A nonparametric test for seasonal unit roots

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Abstract

We consider a nonparametric test for the null of seasonal unit roots in quarterly and monthly time series that builds on the RUR (range unit root) test by Aparicio, Escribano, and Sipols, so we tentatively use the name RURS (for RUR–seasonal). We find that the test concept is more promising than a formalization of visual aids such as rank changes in plots by period. In order to cope with the sensitivity of the original RUR test to autocorrelation under its null of a unit root, we suggest an augmentation step by autoregression. Whereas the statistics for testing unit roots in quarterly time series have essentially the same limiting distribution as the original RUR statistic, the monthly version uses eight test statistics at the seasonal frequencies with hitherto unexplored limiting distributions. We present some evidence on the size and power of our procedure and we provide illustrations by empirical applications.
Parametric and nonparametric tests

Basically, a procedure is called *parametric* if it assumes the ‘window’ of a known collection of distributions \((f_\theta)\), with \(\theta \in \Theta\) and \(\Theta\) isomorphic to a subset of the \(\mathbb{R}^k\) with finite dimension \(k\). If \(\Theta\) is infinite-dimensional or if \((f_\theta)\) is not fully specified, the problem is called *non-parametric*. This definition is usually not followed strictly.
A parametric testing problem considers a decision between $\theta \in \Theta_0$ and $\theta \in \Theta_1$, with $\Theta_0 \cup \Theta_1 = \Theta$ and $\Theta_0 \cap \Theta_1 = \emptyset$. Roughly, tests should fulfil some main conditions. Probability of rejecting $\Theta_0$ should be roughly constant across $\Theta_0$ (similarity), while it should approach 1 for large samples across $\Theta_1$ (consistency), where it should also be larger than for rival tests (most powerful).

For many testing problems, likelihood-ratio tests or approximations (Wald, LM, Hausman tests) obey these conditions.
Embedding of hypotheses

How does the test that was designed for discriminating $\Theta_0$ and $\Theta_1$ behave on $\Theta_0^*$ and $\Theta_1^*$ if $\Theta_j \subset \Theta_j^*$ and the starred sets are ‘plausible’ generalizations?

**Example.** The Dickey-Fuller test for unit roots, designed for autoregressions with Gaussian errors, works well for many non-Gaussian error distributions and dependence. This embedding is successful. The test does not work well for structural breaks and for nonlinear transforms of a random walk. Embedding fails.
Why use nonparametric tests?

Within the parametric problem $\Theta$, parametric tests have more power than nonparametric tests. In parts of $\Theta^*$, nonparametric tests may be non-similar even for large samples, or may have zero power and be inconsistent. If embedding fails, however, parametric tests may suffer from the same problems for parts of $\Theta^* \setminus \Theta$. Nonparametric tests may have more power in parts of $\Theta_1^* \setminus \Theta_1$, which can be traded against a power loss in $\Theta_1$. Therefore, these tests are called ‘robust’.
Nonparametric tests for unit roots

Essentially, parametric unit-roots tests use test statistics that measure the correlation of levels and differences, which are zero for random walks. Nonparametric unit-root tests use test statistics that exploit other characteristic properties of random walks, such as the rate of expansion of trajectories or the number of level crossings. The test by Aparicio, Escribano, and Sipols (2006 *Journal of Time Series Analysis*, AES) counts the frequency of new extrema within the trajectory.
The record count

For a given realization \((x_t, t = 1, \ldots, n)\), define

\[
\begin{align*}
x_{j,j} &= \max_{t=1,\ldots,j} x_t, \\
x_{1,j} &= \min_{t=1,\ldots,j} x_t.
\end{align*}
\]

Then, \(x_{j,j} - x_{1,j}\) defines the sequence of ranges of the series. Any time it increases over \(j = 1, \ldots, n\), this is called a record. The number of records until \(n\) is denoted as \(R^{(x)}(n)\) or \(R(n)\).
The RUR statistic

