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Entangled entanglement: A construction procedure

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ABSTRACT

The familiar Greenberger–Horne–Zeilinger (GHZ) states can be rewritten by entangling the Bell states for two qubits with a third qubit state, which is dubbed *entangled entanglement*. We show that in a constructive way we obtain all eight independent GHZ states that form the simplex of entangled entanglement, the *magic simplex*. The construction procedure allows a generalization to higher dimensions both, in the degrees of freedom (considering qudits) as well as in the number of particles (considering n-partite states). Such bases of GHZ-type states exhibit a cyclic geometry, a *Merry Go Round*, that is relevant for experimental and quantum information theoretic applications.

1. Introduction

Entanglement as one of the most fundamental phenomena in quantum physics has many fascinating aspects. An amazing feature that occurs for multipartite systems is *entangled entanglement*. The term was coined by Krenn and Zeilinger [1] to characterize the phenomenon that the entanglement of two qubits, expressed by the Bell states, can be entangled further with a third qubit, producing such a particular Greenberger–Horne–Zeilinger (GHZ) state. We take up this idea, develop it further and show that all independent (maximally entangled) GHZ states, can be expressed, geometrically quite obviously, in an entangled entanglement form. These basis states configure the *magic simplex* [2]. The word "magic" goes back to Wootter's magic basis to compute the concurrence [3]. We show then explicitly how our construction procedure, which is entirely systematic and intuitive, can be generalized to higher dimensions d and to any finite number of particles n, namely to n-partite qudit states $d \otimes d \otimes d \otimes d \otimes \ldots \otimes d = d^{\otimes n}$.

To obtain an understanding and intuition of the physics behind entangled entanglement we discuss the case of the tripartite GHZ states in Section 2, on one hand, with respect to the Einstein–Podolsky–Rosen (EPR) paradox, and on the other hand, with reference to the mathematical structure, the freedom to factorize a tensor product of algebras (or Hilbert spaces) in different ways, which forms the mathematical basis for the phenomenon of entangled entanglement. In Section 3 we introduce our procedure how to construct systematically the states of entangled entanglement for any higher dimension and number of particles. The use of the unitary Weyl operators [4] turns out to be very helpful (also known under names like "generalized spin operators", "Pauli group" and "Heisenberg group", Refs. [5–8]). In Weyl's book these unitary operators, consisting of phase and (cyclic) ladder operators, were introduced by a "quantization" of classical kinematics that is the reason why the magic simplex is sometimes considered as a phase-space.

We also illustrate the geometric structure of the state space (see Fig. 2), the symmetries inherent in a magic simplex and the cyclicity of the phase operations, when moving from one state to another within the simplex. In particular we discover a *Merry Go Round* of the qutrit GHZ states (see Fig. 3). Finally, conclusions are drawn in Section 4.

2. Physical aspect and mathematical structure

Let us begin with discussing the physics behind the phenomenon of entangled entanglement. We recall the well-known GHZ state [9,10]

$$|\mathrm{GHZ1}^-\rangle_{123} = \frac{1}{\sqrt{2}}\left(|R\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3 +, |L\rangle_1 \otimes |L\rangle_2 \otimes |L\rangle_3\right), \tag{1}$$

where $|R\rangle, |L\rangle$ denote the right- and left-handed circularly polarized photons. Interestingly, expression (1) can be re-expressed by decomposing (1) into linearly polarized states $|H\rangle, |V\rangle$ and Bell states

$$|\mathrm{GHZ1}^-\rangle_{123} = \frac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^-\rangle_{23} - |V\rangle_1 \otimes |\psi^+\rangle_{23}\right), \tag{2}$$

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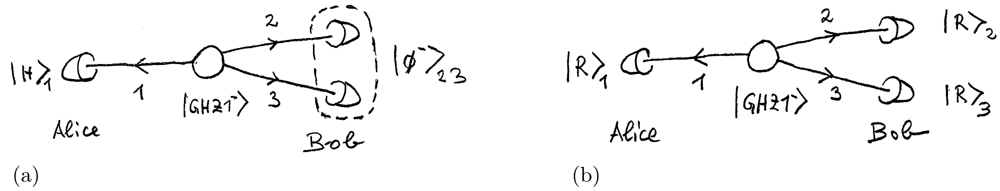

