Chapter 5

Harmonic Oscillator and Coherent States

5.1 Harmonic Oscillator

In this chapter we will study the features of one of the most important potentials in physics, it's the harmonic oscillator potential which is included now in the Hamiltonian

$$V(x) = \frac{m\omega^2}{2} x^2 . \tag{5.1}$$

There are two possible ways to solve the corresponding time independent Schrödinger equation, the algebraic method, which will lead us to new important concepts, and the analytic method, which is the straightforward solving of a differential equation.

5.1.1 Algebraic Method

We start again by using the time independent Schrödinger equation, into which we insert the Hamiltonian containing the harmonic oscillator potential (5.1)

$$H\,\psi = \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2}{2} x^2 \right) \psi = E\,\psi . \tag{5.2}$$

We rewrite Eq. (5.2) by defining the new operator $\bar{x} := m\omega x$

$$H\,\psi = \frac{1}{2m} \left[\left(\frac{\hbar}{i} \frac{d}{dx} \right)^2 + (m\omega x)^2 \right] \psi = \frac{1}{2m} \left[p^2 + \bar{x}^2 \right] \psi = E\,\psi . \tag{5.3}$$

We will now try to express this equation as the square of some (yet unknown) operator

$$p^2 + \bar{x}^2 \quad \rightarrow \quad (\bar{x} + ip)(\bar{x} - ip) = p^2 + \bar{x}^2 + i(p\bar{x} - \bar{x}p) , \tag{5.4}$$

but since x and p do not commute (remember Theorem 2.3), we only will succeed by taking the $x - p$ commutator into account. Eq. (5.4) suggests to factorize our Hamiltonian by defining new operators a and $a^\dagger$ as:

Definition 5.1

$$a \; := \; \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x \, + \, ip) \qquad \textit{annihilation operator}$$

$$a^\dagger \; := \; \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x \, - \, ip) \qquad \textit{creation operator}$$

These operators each create/annihilate a quantum of energy $E = \hbar\omega$, a property which gives them their respective names and which we will formalize and prove later on. For now we note that position and momentum operators are expressed by a's and $a^\dagger$'s like

$$x \, = \, \sqrt{\frac{\hbar}{2m\omega}} \left(a \, + \, a^\dagger \right) \qquad\qquad p \, = \, -i \sqrt{\frac{m\omega\hbar}{2}} \left(a \, - \, a^\dagger \right) \, . \tag{5.5}$$

Let's next calculate the commutator of the creation and annihilation operators. It's quite obvious that they commute with themselves

$$[\, a \, , \, a \,] \, = \, \left[\, a^\dagger \, , \, a^\dagger \,\right] \, = \, 0 \, . \tag{5.6}$$

To find the commutator of a with $a^\dagger$ we first calculate $aa^\dagger$ as well as $a^\dagger a$

$$a \, a^\dagger \;\; = \;\; \frac{1}{2m\omega\hbar} \left[(m\omega x)^2 \, + \, im\omega \, [\, p \, , \, x \,] \, + \, p^2 \right] \tag{5.7}$$

$$a^\dagger a \;\; = \;\; \frac{1}{2m\omega\hbar} \left[(m\omega x)^2 \, - \, im\omega \, [\, p \, , \, x \,] \, + \, p^2 \right] \, . \tag{5.8}$$

Since we know $[\, p \, , \, x \,] \, = \, -i\hbar$ we easily get

$$\left[\, a \, , \, a^\dagger \,\right] \, = \, 1 \, . \tag{5.9}$$

Considering again the Hamiltonian from Eq. (5.3) we use expressions (5.7) and (5.8) to rewrite it as

$$H \, = \, \frac{1}{2m} \left(p^2 \, + \, (m\omega x)^2 \right) \, = \, \frac{\hbar\omega}{2} \left(a^\dagger a \, + \, a \, a^\dagger \right) \, . \tag{5.10}$$

Using the commutator (5.9) we can further simplify the Hamiltonian

$$\left[\, a \, , \, a^\dagger \,\right] \, = \, a \, a^\dagger \, - \, a^\dagger a \, = \, 1 \quad \Rightarrow \quad a \, a^\dagger \, = \, a^\dagger a \, + \, 1 \, , \tag{5.11}$$

$$H \, = \, \hbar\omega(a^\dagger a \, + \, \tfrac{1}{2}) \, . \tag{5.12}$$

For the Schrödinger energy eigenvalue equation we then get

$$\hbar\omega(a^\dagger a \, + \, \tfrac{1}{2}) \, \psi \, = \, E \, \psi \, , \tag{5.13}$$

which we rewrite as an eigenvalue equation for the operator $a^\dagger a$

$$a^\dagger a\,\psi \;=\; \left(\frac{E}{\hbar\omega} \,-\, \frac{1}{2}\right)\psi\;, \tag{5.14}$$

having the following interpretation:

Definition 5.2 $\quad N \;:=\; a^\dagger a \quad$ *occupation (or particle) number operator*

and which satisfies the commutation relations

$$\left[\,N\,,\,a^\dagger\,\right] \;=\; a^\dagger \qquad\qquad [\,N\,,\,a\,] \;=\; -a\;. \tag{5.15}$$

Next we are looking for the eigenvalues ν and eigenfunctions ψ_ν of the occupation number operator N, i.e. we are seeking the solutions of equation

$$N\,\psi_\nu \;=\; \nu\,\psi_\nu\;. \tag{5.16}$$

To proceed we form the scalar product with ψ_ν on both sides of Eq. (5.16), use the positive definiteness of the scalar product (Eq. (2.32)) and the definition of the adjoint operator (Definition 2.5)

$$\nu\,\underbrace{\langle\,\psi_\nu\,|\,\psi_\nu\,\rangle}_{\neq\,0} \;=\; \langle\,\psi_\nu\,|\,N\,\psi_\nu\,\rangle \;=\; \left\langle\,\psi_\nu\,\middle|\,a^\dagger a\,\psi_\nu\,\right\rangle \;=\; \langle\,a\,\psi_\nu\,|\,a\,\psi_\nu\,\rangle \;\geq\; 0\;. \tag{5.17}$$

The above inequality vanishes only if the corresponding vector $a\,\psi$ is equal to zero. Since we have for the eigenvalues $\nu \geq 0$, the lowest possible eigenstate ψ_0 corresponds to the eigenvalue $\nu \;=\; 0$

$$\nu \;=\; 0 \qquad \rightarrow \qquad a\,\psi_0 \;=\; 0\;. \tag{5.18}$$

Inserting the definition of the annihilation operator (Definition 5.1) into condition (5.18), i.e. that the ground state is annihilated by the operator a, yields a differential equation for the ground state of the harmonic oscillator

$$\begin{aligned} a\,\psi_0 \;=\; \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x \,+\, i\,\frac{\hbar}{i}\,\frac{d}{dx})\,\psi_0 \;&=\; 0 \\ \Rightarrow \quad \left(\frac{m\omega}{\hbar}x \,+\, \frac{d}{dx}\right)\psi_0 \;&=\; 0\;. \end{aligned} \tag{5.19}$$

We can solve this equation by separation of variables

$$\int\frac{d\psi_0}{\psi_0} \;=\; -\int dx\,\frac{m\omega}{\hbar}\,x \quad\Rightarrow\quad \ln\psi_0 \;=\; -\,\frac{m\omega}{2\hbar}\,x^2 \,+\, \ln\mathcal{N}\;, \tag{5.20}$$

where we have written the integration constant as $\ln\mathcal{N}$, which we will fix by the normalization condition

$$\ln\left(\frac{\psi_0}{\mathcal{N}}\right) \;=\; -\,\frac{m\omega}{2\hbar}\,x^2 \quad\Rightarrow\quad \psi_0(x) \;=\; \mathcal{N}\,\exp\left(-\tfrac{m\omega}{2\hbar}x^2\right)\;. \tag{5.21}$$

We see that the ground state of the harmonic oscillator is a Gaussian distribution. The normalization $\int\limits_{-\infty}^{\infty} dx\,|\psi_0(x)|^2 \;=\; 1$ together with formula (2.119) for Gaussian functions determines the normalization constant

$$\mathcal{N}^2 \;=\; \sqrt{\frac{m\omega}{\pi\hbar}} \quad\Rightarrow\quad \mathcal{N} \;=\; \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\;. \tag{5.22}$$

We will now give a description of the whole set of eigenfunctions ψ_ν of the operator N based on the action of the creation operator by using the following lemma:

Lemma 5.1 *If ψ_ν is an eigenfunction of N with eigenvalue ν, then $a^\dagger\psi_\nu$ also is an eigenfunction of N with eigenvalue $(\nu+1)$.*

Proof:

$$\begin{aligned} N\,a^\dagger\psi_\nu \;&\overset{\text{Eq. (5.15)}}{=}\; (a^\dagger\,N \,+\, a^\dagger)\,\psi_\nu \;=\; a^\dagger\,(N\,+\,1)\,\psi_\nu \\ &=\; a^\dagger\,(\nu\,+\,1)\,\psi_\nu \;=\; (\nu\,+\,1)\,a^\dagger\,\psi_\nu\;. \qquad\qquad \text{q.e.d.} \end{aligned} \tag{5.23}$$

The state $a^\dagger\psi_\nu$ is not yet normalized, which means it is only proportional to $\psi_{\nu+1}$, to find the proportionality constant we use once more the normalization condition

$$\left\langle\, a^\dagger\psi_\nu \,\middle|\, a^\dagger\psi_\nu \,\right\rangle \;=\; \langle\,\psi_\nu\,|\, \underbrace{a\,a^\dagger}_{a^\dagger a\,+\,1}\;\psi_\nu\,\rangle \;=\; \langle\,\psi_\nu\,|\,(N\,+\,1)\,|\,\psi_\nu\,\rangle \;=\; (\nu\,+\,1)\,\underbrace{\langle\,\psi_\nu\,|\,\psi_\nu\,\rangle}_{1} \tag{5.24}$$

$$\Rightarrow \qquad a^\dagger\psi_\nu \;=\; \sqrt{\nu\,+\,1}\;\psi_{\nu+1}\;. \tag{5.25}$$

This means that we can get any excited state ψ_ν of the harmonic oscillator by successively applying creation operators.