One can show that

$$R(n) = O(n^{1/2})$$

for a random walk with independent increments. One can also show that

$$R(n) = O(\log n)$$

for many stationary processes. This motivate that the statistic

$$J_0^{(n)} = n^{-1/2} R^{(x)}(n)$$

can be the basis for a consistent test, if the null is a random walk and the alternative is stationarity. $J_0^{(n)}$ is called the RUR statistic (range unit root), and its properties were established by AES.
RUR test: good and bad news

**Good news.** The statistic is invariant to monotonic transformations of the observed variable. $x$ and $\log x$ yield the same RUR statistic. It is reasonably robust to level shifts and to distributional assumptions. Under ideal conditions, its power is acceptable.

**Bad news.** The test has less power than the DF test. It is sensitive to correlation in the increments of the random walk under the null. It is not even asymptotically similar for a general I(1) null.
Assume \((x_t)\) is observed quarterly. If its autoregressive representation
\[
\Phi(B)x_t = \varepsilon_t
\]
yields a polynomial \(\Phi(z)\) with \(\Phi(1) = 0\), \((x_t)\) is integrated at frequency zero. If \(\Phi(-1) = 0\), it is integrated at the semi-annual frequency \(\pi\). If \(\Phi(\pm i) = 0\), it is integrated at the annual frequency \(\pi/2\).
Seasonally integrated processes require seasonal differencing to become stationary. The most popular test for seasonal unit roots is the parametric test by Hylleberg, Engle, Granger, Yoo (1990 *Journal of Econometrics*), the HEGY test.
The idea of the nonparametric RURS test: $x^{(1)}$ and $x^{[2]}$

Assume $(x_t)$ follows a *seasonal random walk*

$$x_t = x_{t-4} + \varepsilon_t.$$

Then, the transformed variable

$$x^{(1)}_t = x_t + x_{t-1} + x_{t-2} + x_{t-3}$$

is a random walk. The transformed variable

$$x^{(2)}_t = x_t - x_{t-1} + x_{t-2} - x_{t-3}$$

behaves like $y_t = -y_{t-1} + \varepsilon_t$. Reverting the signs for all observations with $t$ odd yields a random walk $x^{[2]}_t$. 
The idea of the nonparametric RURS test: $x^{[3]}$ and $x^{[4]}$

Recall that $(x_t)$ is generated from a SRW. Then,

$$x_t^{(3)} = x_t - x_{t-2}$$

behaves like $x_t^{(3)} = -x_{t-2}^{(3)} + \varepsilon_t$. It consists of a modified RW for the odd and another one for even $t$. Collecting the even $t$ and the odd $t$ separately and reverting signs within the two time series yields two random walks $x_t^{[3]}$ and $x_t^{[4]}$ with sample size $n/2$. 
The RURS statistics

The statistics

\[ J_1 = n^{-1/2} R_n^{x(1)} , \]
\[ J_2 = n^{-1/2} R_n^{x[2]} , \]
\[ J_3 = (n/2)^{-1/2} R_n^{x[3]} , \]
\[ J_4 = (n/2)^{-1/2} R_n^{x[4]} \]

are the RURS (range unit roots seasonal) statistics. For SRW \((x_t)\), they converge to the AES limit distribution. If \(\phi(1) \neq 0\), \(J_1 \rightarrow 0\). If \(\phi(-1) \neq 0\), \(J_2 \rightarrow 0\), and if \(\phi(\pm i) \neq 0\), then \(J_3 \rightarrow 0\) and \(J_4 \rightarrow 0\).
Handling RUR non-similarity

AES do not address the issue of the null non-similarity of the RUR test. Assume \((x_t)\) is I(1) but not a random walk. We suggest to eliminate serial correlation under the null by regressing \(\Delta x_t\) on \(p\) BIC–selected lags

\[
\Delta x_t = \mu + \sum_{k=1}^{p} \gamma_k \Delta x_{t-k} + \varepsilon_t, \quad t \geq p + 2.
\]