Fig. 1. (a) Bob's photons are in an entangled state. (b) Bob's photons are in a separable state.

where $|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle \otimes |H\rangle \pm |V\rangle \otimes |V\rangle\right)$, $|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle \otimes |V\rangle \pm |V\rangle \otimes |H\rangle\right)$ represent the familiar maximally entangled Bell states. The linearly polarized states $|H/V\rangle$ are related to the circularly polarized states via $|R/L\rangle = \frac{1}{\sqrt{2}}(|H\rangle \pm i|V\rangle)$.

The GHZ state as expressed in Eq. (2) obviously represents entangled entanglement. This feature has been verified experimentally by Zeilinger's group [11] who has performed a Bell-type experiment on three particles, where one part, Alice on line 1, projects onto the horizontally $|H\rangle_1$ or vertically $|V\rangle_1$ polarized state and the other part, Bob on lines 2 and 3, projects onto the maximally entangled states $|\phi^-\rangle_{23}$ or $|\psi^+\rangle_{23}$ via a Bell state measurement based on a polarizing beam splitter [12]. Then the authors test a Clauser–Horne–Shimony–Holt inequality established between Alice and Bob and find a strong violation of the inequality (specifically, of the Bell parameter) by more than five standard deviations. Thus the entangled states of the two photons on Bob's side are definitely entangled again with the single photon on Alice's side.

What is the physical significance of it, in particular, in the light of an EPR reasoning? Let us start with an EPR-like discussion as in Ref. [11]. If Alice is measuring the linearly polarized state $|H\rangle_1$ then Bob will find the Bell state $|\phi^-\rangle_{23}$ for his two photons (see Fig. 1(a)). If she obtains a $|V\rangle_1$ state in her measurement then Bob will get the Bell state $|\psi^+\rangle_{23}$. This perfect correlation between the polarization state of one photon on Alice's side and the entangled state of the two photons on Bob's side implies, under the EPR premises of realism and "no action at a distance", that the entangled state of the two photons must represent an element of reality. Whereas the individual photons of this state, which have no well-defined property, do not correspond to such elements. For a realist this is a surprising feature, indeed.

If, on the other hand, Alice is measuring a right-handed circularly polarized state $|R\rangle_1$ then Bob will find his two photons in a separable state $|R\rangle_2 \otimes |R\rangle_3$ (see Fig. 1(b)), or if Alice measures $|L\rangle_1$ Bob will get $|L\rangle_2 \otimes |L\rangle_3$. Then the two photons of Bob contain individually an element of reality, which is more satisfactory to a realist. Thus by the specific kind of measurement, projecting on linearly or circularly polarized photons, Alice is able to switch on Bob's side the properties of the two photons — and their reality content — between entanglement and separability.

This feature is even more puzzling in case of entangling internal with external degrees of freedom, which is experimentally achieved in neutron interferometry. The experimenters of Ref. [13] produced a GHZ-like state for single neutrons entangled in path–spin–energy. There the above considerations also have to hold.

How can we understand this switching phenomenon between entanglement and separability? A quantum theorist can trace this switch back to two different factorizations of the tensor product of three algebras $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$, where $\mathcal{A}_1$ belongs to Alice and $\mathcal{A}_2 \otimes \mathcal{A}_3$ to Bob. There is total democracy between the different factorizations [14,15], no partition has ontologically a superior status over any other one (if no specific physical realization is taken into account). For an experimentalist, however, a certain factorization is preferred and is clearly fixed by the set-up.