In total analogy to Lemma 5.1 we can formulate the following lemma:

Lemma 5.2 *If ψ_ν is an eigenfunction of N with eigenvalue ν, then $a\psi_\nu$ also is an eigenfunction of N with eigenvalue $(\nu-1)$.*

Proof:

$$\begin{aligned} N\,a\psi_\nu \;&\overset{\text{Eq. (5.15)}}{=}\; (a\,N\,-\,a)\,\psi_\nu \;=\; a\,(N\,-\,1)\,\psi_\nu \\ &=\; a\,(\nu\,-\,1)\,\psi_\nu \;=\; (\nu\,-\,1)\,a\,\psi_\nu\,. \qquad \text{q.e.d.} \end{aligned} \tag{5.26}$$

We again get the proportionality constant from the normalization

$$\langle\, a\psi_\nu \,|\, a\psi_\nu \,\rangle \;=\; \langle\, \psi_\nu \,|\, \underbrace{a^\dagger\, a}_{N}\, \psi_\nu \,\rangle \;=\; \langle\, \psi_\nu \,|\, \underbrace{N\,|\, \psi_\nu \,\rangle}_{\nu\,|\,\psi_\nu\,\rangle} \;=\; \nu\, \underbrace{\langle\, \psi_\nu \,|\, \psi_\nu \,\rangle}_{1} \tag{5.27}$$

$$\Rightarrow \qquad a\psi_\nu \;=\; \sqrt{\nu}\;\psi_{\nu\,-\,1}\,. \tag{5.28}$$

Summary: Eigenvalue equation for the harmonic oscillator

The time independent Schrödinger equation, the energy eigenvalue equation

$$H\,\psi_{\mathrm{n}} \;=\; E_{\mathrm{n}}\,\psi_{\mathrm{n}}\;, \tag{5.29}$$

with the Hamiltonian $H \,=\, \hbar\omega\,(N \,+\, \frac{1}{2})$ and the occupation number operator $N \,=\, a^\dagger a$ provides the *energy eigenvalues*

$$E_{\mathrm{n}} \;=\; \hbar\omega\,(n \,+\, \tfrac{1}{2})\;. \tag{5.30}$$

The corresponding state vectors, the *energy eigenfunctions*, are given by

$$\psi_{\mathrm{n}}(x) \;=\; \frac{1}{\sqrt{n!}}(a^\dagger)^{\mathrm{n}}\,\psi_0(x) \;=\; \frac{1}{\sqrt{n!}}\,\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\,(a^\dagger)^{\mathrm{n}}\,\exp(-\frac{m\omega}{2\hbar}x^2)\;, \tag{5.31}$$

with the *ground state*

$$\psi_0(x) \;=\; \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\,\exp(-\frac{m\omega}{2\hbar}x^2)\;. \tag{5.32}$$

The explicit form of the excited state wave functions will be calculated later on but we can for now reveal that they are proportional to a product of the ground state and a family of functions, the so-called *Hermite polynomials* H_{n}.

The wave functions thus form a ladder of alternating even and odd energy states, see Fig. 5.1, which are each separated by a quantum of energy $\hbar\omega$, i.e. equally spaced. The creation and annihilation operators then "climb" or "descend" this energy ladder step by step, which is why they are also called ladder operators.

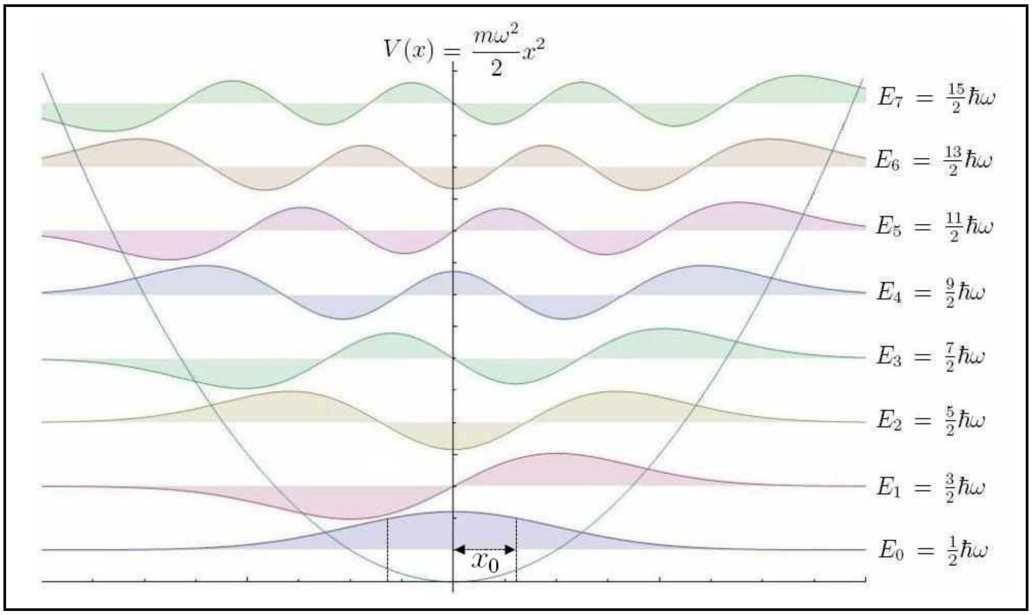

Figure 5.1: Harmonic oscillator: The possible energy states of the harmonic oscillator potential V form a ladder of even and odd wave functions with energy differences of $\hbar\omega$. The ground state is a Gaussian distribution with width $x_0 = \sqrt{\frac{\hbar}{m\omega}}$; picture from http://en.wikipedia.org/wiki/Quantum_mechanical_harmonic_oscillator

5.1.2 Zero Point Energy

We already learned that the lowest possible energy level of the harmonic oscillator is not, as classically expected, zero but $E_0 = \frac{1}{2}\hbar\omega$.

It can be understood in the following way. The ground state is an eigenfunction of the Hamiltonian, containing both kinetic and potential energy contributions, therefore the particle has some kinetic energy in the vicinity of $x = 0$, where the potential energy $V(x \to 0) \to 0$. But this implies according to Heisenberg's uncertainty relation (Eq. (2.80)) that the momentum uncertainty increases pushing the total energy up again until it stabilizes

$$\Delta p \propto \frac{\hbar}{\Delta x} \nearrow \quad \text{for} \quad \Delta x \to 0 \quad \Rightarrow E \neq 0\,. \tag{5.33}$$

We will now illustrate the harmonic oscillator states, especially the ground state and the zero point energy in the light of the uncertainty principle. We start by calculating the position and momentum uncertainties using Definition 2.10

$$\langle\, x \,\rangle \,=\, \langle\, \psi_\mathrm{n} \,|\, x \,|\, \psi_\mathrm{n} \,\rangle \overset{\text{Eq. (5.5)}}{\propto} \langle\, \psi_\mathrm{n} \,|\, a \,+\, a^\dagger \,|\, \psi_\mathrm{n} \,\rangle \,\propto\, \underbrace{\langle\, \psi_\mathrm{n} \,|\, \psi_\mathrm{n-1} \,\rangle}_{0} + \underbrace{\langle\, \psi_\mathrm{n} \,|\, \psi_\mathrm{n+1} \,\rangle}_{0} \,=\, 0 \tag{5.34}$$

$$\left\langle\, x^2 \,\right\rangle \,=\, \langle\, \psi_\mathrm{n} \,|\, x^2 \,|\, \psi_\mathrm{n} \,\rangle \,=\, \frac{\hbar}{2m\omega} \,\langle\, \psi_\mathrm{n} \,|\, a^2 \,+\, aa^\dagger \,+\, a^\dagger a \,+\, (a^\dagger)^2 \,|\, \psi_\mathrm{n} \,\rangle\;. \tag{5.35}$$

Because the the eigenstates ψ_n for different n are orthogonal, the expectation values of a^2 and $(a^\dagger)^2$ vanish identically and we proceed by using Eq. (5.11), where $a^\dagger a = N$.

We get

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \langle \psi_n | 2N + 1 | \psi_n \rangle = \frac{\hbar}{m\omega}(n + \tfrac{1}{2}) = x_0^2 (n + \tfrac{1}{2}), \tag{5.36}$$

where we have introduced a characteristic length of the harmonic oscillator $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ and $n = 0, 1, 2, \cdots$ is a natural number. The position uncertainty is then given by

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = x_0^2 (n + \tfrac{1}{2}), \tag{5.37}$$

which vanishes in the classical limit, i.e. $\hbar \to 0$ or $m \to \infty$.

Let us continue with the momentum uncertainty

$$\langle p \rangle = \langle \psi_n | p | \psi_n \rangle \overset{\text{Eq. (5.5)}}{\propto} \langle \psi_n | a - a^\dagger | \psi_n \rangle \propto \underbrace{\langle \psi_n | \psi_{n-1} \rangle}_{0} - \underbrace{\langle \psi_n | \psi_{n+1} \rangle}_{0} = 0 \tag{5.38}$$

$$\begin{aligned} \langle p^2 \rangle &= \langle \psi_n | p^2 | \psi_n \rangle = -\frac{m\omega\hbar}{2} \langle \psi_n | a^2 - \underbrace{aa^\dagger - a^\dagger a}_{-(2N+1)} + (a^\dagger)^2 | \psi_n \rangle \qquad (5.39) \\ &= \frac{\hbar^2}{x_0^2}(n + \tfrac{1}{2}), \end{aligned}$$

where we have again used the characteristic length x_0, the ladder operator commutation relations (Eq. (5.11)) and the orthogonality of the eigenstates. For the momentum uncertainty we then have

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\hbar^2}{x_0^2}(n + \tfrac{1}{2}). \tag{5.40}$$

Constructing the uncertainty relation from Eq. (5.37) and Eq. (5.40) we finally get

$$\begin{aligned} \Delta x\, \Delta p = \hbar (n + \tfrac{1}{2}) &\geq \tfrac{\hbar}{2} \quad \text{generally} & (5.41) \\ &= \tfrac{\hbar}{2} \quad \text{for the ground state } \ n = 0\,. & (5.42) \end{aligned}$$

The ground state, Eq. (5.32), can be rewritten in terms of the characteristic length x_0

$$\psi_0(x) = \frac{1}{\sqrt{x_0\sqrt{\pi}}} \exp(-\frac{x^2}{2x_0^2}), \tag{5.43}$$

which is a gaussian wave packet with standard deviation x_0 characterizing the position uncertainty in the ground state, see Fig. 5.1.

