modulation over several keys inevitably leads to such a tuning[62]) as to philosophical tradition (now this form of diatonic was sanctified as a structure inherent in the cosmic soul of Plato's *Timaeus*). Still, one could not possibly work out a mathematically satisfying chromatic tetrachord starting from the *leîmma*, while, on the other hand, the lower semitone of the diatonic and the chromatic were identical on the instruments. Eratosthenes solved this contradiction by finding a superparticular ratio which matches the *leîmma* as closely as possible. His solution of 19:20, which corresponds to 88.8 cents, deviates from the 90.2 cents of the *leîmma* only by the 140th part of a tone; Boethius still gives it as an approximation[63]. The philosophical implication is clear: although the chromatic and the diatonic *parypátē* are not mathematically identical, their minute difference cannot be detected by the human ear, so that both notes are naturally played on one and the same string.

The problem of the two chromatic semitones traditionally conceived as equal was addressed in a similar way. Since two similar ratios inevitably lead to a nasty remainder, Eratosthenes made the semitones as equal as possible (and at the same time adhered to the Aristoxenian standard that the larger interval must be the higher one). The difference between 20:19 and 19:18 amounts to 4.8 cents, the 42nd part of a tone, which it is once more impossible to ascertain by ear. A pure minor third emerges as the highest chromatic interval. This may have been in accordance with contemporary lyre tunings; even so, since we were able to derive the figures from basic assumptions, they cannot be taken simply as first-hand evidence for musical practice.

For his enharmonic, Eratosthenes adhered to the traditional equation of two enharmonic quartertones adding up to a diatonic/chromatic semitone[64]. Consequently, he took the superparticular chromatic one and divided it into two parts as nearly equal as possible (with a difference of only the 176th part of a tone).

Still, it proved impossible to reconcile all demands of musical practice and mathematical aesthetics. Apart from accepting the traditional *leîmma*[65], Eratosthenes sacrificed exactly those two features which Archytas had been so eager to preserve: a resonant enharmonic third, and a chromatic *likhanós* standing exactly one whole tone above *hypátē*. Obviously, a different musical environment triggered different choices. Archytas had seen the glory days of enharmonic music. Without doubt he felt obliged to consider this genus, then held in the highest esteem, in first place, perhaps together with the diatonic. Yet in Eratosthenes' time, the enharmonic seems to have been practically dead. In Hellenistic music the chromatic was

61 It will be noticed that the present reconstruction is both simpler and more specific than the suggested employment of mathematical means (Tannery 1915, 81; van der Waerden 1943, 185–187; Barker 1989, 48f.; Barker 2000, 123–125), which produces merely an unspecific enumeration of superparticulars (more easily obtained by simply counting); Archytas' newly developed theory of means was important mainly for the basic structure of 'fixed' notes (cf. Hagel 2005a).

62 Cf. Ptol., *Harm.* 1.16, 40.8–13 Düring; hence even Ptolemy is forced to arrange his keys in these, partially non-superparticular, intervals. For identical *synēmménon* and *diezeugménon* tetrachords inevitably enforcing 'Pythagorean' tuning, cf. Tannery 1915, 94-95.

63 Boethius, *Inst. mus.* 3.13.

64 Cf. Winnington-Ingram 1932, n. 2; Barbera 1977, 302.

65 Note that it is impossible to find a neat diatonic tetrachord division with 20:19 as the lowest interval, mainly because 19 is a prime number. Although 20:19 – 19:17 – 17:15 (89 – 193 – 217 cents) might have been musically acceptable, the ratios are not superparticular.

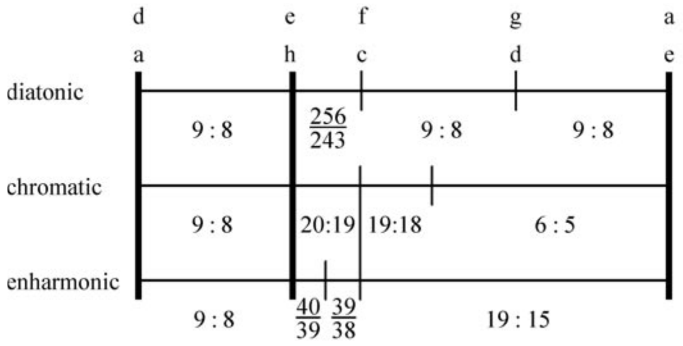

Diagram 8 Eratosthenes' tunings.

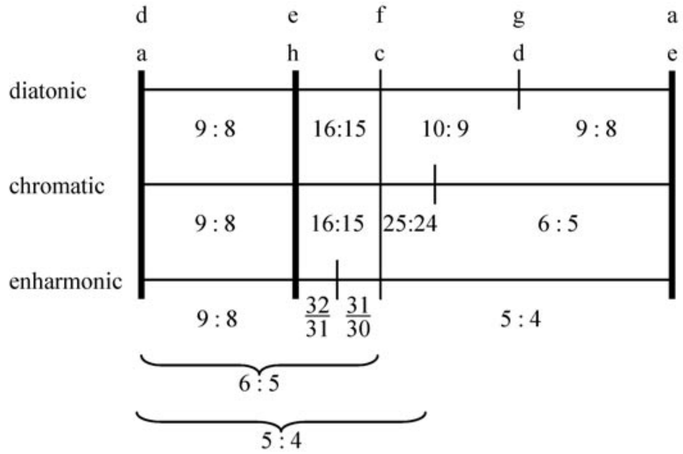

Diagram 9 Didymus' tunings.

clearly dominant, especially since it ideally accommodated modulations between remote keys. Consequently, a beautiful chromatic tetrachord division was called for, even if this precluded the old notion of the whole-tone *khrōmatikḗ*. As a result, just as with Archytas' tunings, all of Eratosthenes' figures can be explained on simple basic assumptions: for the Hellenistic philosopher we still need not postulate any experiments. Of course Eratosthenes may have constructed his tunings on the monochord and enjoyed a resonant chromatic third and the impossibility of distinguishing the *leîmma* from his 20:19 semitone; but he certainly did not proceed the other way round, by tuning intervals heard in music-making on an experimental device and subsequently 'measuring' them by means of a ruler to obtain ratios.