For tripartite states, the GHZ states, which are defined on a tensor product of three algebras, there exists the following theorem [14], where $\rho = |\psi\rangle\langle\psi|$ denotes the corresponding density matrix of the quantum state $|\psi\rangle$:

Theorem 1 *(Factorization algebra). For any pure tripartite state ρ one can find a factorization $M = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ such that ρ is separable with respect to this factorization and another factorization $M = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3$ where ρ appears to be maximally entangled.*

For mixed states, however, such a unitary switching between separable and entangled states exists only beyond a certain bound of mixedness [14].

Example. To illustrate Theorem 1 we consider the circularly polarized states $\{|R\rangle, |L\rangle\}$. We find, for example, the following unitary matrix $U^\dagger$ that transforms the separable state $|R\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3$ into the entangled state $|\mathrm{GHZ1}^-\rangle_{123}$ of Eq. (1)

$$U^\dagger |R\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3 = |\mathrm{GHZ1}^-\rangle_{123}\,, \tag{3}$$

where

$$U = U_0^{\mathrm{T}\otimes 3} \cdot U_{\mathrm{ent}} \cdot U_0^{\otimes 3} \qquad \text{with} \qquad U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{4}$$

and

$$U_{\mathrm{ent}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{5}$$

Having found the structure of entangled entanglement, it is quite natural to ask if other GHZ states can be expressed in a similar way. The answer is yes, we can construct a complete orthonormal system. Geometrically it is quite obvious how to proceed. We just have to entangle the opposite states $|\phi^-\rangle$ and $|\psi^+\rangle$ or $|\phi^+\rangle$ and $|\psi^-\rangle$ of the $2 \otimes 2$ dimensional tetrahedron of Bell states [16–18] with $|H\rangle$ and $|V\rangle$, and respect the symmetric and antisymmetric property respectively. In this way we immediately find an orthonormal basis of eight states

$$\begin{aligned}
|\mathrm{GHZ1}^+\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^-\rangle_{23} + |V\rangle_1 \otimes |\psi^+\rangle_{23}\right) \\
|\mathrm{GHZ1}^-\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^-\rangle_{23} - |V\rangle_1 \otimes |\psi^+\rangle_{23}\right) \\
|\mathrm{GHZ2}^+\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^+\rangle_{23} + |V\rangle_1 \otimes |\psi^-\rangle_{23}\right) \\
|\mathrm{GHZ2}^-\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^+\rangle_{23} - |V\rangle_1 \otimes |\psi^-\rangle_{23}\right) \\
|\mathrm{GHZ3}^+\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|V\rangle_1 \otimes |\phi^-\rangle_{23} + |H\rangle_1 \otimes |\psi^+\rangle_{23}\right) \\
|\mathrm{GHZ3}^-\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|V\rangle_1 \otimes |\phi^-\rangle_{23} - |H\rangle_1 \otimes |\psi^+\rangle_{23}\right) \\
|\mathrm{GHZ4}^+\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|V\rangle_1 \otimes |\phi^+\rangle_{23} + |H\rangle_1 \otimes |\psi^-\rangle_{23}\right) \\
|\mathrm{GHZ4}^-\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|V\rangle_1 \otimes |\phi^+\rangle_{23} - |H\rangle_1 \otimes |\psi^-\rangle_{23}\right).
\end{aligned} \tag{6}$$

These eight states form the vertices of a simplex $\mathbb{S}$ in the corresponding eight dimensional Hilbert space of the tensor product $2\otimes2\otimes2$. It is the analogue to the tetrahedron of Bell states in $2\otimes2$ dimensions. The set $\mathbb{S}$ itself, the *magic simplex of entangled entanglement*, consists of the convex combinations of all the corresponding density matrices $\rho_{\mathrm{GHZ}i^{\pm}}$

$$\mathbb{S} := \left\{ \rho = \sum_{i=1,\dots,4;k=+,-} \lambda_i^k \rho_{\mathrm{GHZ}i^k} \,|\; \lambda_i^{\pm} \geq 0, \quad \sum \lambda_i^{\pm} = 1 \right\}, \tag{7}$$

where

$$\rho_{\mathrm{GHZ}i^{\pm}} = |\mathrm{GHZ}i^{\pm}\rangle\langle \mathrm{GHZ}i^{\pm}|, \quad i = 1, \dots, 4\,. \tag{8}$$