Since we have made clear that the zero point energy is in accordance with the uncertainty relation we can even strengthen this correlation in so far as the non-vanishing zero point energy is a direct consequence of the uncertainty principle. Starting from the uncertainty relation (Eq. (2.80)) and keeping in mind that here the expectation values of the position- and momentum operator vanish identically, see Eq. (5.34) and Eq. (5.38), we can rewrite the uncertainty relation as

$$\langle\, x^2 \,\rangle\, \langle\, p^2 \,\rangle \,\geq\, \frac{\hbar^2}{4}\,. \tag{5.44}$$

We now calculate the mean energy of the harmonic oscillator, which is the expectation value of the Hamiltonian from Eq. (5.2)

$$E \,=\, \langle\, H \,\rangle \,=\, \frac{\langle\, p^2 \,\rangle}{2m} + \frac{m\omega^2}{2}\,\langle\, x^2 \,\rangle \;\overset{\text{Eq. (5.44)}}{\geq}\; \frac{\langle\, p^2 \,\rangle}{2m} + \frac{m\omega^2}{2}\,\frac{\hbar^2}{4}\,\frac{1}{\langle\, p^2 \,\rangle}\,. \tag{5.45}$$

To find the minimal energy we calculate the variation of Eq. (5.45) with respect to $\langle\, p^2 \,\rangle$, which we then set equal to zero to find the extremal values

$$\frac{1}{2m} - \frac{m\omega^2}{2}\,\frac{\hbar^2}{4}\,\frac{1}{\langle\, p^2 \,\rangle^2} \,=\, 0 \quad\Rightarrow\quad \langle\, p^2 \,\rangle_{\text{min}} \,=\, \frac{m\omega\hbar}{2}\,. \tag{5.46}$$

Inserting the result of Eq. (5.46) into the mean energy (Eq. (5.45)) we get

$$E \,\geq\, \frac{1}{2m}\,\frac{m\omega\hbar}{2} + \frac{m\omega^2\hbar^2}{8}\,\frac{2}{m\omega\hbar} \,=\, \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} \,=\, \frac{\hbar\omega}{2}\,, \tag{5.47}$$

thus

$$E \,\geq\, \frac{\hbar\omega}{2}\,. \tag{5.48}$$

Theorem 5.1 (Zero point energy)

The zero point energy is the smallest possible energy a physical system can possess, that is consistent with the uncertainty relation. It is the energy of its ground state.

5.1.3 Comparison with the Classical Oscillator

In this section we want to compare the quantum oscillator with predictions we would get from classical physics. There the classical motion is governed by the position function

$$x(t) \,=\, q_0\,\sin(\omega t)\,, \tag{5.49}$$

and the total energy of the system is constant and given by

$$E \,=\, \frac{m\omega^2}{2}\,q_0^2\,, \tag{5.50}$$

where in both cases, the angular frequency ω is fixed and q_0 is a given starting value. To make a comparison with quantum mechanics we can construct a classical probability density $W_{\text{class}}(x)$ in the following way

$$W_{\text{class}}(x)\, dx \;=\; \frac{dt}{T}\,, \tag{5.51}$$

where dt is the time period where the classical oscillator can be found in the volume element dx, and $T = \frac{2\pi}{\omega}$ is the full oscillation period. We then calculate the volume element dx by differentiating equation (5.49)

$$dx \;=\; q_0\, \omega\, \cos(\omega t)\, dt \;=\; q_0\, \omega \sqrt{1 \,-\, \sin^2(\omega t)}\; dt \;\stackrel{\text{Eq. (5.49)}}{=}\; q_0\, \omega \sqrt{1 \,-\, (\tfrac{x}{q_0})^2}\; dt\,. \tag{5.52}$$

Finally inserting Eq. (5.52) into Eq. (5.51) we get the classical probability density of the oscillator as a function of x, see Fig. 5.2 (dashed curve)

$$W_{\text{class}}(x) \;=\; \Big(2\pi\, q_0 \sqrt{1 \,-\, (\tfrac{x}{q_0})^2}\Big)^{-1}\,. \tag{5.53}$$

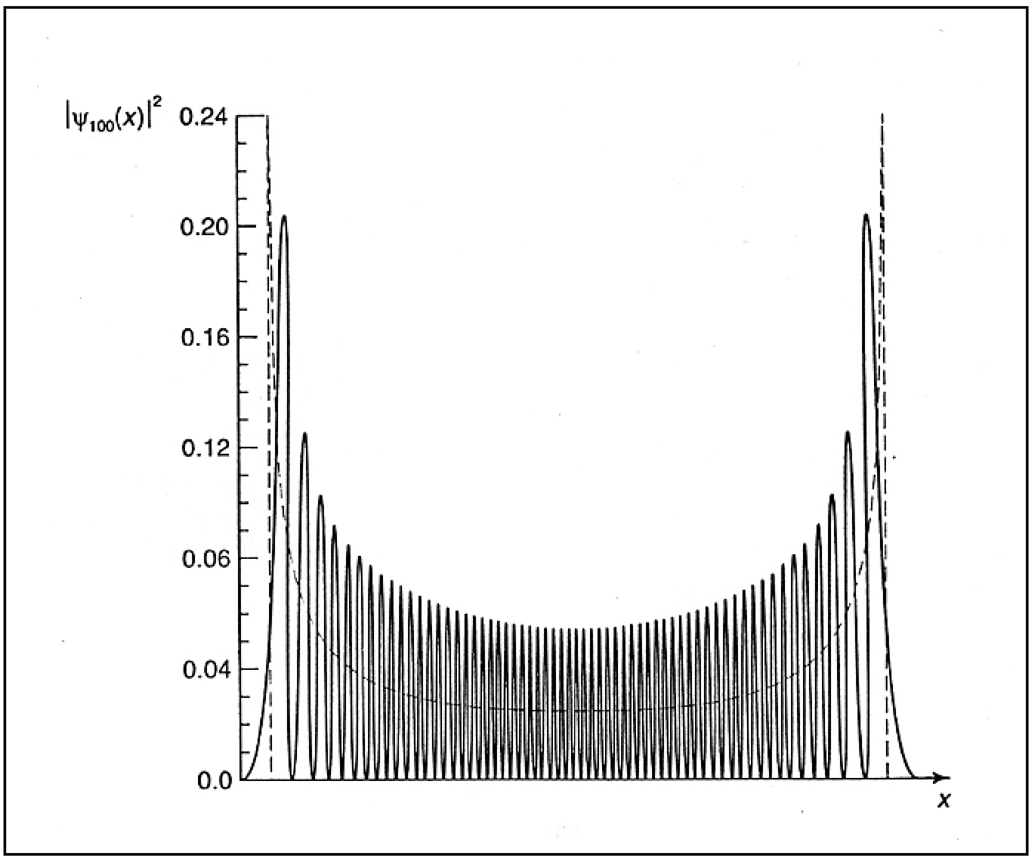

Figure 5.2: Oscillator probabilities: Comparison of the quantum probability (solid curve), $|\, \psi_n \,|^2$ in case of $n = 100$, with a classical oscillator probability (dashed curve), Eq. (5.53).

The starting value q_0 is fixed by the comparison of the energy of the quantum oscillator with the corresponding classical energy

$$E_{\mathrm{n}} = \frac{2n+1}{2}\,\hbar\omega \quad \longleftrightarrow \quad E_{\text{class}} = \frac{m\omega^2}{2}\,q_0^2\ . \tag{5.54}$$

It allows us to relate the starting value q_0 of the classical oscillator to the characteristic length x_0 of the quantum mechanical oscillator

$$\Rightarrow \quad q_0 = \sqrt{\frac{(2n+1)\hbar}{m\omega}} = \sqrt{2n+1}\,x_0\ . \tag{5.55}$$

In the comparison quantum versus classical oscillator we find the following. For the low-lying states, of course, the two probabilities for finding the particle differ considerably but for the high-lying states the smoothed quantum probability resembles very much the classical one, as can be seen in Fig. 5.2 for the case $n = 100$.

5.1.4 Matrix Representation of the Harmonic Oscillator

In this section we now want to briefly sketch how the harmonic oscillator problem can be written in a matrix formulation, which introduces us to the concept of the *Fock space* and the *occupation number representation*. It is the appropriate formalism for relativistic quantum mechanics, i.e. Quantum Field Theory (QFT).

Starting from the eigenfunctions of the harmonic oscillator $\{\,\psi_{\mathrm{n}}(x)\,\}$, which form a complete orthonormal system of the corresponding Hilbert space, we change our notation and label the states only by their index number n, getting the following vectors[1]

$$\psi_0 \;\rightarrow\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = |\,0\,\rangle\ , \quad \psi_1 \;\rightarrow\; \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = |\,1\,\rangle\ , \quad \psi_{\mathrm{n}} \quad \rightarrow \underset{\text{n−th row}}{\nearrow} \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = |\,n\,\rangle\ . \tag{5.56}$$

The vectors $\{\,|\,n\,\rangle\,\}$ now also form a complete orthonormal system, but one of the so called *Fock space* or *occupation number space*[2]. We will not give a mathematical definition of this space but will just note here, that we can apply our rules and calculations as before on the Hilbert space. The ladder operators however have a special role in this space as they allow us to construct any vector from the ground state[3], see Eq. (5.58).

[1] Technically the vectors of Eq. (5.56) are infinite-dimensional, which means that the column-matrix notation is problematic, the ket-notation however is fine and we accept the column-vectors as visualization.

[2] In many body quantum mechanics and QFT the occupation number n is no mere label for the physical states, but describes the particle number of the considered state. This means that in QFT particles are described as excitations of a field, composed of a multitude of harmonic oscillators. The creation- and annihilation operators thus create and annihilate particles.

[3] which in QFT is the vacuum state, often also denoted by $|\,\Omega\,\rangle$.

