Chapter 10

Entanglement of Quantum Systems

At the end we want to turn to an area which arose quite recently, in the late 80ties and 90ties, which received much attention since it opens a new kind of communication which wasn't possible before. This is quantum information, quantum communication and quantum computation. The foundation came from John Bell and his famous inequalities which focused on a certain feature of the quantum states, on the so-called nonlocality or entanglement of the states. Entanglement is the basic ingredient of quantum information theory and became nowadays the most used term in quantum physics, a kind of "magic word". In this chapter we will give an introduction into the entanglement concept, into Bell inequalities and discuss finally its most exciting consequence, namely quantum teleportation.

10.1 Entanglement and Separability

The term "entanglement" ("Verschränkung" in the original German phrasing) was introduced by Erwin Schrödinger in 1935 in order to describe an intrinsic feature of quantum mechanics, that arises from the structure of Hilbert space, i.e. the superposition principle of the states therein. He stated that:

> A composite quantum system, whose subsystems are distant from each other, is in an "***entangled state***" if the total system is in a well defined state, but the subsystems themselves are not.

To formulate this statement more rigorously let us look at systems of two components, e.g. two spin $\frac{1}{2}$ particles or two photons, and construct the four possible product states (eigenstates of σ_z), already known from Eqs. (7.43) - (7.46)

$$|\uparrow\rangle \otimes |\uparrow\rangle \quad , \; |\uparrow\rangle \otimes |\downarrow\rangle \quad , \; |\downarrow\rangle \otimes |\uparrow\rangle \quad , \; |\downarrow\rangle \otimes |\downarrow\rangle \; . \tag{10.1}$$

These states are of the class of the so-called "separable" states, or just "product" states. In quantum information theory the left subsystem is often called ***Alice*** and the right one ***Bob***. For photons the arrow notation of the σ_z eigenstates is usually replaced by the letters H and V, standing for horizontal and vertical polarization respectively

$$|\,H\,\rangle \otimes |\,H\,\rangle \quad , \; |\,H\,\rangle \otimes |\,V\,\rangle \quad , \; |\,V\,\rangle \otimes |\,H\,\rangle \quad , \; |\,V\,\rangle \otimes |\,V\,\rangle \; . \tag{10.2}$$

Generally, instead of the states of Eq. (10.1) and Eq. (10.2), one could use any quantum system consisting of two subsystems, i.e. a bipartite system with each two degrees of freedom, called a 2-qubit state. But whichever of the above product states you choose, the states of the individual subsystems Alice and Bob are well defined. For example, for the state $|\!\uparrow\rangle \otimes |\!\downarrow\rangle$, when measuring the spin along the z-direction, the outcome of the measurements of Alice will be "up", i.e. determined by the state $|\!\uparrow\rangle$, while Bob's result will be always "down", i.e. determined by the state $|\!\downarrow\rangle$.

From these product states we can form linear combinations, e.g.

$$\left|\,\psi^{\pm}\right\rangle \;=\; \frac{1}{\sqrt{2}}\left(\,|\!\uparrow\rangle\,|\!\downarrow\rangle \pm |\!\downarrow\rangle\,|\!\uparrow\rangle\,\right) \tag{10.3}$$

$$\left|\,\phi^{\pm}\right\rangle \;=\; \frac{1}{\sqrt{2}}\left(\,|\!\uparrow\rangle\,|\!\uparrow\rangle \pm |\!\downarrow\rangle\,|\!\downarrow\rangle\,\right) \; . \tag{10.4}$$

These are the four (maximally) entangled 2-qubit states, called the ***Bell states***. We also recognize that the Bell states are not only artificial constructions but we have already encountered them when studying composite quantum systems with spin, where the $|\,\psi^{\pm}\,\rangle$ were found to be simultaneous eigenstates of $\vec{S}$ and S_z, see Eqs. (7.61) - (7.63).