The convex combination (7) of the GHZ states builds up a simplex with the maximally mixed state $\frac{1}{8}\mathbb{1}$ in its center. All the density matrices inside this simplex represent valid quantum states: The eigenvalues of every state $\rho = \sum_{i=1,\dots,4;k=+,-} \lambda_i^k \rho_{\mathrm{GHZ}i^k}$ in the simplex are just equal to the eight coefficients λ_i^k [19]. For the elements of $\mathbb{S}$, we know, by construction, that $\lambda_i^{\pm} \geq 0$ and $\sum \lambda_i^{\pm} = 1$, so that inside the simplex all the eigenvalues are non-negative and for the trace we have $\mathrm{Tr}\rho = 1$. Therefore, any element $\rho \in \mathbb{S}$ corresponds to a density matrix.

On the other hand, we can show that the matrices outside the simplex $\mathbb{S}$ have to violate one of the above properties and therefore do not form density matrices of physical states.

All states of entangled entanglement (6) can certainly be re-expressed by tensor products of right-handed $|R\rangle$ and left-handed $|L\rangle$ circularly polarized photon states [19]

$$\begin{aligned}
&|\mathrm{GHZ}1^+\rangle_{123} \\
&\quad = \tfrac{1}{\sqrt{2}}\big(|R\rangle_1 \otimes |L\rangle_2 \otimes |L\rangle_3 + |L\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3\big) \\
&|\mathrm{GHZ}1^-\rangle_{123} \\
&\quad = \tfrac{1}{\sqrt{2}}\big(|R\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3 + |L\rangle_1 \otimes |L\rangle_2 \otimes |L\rangle_3\big) \\
&|\mathrm{GHZ}2^+\rangle_{123} \\
&\quad = \tfrac{1}{\sqrt{2}}\big(|R\rangle_1 \otimes |R\rangle_2 \otimes |L\rangle_3 + |L\rangle_1 \otimes |L\rangle_2 \otimes |R\rangle_3\big) \\
&|\mathrm{GHZ}2^-\rangle_{123} \\
&\quad = \tfrac{1}{\sqrt{2}}\big(|R\rangle_1 \otimes |L\rangle_2 \otimes |R\rangle_3 + |L\rangle_1 \otimes |R\rangle_2 \otimes |L\rangle_3\big) \\
&|\mathrm{GHZ}3^+\rangle_{123} \\
&\quad = -\tfrac{i}{\sqrt{2}}\big(|R\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3 - |L\rangle_1 \otimes |L\rangle_2 \otimes |L\rangle_3\big) \\
&|\mathrm{GHZ}3^-\rangle_{123} \\
&\quad = -\tfrac{i}{\sqrt{2}}\big(|R\rangle_1 \otimes |L\rangle_2 \otimes |L\rangle_3 - |L\rangle_1 \otimes |L\rangle_2 \otimes |R\rangle_3\big) \\
&|\mathrm{GHZ}4^+\rangle_{123} \\
&\quad = -\tfrac{i}{\sqrt{2}}\big(|R\rangle_1 \otimes |L\rangle_2 \otimes |R\rangle_3 - |L\rangle_1 \otimes |R\rangle_2 \otimes |L\rangle_3\big) \\
&|\mathrm{GHZ}4^-\rangle_{123} \\
&\quad = -\tfrac{i}{\sqrt{2}}\big(|R\rangle_1 \otimes |R\rangle_2 \otimes |L\rangle_3 - |L\rangle_1 \otimes |L\rangle_2 \otimes |R\rangle_3\big)\,.
\end{aligned} \tag{9}$$

Of course, via local unitary transformations the eight GHZ states (9) can be transformed into the corresponding states containing only linearly polarized $|H\rangle$ and $|V\rangle$ states.