Estimation residuals \(u_t\) are accumulated according to

\[
\tilde{x}_t = \sum_{j=p+2}^{t} u_j + x_{p+1},
\]

such that \(\tilde{x}_t\) is ideally a pure random walk without drift. The same can be done for the constructed variables \(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\).
Handling deterministic terms

AES suggest to conduct a two-sided test. In its left tail, the RUR test rejects in favor of the stationary alternative. In its right tail, the RUR test rejects in favor of the ‘drifting’ alternative, as I(1) with drift yields $O(n)$ expansion of the RUR statistic. The test has little power in the presence of sub-linear trends.

In the RURS construction, drifts and deterministic seasonal patterns are eliminated. The RURS test is conducted as a one-sided test. The right tail of the AES limit distribution can only materialize in the presence of super-linear trends.
Simulated quantiles

<table>
<thead>
<tr>
<th>model</th>
<th>1%</th>
<th>5%</th>
<th>10%</th>
<th>median</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>SRW $n = 1000$</td>
<td>±1</td>
<td>0.95</td>
<td>1.14</td>
<td>1.23</td>
<td>1.67</td>
<td>2.24</td>
<td>2.40</td>
</tr>
<tr>
<td></td>
<td>±i</td>
<td>1.07</td>
<td>1.25</td>
<td>1.38</td>
<td>2.06</td>
<td>2.99</td>
<td>3.35</td>
</tr>
<tr>
<td>SRW $n = 500$</td>
<td>±1</td>
<td>0.94</td>
<td>1.11</td>
<td>1.20</td>
<td>1.65</td>
<td>2.18</td>
<td>2.41</td>
</tr>
<tr>
<td></td>
<td>±i</td>
<td>1.01</td>
<td>1.26</td>
<td>1.39</td>
<td>2.02</td>
<td>2.97</td>
<td>3.28</td>
</tr>
<tr>
<td>SRW $n = 100$</td>
<td>±1</td>
<td>0.78</td>
<td>0.98</td>
<td>1.08</td>
<td>1.57</td>
<td>2.16</td>
<td>2.26</td>
</tr>
<tr>
<td></td>
<td>±i</td>
<td>0.84</td>
<td>1.12</td>
<td>1.26</td>
<td>1.82</td>
<td>2.66</td>
<td>3.08</td>
</tr>
</tbody>
</table>

Note: Rows ±1 correspond to the $J_1$ and $J_2$ statistics, while rows ±i correspond to $J_3$ and $J_4$.

Left tails to be used for testing. Values are close to AES values but they are not identical.
Correction for autocorrelation works: AR disturbances

10%, 50%, and 90% quantiles for the uncorrected (solid) and for the augmentation-corrected (dashed) RURS statistic $J_2^{(n)}$ if it is calculated from trajectories of length $n = 100$ from the data-generation process $\Delta_4 x_t = \phi \Delta_4 x_{t-1} + \varepsilon_t$ and $\phi$ is varied over the interval $[-1, 1]$. $\phi$ values on the abscissa.
Correction for autocorrelation works: MA disturbances

10%, 50%, and 90% quantiles for the uncorrected (solid) and for the augmentation-corrected (dashed) RURS statistic $J_2^{(n)}$ if it is calculated from trajectories of length $n = 100$ from the data-generation process $\Delta_4 x_t = \varepsilon_t + \theta \varepsilon_{t-1}$ and $\theta$ is varied over the interval $[-1, 1]$. $\theta$ values on the abscissa.
The main purpose of seasonal unit-root tests is to discriminate between deterministic and unit-root stochastic seasonality. The most important alternative is not a stationary process but a stable process with added deterministic cycles. If the generating model is $x_t = x_{t-1} + \sum_{j=1}^{4} \delta_j D_{j,t} + \varepsilon_t$, $J_1$ should not reject but $J_k$ should do so for $k = 2, \ldots, 4$. $D_{j,t}$ are seasonal dummy constants.
Power simulations for the RURS test