Let us state the most important relations here:

$$\langle\, n \,|\, m \,\rangle \;=\; \delta_{\mathrm{n,m}} \tag{5.57}$$

$$|\, n \,\rangle \;=\; \frac{1}{\sqrt{n!}}\,(a^{\dagger})^{\mathrm{n}}\;|\, 0 \,\rangle \tag{5.58}$$

$$a^{\dagger}\;|\, n \,\rangle \;=\; \sqrt{n\,+\,1}\;|\, n\,+\,1 \,\rangle \tag{5.59}$$

$$a\;|\, n \,\rangle \;=\; \sqrt{n}\;|\, n\,-\,1 \,\rangle \tag{5.60}$$

$$N\;|\, n \,\rangle \;=\; a^{\dagger}a\;|\, n \,\rangle \;=\; n\;|\, n \,\rangle \tag{5.61}$$

$$H\;|\, n \,\rangle \;=\; \hbar\omega\,(N\,+\,\tfrac{1}{2})\;|\, n \,\rangle \;=\; \hbar\omega\,(n\,+\,\tfrac{1}{2})\;|\, n \,\rangle\;. \tag{5.62}$$

With this knowledge we can write down matrix elements as transition amplitudes. We could, for example, consider Eq. (5.57) as the matrix element from the n-th row and the m-th column

$$\langle\, n \,|\, m \,\rangle \;=\; \delta_{\mathrm{n,m}} \quad\longleftrightarrow\quad \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ \vdots & & \ddots \end{pmatrix}\;. \tag{5.63}$$

In total analogy we can get the matrix representation of the creation operator

$$\langle\, k \,|\; a^{\dagger}\;|\, n \,\rangle \;=\; \sqrt{n\,+\,1}\;\underbrace{\langle\, k \,|\, n\,+\,1 \,\rangle}_{\delta_{\mathrm{k,n+1}}} \quad\longleftrightarrow\quad a^{\dagger} \;=\; \begin{pmatrix} 0 & 0 & 0 & \cdots & \\ \sqrt{1} & 0 & 0 & & \\ 0 & \sqrt{2} & 0 & & \\ \vdots & & \sqrt{3} & & \\ & & & \ddots & \end{pmatrix} \tag{5.64}$$

as well as of the annihilation operator, occupation number operator and Hamiltonian

$$\langle\, k \,|\; a\;|\, n \,\rangle \;=\; \sqrt{n}\;\underbrace{\langle\, k \,|\, n\,-\,1 \,\rangle}_{\delta_{\mathrm{k,n-1}}} \quad\longleftrightarrow\quad a \;=\; \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots & \\ 0 & 0 & \sqrt{2} & & \\ 0 & 0 & 0 & \sqrt{3} & \\ \vdots & & & & \ddots \\ & & & & \end{pmatrix} \tag{5.65}$$

$$\langle\, k \,|\; N\;|\, n \,\rangle \;=\; n\;\underbrace{\langle\, k \,|\, n \,\rangle}_{\delta_{\mathrm{k,n}}} \quad\longleftrightarrow\quad N \;=\; \begin{pmatrix} 0 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & & \\ 0 & 0 & 2 & & \\ \vdots & & & 3 & \\ & & & & \ddots \end{pmatrix} \tag{5.66}$$

$$\langle\, k \,|\; H \;|\, n \,\rangle \;=\; \hbar\omega\,(n \,+\, \tfrac{1}{2})\,\underbrace{\langle\, k \,|\, n \,\rangle}_{\delta_{\mathrm{k,n}}} \qquad \longleftrightarrow \qquad H \;=\; \hbar\omega \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots & \\ 0 & \frac{3}{2} & 0 & & \\ 0 & 0 & \frac{5}{2} & & \\ \vdots & & & \frac{7}{2} & \\ & & & & \ddots \end{pmatrix} . \tag{5.67}$$

5.1.5 Analytic Method

In this last section on the harmonic oscillator we want to explore the straightforward method of solving the Schrödinger equation as a differential equation

$$H\,\psi \;=\; \left(-\,\frac{\hbar^2}{2m}\,\frac{d^2}{dx^2} \,+\, \frac{m\omega^2}{2}x^2\right)\,\psi \;=\; E\,\psi\,. \tag{5.68}$$

Multiplying both sides by $\frac{2}{\hbar\omega}$ and introducing the dimensionless variables $\xi = \frac{x}{x_0}$ and $K = \frac{2E}{\hbar\omega}$, where we again have $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, we obtain the differential equation

$$\frac{d^2}{d\xi^2}\,\psi(\xi) \;=\; \big(\xi^2 \,-\, K\big)\;\psi(\xi)\,. \tag{5.69}$$

To get an idea how the solution could look like, we first try to solve this equation for large ξ, which means we can neglect the K-terms

$$\frac{d^2}{d\xi^2}\,\psi(\xi) \;=\; \xi^2\,\psi(\xi)\,. \tag{5.70}$$

For this differential equation we try the ansatz $\psi = \exp(-\frac{\xi^2}{2})$, providing

$$\frac{d}{d\xi}\,\psi(\xi) \;=\; -\xi\,\exp(-\frac{\xi^2}{2}) \quad , \quad \frac{d^2}{d\xi^2}\,\psi(\xi) \;=\; \big(\xi^2 \,-\, 1\big)\;\psi(\xi)\,. \tag{5.71}$$

Since we are in the limit of large ξ, we can drop 1 in the last term on the right hand side of Eq. (5.71) and we have made the correct ansatz (in this approximation). To find the general solution we can use a similar ansatz, differing only insofar as we multiply it with a function of ξ, a method known as variation of the constants

$$\psi(\xi) \;=\; h(\xi)\,\exp(-\frac{\xi^2}{2})\,. \tag{5.72}$$

Differentiating our ansatz (5.72) we obtain

$$\frac{d}{d\xi}\,\psi(\xi) \;=\; \left(\frac{d\,h(\xi)}{d\xi} \,-\, \xi\,h(\xi)\right)\,\exp(-\frac{\xi^2}{2}) \tag{5.73}$$

$$\frac{d^2}{d\xi^2}\,\psi(\xi) \;=\; \left(\frac{d^2h(\xi)}{d\xi^2} \,-\, 2\,\xi\,\frac{d\,h(\xi)}{d\xi} \,+\, (\xi^2 \,-\, 1)\,h(\xi)\right)\,\exp(-\frac{\xi^2}{2})\,. \tag{5.74}$$

On the other hand we know the second derivative of ψ from the Schrödinger equation (5.69), leading to a differential equation for the function $h(\xi)$

$$\frac{d^2 h(\xi)}{d\xi^2} - 2\,\xi\,\frac{d\,h(\xi)}{d\xi} + (K - 1)\,h(\xi) = 0\,, \tag{5.75}$$

which can be solved by a power series ansatz. We are not computing this here, but direct the interested reader to a lecture and/or book about differential equations[4]. Equation (5.75) is the differential equation for the so called *Hermite polynomials* H_{n}, thus we replace our function by

$$h(\xi) \quad \rightarrow \quad H_{\mathrm{n}}(\xi) \quad , \quad n = 0, 1, 2, \cdots \tag{5.76}$$

where $(K - 1) = 2n$ or $K = \frac{2E}{\hbar\omega} = 2n + 1$ respectively. With this substitution we directly find the energy levels of the harmonic oscillator, as calculated earlier in Eq. (5.30)

$$E_{\mathrm{n}} = \hbar\omega\,(n + \tfrac{1}{2})\,. \tag{5.77}$$

With the help of the Hermite polynomials we can now also give the exact expression for the eigenfunctions of the harmonic oscillator

$$\psi_{\mathrm{n}} = \frac{1}{\sqrt{2^{\mathrm{n}}(n!)}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_{\mathrm{n}}(\frac{x}{x_0})\ \exp(-\frac{x^2}{2x_0^2})\,. \tag{5.78}$$

The Hermite Polynomials can be easiest calculated with the *Formula of Rodrigues*

$$H_{\mathrm{n}}(x) = (-1)^{\mathrm{n}}\, e^{x^2}\left(\frac{d^{\mathrm{n}}}{dx^{\mathrm{n}}}\, e^{-x^2}\right)\,. \tag{5.79}$$

To give some examples we list some of the first Hermite Polynomials

$$\begin{aligned} H_0(x) &= 1\,, \quad H_1(x) = 2x\,, \quad H_2(x) = 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x\,, \quad H_4(x) = 16x^4 - 48x^2 + 12\,. \end{aligned} \tag{5.80}$$

A last interesting property is the orthogonality relation of the Hermite Polynomials

$$\int_{-\infty}^{\infty} dx\ H_{\mathrm{m}}(x)\ H_{\mathrm{n}}(x)\ e^{-x^2} = \sqrt{\pi}\,2^{\mathrm{n}}\,(n!)\,\delta_{\mathrm{mn}}\,. \tag{5.81}$$

5.2 Coherent States

Coherent states play an important role in quantum optics, especially in laser physics and much work was performed in this field by Roy J. Glauber who was awarded the 2005 Nobel prize for his contribution to the quantum theory of optical coherence. We will try here to give a good overview of coherent states of laser beams. The state describing a laser beam can be briefly characterized as having

[4] See for example [14].

1. an indefinite number of photons,
2. but a precisely defined phase,

in contrast to a state with fixed particle number, where the phase is completely random.

There also exists an uncertainty relation describing this contrast, which we will plainly state here but won't prove. It can be formulated for the uncertainties of amplitude and phase of the state, where the inequality reaches a minimum for coherent states, or, as we will do here, for the occupation number N and the phase Φ [5]

$$\Delta N \, \Delta(\sin\Phi) \, \geq \, \frac{1}{2} \cos\Phi \, , \tag{5.82}$$

which, for small Φ, reduces to

$$\Delta N \, \Delta\Phi \, \geq \, \frac{1}{2} \, . \tag{5.83}$$

5.2.1 Definition and Properties of Coherent States

Since laser light has a well-defined amplitude – in contrast to thermal light which is a statistical mixture of photons – we will define coherent states as follows:

Definition 5.3 *A **coherent state** $|\,\alpha\,\rangle$, also called **Glauber state**, is defined as eigenstate of the amplitude operator, the annihilation operator a, with eigenvalues $\alpha \in \mathbb{C}$*

$$a \, |\, \alpha \,\rangle \; = \; \alpha \, |\, \alpha \,\rangle \quad .$$

Since a is a non-hermitian operator the phase $\alpha \, = \, |\alpha| \, e^{i\varphi} \, \in \, \mathbb{C}$ is a complex number and corresponds to the complex wave amplitude in classical optics. Thus coherent states are wave-like states of the electromagnetic oscillator.

Properties of coherent states:

Let us study the properties of coherent states.

Note: The vacuum $|\,0\,\rangle$ is a coherent state with $\alpha = 0$.

Mean energy:

$$\langle\, H \,\rangle \, = \, \langle\, \alpha \,|\; H \;|\, \alpha \,\rangle \, = \, \hbar\omega \; \langle\, \alpha \,|\; a^\dagger a \, + \, \tfrac{1}{2} \; |\, \alpha \,\rangle \, = \, \hbar\omega \, (\, |\alpha|^2 \, + \, \tfrac{1}{2} \,) \, . \tag{5.84}$$

The first term on the right hand side represents the classical wave intensity and the second the vacuum energy.

[5] See for example [15].

Next we introduce the *phase shifting operator*

$$U(\theta) \,=\, e^{-i\theta N}\,, \tag{5.85}$$

where N represents the occupation number operator (see Definition 5.2). Then we have

$$U^\dagger(\theta)\, a\, U(\theta) \,=\, a\, e^{-i\theta} \quad \text{or} \quad U(\theta)\, a\, U^\dagger(\theta) \,=\, a\, e^{i\theta}\,, \tag{5.86}$$

i.e. $U(\theta)$ gives the amplitude operator a phase shift θ.