The appeal and importance of the construction of the entangled entanglement is that this procedure can be easily generalized to construct the corresponding states of higher dimensions d and arbitrary number of particles n. Entangling the GHZ states (6) again with $|H\rangle$ and $|V\rangle$ we obtain the corresponding simplex in the $2\otimes2\otimes2\otimes2$ tensor space and so on. In this way we can construct all higher dimensional simplices of entangled entanglement states in a straightforward way just by entangling again the vertices of the simplex with $|H\rangle$ and $|V\rangle$. Thus we find the magic simplex for any particle number. The extension to higher dimensional system is obtained by generalization of the Pauli matrices to the unitary Weyl operators. The procedure that constructs the simplices in a systematic and straightforward way, where the Weyl operators simplify the method, we are going to present in the next section.

3. Construction procedure of entangled entanglement

We now construct a set of complete orthonormal basis states for any finite dimension and number of particles. Our point is that these basis states exhibit already the structure of entangled entanglement. A similar construction to our's is the one of the authors of Ref. [20], who also used Weyl operators to classify the basis states. Whereas our construction is directed to show explicitly entangled entanglement, their's, in contrast, is a classification in terms of so-called "cluster sums".

We proceed in two steps. Firstly, we construct one entangled entanglement state, and secondly, we find the remaining entangled entanglement states of the simplex basis. This we do by acting with Weyl operators in one of the subsystems.

Let us begin with qubits and choose without loss of generality the state

$$|\Phi_1\rangle := |0\rangle\,. \tag{10}$$

From now on we use the convenient notation $|0\rangle$ and $|1\rangle$ of quantum information instead of the experimental notation $|H\rangle$ and $|V\rangle$, since it can easily be generalized for the higher dimensional cases.

Inspired by the construction in Eq. (6) we carry on the subsequent strategy, where we apply the Weyl operators, generally defined by

$$W_{k,l} = \sum_{s=0}^{d-1} w^{(s-l)k} |s-l\rangle\langle s| \text{ with } w = e^{\frac{2\pi i}{d}} \text{ and } k,l = 0,\dots,d-1\,,$$

onto the subsystems.

Then we consider a two particle state $|\Phi_2\rangle$ and apply the Weyl operator $W_{1,1}$ in the following way to obtain the Bell state $|\phi^-\rangle$

$$\begin{aligned}
|\Phi_2\rangle &= \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |\Phi_1\rangle + |1\rangle \otimes W_{1,1}|\Phi_1\rangle\big) \\
&= \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |0\rangle + |1\rangle \otimes W_{1,1}|0\rangle\big) \\
&= \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle\big) \\
&= |\phi^-\rangle\,.
\end{aligned} \tag{11}$$

Next we iterate the state and get a GHZ state, specifically $|\mathrm{GHZ}1^-\rangle_{123}$ of Eq. (6)

$$\begin{aligned}
|\Phi_3\rangle &= \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |\Phi_2\rangle + |1\rangle \otimes (\mathbb{1} \otimes W_{1,1})|\Phi_2\rangle\big) \\
&= \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |\phi^-\rangle - |1\rangle \otimes |\psi^+\rangle\big)\,.
\end{aligned} \tag{12}$$

Thus by iterating we find quite generally the n-partite qubit state already in its entangled entanglement form

$$|\Phi_n\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |\Phi_{n-1}\rangle + |1\rangle \otimes (\underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{n-2} \otimes W_{1,1})|\Phi_{n-1}\rangle\right) \tag{13}$$

$$= \frac{1}{\sqrt{2}} \sum_{i=0}^{1} \left(\mathbb{1}^{\otimes(n-1)} \otimes W_{i,i}\right) |i\rangle \otimes |\Phi_{n-1}\rangle\,. \tag{14}$$

Finally, we generalize the states to higher dimensions d, what we can do in a similar way

$$|\Phi_n^d\rangle = \frac{1}{\sqrt{d}}\Bigg(|0\rangle\otimes|\Phi_{n-1}\rangle + |1\rangle\otimes(\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n-2}\otimes W_{1,1})|\Phi_{n-1}\rangle$$

$$\ldots + |d-1\rangle\otimes(\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n-2}\otimes W_{d-1,d-1})|\Phi_{n-1}\rangle\Bigg)$$

$$= \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\left(\mathbb{1}^{\otimes(n-1)}\otimes W_{d-1,d-1}\right)\,|i\rangle\otimes|\Phi_{n-1}\rangle\,. \tag{15}$$