<table>
<thead>
<tr>
<th>model</th>
<th>$r(0.01)$</th>
<th>$r(0.05)$</th>
<th>$r(0.1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RW+ $n = 100$</td>
<td>$J_1$</td>
<td>0.02</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>$J_2$</td>
<td>0.10</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>$J_3, J_4$</td>
<td>0.27</td>
<td>0.62</td>
</tr>
<tr>
<td>RW+ $n = 500$</td>
<td>$J_1$</td>
<td>0.13</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>$J_2$</td>
<td>0.70</td>
<td>0.88</td>
</tr>
<tr>
<td></td>
<td>$J_3, J_4$</td>
<td>0.98</td>
<td>1.00</td>
</tr>
<tr>
<td>RW+ $n = 1000$</td>
<td>$J_1$</td>
<td>0.14</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>$J_2$</td>
<td>0.92</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>$J_3, J_4$</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

RW+ denotes that the generating model is a random walk with added seasonal constants drawn from a $N(0,1)$ distribution. $r(p)$ denotes rejection frequency when the RURS test is used at a significance level of $p$. 
Figure: Belgian barley prices, 1971:1–2003:1, quarterly observations.
RURS and HEGY yield different results

Traditional parametric HEGY tests reject unit roots at $-1$ and at $\pm i$ and indicate purely deterministic seasonal variation. RURS statistics are $J_1 = 2.03$, $J_2 = 1.85$, and $J_3 = J_4 = 1.00$. Unit roots at $\pm 1$ are supported, while $\pm i$ is rejected at the 5% level. The RURS test finds a deterministic annual cycle and a persistently changing semi-annual pattern.
A nonparametric test for seasonal unit roots

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Among the 12 frequencies, the transformation to random walks works for 4 cases only: \( \pm 1 \), i.e. 0 and \( \pi \), and \( \pm i \), i.e. \( \pi/2 \) and \( 3\pi/2 \). At all other frequencies, the asymptotic distribution of the RURS statistic is unknown. The limit trajectories are not Brownian motion.

If \((x_t)\) is a monthly seasonal random walk, i.e. \( x_t = x_{t-12} + \varepsilon_t \), sampling every 12th observation only yields a random walk for any moving-average transformation of \( x \) of order less than 12. Therefore, the expansion rate of any RURS statistic will again be \( O(n^{1/2}) \).
Example: testing at frequency $\pi/6$

Assume $(x_t)$ is a monthly SRW $x_t = x_{t-12} + \varepsilon_t$. Then, the dynamic transformations

$$
x_t^{(2)} = (\sqrt{3} + B - B^3 - \sqrt{3}B^4 - 2B^5 - \sqrt{3}B^6 - B^7 + B^9 + \sqrt{3}B^{10} + 2B^{11})x_t,
$$

$$
x_t^{(3)} = (1 + \sqrt{3}B + 2B^2 + \sqrt{3}B^3 + B^4 - B^6 - \sqrt{3}B^7 - 2B^8 - \sqrt{3}B^9 - B^{10})x_t
$$

will be pure unit-root processes of the form $(1 - B + B^2)y_t = \varepsilon_t$ at the angular frequency $\pi/12$.

For these processes, a parallel procedure can be used to purge them from serial correlation under the null. Consider the auxiliary regression

$$
(1 - \sqrt{3}B + B^2)y_t = \mu + \sum_{j=1}^{p} (1 - \sqrt{3}B + B^2)y_{t-j} + \varepsilon_t, \; t \geq p + 3.
$$

Purged trajectories evolve from accumulating estimation residuals

$$
\tilde{y}_t = y_{t-1} + y_{t-2} + u_t, \; t \geq p + 3.
$$

The same can be done for frequencies $\pi/6$, $\pi/3$, $5\pi/12$. 
Simulated quantiles for the monthly RURS test