Let us now try to grasp what entanglement in terms of e.g. the Bell state $|\,\psi^{\pm}\,\rangle$ means. Consider two observers, Alice and Bob, who are far apart, both performing measurements on their part of the entangled state. They will both get the results $\uparrow$ or $\downarrow$ equally often, i.e. with a probability of $\frac{1}{2}$, which means that neither of the subsystems is in a well defined state. But on the other hand, everytime Alice measures $\uparrow$, Bob will with certainty measure the result $\downarrow$ and vice versa, i.e. the measurement results are perfectly correlated.

This has some far reaching consequences for the structure of the theory, i.e. it leads to a loss of locality and/or realism of the theory, which is backed up by experiments and it creates interesting possibilities for information processing, such as quantum cryptography, quantum teleportation and many more.

10.2 EPR Paradox

The EPR (Albert Einstein, Boris Podolsky and Nathan Rosen) paradox was formulated in 1935 (see Ref. [18]) as a gedankenexperiment to strengthen the claim that quantum mechanics in its probabilistic character was somehow incomplete. The argumentation was later on reformulated by David Bohm in 1952 (see Ref. [19]) for the simple quantum mechanical system of two spin $\frac{1}{2}$ particles, which is the way we will present the problem here.

The main argument of EPR is based on three requirements (see, e.g. Ref. [20]):

1. ***Completeness:*** Every element of physical reality must have a counterpart in the physical theory in order for the theory to be complete.
2. ***Realism:*** If the value of a physical quantity can be predicted with certainty, i.e. probability 1, without disturbing the system, then the quantity has physical reality.
3. ***Locality:*** There is no action at a distance. Measurements on a (sub)system do not affect measurements on (sub)systems that are far away.

EPR then conclude that under certain circumstances quantum mechanics is *not* a complete theory.

To understand this claim, let us consider two spin $\frac{1}{2}$ particles in the spin singlet state $|\,\psi^-\,\rangle$, the antisymmetric Bell state, and let them propagate in opposite direction along the x-axis from their source. Let then two observers, Alice and Bob, perform spin measurement along the z-direction.

Quantum mechanics tells us that for the state

$$\left|\,\psi^-\,\right\rangle \;=\; \frac{1}{\sqrt{2}}\,(\,|\!\uparrow\rangle\,|\!\downarrow\rangle \,-\, |\!\downarrow\rangle\,|\!\uparrow\rangle\,) \tag{10.5}$$

the result measured by Alice will be undetermined, i.e. either $\uparrow$ or $\downarrow$, but if Alice measures $\uparrow$, then Bob will measure $\downarrow$ with certainty and vice versa, which assigns physical reality to the spin of Bob's particle in the sense of EPR. Since there is no disturbance or action at a distance, EPR conclude, quantum mechanics does not contain any information about predetermined measurement outcomes and is therefore incomplete.

To account for the missing information, there must be some inaccessible parameter, a hidden variable, to determine which spin eigenvalue is realized in the measurement. EPR thus demand a hidden variable theory (HVT) to explain this problem.

In the same year as the EPR paper was published, Bohr replied (using the same title for his paper as EPR, i.e. "Can quantum-mechanical description of physical reality be considered complete ?", see Ref. [21]) criticizing their perception of reality and sweeping away their arguments without really answering the interesting paradox presented in the EPR paper. But due to Bohr's great authority the physical community followed his view that quantum mechanics *is* complete and the case rested for nearly 30 years, until John Bell published his famous article in 1964, presenting a way to solve the debate.

10.3 Bell Inequalities

In 1964 John S. Bell published an article (see Ref. [22]), proposing a solution to the (until then purely philosophical) Bohr-Einstein debate, which made it possible to determine experimentally, whether or not the requirements of the EPR paradox are fulfilled in Nature. The essence of Bell's proposal can be formulated in the following theorem:

Theorem 10.1 (Bell's theorem) *In certain experiments all local realistic theories (LRT) are incompatible with quantum mechanics.*