Having obtained now an entangled entanglement state for any finite dimension d and number n of particles, Eq. (15), we proceed to construct the d^n-1 remaining entangled entanglement states of the simplex by acting in one of the subsystems with all d^2 Weyl operators and in $n-2$ subsystems with Weyl operators $W_{s,0}$ that change the phase, i.e.

$$|\Phi_n^d(s_1,s_2,\ldots,s_{n-2},k,l)\rangle$$
$$= \mathbb{1}\otimes W_{s_1,0}\otimes\cdots\otimes W_{s_{n-2},0}\otimes W_{k,l}\,|\Phi_n^d\rangle. \tag{16}$$

Let us remark that a certain selection of all possible locally unitaries $W_{a_1,b_1}\otimes W_{a_2,b_2}\otimes\cdots\otimes W_{a_n,b_n}$ does the job of defining a complete orthogonal basis. Such a selection can be illustrated as a "Merry Go Round" (see Fig. 2 and Fig. 3) and is discussed later. Herewith the construction of our entangled entanglement simplex $\mathcal{W}_n^d$ for n particles with dimension d is as follows

$$\mathcal{W}_n^d := \Bigg\{\sum_{s_1,\ldots,s_{n-2},k,l=0}^{d-1} c_{s_1,\ldots,s_{n-2},k,l}$$
$$\cdot\,|\Phi_n^d(s_1,s_2,\ldots,s_{n-2},k,l)\rangle\langle\Phi_n^d(s_1,s_2,\ldots,s_{n-2},k,l)|$$
$$\text{with}\quad c_{s_1,\ldots,s_{n-2},k,l}\geq 0\quad\text{and}\quad\sum c_{s_1,\ldots,s_{n-2},k,l}=1\Bigg\}\,. \tag{17}$$

Due to our construction the states can be explicitly written as

$$|\Phi_n^d(s_1,s_2,\ldots,s_{n-2},k,l)\rangle = \frac{1}{\sqrt{d^{n-1}}}$$
$$\Big(\gamma(0,\ldots,0,0)\,|0\ldots00z(0,\ldots,0,0)\rangle +$$
$$\gamma(0,\ldots,0,1)\,|0\ldots01z(0,\ldots,0,1)\rangle\ldots +$$
$$\gamma(0,\ldots,0,d-1)\,|0\ldots0(d-1)z(0,\ldots,0,d-1)\rangle +$$
$$\ldots\ldots$$
$$\gamma(d-1,\ldots,d-1,0)$$
$$|(d-1)\ldots(d-1)0z(d-1,\ldots,d-1,0)\rangle +$$
$$\ldots\ldots$$
$$\gamma(d-1,\ldots,d-1,d-1)$$
$$|\underbrace{(d-1),\ldots,(d-1)}_{(n-1)-times}z(d-1,\ldots,d-1,d-1)\rangle\Big)\,, \tag{18}$$

where the coefficients $\gamma(i_1,\ldots,i_{n-1})$ are obtained from the different Weyl transformations, meaning that they are powers of the Weyl factor $e^{\frac{2\pi i}{d}}$ and the entries $z(i_1,\ldots,i_{n-1})$ are the results of the Weyl transformations $W_{k,l}$ for the last particle. The $z(i_1,\ldots,i_{n-1})$ take different values, but all in all there are equally many results giving the "digits" $0,1,\ldots,(d-1)$.

