Seemingly unrelated Regressions (SUR)

Baltagi, Ch. 6

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Overview

- What are SUR-models?
- 1-Way Model
- 2-Way Model
- Example
- Simultaneous Equations

SUR in general

- Concept by Arnold Zellner from 1962
- Basic idea: Error terms of different equations are correlated amongst each other

  → *f.ex.: demand of different firms*
  or for different products (substitutes)

- Same number of explanatory variables in each equation, not necessarily same X

SUR with panel-data

- *"Same same but different!"*
- Feasible GLS is used to estimate the variance-covariance matrix and parameter estimates
- The process is iterated until the errors are minimized
- Variables could be transformed with a form of Cochrane-Orchutt correction to model autocorrelation

1-Way model

A set of equations …

$$y_1 = Z_1 \delta_1 + u_1$$
$$\vdots$$
$$y_M = Z_M \delta_M + u_M$$

$$y_j = Z_j \delta_j + u_j$$

1-Way model

… with error terms …

$$u_j = Z_\mu \mu_j + \nu_j$$

where:

$$Z_\mu = (I_N \otimes l_T)$$

$$\mu_j = (\mu_{1j}, \mu_{2j} \ldots, \mu_{Nj})$$

$$\nu_j = (\nu_{11j}, \ldots, \nu_{1Tj}, \ldots \nu_{N1j}, \ldots ; \nu_{NTj})$$

1-Way model

… has the following covariance matrices:

$$\Omega_{jl} = E(u_j u_l^{'})$$
$$= \sigma^2_{\mu_{jl}}(I_N \otimes J_T) + \sigma^2_{\nu_{jl}}(I_N \otimes I_T)$$

Variance-Covariance Matrix:

$$\Omega = E(uu')$$
$$= \sum\nolimits_{\mu} \otimes (I_N \otimes J_T) + \sum\nolimits_{\nu} \otimes (I_N \otimes I_T)$$

1-Way model

$$
= \begin{pmatrix} \sigma^2_{\mu_{11}} & \cdots & \cdots & \sigma^2_{\mu_{1j}} \\ \sigma^2_{\mu_{21}} & \ddots & & \\ \vdots & & \ddots & \\ \sigma^2_{\mu_{j1}} & & & \sigma^2_{\mu_{jl}} \end{pmatrix} \otimes \begin{pmatrix} 1 & \cdots & 1 & & & \\ \vdots & & \vdots & & 0 & \\ 1 & \cdots & 1 & & & \\ & & & 1 & \cdots & 1 \\ & 0 & & \vdots & & \vdots \\ & & & 1 & \cdots & 1 \end{pmatrix}
$$

$$
+ \begin{pmatrix} \sigma^2_{\nu_{11}} & \cdots & \cdots & \sigma^2_{\nu_{1j}} \\ \sigma^2_{\nu_{21}} & \ddots & & \\ \vdots & & \ddots & \\ \sigma^2_{\nu_{j1}} & & & \sigma^2_{\nu_{jl}} \end{pmatrix} \otimes \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}
$$

1-Way model

now we replace $J_T = T\bar{J}_T \ and \ I_T = E_T + \bar{J}_T$

$$\Omega = T\Sigma_\mu \otimes (I_N \otimes \bar{J}_T) + \Sigma_\nu \otimes (I_N \otimes (E_T + \bar{J}_T))$$

$$= (T\Sigma_\mu + \Sigma_\nu) \otimes (I_N \otimes \bar{J}_T) + \Sigma_\nu \otimes (I_N \otimes E_T)$$

$$= \Sigma_1 \otimes P + \Sigma_\nu \otimes Q, \qquad with \qquad Q = I_{NT} - P$$

in order to be able to invert Ω

$$\Omega^r = \Sigma_1^r \otimes P + \Sigma_\nu^r \otimes Q$$

$$\Omega^{-1} = \Sigma_1^{-1} \otimes P + \Sigma_\nu^{-1} \otimes Q$$

$$\Omega^{-1/2} = \Sigma_1^{-1/2} \otimes P + \Sigma_\nu^{-1/2} \otimes Q$$

1-Way model

VC matrix can be estimated using OLS or within-type residuals

$$\hat{\Sigma}1 = \frac{U'PU}{N}, \quad \hat{\Sigma}_{\mu} = \frac{U'QU}{N(T-1)}$$

if N, T → ∞, the Within estimator is asymptotically efficient and has the same asymptotic VC matrix as the GLS estimator