Proof: The l.h.s. $\frac{d}{d\theta} U^\dagger(\theta)\, a\, U(\theta) \,=\, i\, U^\dagger(\theta)\, [N, a]\, U(\theta) \,=\, -i\, U^\dagger(\theta)\, a\, U(\theta)$ and the r.h.s. $\frac{d}{d\theta}\, a\, e^{-i\theta} \,=\, -i\, a\, e^{-i\theta}$ obey the same differential equation.

The phase shifting operator also shifts the phase of the coherent state

$$U(\theta)\, |\, \alpha\, \rangle \,=\, \left|\, \alpha\, e^{-i\theta}\, \right\rangle\,. \tag{5.87}$$

Proof: Multiplying Definition 5.3 with the operator U from the left and inserting the unitarian relation $U^\dagger U \,=\, \mathbb{1}$ on the l.h.s we get

$$\begin{aligned} \underbrace{U\, a\, U^\dagger}_{a\, e^{i\theta}}\, U\, |\, \alpha\, \rangle \quad &= \quad \alpha\, U\, |\, \alpha\, \rangle \\ \Rightarrow \quad a\, U\, |\, \alpha\, \rangle \quad &= \quad \alpha\, e^{-i\theta}\, U\, |\, \alpha\, \rangle\,. \end{aligned} \tag{5.88}$$

Denoting $U\, |\, \alpha\, \rangle \,=\, \left|\, \alpha'\, \right\rangle$ we find from Definition 5.3 that $a \left|\, \alpha'\, \right\rangle \,=\, \alpha' \left|\, \alpha'\, \right\rangle$ and thus $\alpha' = \alpha\, e^{-i\theta}$, q.e.d.

Now we want to study the coherent states more accurately. In order to do so we introduce the so called *displacement operator* which generates the coherent states, similar to how the creation operator $a^\dagger$, see Eq. (5.58), creates the occupation number states $|\, n\, \rangle$.

Definition 5.4 *The* ***displacement operator*** $D(\alpha)$ *is defined by*

$$D(\alpha) \,=\, e^{\,\alpha\, a^\dagger \,-\, \alpha^* a}$$

where $\alpha \,=\, |\alpha|\, e^{i\varphi} \,\in\, \mathbb{C}$ *is a complex number and* $a^\dagger$ *and* a *are the creation - and annihilation - operators.*

The displacement operator is an unitary operator, i.e. $D^\dagger D \,=\, \mathbb{1}$ and we can also rewrite it using Eq. (2.72). To use this equation we have to ensure that the commutators $[\,[\,A\,,\,B\,]\,,\,A\,]$ and $[\,[\,A\,,\,B\,]\,,\,B\,]$ vanish, where $A \,=\, \alpha\, a^\dagger$ and $B \,=\, \alpha^* a$. We therefore start by calculating the commutator of A and B

$$[\,A\,,\,B\,] \,=\, \left[\,\alpha\, a^\dagger\,,\,\alpha^* a\,\right] \,=\, \alpha\,\alpha^*\, \underbrace{\left[\,a^\dagger\,,\,a\,\right]}_{-1} \,=\, -\,|\alpha|^2\,. \tag{5.89}$$

Since the result is a real number, it commutes with A and B and we are allowed to use Eq. (2.72) where we can immediately insert the result of Eq. (5.89) to get the displacement operator in the following form:

$$D(\alpha) \,=\, e^{-\frac{1}{2}|\alpha|^2}\, e^{\alpha\, a^\dagger}\, e^{\alpha^* a}\,. \tag{5.90}$$

Properties of the displacement operator:

$$\text{I)} \qquad D^\dagger(\alpha) \,=\, D^{-1}(\alpha) \,=\, D(-\alpha) \quad \text{unitarity} \tag{5.91}$$

$$\text{II)} \qquad D^\dagger(\alpha)\, a\, D(\alpha) \,=\, a \,+\, \alpha \tag{5.92}$$

$$\text{III)} \qquad D^\dagger(\alpha)\, a^\dagger\, D(\alpha) \,=\, a^\dagger \,+\, \alpha^* \tag{5.93}$$

$$\text{IV)} \qquad D(\alpha \,+\, \beta) \,=\, D(\alpha)\, D(\beta)\, e^{-i\,\mathrm{Im}(\alpha\,\beta^*)} \tag{5.94}$$

We will prove two of these statements and leave the other two as an exercise.

Proof: of II)

$$\begin{aligned} D^\dagger(\alpha)\, a\, D(\alpha) \;&=\; e^{\alpha^* a \,-\, \alpha\, a^\dagger}\, a\, e^{\alpha\, a^\dagger \,-\, \alpha^* a} \,=\, a \,+\, \left[\, \alpha^* a \,-\, \alpha\, a^\dagger \,,\, a \,\right] \,= \\ &=\; a \,+\, \alpha^* \underbrace{\left[\, a \,,\, a \,\right]}_{0} - \alpha \underbrace{\left[\, a^\dagger \,,\, a \,\right]}_{-1} \,=\, a \,+\, \alpha \qquad \text{q.e.d.} \end{aligned} \tag{5.95}$$

We have used the Baker-Campbell-Hausdorff formula (Eq. (2.71)), where $A = \alpha^* a \,-\, \alpha\, a^\dagger$ and $B = a$. The higher order commutators vanish, since the commutator of A and B is a complex number, that commutes with the other operators.

Proof: of IV)

$$\begin{aligned} D(\alpha \,+\, \beta) \;&=\; e^{\alpha\, a^\dagger \,-\, \alpha^* a \,+\, \beta\, a^\dagger \,-\, \beta^* a} \,=\, e^{\alpha\, a^\dagger \,-\, \alpha^* a}\, e^{\beta\, a^\dagger \,-\, \beta^* a}\, e^{-\frac{1}{2}\left[\, \alpha\, a^\dagger \,-\, \alpha^* a \,,\, \beta\, a^\dagger \,-\, \beta^* a \,\right]} \,= \\ &=\; D(\alpha)\, D(\beta)\, e^{-\frac{1}{2}\,(\alpha\beta^* \,-\, \alpha^*\beta)} \,=\, D(\alpha)\, D(\beta)\, e^{-i\,\mathrm{Im}(\alpha\,\beta^*)} \qquad \text{q.e.d.} \end{aligned} \tag{5.96}$$

Here we have used Eq. (2.72), with $A = \alpha\, a^\dagger \,-\, \alpha^* a$ and $B = \beta\, a^\dagger \,-\, \beta^* a$ which is justified because the condition that the higher commutators vanish is satisfied since $[\, A \,,\, B \,] \in \mathbb{C}$. The result of this last commutator can also be seen easily by noting that the creation- and annihilation-operators commute with themselves but not with each other, giving ± 1.

With this knowledge about the displacement operator we can now create coherent states.

Theorem 5.2 *The coherent state $|\,\alpha\,\rangle$ is generated from the vacuum $|\,0\,\rangle$ by the displacement operator $D(\alpha)$*

$$|\,\alpha\,\rangle \;=\; D(\alpha)\;|\,0\,\rangle\;.$$

The "vacuum" $|\,0\,\rangle$ is the ground state with occupation number $n = 0$, which is defined by $a\;|\,0\,\rangle \;=\; 0$, see also Eq. (5.18).

Proof: Applying a negative displacement to $|\,\alpha\,\rangle$ we find from properties (5.91) and (5.92) that

$$\begin{aligned} & a\,D(-\alpha)\;|\,\alpha\,\rangle = D(-\alpha)\;\underbrace{D^\dagger(-\alpha)\,a\,D(-\alpha)}_{a-\alpha}\;|\,\alpha\,\rangle \\ \Rightarrow \quad & a\,D(-\alpha)\;|\,\alpha\,\rangle \;=\; D(-\alpha)(a-\alpha)\;|\,\alpha\,\rangle \;=\; 0\,. \end{aligned} \tag{5.97}$$

The r.h.s. vanishes because of Definition 5.3. This implies that $D(-\alpha)\;|\,\alpha\,\rangle$ is the vacuum state $|\,0\,\rangle$

$$D(-\alpha)\;|\,\alpha\,\rangle \;=\; |\,0\,\rangle \qquad \Rightarrow \qquad |\,\alpha\,\rangle \;=\; D(\alpha)\;|\,0\,\rangle \quad \text{q.e.d.} \tag{5.98}$$

On the other hand, we also could use Theorem 5.2 as a definition of the coherent states then our Definition 5.3 would follow as a theorem.

Proof:

$$\begin{aligned} \underbrace{D(\alpha)\,D^\dagger(\alpha)}_{\mathbb{1}}\;a\;|\,\alpha\,\rangle \quad &= \quad D(\alpha)\;\underbrace{D^\dagger(\alpha)\,a\,D(\alpha)}_{a+\alpha}\;|\,0\,\rangle \;=\; D(\alpha)\;\Bigg(\underbrace{a\;|\,0\,\rangle}_{0} + \alpha\;|\,0\,\rangle\Bigg) \;= \\ &= \quad \alpha\,D(\alpha)\;|\,0\,\rangle \;=\; \alpha\;|\,\alpha\,\rangle \\ &\Rightarrow \quad a\;|\,\alpha\,\rangle \;=\; \alpha\;|\,\alpha\,\rangle \quad \text{q.e.d.} \end{aligned} \tag{5.99}$$

The meaning of our Definition 5.3 becomes apparent when describing a coherent photon beam, e.g. a laser beam. The state of the laser remains unchanged if one photon is annihilated, i.e. if it is detected.