For all dimensions d and particle numbers n, the set of d^n states

$$\Big\{|\Phi_n^d(s_1,s_2,\ldots,s_{n-2},k,l)\rangle,\ 0\leq s_1,s_2,\ldots,s_{n-2},k,l\leq d-1\Big\} \tag{19}$$

form an orthonormal system, which can be proved straightforwardly

$$\langle\Phi_n^d(s'_1,s'_2,\ldots,s'_{n-2},k',l')|\Phi_n^d(s_1,s_2,\ldots,s_{n-2},k,l)\rangle$$
$$= \delta_{s'_1,s_1}\ldots\delta_{s'_{n-2},s_{n-2}}\delta_{k',k}\delta_{l',l}\,. \tag{20}$$

Summarizing, our construction mechanism works generally, i.e., for any number of particles n and any dimension d. The GHZ-type states, forming an orthonormal basis, have already the structure of entangled entanglement and reveal a particular geometry, which is quite remarkable.

In Fig. 2 we have depicted the geometry of three qubit states. We find squares of states which are obtained by applying the four different Weyl operators $W_{0,0}, W_{0,1}, W_{1,0}, W_{1,1}$ to one subsystem – in our choice to the third subsystem – of a reference GHZ-type state, denoted by GHZ000 in Fig. 2. Two squares are connected by applying a Weyl operator to another subsystem. In our choice the phase operation $W_{1,0}$ is acting on the second subsystem and moves a certain GHZ state to a GHZ state in the other square.

The geometric structure generalizes for higher dimensions in an obvious way as illustrated for three qutrits in Fig. 3. We find three squares with each nine GHZ-type state related by the application of the nine Weyl operators to the third subsystem. To move from one square to another we have to apply the phase shift operations $W_{1,0}$ or $W_{2,0}$ to the second subsystem (as in the qubit case). In this way it yields a cyclic property – a *Merry Go Round* – of the GHZ states.

4. Conclusions

GHZ-type entangled states are from their physics content fundamentally different to other (genuine) multi-partite entangled states. They show the interesting feature of entangled entanglement, where an entangled state of two qubits, a Bell state, is further entangled with a third qubit state. We have shown that any GHZ state can be expressed in this way and we have given a generalization to higher dimensions. We have presented a procedure how to construct systematically, with help of Weyl operators, the entangled entanglement states for any higher dimension *and* number of particles. These states form the vertices of a "magic simplex" in the corresponding Hilbert–Schmidt space. Our construction procedure reveals an interesting cyclic geometric structure, a "Merry Go Round" between the GHZ states among a simplex, as depicted in Fig. 2 for qubits and in Fig. 3 for qutrits.

The physical significance of entangled entanglement is that the naive concept of reality, that these GHZ-type entangled states always have well-defined local properties, fails promptly. When Alice is measuring one particle the reality content of the remaining photons on Bob's side switches accordingly (see Fig. 1). If Alice projects, for instance, on linearly polarized photons Bob finds his two photons in a Bell state. According to Einstein–Podolsky–Rosen we have to attribute an element of reality to the entangled state on Bob's side but not to each individual photon separately. On the other hand, if she measures a circularly polarized photons Bob detects his two photons in a separable state, that means, again according to EPR, they contain individually an element of reality. Thus Alice is able to switch the properties, the reality content,