10. DIDYMUS

In the first century AD, the theorist Didymus brings about the triumph of the resonant third. As can be gathered from Diagram 9, each of his genera embodies at least one major and one minor third, his chromatic even two of both. Didymus is the first to strictly implement the traditional equations of diatonic and chromatic *parypátē* and enharmonic *likhanós*, which he locates at a 16:15 semitone above *hypátē*. Thus having disposed of the *leîmma*, he is, as well, the first to define all involved intervals as superparticulars, obtaining a mathematically perfect layout with a beautiful resonant diatonic.

The major shortcoming concerns the chromatic, where the two 'semitones' are markedly dissimilar (112 against 71 cents), with the larger one below, thus contradicting the general rule stated by Aristoxenus and still upheld by Ptolemy[66]. Furthermore, the two intervals do not add up to the 9:8 tone of the *khrōmatikḗ*, but to the small 10:9 tone, just as in Eratosthenes' system. It must be significant that here again, an author transfers the inevitable mathematical troubles to the realm of obsolescent musical styles: in Roman times, the chromatic was already disappearing, and the diatonic had taken the lead, although apparently in a modal form that often differed significantly from

66 Ptolemy explicitly criticizes Didymus on that point: *Harm.* 2.13, 68.27–29 Düring.

earlier Greek music[67]. At the same time, the extensive modulations of Hellenistic music had apparently become obsolete. Consequently, the notes of Didymus' diatonic no longer serve as the framework for modulations following the circle of the fifths, as those of Eratosthenes' still did – although the 9:8 tone at the top of the tetrachord still assures that the most basic modulations can be carried out. Instead, it ensures a maximum of resonance within a more limited tonal space.

Didymus did make use of experimental instruments[68], and we should therefore assume that his tunings do not deviate all too much from those heard at his time. And indeed it seems perfectly plausible that the resonant third, already employed at least in enharmonic music at quite early times, had gained ground by now. A renewed influence of Near Eastern music, which had obviously appreciated resonant thirds as early as in the 2nd millennium BC, may have played its part[69]. In any case, Didymus' account proves beyond doubt that the identification of chromatic and diatonic *parypátē* with the enharmonic *likhanós* had become an integral part of musical reasoning. And once more, there is no trace of septimal intervals, while all theorists before Ptolemy employed the figures for resonant thirds at least in the dominant genus of their respective times.

11. PTOLEMY

Nevertheless Ptolemy not only adopts Archytas' diatonic tetrachord as one among other possible solutions, but holds that it is part of all cithara tunings contemporary music knew. Should we have to reckon with a revival of a musical style that had continued to exist in the shadow of a mainstream characterised by resonant thirds? This is hardly an option, not only because we have seen that Archytas' system is most probably based more on mathematical reasoning than on experiment, but also because it is conspicuous especially for its unique relations between the three genera, so that the continuation of an isolated diatonic could hardly be called the continuation of 'Archytean' style. Might it be possible, on the other hand, that Ptolemy adopted this specific division for some aesthetic reason, perhaps because he wanted to attribute the number seven a musical role[70], or even simply out of reverence for his great predecessor[71]?

In any case, it is not easy to distrust Ptolemy, since the tests he proposes for each of his divisions are devised in a perfectly scientific manner. Unfortunately, we cannot reproduce them, since they require knowledge of contemporary tunings, which Ptolemy could assume his readers to have, but which we naturally lack. However, to convert the relationships that are thus established by musical ear into figures, Ptolemy has to rely on one additional a priori assumption, namely that acceptable melodic intervals always correspond to superparticular ratios. To prove this, converse tests are offered: the scales are constructed experimentally and consequently found to correspond exactly to those of musical practice. Therefore only one question remains open: how exact is 'exactly'?

But let us consider the experiment by which Ptolemy assesses the 'Archytean' diatonic of 28:27 – 8:7 – 9:8, which he calls the 'tonic diatonic'[72]. It is based on another division, the 'tense chromatic' tetrachord of the *trópoi* tuning, which Ptolemy has established previously as including a septimal tone between its lowest and second but highest notes[73]. Here it will suffice to investigate the final steps, which include the following construction on the eight-stringed experimental instrument (cf. Diagram 10):

- Tune a tetrachord like the higher one of the *trópoi* tuning[74].
- Tune a second tetrachord like the lower one in the *stereá* tuning, so that its second lowest note is identical with the lowest of the other tetrachord.

As a consequence, Ptolemy claims, one will find that the second highest notes of both tetrachords coincide, too – and consequently the 8:7 septimal tone between these notes is established for the second tetrachord, as well.

Although the procedure seems quite straightforward, it must not escape our attention that the

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67 Cf. West 1992, 184–189; 383f.; Hagel 2005b. I plan to give figures for this development in a future publication.