Expansion of coherent state in Fock space: The phase $|\,\alpha\,\rangle$ describes the wave aspect of the coherent state. Next we want to study the particle aspect, the states in Fock space. A coherent state contains an indefinite number of photons, which will be evident from the expansion of the coherent state $|\,\alpha\,\rangle$ into the CONS of the occupation number states $\{\,|\,n\,\rangle\,\}$. Let us start by inserting the completeness relation of the occupation number states

$$|\,\alpha\,\rangle \;=\; \sum_n |\,n\,\rangle\langle\,n\,|\,\alpha\,\rangle\;. \tag{5.100}$$

Then we calculate the transition amplitude $\langle\,n\,|\,\alpha\,\rangle$ by using Definition 5.3, where we multiply the whole eigenvalue equation with $\langle\,n\,|$ from the left

$$\langle\,n\,|\;a\;|\,\alpha\,\rangle \;=\; \langle\,n\,|\;\alpha\;|\,\alpha\,\rangle\;. \tag{5.101}$$

Using the adjoint of relation (5.59) we rewrite the left side of Eq. (5.101)

$$a^\dagger\;|\,n\,\rangle \;=\; \sqrt{n\,+\,1}\;|\,n\,+\,1\,\rangle \quad\Rightarrow\quad \langle\,n\,|\;a \;=\; \sqrt{n\,+\,1}\;\langle\,n\,+\,1\,| \tag{5.102}$$

$$\sqrt{n\,+\,1}\;\langle\,n\,+\,1\;|\,\alpha\,\rangle \;=\; \alpha\;\langle\,n\,|\,\alpha\,\rangle\;, \tag{5.103}$$

and replace the occupation number n with $n-1$ to obtain

$$\langle\,n\,|\,\alpha\,\rangle \;=\; \frac{\alpha}{\sqrt{n}}\;\langle\,n\,-\,1\;|\,\alpha\,\rangle\;. \tag{5.104}$$

By iterating the last step, i.e. again replacing n with $n-1$ in Eq. (5.104) and reinserting the result on the right hand side, we get

$$\langle\,n\,|\,\alpha\,\rangle \;=\; \frac{\alpha^2}{\sqrt{n(n\,-\,1)}}\;\langle\,n\,-\,2\;|\,\alpha\,\rangle \;=\; \cdots \;=\; \frac{\alpha^n}{\sqrt{n!}}\;\langle\,0\,|\,\alpha\,\rangle\;, \tag{5.105}$$

which inserted into the expansion of Eq. (5.100) results in

$$|\,\alpha\,\rangle \;=\; \langle\,0\,|\,\alpha\,\rangle \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\;|\,n\,\rangle\;. \tag{5.106}$$

The remaining transition amplitude $\langle\,0\,|\,\alpha\,\rangle$ can be calculated in two different ways, first with the normalization condition and second with the displacement operator. For the sake of variety we will use the second method here and leave the other one as an exercise

$$\langle\,0\,|\,\alpha\,\rangle \;=\; \langle\,0\,|\;D(\alpha)\;|\,0\,\rangle \;\overset{\text{Eq. (5.90)}}{=}\; e^{-\frac{1}{2}|\alpha|^2}\;\langle\,0\,|\;e^{\alpha\,a^\dagger}\;e^{\alpha^* a}\;|\,0\,\rangle\;. \tag{5.107}$$

We expand the exponentials of operators in the scalar product into their Taylor series

$$\langle\,0\,|\;e^{\alpha\,a^\dagger}\;e^{\alpha^* a}\;|\,0\,\rangle \;=\; \langle\,0\,|\;(1\,+\,\alpha\,a^\dagger\,+\,\cdots)(1\,+\,\alpha^*\,a\,+\,\cdots)\;|\,0\,\rangle \;=\; 1\;, \tag{5.108}$$

and we let the left bracket act to the left on the bra and right bracket to the right on the ket. In this way we can both times use the vacuum state property $a\;|\,0\,\rangle \,=\, 0$ to eliminate all terms of the expansion except the first. Therefore the transition amplitude is given by

$$\langle\,0\,|\,\alpha\,\rangle \;=\; e^{-\frac{1}{2}|\alpha|^2}\;, \tag{5.109}$$

and we can write down the coherent state in terms of an exact expansion in Fock space

$$| \alpha \rangle \;=\; e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, | n \rangle \;=\; e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha \, a^\dagger)^n}{n!} \, | 0 \rangle \; . \tag{5.110}$$

Probability distribution of coherent states: We can subsequently analyze the probability distribution of the photons in a coherent state, i.e. the probability of detecting n photons in a coherent state $| \alpha \rangle$, which is given by

$$P(n) \;=\; |\langle n \,|\, \alpha \rangle|^2 \;=\; \frac{|\alpha|^{2n} \, e^{-|\alpha|^2}}{n!} \; . \tag{5.111}$$

By noting, that the *mean photon number* is determined by the expectation value of the particle number operator

$$\bar{n} \;=\; \langle \alpha | \; N \; | \alpha \rangle \;=\; \langle \alpha | \; a^\dagger a \; | \alpha \rangle \;\stackrel{\text{Def. 5.3}}{=}\; |\alpha|^2 \; , \tag{5.112}$$

we can rewrite the probability distribution to get

$$P(n) \;=\; \frac{\bar{n}^n \, e^{-\bar{n}}}{n!} \; , \tag{5.113}$$

which is a *Poissonian distribution*.

Remarks: First, Eq. (5.112) represents the connection between the mean photon number – the particle view – and the complex amplitude squared, the intensity of the wave – the wave view.

Second, classical particles obey the same statistical law, the Poisson formula (5.111), when they are taken at random from a pool with $|\alpha|^2$ on average. Counting therefore the photons in a coherent state, they behave like randomly distributed classical particles, which might not be too surprising since coherent states are wave-like. Nevertheless, it's amusing that the photons in a coherent state behave like the raisins in a Gugelhupf [6], which have been distributed from the cook randomly with $|\alpha|^2$ on average per unit volume. Then the probability of finding n raisins per unit volume in the Gugelhupf follows precisely law (5.111).

Scalar product of two coherent states: From Theorem 4.3 we can infer some important properties of the particle number states, which are the eigenstates of the particle number operator $N = a^\dagger a$. Since this operator is hermitian, the eigenvalues n are real and the eigenstates $| \, n \, \rangle$ are orthogonal. But the coherent states are eigenstates of the annihilation operator a, which is surely not hermitian and as we already know, the eigenvalues α are complex numbers. Therefore we cannot automatically assume the coherent states

[6] A traditional Austrian birthday cake.

to be orthogonal, but have to calculate their scalar product, using again Theorem 5.2 and Eq. (5.90)

$$\langle\,\beta\,|\,\alpha\,\rangle \;=\; \langle\,0\,|\; D^\dagger(\beta)\, D(\alpha)\;|\,0\,\rangle \;=\; \langle\,0\,|\; e^{-\beta\, a^\dagger}\, e^{\beta^*\, a}\, e^{\alpha\, a^\dagger}\, e^{-\alpha^* a}\;|\,0\,\rangle\; e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)}\,. \tag{5.114}$$

We now use the same trick as in Eq. (5.108), i.e., expand the operators in their Taylor series, to see that the two "outer" operators $e^{-\beta\, a^\dagger} = (1 - \beta\, a^\dagger + \cdots)$ and $e^{-\alpha^* a} = (1 - \alpha^* a + \cdots)$, acting to left and right respectively, annihilate the vacuum state, save for the term "1" and we can thus ignore them. We then expand the remaining operators as well and consider their action on the vacuum

$$\begin{aligned} &\langle\,0\,|\;(1 \,+\, \beta^* a \,+\, \frac{1}{2!}(\beta^* a)^2 \,+\, \cdots)\,(1 \,+\, \alpha\, a^\dagger \,+\, \frac{1}{2!}(\alpha\, a^\dagger)^2 \,+\, \cdots)\;|\,0\,\rangle \\ = &\;(\cdots \,+\, \langle\,2\,|\,\frac{1}{2!}\sqrt{2!}\,(\beta^*)^2 \,+\, \langle\,1\,|\;\beta^* \,+\, \langle\,0\,|\,)\,(\,|\,0\,\rangle \,+\, \alpha\;|\,1\,\rangle \,+\, \frac{1}{2!}\sqrt{2!}\,\alpha^2\;|\,2\,\rangle \,+\, \cdots)\,. \end{aligned} \tag{5.115}$$

Using the orthogonality of the particle number states $\langle\,n\,|\,m\,\rangle \;=\; \delta_{\rm nm}$ we get

$$\langle\,\beta\,|\,\alpha\,\rangle \;=\; e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)}\,(\,1 \,+\, \alpha\,\beta^* \,+\, \frac{1}{2!}\,(\alpha\,\beta^*)^2 \,+\, \cdots) \;=\; e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha\,\beta^*}\,. \tag{5.116}$$

We further simplify the exponent, by noting that

$$|\,\alpha \,-\, \beta\,|^2 \;=\; (\alpha \,-\, \beta)(\alpha^* \,-\, \beta^*) \;=|\alpha|^2 \,+\, |\beta|^2 \,-\alpha\beta^* \,-\, \alpha^*\beta\;, \tag{5.117}$$

and we can finally write down the transition probability in a compact way

$$|\,\langle\,\beta\,|\,\alpha\,\rangle\,|^2 \;=\; e^{-|\,\alpha\,-\,\beta\,|^2}\,. \tag{5.118}$$

That means, the coherent states indeed are *not orthogonal* and their transition probability only vanishes in the limit of large differences $|\,\alpha \,-\, \beta\,| \gg 1$.

Completeness of coherent states: Although the coherent states are not orthogonal, it is possible to expand coherent states in terms of a complete set of states. The *completeness relation* for the coherent states reads

$$\frac{1}{\pi}\int d^2\alpha\;\;|\,\alpha\,\rangle\,\langle\,\alpha\,| \;=\; \mathbb{1}\;. \tag{5.119}$$

In fact, the coherent states are "overcomplete", which means that, as a consequence of their nonorthogonality, any coherent state can be expanded in terms of all the other coherent states. So the coherent states are not linearly independent

$$|\,\beta\,\rangle \;=\; \frac{1}{\pi}\int d^2\alpha\;\;|\,\alpha\,\rangle\,\langle\,\alpha\,|\,\beta\,\rangle \;=\; \frac{1}{\pi}\int d^2\alpha\;\;|\,\alpha\,\rangle\; e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha\,\beta^*}\,. \tag{5.120}$$

Proof of completeness:

$$\frac{1}{\pi}\int d^2\alpha \;|\,\alpha\,\rangle\,\langle\,\alpha\,| \overset{Eq.\ (5.110)}{=} \frac{1}{\pi}\sum_{n,m}\frac{1}{\sqrt{n!\,m!}}\;|\,n\,\rangle\,\langle\,m\,|\int d^2\alpha\; e^{-|\alpha|^2}\,\alpha^{\mathrm{n}}(\alpha^*)^{\mathrm{m}}\,. \tag{5.121}$$