2-Way model

Now we include a time effect:

$$u_j = Z_\mu \mu_j + Z_\lambda \lambda_j + \nu_j$$

where

$$Z_\mu = (I_N \otimes l_T)$$
$$Z_\lambda = (l_T \otimes I_N)$$
$$\mu_j = (\mu_{1j}, \mu_{2j} ..., \mu_{Nj})$$
$$\lambda_j = (\lambda_{1j}, \lambda_{2j} ..., \lambda_{Tj})$$
$$\nu_j = (\nu_{11j}, ..., \nu_{1Tj}, ... \nu_{N1j}, ...; \nu_{NTj})$$

2-Way model

The error term matrix has then the following form:

$$\Omega = E(uu') = \sum_{\mu} \otimes (I_N \otimes J_T) + \sum_{\lambda} \otimes (J_N \otimes I_T) + \sum_{\nu} \otimes (I_N \otimes I_T)$$

Ω has to be decomposed to invert:

$$\Omega = T\sum_{\mu} \otimes (I_N \otimes \bar{J}_T) + N\sum_{\lambda} \otimes (\bar{J}_N \otimes I_T) + \sum_{\nu} \otimes ((E_N + \bar{J}_N) \otimes (E_T + \bar{J}_T))$$

$$= (T\sum_{\mu} + \sum_{\nu}) \otimes (I_N \otimes \bar{J}_T) + (N\sum_{\lambda} + \sum_{\nu}) \otimes (\bar{J}_N \otimes I_T) + \sum_{\nu} \otimes (I_N \otimes E_T)$$

$$\Rightarrow \sum_{i=1}^{4} \Lambda_i \otimes Q_i$$

$$\Lambda_1 = \sum_{\nu}, \quad \Lambda_2 = T\sum_{\mu} + \sum_{\nu}, \quad \Lambda_3 = N\sum_{\lambda} + \sum_{\nu}, \quad \Lambda_4 = T\sum_{\mu} + N\sum_{\lambda} + \sum_{\nu}$$

2-Way model

Having done this, we can introduce any arbitrary exponent *r*

$$\Omega^r = \sum_{i=1}^{4} \Lambda_i^r \otimes Q_i$$

$$\Rightarrow \sum_{i=1}^{4} \Lambda_i^{-1} \otimes Q_i, \qquad or \qquad \Rightarrow \sum_{i=1}^{4} \Lambda_i^{-1/2} \otimes Q_i$$

individual components of Ω can be estimated

$$\hat{\Sigma}_\nu = \frac{U'Q_1U}{(N-1)(T-1)}, \quad \hat{\Lambda}_2 = \frac{U'Q_2U}{(N-1)}, \quad \hat{\Lambda}_3 = \frac{U'Q_3U}{(T-1)}$$

with *U* derived from OLS or within estimation

Example

Stata commands

- ***xtsur*** → *1-way random effects estimation*

- Alternatives:

  sureg + dummies for individual and/or time effects + constraints ($\sum \mu_i = 0, \sum \lambda_t = 0$)

→ *noconstant*-option does not work with ***sureg***

Source: FAOSTAT (http://faostat.fao.org/)

Example: 1-Way estimation

Variable	egg1	milk1
lchick	.41690575	
	0.0000	
lchick_meat	.38514723	
	0.0000	
lgdp	-.02212248	.1390674
	0.4905	0.0000
lcat		.54145784
		0.0000
lcat_meat		.32742093
		0.0000
_cons	5.4959816	.66752561
	0.0000	0.3943
N	354	354
theta	.94029847	.96263607

legend: b/p

Example: 1-Way estimation

```
. xtsur (legg lchick lchick_meat lgdp) ///
>       (lmilk lcat lcat_meat lgdp)

Seemingly unrelated regression (SUR) in unbalanced panel data set
 One-way random effect estimation:
------------------------------------------------------------------------------
Number of Group variable:    1                 Number of obs      =       354
Panel variable: code                           Number of eqn      =         2
Time variable : year                           Number of groups   =         1
 ------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
legg         |
      lchick |   .7816699   .0517333    15.11   0.000     .6802745    .8830653
 lchick_meat |   .3962238   .0507861     7.80   0.000     .2966848    .4957627
        lgdp |   .1567003   .0329094     4.76   0.000     .0921991    .2212014
-------------+----------------------------------------------------------------
lmilk        |
        lcat |   .5815703   .0433592    13.41   0.000     .4965877    .6665528
   lcat_meat |    .312817   .0528716     5.92   0.000     .2091906    .4164434
        lgdp |   .1613396   .0306559     5.26   0.000     .1012552    .2214241
------------------------------------------------------------------------------
```