68 Ptol., *Harm.* 2.13.

69 Cf. Hagel 2005b.

70 The third book of Ptolemy's *Harmonics* is dedicated to assigning astronomical movements musical meaning. Since there were seven planets, this number is bound to play a substantial role, which Ptolemy derives from heptatony by making the number of keys equal to the number of notes in the octave – which is by no means necessary or especially useful for musical practice.

71 In one of the last chapters of his *Harmonics* – which is, unfortunately, almost entirely lost – as well as in his *Canobic Inscription*, Ptolemy took over a 'cosmic scale' which may go back to Archytas; cf. Hagel 2005a.

72 Ptol., *Harm.* 2.1, 43.9–18 Düring

73 Ptol., *Harm.* 2.1, 42.10–43.8 Düring. The point is that this interval and the highest one of the tetrachord are both larger than a 9:8 tone, and that this is true only for one pair of superparticulars, namely 7:6 and 8:7. Of these the higher one is shown to be the larger.

74 It is of the essence that in this chapter Ptolemy uses several terms (names of notes and tunings) that he has not defined nor explained in advance, and which are therefore certain to reflect common musical practice directly.

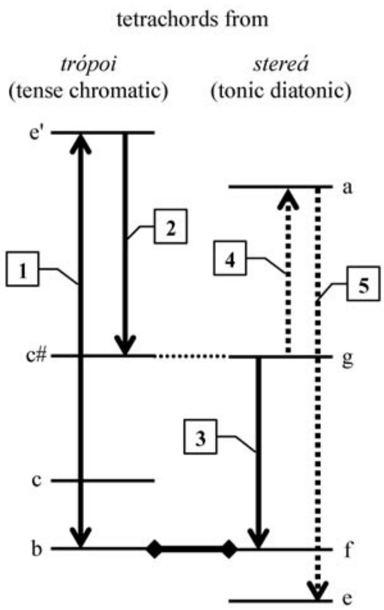

Diagram 10 Determining the tonic diatonic on the canon.

musical expertise demanded here substantially exceeds that which is needed for tuning a lyre. For, as J. C. Franklin has shown, practically all tetrachord divisions of ancient Pythagorean theory from Archytas to Ptolemy can be established, on the instrument, by the help of lesser resonant intervals[75]. Still, for the tunings that include septimal intervals the ability of establishing a septimal third by ear is necessary; but never a smaller interval. Ptolemy's setup requires more:

- To establish the *trópoi* tetrachord, as far as it is needed here, a fourth ([1] in Diagram 10) and a 7:6 septimal third [2] have to be tuned.
- To establish the tonic diatonic of *stereá*, however, from the fixed second but lowest note, it is necessary to tune an 8:7 septimal tone [3], as well as a 9:8 whole tone [4], without the help of larger resonant intervals.

Whether these intervals could be achieved with anything like the exactness Ptolemy has commonly been taken to imply, is more than doubtful. In any case, the two pitches that are finally to be compared, and whose 'identity' rests on the two septimal intervals tuned by ear, cannot have been identical (ἰσότονος εὑρεθήσεται, says Ptolemy), but at most indistinguishable, and this only if human perception were at least as exact in establishing septimal tones as it is in judging the identity of two pitches. This is, however, not the case – and so it must be admitted that Ptolemy's experiment, taken literally, is impossible. Still, Ptolemy might have demonstrated the validity of his approach by a slightly different preparation of his experimental setting. If he tuned not one but both 'identical' notes to the pitches of their counterparts from the very beginning, he could then proceed to show (a) that the first tetrachord is an acceptable *trópoi* tuning, (b) that the two lower identical notes are identical, (c) that the second tetrachord is acceptable as *stereá* and (d) finally that the other pair is identical. Only this is in accordance with human capabilities, and consequently we must dispose of the idea of 'exactness' in the context of Ptolemy's tests.

Indeed this accords with explicit ancient evidence. Modern scholarship, guided by a vision of the exalted Greek musical culture (although regularly frustrated by finds of musical fragments), has often adhered to a principle most plainly formulated by R. P. Winnington-Ingram:

> "I believe … that the Greeks used intervals strange to us with precision."[76]

This belief makes a strange contrast with the confession of the philosopher Porphyry in his commentary on Ptolemy's *Harmonics*:

> "We are unable to perceive exactly the difference between notes beyond 4:3 [the fourth]. For the senses apprehend but roughly. And how would they apprehend 5:4 [the major third], or 6:5 [the minor third], or 21:20 ? But below 4:3, they will apprehend: so 3:2 [the fifth] and 2:1 [the octave]…"[77]

This was written about a century after Ptolemy: should such a catastrophic decline in musical perception have taken place? No, for we must confer a celebrated passage from Ptolemy himself, where he admits that the citharodes tune their instruments in the traditional 'Pythagorean' way (256:243 – 9:8 – 9:8), while they sing in what Ptolemy describes as his 'tense diatonic' (16:15 – 9:8 – 10:9)[78]. Although this gives substantial differences of 22 cents between singer and instrument for two out of four notes in the tetrachord, Ptolemy does not see much of a problem:

75 Franklin 2005.
76 Winnington-Ingram 1932, 206 n. 2.
77 Porphyry, *Comm. Harm.* 152.18–21 Düring: οὐδὲ γὰρ δυνάμεθα ἐπέκεινα τοῦ ἐπὶ γ′ διαισθανθῆναι ἄλλως διαφορὰν φθόγγου πρὸς φθόγγον· παχυμερῶς γὰρ αἰσθήσεις ἀντιλαμβάνονται. καὶ πῶς ἂν ἐπὶ ἢ ἐπὶ ε′ ἢ ἐπὶ κ′ ἀντιλήψονται; κάτωθεν δὲ τοῦ ἐπὶ γ′ ὡς ἐπὶ τοῦ ἡμιολίου καὶ τοῦ διπλασίου αἰσθανθήσονται.…
78 Ptol., *Harm.* 1.16, 39.14–40.8 Düring.