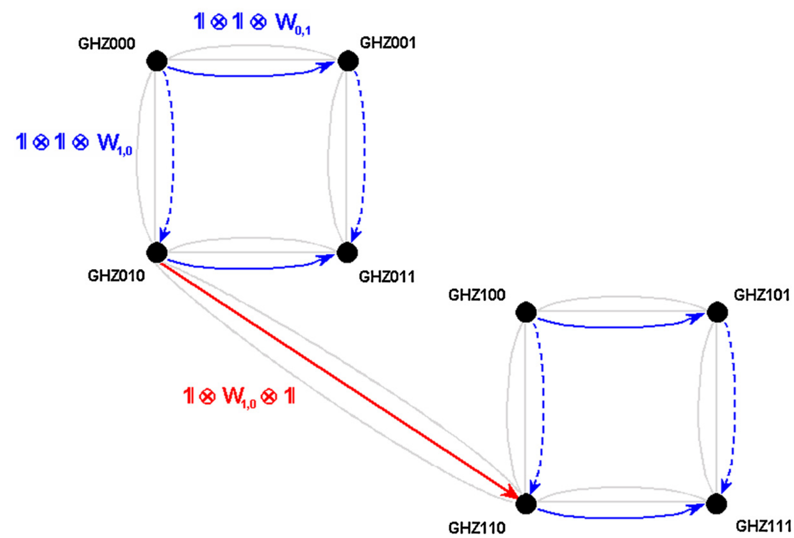

Fig. 2. (Color online.) Geometry of the basis states forming the magic simplex $\mathbb{S}$ for three qubits: Here we simplify the notation by $|\Phi_3^2(s,k,l)\rangle = \mathrm{GHZ}_3^2(s,k,l) =: \mathrm{GHZ}skl$. To change a certain GHZ state to another one within the square one needs to apply either a flip or phase operation in the last subsystem, whereas the phase operation $W_{1,0}$ applied to the second subsystem moves a certain GHZ state to a GHZ state in the other square.

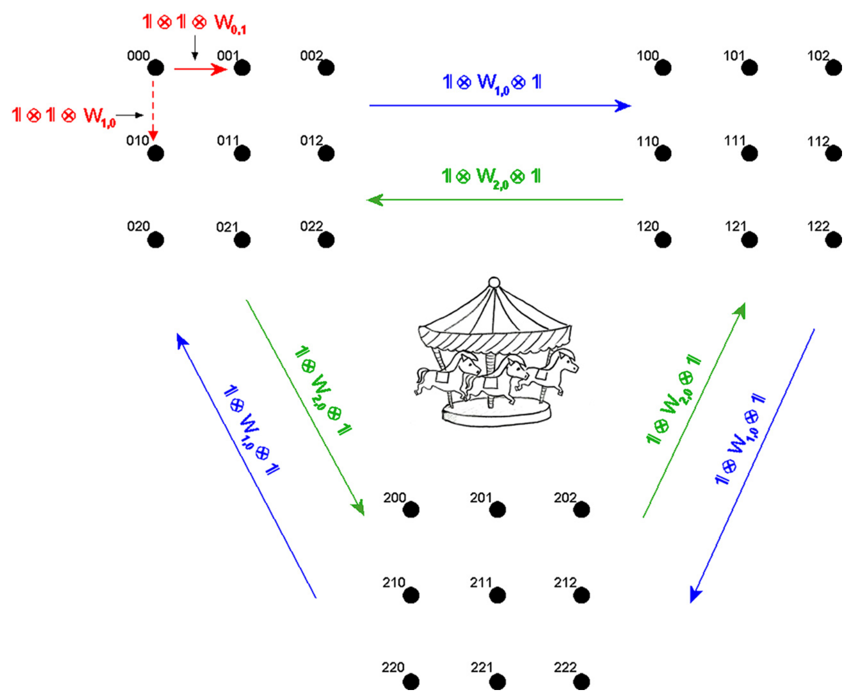

Fig. 3. (Color online.) *"Schön ist so ein Ringelspiel ..."* [21] or the *Merry Go Round* of the GHZ states. In our construction the GHZ states possess a cyclic property that allows to move from square to the next one like in the Carousel of the famous Viennese Prater [22]. The geometry for three qutrits is a generalization of the one for three qubits, i.e. Fig. 2. Here we simplify the notation by $|\Phi_3^3(s,k,l)\rangle = \mathrm{GHZ}_3^3(s,k,l) =: skl$. To change a certain GHZ state to another one within the square one needs to apply either a flip or phase operation in the third subsystem, whereas the phase operation $W_{1,0}$ or $W_{2,0}$ applied to the second subsystem moves a certain GHZ state from one square to a GHZ state in another square, yielding in this way the cyclic property in our construction of GHZ states. This can be extended straightforwardly to higher dimensions.

of the photons on Bob's side. Mathematically, this switching phenomenon between entanglement and separability can be traced back to different factorizations of the tensor product of algebras of the quantum states for the involved particles.

An investigation of the separability and entanglement properties of the magic simplices within the HMGH-framework [23] can be found in Ref. [24].

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