The integral on the right hand side of Eq. (5.121) can be solved using polar coordinates $\alpha = |\alpha|e^{i\varphi} = r\,e^{i\varphi}$

$$\int d^2\alpha\; e^{-|\alpha|^2}\,\alpha^{\mathrm{n}}(\alpha^*)^{\mathrm{m}} \;=\; \int_0^\infty r\,dr\,e^{-r^2}\,r^{n+m}\underbrace{\int_0^{2\pi} d\varphi\,e^{i(n-m)\varphi}}_{2\pi\,\delta_{\mathrm{nm}}} \;=\; 2\pi\int_0^\infty r\,dr\,e^{-r^2}\,r^{2n}\,. \tag{5.122}$$

Substituting $r^2 = t$, $2rdr = dt$ to rewrite the integral

$$2\int_0^\infty r\,dr\,e^{-r^2}\,r^{2n} \;=\; \int_0^\infty e^{-t}\,t^n\,dt \;=\; \Gamma(n\,+\,1) \;=\; n!\;, \tag{5.123}$$

we recover the exact definition of the Gamma function, see [13, 14]. With this we can finally write down the completeness relation

$$\frac{1}{\pi}\int d^2\alpha\;|\,\alpha\,\rangle\,\langle\,\alpha\,| \;=\; \frac{1}{\pi}\sum_n\frac{1}{n!}\;|\,n\,\rangle\,\langle\,n\,|\;\pi\,n! \;=\; \sum_n|\,n\,\rangle\,\langle\,n\,| \;=\; \mathbb{1} \quad \text{q.e.d.} \tag{5.124}$$

5.2.2 Coordinate Representation of Coherent States

Recalling the coherent state expansion from Eq. (5.110)

$$|\,\alpha\,\rangle \;=\; e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\;|\,n\,\rangle \;=\; e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha\,a^\dagger)^n}{n!}\;|\,0\,\rangle\;\;, \tag{5.125}$$

and the coordinate representation of the harmonic oscillator states (Eq. (5.78))

$$\psi_{\mathrm{n}}(x) \;=\; \langle\,x\,|\,n\,\rangle \;\;=\;\; \psi_{\mathrm{n}}(x) \;=\; \frac{1}{\sqrt{2^{\mathrm{n}}(n!)\sqrt{\pi}x_0}}\,H_{\mathrm{n}}(\xi)\;e^{-\frac{\xi^2}{2}} \tag{5.126}$$

$$\psi_0(x) \;=\; \langle\,x\,|\,0\,\rangle \;\;=\;\; \psi_0(x) \;=\; \frac{1}{\sqrt{\sqrt{\pi}x_0}}\;e^{-\frac{\xi^2}{2}}\;, \tag{5.127}$$

where $\xi = \frac{x}{x_0}$ and $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, we can easily give the coordinate representation of the coherent states

$$\langle\,x\,|\,\alpha\,\rangle \;=\; \phi_\alpha(x) \;=\; e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,\psi_{\mathrm{n}}(x) \;=\; e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha\,a^\dagger)^n}{n!}\,\psi_0(x)\;. \tag{5.128}$$

Time evolution: Let's first take a look at the time evolution of the harmonic oscillator states and at the end we recall Theorem 4.1 and Definition 4.1. Using the harmonic oscillator energy (Eq. (5.77)) we easily get

$$\psi_{\mathrm{n}}(t,x) \;=\; \psi_{\mathrm{n}}(x)\, e^{-\frac{i}{\hbar}E_{\mathrm{n}}t} \;=\; \psi_{\mathrm{n}}(x)\, e^{-in\,\omega t}\, e^{-\frac{i\omega t}{2}}\ . \tag{5.129}$$

We are then in the position to write down the time evolution of the coherent state

$$\phi_\alpha(t,x) \;=\; e^{-\frac{1}{2}|\alpha|^2}\, e^{-\frac{i\omega t}{2}} \sum_{n=0}^{\infty} \frac{(\alpha\, e^{-i\omega t})^n}{\sqrt{n!}}\, \psi_{\mathrm{n}}(x)\ . \tag{5.130}$$

With a little notational trick, i.e. making the label α time dependent $\alpha(t) = \alpha\, e^{-i\omega t}$, we can bring Eq. (5.130) into a more familiar form

$$\phi_\alpha(t,x) \;=\; \phi_{\alpha(\mathrm{t})}(x)\, e^{-\frac{i\omega t}{2}}\ , \tag{5.131}$$

which we can identify as a solution of the time dependent Schrödinger equation.

Expectation value of x for coherent states: Subsequently, we want to compare the motion of coherent states to that of the quantum mechanical (and classical) harmonic oscillator, which we will do by studying the expectation value of the position operator. For the quantum mechanical case we have already calculated the mean position, which does not oscillate (see Eq. (5.34))

$$\langle\, x \,\rangle_{\text{oscillator}} \;=\; 0\,. \tag{5.132}$$

Recalling the expression of x in terms of creation and annihilation operators, Eq. (5.5), and the eigenvalue equation of the annihilation operator, Definition 5.3, we can easily compute the expectation value of the position for the coherent states

$$\begin{aligned} \langle\, x \,\rangle_{\text{coherent}} \;&=\; \left\langle\, \phi_{\alpha(\mathrm{t})} \,\middle|\, x \,\middle|\, \phi_{\alpha(\mathrm{t})} \,\right\rangle \;=\; \frac{x_0}{\sqrt{2}} \left\langle\, \phi_{\alpha(\mathrm{t})} \,\middle|\, a \,+\, a^\dagger \,\middle|\, \phi_{\alpha(\mathrm{t})} \,\right\rangle \;= \\ &=\; \frac{x_0}{\sqrt{2}}\,(\alpha(t) \,+\, \alpha^*(t)) \;=\; \sqrt{2}\,x_0\,\mathrm{Re}(\alpha(t)) \;=\; \sqrt{2}\,x_0\,|\alpha|\cos(\omega \mathrm{t} \,-\, \varphi)\ , \end{aligned} \tag{5.133}$$

where we used $\alpha(t) = \alpha\, e^{-i\omega t} = |\alpha|\, e^{-i(\omega t - \varphi)}$. To summarize the calculation, we conclude that the coherent state, unlike the quantum mechanical harmonic oscillator, *does oscillate*, similar to its classical analogue

$$\langle\, x \,\rangle_{\text{coherent}} \;=\; \sqrt{2}\,x_0\,|\alpha|\cos(\omega \mathrm{t} \,-\, \varphi)\ . \tag{5.134}$$

Coordinate representation in terms of displacement operator: In this section we want to express and study the coordinate representation of coherent states via the displacement operator D

$$\phi_\alpha(x) \,=\, \langle\, x \,|\, \alpha \,\rangle \,=\, \langle\, x \,|\; D(\alpha) \,|\, 0 \,\rangle \,=\, \langle\, x \,|\; e^{\alpha\, a^\dagger \,-\, \alpha^* a} \;|\, 0 \,\rangle \;. \tag{5.135}$$

To simplify further calculations we now rewrite the creation and annihilation operators from Definition 5.1 in terms of the dimensionless variable $\xi = \frac{x}{x_0}$,

$$a \,=\, \tfrac{1}{\sqrt{2}}(\xi \,+\, \frac{d}{d\xi}) \quad , \quad a^\dagger \,=\, \tfrac{1}{\sqrt{2}}(\xi \,-\, \frac{d}{d\xi}) \;. \tag{5.136}$$

This rewritten operators (5.136) we insert into the exponent of the displacement operator from Eq. (5.135)

$$\begin{aligned}
\alpha\, a^\dagger \,-\, \alpha^* a \;&=\; \frac{\alpha}{\sqrt{2}}(\xi \,-\, \frac{d}{d\xi}) \,-\, \frac{\alpha^*}{\sqrt{2}}(\xi \,+\, \frac{d}{d\xi}) \,= \\
&=\; \frac{\xi}{\sqrt{2}}(\underbrace{\alpha \,-\, \alpha^*}_{2i\,\mathrm{Im}(\alpha)}) \,-\, \frac{1}{\sqrt{2}}\frac{d}{d\xi}(\underbrace{\alpha \,+\, \alpha^*}_{2\,\mathrm{Re}(\alpha)}) \,= \\
&=\; \sqrt{2}\, i\, \mathrm{Im}(\alpha)\, \xi \,-\, \sqrt{2}\, \mathrm{Re}(\alpha)\, \frac{d}{d\xi} \;,
\end{aligned} \tag{5.137}$$

and get for the wavefunction (5.135) the following expression

$$\phi_\alpha(x) \,=\, \langle\, x \,|\, \alpha \,\rangle \,=\, e^{\sqrt{2}\, i\, \mathrm{Im}(\alpha)\, \xi \,-\, \sqrt{2}\, \mathrm{Re}(\alpha)\, \frac{d}{d\xi}} \;\psi_0(\xi) \;, \tag{5.138}$$

where ψ_0 is the harmonic oscillator ground state (Eq. (5.127)).

The time dependent version follows outright

$$\phi_\alpha(t,x) \,=\, \phi_{\alpha(\mathrm{t})}\, e^{-\frac{i\omega t}{2}} \,=\, \langle\, x \,|\, \alpha(t) \,\rangle\; e^{-\frac{i\omega t}{2}} \,=\, e^{-\frac{i\omega t}{2}} \;\langle\, x \,|\; e^{\alpha(t)\, a^\dagger \,-\, \alpha^*(t) a} \;|\, 0 \,\rangle$$

$$\Rightarrow \quad \phi_\alpha(t,x) \,=\, e^{-\frac{i\omega t}{2}}\, e^{\sqrt{2}\, i\, \mathrm{Im}(\alpha(t))\, \xi \,-\, \sqrt{2}\, \mathrm{Re}(\alpha(t))\, \frac{d}{d\xi}} \;\psi_0(\xi) \;. \tag{5.139}$$

On the other hand, as a check, we can also arrive at this result by starting from expansion (5.125), where we interpret the sum over n as the power series for the exponential map, i.e.

$$\phi_\alpha(t,x) \,=\, e^{-\frac{1}{2}|\alpha|^2}\, e^{-\frac{i\omega t}{2}} \sum_{n=0}^{\infty} \frac{(\alpha(t)\, a^\dagger)^n}{n!}\, \psi_0(x) \,=\, e^{-\frac{1}{2}|\alpha|^2}\, e^{-\frac{i\omega t}{2}}\, e^{\alpha(t)\, a^\dagger}\, \psi_0(x) \;. \tag{5.140}$$