> "The difference between the great and the small tone […] is not worth mentioning, […] and the same is true for the ditone and the pure third […]"[79].

Obviously, for Ptolemy a divergence of a tenth of a tone already fell into a grey area, still recognisable under experimental conditions, but practically negligible in real music-making[80]. This leads us to a most important conclusion: even Ptolemy's tunings are mathematical idealizations. We must expect that musical practice deviated from them by a certain amount. Instrumental tunings were oriented not only towards optimally sounding intervals, as adequately described by the superparticular account, but also at the demands of modulation between genera and to different keys. While we have seen that earlier theorists took these necessities into consideration, Ptolemy is less interested in them, especially not in modulation between the genera at the same place in the scale[81]. Only in the passage quoted can we get a glimpse of these centuries old practical compromises still existing. Yet Ptolemy designs and offers to the readers' judgement perfect stand-alone octaves for non-modulating melodies. These may even have produced a more pleasant sound than those actually in use; but for musicians of the Roman era, they were perhaps of not much more use than a piano in just tuning for a nineteenth-century player.

But even if we cannot postulate the plain identity of Ptolemy's 'tonic diatonic' with the tonal structures of musical practice, we are bound to search for the musical reality behind it. For an investigation of the introduction of this specific tuning in the context of Ptolemy's general methodology shows that this tuning was apparently indeed of considerable practical importance[82]. In principle, Ptolemy uses a fixed algorithm to produce possible mathematically acceptable tetrachord divisions: the fourth is bisected into two superparticular intervals, one of which is further divided into two parts, one being about half the size of the other, while both are again superparticular[83]. The divisions are constructed experimentally and subjected to the judgement of the ear. In this way, some of them are found to correspond to familiar tunings, others not so: there is also a cultural factor in human music. Only in the case of the 'tonic diatonic', Ptolemy deviates from his procedure. Although a whole tone at the top of the tetrachord is not in accord with his method, Ptolemy claims:

> "… the ratio 9:8 is found, in its own right, to describe the whole tone by the difference of the two first consonances [3:2 : 4:3 = 9:8], which, according to good reason and necessity, must also occupy the highest position [in the tetrachord], conjoined with those closest to it, since none of the superparticulars complements it to the epitritic ratio [4:3 : 9:8 = 32:27]."[84]

Such a deviation from methodological principles must indicate that the 'tonic diatonic' incorporates one or more features of highest practical importance, which obviously fail to be reflected in the other divisions. A first point is certainly the whole tone at the top, which is so important for modulation, and which Ptolemy explicitly mentions as the cause for introducing the division, although his exact reasons are hidden behind a rather nebulous wording[85]. Whether the other two intervals of the tuning, which define the unusually low pitch of the lower 'movable' note (*parypátē*), are close to practice, as well, depends, *inter alia*, on the correct interpretation of Ptolemy's next sentences:

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79 Harm. 1.16, p. 39.19–40.2 Düring: προχωρεῖ δ' αὐτοῖς τὸ τοιοῦτο διὰ τὸ μηδενὶ ἀξιολόγῳ διαφέρειν μήτε … τὸν ἐπὶ η′ τοῦ ἐπὶ θ′, μήτε … τὸν ἐπὶ ιε′ τοῦ λείμματος. … ὁ αὐτὸς δὲ οὗτός ἐστι λόγος καὶ τοῦ διτόνου, τουτέστι τοῦ δὶς ἐπὶ η′, πρὸς τὸν ἐπὶ δ′…. For the meaning of ἀξιόλογος, cf. 1.14, 32.20 Düring, here applied to the twenty-fourth part of a tone (8 cents).

80 Barker 2000, 153–155; 241 (also discussing Ptolemy's credibility on that point). If this important difference is born in mind one need not assume a total methodological disaster (cf. Raffa 2002, 365). For general remarks on the possible precision of lyre tunings, cf. Mathiesen 1999, 476.

81 His six cithara tunings are all diatonic except for *trópoi*, which contains an upper chromatic tetrachord that is entirely incompatible with any of his diatonic divisions. Modulation between chromatic and diatonic at this position would require, according to Ptolemy's figures, two different strings only 18 or 9 cents apart, depending on the diatonic tuning used.

82 For a detailed investigation of Ptolemy's methodology, cf. Barker 2000, 132–157; 243–249.

83 For example, 4:3 = 10:9 × 6:5; 6:5 = 18:15 (tripling the boundaries), whence 6:5 = 18:16 × 16:15 = 9:8 × 16:15 can easily be found; the resulting diatonic division is therefore 16:15 × 9:8 × 10:9, because 10:9 as the minor part of the first bisection belongs at the highest place; 16:15, as the smallest interval, at the lowest.