<table>
<thead>
<tr>
<th>statistic</th>
<th>frequency</th>
<th>n</th>
<th>1%</th>
<th>5%</th>
<th>10%</th>
<th>median</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>0</td>
<td>100</td>
<td>0.995</td>
<td>1.194</td>
<td>1.293</td>
<td>2.059</td>
</tr>
<tr>
<td>$J_2$</td>
<td>$\pi/6$</td>
<td>100</td>
<td>0.697</td>
<td>0.995</td>
<td>1.095</td>
<td>1.806</td>
</tr>
<tr>
<td>$J_3$</td>
<td>$\pi/6$</td>
<td>100</td>
<td>0.792</td>
<td>1.089</td>
<td>1.287</td>
<td>1.972</td>
</tr>
<tr>
<td>$J_4$</td>
<td>$\pi/3$</td>
<td>100</td>
<td>0.498</td>
<td>0.796</td>
<td>0.995</td>
<td>1.678</td>
</tr>
<tr>
<td>$J_5$</td>
<td>$\pi/3$</td>
<td>100</td>
<td>0.594</td>
<td>0.792</td>
<td>0.990</td>
<td>1.695</td>
</tr>
<tr>
<td>$J_6$</td>
<td>$\pi/2$</td>
<td>100</td>
<td>0.885</td>
<td>1.180</td>
<td>1.327</td>
<td>1.959</td>
</tr>
<tr>
<td>$J_7$</td>
<td>$\pi/2$</td>
<td>100</td>
<td>0.885</td>
<td>1.032</td>
<td>1.180</td>
<td>1.956</td>
</tr>
<tr>
<td>$J_8$</td>
<td>$2\pi/3$</td>
<td>100</td>
<td>0.597</td>
<td>0.896</td>
<td>0.995</td>
<td>1.816</td>
</tr>
<tr>
<td>$J_9$</td>
<td>$2\pi/3$</td>
<td>100</td>
<td>0.495</td>
<td>0.891</td>
<td>0.990</td>
<td>1.857</td>
</tr>
<tr>
<td>$J_{10}$</td>
<td>$5\pi/6$</td>
<td>100</td>
<td>0.498</td>
<td>0.697</td>
<td>0.896</td>
<td>1.601</td>
</tr>
<tr>
<td>$J_{11}$</td>
<td>$5\pi/6$</td>
<td>100</td>
<td>0.495</td>
<td>0.792</td>
<td>0.891</td>
<td>1.616</td>
</tr>
<tr>
<td>$J_{12}$</td>
<td>$\pi$</td>
<td>100</td>
<td>0.995</td>
<td>1.194</td>
<td>1.393</td>
<td>2.058</td>
</tr>
</tbody>
</table>

Note: Empirical quantiles from 10,000 replications. Generating model is the SRW $x_t = x_{t-12} + \varepsilon_t$. 

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A nonparametric test for seasonal unit roots
Power of the monthly RURS test

Rejection frequency for 10% tests using the statistics $J_j, j = 1, \ldots, 12$ if $n = 500$. Generating processes are $x_t = \phi x_{t-12} + \epsilon_t$. $\phi$ on the $x$–axis.
What do the monthly power experiments show?

Power is higher at the ‘strange’ frequencies. The most difficult roots are at $\pm i$, where information is limited to $n/2$ if the sample size is $n$.

Generally, power of the monthly RURS test is not very satisfactory.
AES suggested to run their RUR test forward and also backward in order to increase power. The same can be done with the RURS test.

Often, applied researchers are not really interested in seasonal unit roots. They would like to know whether to apply the filter $1 - B^S$ for $S = 4, 12$ or not. The focus could be on the joint application of $J_j$, $j = 2, \ldots, S$ in analogy to $F$-tests.

Empirical examples for the monthly version.

Experiments for $\Theta^*$ designs, such as processes with unusual error distributions, outliers, breaks, and nonlinear transformations of SRW.
Thank you for your attention