Example: 2-Way estimation

Variable	egg2	milk2
lchick	.42192118	
	0.0000	
lchick_meat	.37935621	
	0.0000	
lgdp	.00337928	.34067258
	0.9396	0.0000
t2	-.00512125	-.04554634
	0.6833	0.0003
t3	-.013413	-.08610724
	0.4128	0.0000
lcat		.59157518
		0.0000
lcat_meat		.3033061
		0.0000
_cons	5.3030452	-1.4259562
	0.0000	0.1074
N	354	354
theta	.94067432	.96258285

legend: b/p

Example: 2-Way estimation

```
. xtsur (legg lchick lchick_meat lgdp t*) ///
>       (lmilk lcat lcat_meat lgdp t*)

 One-way random effect estimation:
 ------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
legg         |
      lchick |    .426133   .0479355     8.89   0.000     .3321812    .5200847
 lchick_meat |   .3713545   .0403583     9.20   0.000     .2922537    .4504552
        lgdp |  -.0648342   .0407461    -1.59   0.112    -.1446952    .0150268
          t1 |   5.912767   .4476631    13.21   0.000     5.035363     6.79017
          t2 |   5.915545   .4509307    13.12   0.000     5.031737    6.799353
          t3 |   5.918376   .4555626    12.99   0.000      5.02549    6.811262
-------------+----------------------------------------------------------------
lmilk        |
        lcat |   .4845353   .0602053     8.05   0.000     .3665351    .6025355
   lcat_meat |   .2245834   .0509356     4.41   0.000     .1247515    .3244153
        lgdp |   .2701145   .0545115     4.96   0.000     .1632738    .3769551
          t1 |   1.585068   .8622317     1.84   0.066    -.1048753    3.275011
          t2 |   1.548979   .8661407     1.79   0.074    -.1486259    3.246583
          t3 |   1.522254    .871834     1.75   0.081    -.1865097    3.231017
------------------------------------------------------------------------------
```

Simultaneous Equations

- Differs from SUR, as there are rhs endogenous variables in the system of equations

$$Z = diag[Z_j] \quad \Rightarrow \qquad Z_j = [Y_j, X_j]$$

$$\Rightarrow \begin{pmatrix} Z_1 & & & \\ & Z_2 & & \\ & & \ddots & \\ & & & Z_M \end{pmatrix} = \begin{pmatrix} [Y_1, X_1] & & & \\ & \ddots & & \\ & & \ddots & \\ & & & [Y_M, X_M] \end{pmatrix}$$

- The VC matrix has the same form as in SUR!

Simultaneous Equations

System of equations:

$$\tilde{y} = \tilde{Z}\delta + \tilde{u} \qquad \bar{y} = \bar{Z}\delta + \bar{u}$$

where

$$\tilde{y} = (I_M \otimes Q)y \qquad \bar{y} = (I_M \otimes P)y$$

$$\tilde{Z} = (I_M \otimes Q)Z \qquad \bar{Z} = (I_M \otimes P)Z$$

$$\tilde{u} = (I_M \otimes Q)u \qquad \bar{u} = (I_M \otimes P)u$$

$$Q = I_{NT} - P, \qquad P = I_N \otimes \bar{J}_T$$

Simultaneous Equations

- Within 3SLS-estimator

$$\tilde{y} = \tilde{Z}\delta + \tilde{u} \Rightarrow \tilde{\delta}_{W3SLS}$$

- Between 3SLS-estimator

$$\bar{y} = \bar{Z}\delta + \bar{u} \Rightarrow \hat{\delta}_{B3SLS}$$

- (Random) Error component 3SLS-estimator

$$\hat{\delta}_{EC3SLS} = W_1\tilde{\delta}_{W3SLS} + W_2\hat{\delta}_{B3SLS}$$