84 Ptol., *Harm.* 1.15, 36.21–25 Düring. Cf. Barker 1989, 309f. n. 135; Redondo Reyes 2002, 488f. n. 304; Raffa 2002, 360: "L'operazione con la quale Tolomeo decide di porre al primo posto il rapporto 9/8 è, rispetto al contesto, assolutamente arbitraria: la motivazione adotta – κατὰ τὸ εὔλογον καὶ τὸ ἀναγκαῖον, «secondo la correttezza matematica e la necessità», 36.22 – è formulata in un modo sbrigativo e apodittico che stride con la meticolosità e precisione di passaggi del discorso precedente, in cui tutto era rispondente al sistema".

85 Many of Ptolemy's original readers, however, who were certainly familiar with the cithara tunings, would immediately recognize the necessity (τὸ ἀναγκαῖον) of identifying the upper interval of the tetrachord with the disjunctive tone in the *trítai*, *trópoi*, and *hypértropa* tunings, and probably also with the *khrōmatikḗ* tone of *hypértropa* and *iástia*.

> "While 10:9 is already conjoined with it [9:8] in the division lined out immediately before, 8:7 is not so. Therefore we will conjoin it with this ratio put in the central position, and what is needed to complement the fourth, namely 28:27, we will assign to the lowest position."[86]

It is certainly fascinating how Ptolemy manages to present the methodological shortcoming as part of his methodology: the fact that a whole tone at the top is incompatible with his general procedure is mentioned only as the reason why a different procedure must be adopted for determining the other intervals, and moreover it is put forth as if this procedure would follow from it[87]. What ought to have been a principal objection has become a fictitious reason.

In any case, the obvious tetrachord with a 9:8 tone at the top would have been 16:15 – 10:9 – 9:8, the diatonic given by Didymus only about a century earlier. Ptolemy rejects it, once more with insufficient argument: the ratios 10:9 and 9:8 were already found side by side in another division, established before (16:15 – 9:8 – 10:9). There is no explanation why two divisions with equal intervals in different order are not allowed – both are in best accord with Ptolemy's principles. Perhaps he felt that an assumption of four mathematically constructed diatonic shades complicated the matter too much, even if he was free to dispose of the redundant one in the next chapter as unfamiliar to the musical ear. Or perhaps Ptolemy indeed regarded divisions that differ only in the order of their intervals as an aesthetical flaw, so that the additional adoption of Didymus' diatonic was out of question for him. In neither case can we conclude that the selection of the 'tonic diatonic' ratios was a concession to musical practice rather than to aesthetical considerations. On the other hand, if Ptolemy choose his ratios with a view to practice, it is hard to understand why he does not make this explicit. A thorough discussion of both options, followed by the rejection of the Didymean division because of its unfamiliar sound, would have suited Ptolemy's overall approach much better. But this was obviously impossible, the difference between his accepted 'tense diatonic' and Didymus' diatonic being smaller than that between 'tense diatonic' and Pythagorean tuning, which Ptolemy himself classifies as "not worth mentioning"[88].

Consequently, although we cannot exclude the possibility that the septimal tone and the curious small semitone of Ptolemy's 'tonic diatonic' reflected a tuning of musical practice, from the methodological context mathematical/aesthetical reasons emerge as equally plausible. In this case, as regards the current passage, the practical background can be restricted to the two principles explicitly mentioned in the text:

(1) a whole tone at the top (as constructed traditionally by tuning in fifths and fourths and required for modulation to the neighbouring keys), and
(2) two tones rather similar in size.

The latter is nothing else than the principle of a diatonic with two identical tones, found both in Aristoxenian and Pythagorean sources, but in new cloth, identity sacrificed on the altar of superparticularity and replaced by maximal closeness in size (οἳ ἔγγιστα πρὸς αὐτόν).

Another detail confirms our analysis. From an unprejudiced point of view, Ptolemy's 'tonic diatonic' shows two distinctive features, if compared with other diatonic divisions: the 9:8 whole tone at its upper end, which aligns it with all the standard diatonics, on the one hand, and the unusually small half tone at the lower end, on the other. Both are addressed in the name Porphyry uses for this division in his commentary on the *Harmonics*: *malakón éntonon*, 'soft [tuning], including the whole tone'. It is certainly of highest significance that Porphyry here deviates from Ptolemy's own nomenclature. Either musical practice did not provide a name for such a tuning, at least not in Ptolemy's time, or Ptolemy deliberately invented a new term. This is clear from the passage in which he assigns names to his mathematical structures:

> "And here again, in conformity with the size of the highest intervals we will connect the tetrachord consisting of the ratios 21:10 – 10:9 – 8:7 with the soft diatonic, and the one consisting of 16:15 – 9:8 – 10:9 with the tense diatonic, and the one consisting of 28:27 – 8:7 – 9:8 with that one which is intermediate in some way between the soft and the tense, and which could reasonably be called 'tonic', because such is the size of its highest position."[89]

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86 Ptol., *Harm.* 1.15, 36.25–28 Düring.

87 By the construction with διὰ τὸ + *AcI*.

88 The differences are the same in the higher 'movable' note, while the lower notes of Didymus' diatonic and Ptolemy's 'tense diatonic' are identical.