To show that this result is the same as Eq. (5.139) we simply calculate

$$
\begin{aligned}
e^{\alpha(t)\,a^\dagger - \alpha^*(t)\,a}\,\psi_0(x) \;&\overset{\text{Eq. (2.72)}}{=}\; e^{\alpha(t)\,a^\dagger}\; e^{-\alpha^*(t)\,a}\; e^{\frac{1}{2}\alpha\,\alpha^* \overbrace{[\,a^\dagger\,,\,a\,]}^{-1}}\;\psi_0(x) \;= \\
&= \; e^{-\frac{1}{2}|\alpha|^2}\; e^{\alpha(t)\,a^\dagger}\;(1\,-\,\alpha^*(t)\;\overset{\longrightarrow 0}{a}\;+\,\cdots)\;\psi_0(x)\;= \\
&= \; e^{-\frac{1}{2}|\alpha|^2}\; e^{\alpha(t)\,a^\dagger}\;\psi_0(x)\;, \qquad (5.141)
\end{aligned}
$$

where we once again used the fact, that the action of the annihilation operator on the ground state vanishes. Inserting the result into Eq. (5.140) we get

$$
\phi_\alpha(t,x)\;=\;e^{-\frac{i\omega t}{2}}\,e^{\alpha(t)\,a^\dagger - \alpha^*(t)\,a}\,\psi_0(x)\;. \qquad (5.142)
$$

Explicit calculation of the wave function: Our aim is now to explicitly compute the wave function from expression (5.138). We first state the result and derive it in the following

$$
\begin{aligned}
\phi_\alpha(t,x) \;&=\; \frac{1}{\sqrt{x_0\sqrt{\pi}}}\,e^{-\frac{i\omega t}{2}}\,e^{\sqrt{2}\,i\,\mathrm{Im}(\alpha(t))\,\xi\,-\,\sqrt{2}\,\mathrm{Re}(\alpha(t))\,\frac{d}{d\xi}}\;e^{-\frac{\xi^2}{2}} \\
&=\; \frac{1}{\sqrt{x_0\sqrt{\pi}}}\,e^{-\frac{i\omega t}{2}}\,e^{\sqrt{2}\,\alpha(t)\,\xi\,-\,\frac{\xi^2}{2}\,-\,\mathrm{Re}(\alpha(t))\,\alpha(t)}\;. \qquad (5.143)
\end{aligned}
$$

Proof: Let us now perform the explicit calculation. That means, we apply the differential operator[7] $e^{\mathcal{D}} = e^{\sqrt{2}\,i\,\mathrm{Im}(\alpha(t))\,\xi\,-\,\sqrt{2}\,\mathrm{Re}(\alpha(t))\,\frac{d}{d\xi}}$ to the ground state of the harmonic oscillator $\psi_0(\xi) = \frac{1}{\sqrt{x_0\sqrt{\pi}}}e^{-\frac{\xi^2}{2}}$, but we will only consider the linear and quadratic term of the operator expansion, as sketched in Eq. (5.144)

$$
e^{\mathcal{D}}\,e^{-\frac{\xi^2}{2}}\;=\;(1\,+\,\mathcal{D}\,+\,\frac{1}{2!}\,\mathcal{D}^2\,+\,\cdots)\,e^{-\frac{\xi^2}{2}}\;, \qquad (5.144)
$$

$$
\begin{aligned}
\text{linear term:} \qquad \mathcal{D}\,e^{-\frac{\xi^2}{2}} \;&=\; \Big(\sqrt{2}\,i\,\mathrm{Im}(\alpha(t))\,\xi\,-\,\sqrt{2}\,\mathrm{Re}(\alpha(t)\,\tfrac{d}{d\xi}\Big)\;e^{-\frac{\xi^2}{2}} \\
&=\; \Big(\sqrt{2}\,i\,\mathrm{Im}(\alpha(t))\,\xi\,+\,\sqrt{2}\,\mathrm{Re}(\alpha(t)\,\xi\Big)\;e^{-\frac{\xi^2}{2}} \\
&=\; \sqrt{2}\,\xi\;\underbrace{(\mathrm{Re}(\alpha(t))\,+\,i\,\mathrm{Im}(\alpha(t)))}_{\alpha(t)}\;e^{-\frac{\xi^2}{2}} \\
&=\; \sqrt{2}\,\alpha(t)\,\xi\,e^{-\frac{\xi^2}{2}}\;, \qquad (5.145)
\end{aligned}
$$

[7]The operator $\mathcal{D}$, which is a notation to shorten the cumbersome calculations, is not to be confused with the displacement operator D.

quadratic term:

$$
\begin{aligned}
\mathcal{D}^2 e^{-\frac{\xi^2}{2}} &= \sqrt{2}\,\alpha(t)\,\mathcal{D}\,\xi\, e^{-\frac{\xi^2}{2}} \\
&= \sqrt{2}\,\alpha(t)\left(\sqrt{2}\,i\,\mathrm{Im}(\alpha(t))\,\xi \,-\, \sqrt{2}\,\mathrm{Re}(\alpha(t))\,\tfrac{d}{d\xi}\right)\xi\, e^{-\frac{\xi^2}{2}} \\
&= \left(2\,\alpha(t)\,i\,\mathrm{Im}(\alpha(t))\,\xi^2 \,-\, 2\,\alpha(t)\,\mathrm{Re}(\alpha(t)) \,+\, 2\,\alpha(t)\,\mathrm{Re}(\alpha(t))\,\xi^2\right) e^{-\frac{\xi^2}{2}} \\
&= \Big(2\,\alpha(t)\,\underbrace{(\mathrm{Re}(\alpha(t)) \,+\, i\,\mathrm{Im}(\alpha(t)))}_{\alpha(t)}\,\xi^2 \,-\, 2\,\alpha(t)\,\mathrm{Re}(\alpha(t))\Big)\, e^{-\frac{\xi^2}{2}}\,.
\end{aligned} \tag{5.146}
$$

Inserting the results of Eq. (5.145) and Eq. (5.146) into the expansion (5.144) we get

$$
e^{\mathcal{D}}\, e^{-\frac{\xi^2}{2}} = (1 \,+\, \underbrace{\sqrt{2}\,\alpha(t)\xi \,-\, \alpha(t)\,\mathrm{Re}(\alpha(t))}_{\text{linear term}} + \frac{2\,\alpha(t)\,\alpha(t)\,\xi^2}{2!} \,+\, \cdots)\, e^{-\frac{\xi^2}{2}}\,, \tag{5.147}
$$

where we can already identify the linear term and the first part of the quadratic term in the end result of the expansion. Finally, rewriting the whole expansion as exponential function and including the factor $e^{-\frac{\xi^2}{2}}$, we arrive at the proposed result of Eq. (5.143).

Probability distribution of the wave packet: As last computation in this section we want to study the probability density of the wave function of the coherent state, thus we take the modulus squared of expression (5.143)

$$
\begin{aligned}
|\phi_\alpha(t,x)|^2 &= \phi_\alpha^*\,\phi_\alpha(t,x) \\
&= \frac{1}{x_0\sqrt{\pi}}\; e^{-\xi^2 + \sqrt{2}\,\overbrace{(\alpha^* \,+\, \alpha)}^{2\mathrm{Re}(\alpha(t))}\,\xi \,-\, \mathrm{Re}(\alpha(t))\,\overbrace{(\alpha^* \,+\, \alpha)}^{2\mathrm{Re}(\alpha(t))}} \\
&= \frac{1}{x_0\sqrt{\pi}}\; e^{-\left(\xi \,-\, \sqrt{2}\,\mathrm{Re}(\alpha(t))\right)^2} = \frac{1}{x_0\sqrt{\pi}}\; e^{-\left(\xi \,-\, \sqrt{2}\,|\alpha|\,\cos(\omega t \,-\, \varphi)\right)^2}\,.
\end{aligned} \tag{5.148}
$$

Recalling the result for the expectation value (the mean value) of the position (Eq. (5.134)) and our choice of $\xi = \frac{x}{x_0}$ we finally find the probability density

$$
|\phi_\alpha(t,x)|^2 = \frac{1}{x_0\sqrt{\pi}}\,\exp\left(-\frac{(x \,-\, \langle\, x\,\rangle\,(t))^2}{x_0^2}\right)\,, \tag{5.149}
$$

which is a Gaussian distribution with constant width.

Result: The coherent states are oscillating Gaussian wave packets with constant width in a harmonic oscillator potential, i.e., the wave packet of the coherent state is not spreading (because all terms in the expansion are in phase). It is a wave packet with minimal uncertainty. These properties make the coherent states the closest quantum mechanical analogue to the free classical single mode field. For an illustration see Fig. 5.3.

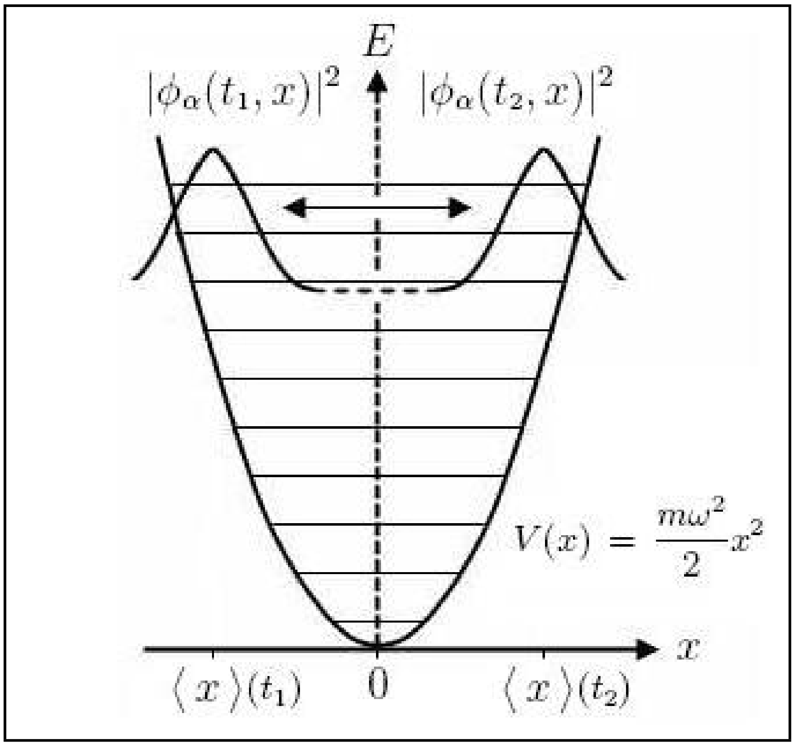

Figure 5.3: Coherent state: The probability density of the coherent state is a Gaussian distribution, whose center oscillates in a harmonic oscillator potential. Due to its being a superposition of harmonic oscillator states, the coherent state energy is not restricted to the energy levels $\hbar\omega(n + \frac{1}{2})$ but can have any value (greater than the zero point energy).