89 Ptol., *Harm.* 1.15, 36.28–35 Düring: κἀνταῦθα δὴ πάλιν ἀκολούθως τῷ μεγέθει τῶν ἡγουμένων λόγων τό μὲν συντιθέμενον τετράχορδον ἔκ τε τοῦ ἐπὶ ζ′ καὶ τοῦ ἐπὶ θ′ καὶ τοῦ ἐπὶ κ′ προσάψομεν τῷ μαλακῷ διατονικῷ, τὸ δὲ συντιθέμενον ἔκ τε τοῦ ἐπὶ θ′ καὶ τοῦ ἐπὶ η′ καὶ τοῦ ἐπὶ ιε′ τῷ συντόνῳ διατονικῷ, τὸ δὲ συντιθέμενον ἔκ τε τοῦ ἐπὶ η′ καὶ τοῦ ἐπὶ ζ′ καὶ τοῦ ἐπὶ κζ′, τῷ μεταξύ πως τοῦ μαλακοῦ καὶ τοῦ συντόνου, κληθέντι δ' ἂν εὐλόγως τονιαίῳ διὰ τὸ τηλικοῦτον εἶναι τὸν ἡγούμενον αὐτοῦ τόνον.

Although generally mistranslated[90], the Greek leaves no doubt that Ptolemy, while addressing the two other shades with names of common use, introduces a new term for our curious division[91]. Interestingly, his name, as well, refers only to the whole tone, just as in his account the small semitone emerges merely as the by-product of primary assumptions. But did Ptolemy have to invent a name? In principle, there are three possibilities:

(1) There existed a common name for the tuning Ptolemy has in mind, but for some reason he chose not to use it. If this is true, a very likely candidate is Porphyry's *malakòn éntonon*. But it is hard to see why Ptolemy would have avoided it, and why he would have avoided any commonly known name in an argument that targets at the readers' recognition of common musical features in figures produces by mathematical algorithms. In any case, *éntonon* provides the same semantic information as Ptolemy's *toniaîon*, and even if he did not want to introduce the notion of softness for the basic division of his tunings, why would he have altered a familiar *éntonon*?
(2) There was no common name because such a kind of tuning simply did not exist. This is highly implausible, since we have seen that Ptolemy introduces his tonic diatonic at high methodological costs: at least the whole tone was part of common practice. Moreover, why would Ptolemy claim this structure to represent the basis of all lyra and cithara tunings, if not because it indeed incorporated some basic features of practice?
(3) There was no distinctive name because, unlike the 'soft' and the 'tense' modifications, this division represented just the regular diatonic, thus requiring no distinctive epithet. Consequently, Ptolemy describes it as "lying in some way between the soft and the tense". This last option concords best with its basic role in the lyre tunings, and must come closest to the truth.

Note that a confirmation of Ptolemy's 'tense diatonic' may be derived from the present conclusion. Up to Didymus, all writers were unanimous in allowing no smaller interval than the whole tone for the highest position of the tetrachord. For Aristoxenus, the usual 'Pythagorean' shade was the 'tense' as opposed to the 'soft' diatonic. Now Ptolemy makes the 'tense' *likhanós* even higher, by recognizing a small 10:9 tone at this position, with a pure minor third below. The differences are those he calls "not worth mentioning", and indeed he testifies to the continued application of the old Pythagorean instrumental 'tense diatonic' – but the insistence on a difference may nevertheless be significant[92].

Still, some questions are left open: how exact is the correspondence between Ptolemy's figures and the actual instrumental tunings, and how comes it that Porphyry provides a different name? Only the second is easily answered. Obviously Porphyry was not at all happy with Ptolemy's terminology. Whether or not he knew such a tuning and its name from the music of his time, Ptolemy's simple *toniaîon* was seemingly misleading, as it concentrated on a common characteristic of diatonic tunings while dismissing the conspicuous feature, the unusually low *parypátē*. Consequently Porphyry added the qualification of 'soft', which was traditional for tetrachordal shades created by some process of down-tuning, and supplanted the term *toniaîon* with his *éntonon* of similar meaning: otherwise, readers of Ptolemy's text together with his commentary might be confused into trying to distinguish between a *diátonon toniaîon* and a *toniaîon malakón*[93].

At the same time, Porphyry testifies to the fact that the small semitone of Ptolemy's 'standard' diatonic was no more standard a century after Ptolemy than it seems to have been a century before Ptolemy, when Didymus wrote. We must therefore seriously face the possibility that the lower part of Ptolemy's 'tonic diatonic' constitutes more of a mathematical illusion than one would like to assume. Diagram 11 provides a visual representation of the 'standard' diatonic tunings given throughout antiquity. All authors before Ptolemy differ only by amounts smaller than those 22 cents, "not worth mentioning": by Ptolemy's standards, they are similar for all practical purposes. Ptolemy's low *parypátē* clearly stands out, differing from Didymus' diatonic by a quartertone, and even from the 'Pythagorean' tuning by 27 cents.

As shown above, Ptolemy's test would hardly have produced exact intervals. Nor does his test for the equality of the middle interval of the 'tonic

90 The crucial ἄν is correctly translated only by Raffa 2002, 137, but neglected or inadequately rendered by Düring 1934, 52; Barker 1989, 310; Solomon 2000, 52; Redondo Reyes 2002, 178.

91 Note that although the position of the *likhanós* is envisaged as the criterion, the three shades are not taken in this order. First come the two commonly known names, 'soft' and 'tense diatonic', because they are needed to define the third shade as "intermediate in some sense" (πως because only the *likhanós* occupies an intermediate position, while the 'tonic' *parypátē* lies below the two other ones).

92 On the possible musical significance of this newly acknowledged order of resonant thirds, cf. Hagel 2005b.

93 Furthermore, the suffix-less composite *éntonon* can, just as the common *diátonon*, receive an additional qualifying adjective easily, whereas a full '*diátonon toniaîon malakón*' is clumsy and a '*toniaîon malakón*' awkward.

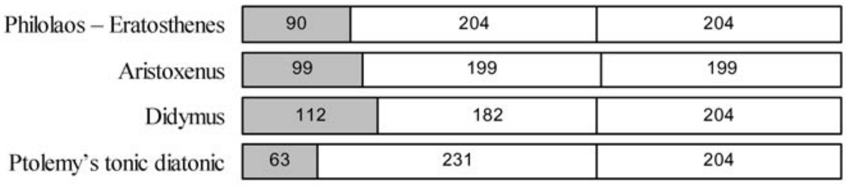

Diagram 11 Standard diatonic tunings throughout antiquity (cents).

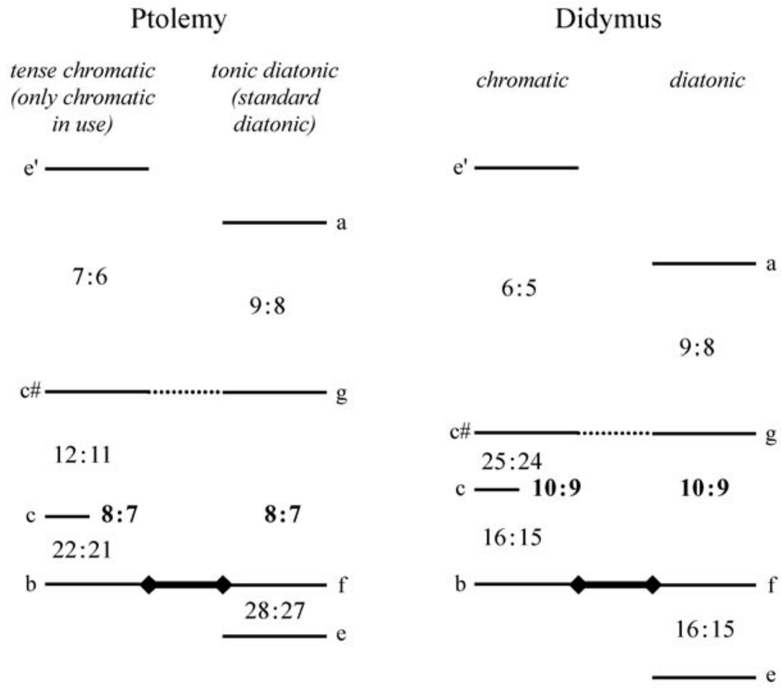

Diagram 12 Ptolemy's test with Didymus' ratios.

diatonic' with the sum of the two lower 'tense chromatic' intervals tell anything about their absolute sizes: Didymus' divisions, although entirely different, testify to the same equation (cf. Diagram 12)[94]. Here, everything rests on Ptolemy's statement that the corresponding 'tense chromatic' interval "will be found to establish a size larger than a [9:8] tone"[95]. Unfortunately, no experimental proof for this relation is offered. This is all the more noteworthy because, only a few lines below, Ptolemy conceives a test for establishing the relative size of two more markedly dissimilar intervals[96]. But perhaps the author just passed over the more obvious construction of a whole tone at the appropriate pitch[97]; at any rate such an assumption is by far more plausible than that of deliberative fraud on Ptolemy's part, especially on an occasion where he could so easily be refuted[98].

Only once does Ptolemy focus directly on the size of the semitone as compared to other tunings, although in such abridged and enigmatic form that the apparent meaning emerges only by meticulous interpretation of what he must have meant. As his last argument against the validity of Aristoxenus' description of tuning shades Ptolemy puts forth that "he [Aristoxenus] makes similar the intervals at the lowest note of the tense diatonic and the tonic chromatic, while the chromatic one is actually larger"[99]. As has been pointed out, this makes sense only if Ptolemy equates Aristoxenus' 'tense diatonic' not with his own division of this name,

94 Note that this correspondence occurs between the 'standard' divisions of both authors: while Didymus provides but one division for each genus anyway, Ptolemy explains that only his 'tense' chromatic is actually in use at his time (*Harm.* 1.16, 38.2–6; cf. 2.16, 80 Düring).

95 Ptol., *Harm.* 2.1, 42.15f Düring.

96 Ptol., *Harm.* 2.1, 43.3–8 Düring. The difference between the 9:8 tone and the 8:7 septimal tone, which is only stated, amounts to 27 cents, that between the septimal tone and the 7:6 septimal third, which is proven, to 36 cents.

97 Cf. Barker 2000, 245f. Note that, at this point, the necessary additional strings for such a construction are still available on the eight-stringed experimental instrument.

98 Cf. Barker 2000, 247–248.

99 Ptol., *Harm.* 1.14, 32.25–27 Düring.

but with his 'tonic diatonic', and thus refers to its small diatonic semitone[100]. This can, of course, be taken as additional evidence only if we take it for granted that Ptolemy did not derive the relation from his own tables. And this is indeed unlikely, exactly because of the apparent terminological confusion – which does not arise when taking into account the original readers' perspective. At this point in his work, Ptolemy has not yet developed his divisions. If he can reasonably expect his readers to extract any meaning from his argument, they must have been aware of two pieces of information: firstly, that Aristoxenus' 'tense diatonic' was actually his standard form of diatonic – which was common knowledge, to be read in any handbook –, and secondly, that the standard diatonic of musical practice employed a smaller semitone. Not having proposed his idiosyncratic terminology at this point, Ptolemy was still in the position to state his argument just in the form he does.

All in all, it seems probable that by Ptolemy's time (and in the musical culture accessible to him) some form of diatonic with an unusually small semitone was current. Perhaps the septimal interpretation of this tuning merely reflects Ptolemy's mathematical principles. Yet if the low *parypátē* had to do with some form of resonance, Ptolemy's analysis is certainly correct, because no 'pure' interval except for the septimal third (and tone) accords with the evidence. Even so, it is extremely improbable that this second century tuning stood in any historical continuity with the music of Archytas' time, whose identical figures could be shown to result from mathematical considerations.

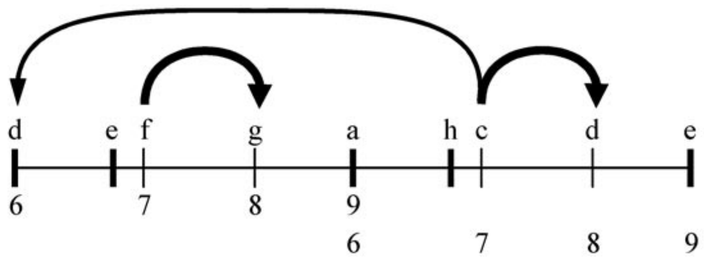

Diagram 13 Focal notes of septimal intervals.

12. MUSICAL SIGNIFICANCE

It remains to ask what musical purpose such a 'deviant' tuning could serve. If the 'septimality' of the tuning is a mathematical chimera, which means that resonance played no role, I can think only of two possibilities: either there was a predilection for very small semitones per se, or the major and minor thirds were deliberately brought 'out of tune'. The latter is hardly plausible in a tradition in which resonance played such an important role in theory. At least, we would have to expect Ptolemy to criticize musicians for not adhering to the 'natural' consonances; yet he is entirely convinced that the music of his time is adequately described by small superparticular ratios, and that his divisions are recognized by any musician as identical with the tetrachords of musical practice. On the other hand, small semitones seem not to have been enjoyed for their own sake in the Roman period: the 'soft' variant of the chromatic, whose lower semitone is identical with the 'tonic diatonic' one in Ptolemy's account, was no longer heard (not to mention the enharmonic).

This leaves us with the other option: that Ptolemy's numeric representation went to the heart of the matter, and there was indeed a predilection for septimal intervals in the elevated strata of musical culture Ptolemy was part of. Once more, we must wonder what this can have meant in practice. The simple melodic presence of septimal intervals can hardly have justified the sacrifice of resonant major and minor thirds, or that of a larger semitone, as was usual even in the chromatic. Were the septimal thirds therefore enjoyed as simultaneous sounds? Or were they conceivably combined with the septimal tones to form 'chords' of the extension of a fourth[101]?

Or did the septimal intervals derive their musical meaning from their function in the overtone series, according to the mechanism we have considered at the outset of our investigation? The scalar structure supports such a usage, the smaller septimal tone being situated above the septimal third, according to their order in the overtone series. Such triplets, as we have seen, can establish their highest member as the focal pitch, insofar it corresponds to the basic note of the harmonic series. Diagram 13 shows what this implies for a

[100] Barker 2000, 119–120; 131; 152.

[101] Such 'septimal chords' in their 'major' variant would have been possible, according to Ptolemy's octachord cithara tunings, e.g. between (thetic) *hypátē* and *mésē* in *parypátai*, *trítai*, *trópoi* and *hypértropa*.

diatonic scale in 'tonic diatonic' throughout. The two focal notes, defined by the two tetrachords of the octave, are g and d, corresponding to Greek *likhanós* and *paranḗtē*, the latter being identical with *nḗtē synēmménōn* and standing one octave above *hyperypátē*, which might therefore share its functional status.

Interestingly, these two notes supersede the old focal notes of a and e (*mésē* and *hypátē*) in some of the post-Hellenistic musical fragments[102]. The obvious examples are the Seikilos song and Mesomedes' *Hymn to Nemesis*[103], which are known to date from after Didymus, but from before Ptolemy – but there are numerous others, all later than Didymus, as well[104]. Perhaps it is too speculative, but I find it tempting to relate the introduction of the septimal diatonic with the evolution (or import[105]?) of this new style of music.

ACKNOWLEDGEMENTS

I wish to express my thanks to the APART programme of the Austrian Academy of Sciences for supporting research for this paper, and to Scott Wallace for editing my English.

102 Cf. West 1992, 186–189; 383, subsuming what I call here music focussing on g and d under "G mode".

103 DAGM nrs. 23 and 28.

104 Pap. Michigan 2958.1–18 (DAGM nr. 42); Pap. Oxy. 4463 (DAGM nr. 47); Pap. Oxy. 4465 (DAGM nr. 49); Pap. Berlin 6870+14097,1–12 (DAGM nr. 50); Pap. Berlin 6870,13–15 (DAGM nr. 51); Pap. Berlin 6870, 20–22 (DAGM nr. 52); probably Pap. Oxy. 3161 recto (DAGM nr. 53); Pap. Oxy. 1786 (DAGM nr. 59).

105 For a possible connection of the triumph of diatonicism in the Roman period with a new wave of Oriental influence, cf. Hagel 2005b. There I have considered Ptolemy's 'tense diatonic' as another possible reaction of tuning practice on the new G mode.

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