To formulate this mathematically, let us go back to the set-up discussed in Section 10.2, Alice and Bob performing spin measurements on an entangled 2-qubit state. For the schematic set-up of such an experiment see Fig. 10.1. The spin measurement along an arbitrary direction $\vec{a}$ is represented by the operator $\vec{\sigma}\vec{a}$ in quantum mechanics, but let us stay more general for now and assume that both Alice and Bob measure some observable $A^{\text{obs.}}(\vec{a})$ and $B^{\text{obs.}}(\vec{b})$ respectively, where the possible outcomes both also depend on some internal hidden parameter λ. The reality assumption of the EPR paradox then requires, that the parameter λ is the same for both subsystems

$$\begin{aligned}
\text{Alice:} \quad & A^{\text{obs.}}(\vec{a}) \quad \longrightarrow \quad A(\vec{a},\, \lambda) \;=\; \pm 1 \; (\uparrow \;\text{ or }\; \downarrow) \text{ or } 0 \text{ (no detection)} \\
\text{Bob:} \quad & B^{\text{obs.}}(\vec{b}) \quad \longrightarrow \quad B(\vec{b},\, \lambda) \;=\; \pm 1 \; (\uparrow \;\text{ or }\; \downarrow) \text{ or } 0 \text{ (no detection)}
\end{aligned}$$

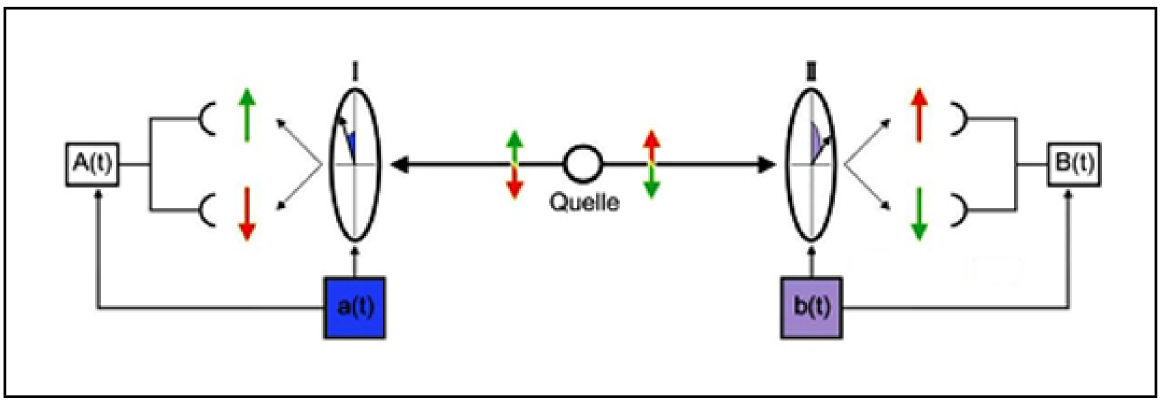

Figure 10.1: Set-up of a Bell-type experiment: Two entangled spin $\frac{1}{2}$ particles in the spin singlet state, the antisymmetric Bell state, are produced at a source and propagate in opposite direction along the x-axis. Two observers, Alice and Bob, perform a spin measurement along the z-direction. Whenever Alice measures $\uparrow$, Bob will measure $\downarrow$ with certainty, and vice versa. The detector orientations $a(t)$ and $b(t)$ will be fixed at random at each time t and registered by Alice and Bob together with the joint measurement results $A(t)$ and $B(t)$.

Bell's locality assumption requires that neither result depends on the measuring direction of the other side, i.e.

$$A(\vec{a},\, \cancel{\vec{b}},\, \lambda)\;, \quad B(\cancel{\vec{a}},\, \vec{b},\, \lambda)\;. \tag{10.6}$$

Furthermore we can assume that there is some normalized distribution function $\rho(\lambda)$,

$$\int d\lambda\, \rho(\lambda) \;=\; 1\;, \tag{10.7}$$

which determines the outcome for a given lambda. The expectation value of the combined measurement on both Alice's and Bob's side is then

$$E(\vec{a},\, \vec{b}) \;=\; \int d\lambda\, \rho(\lambda)\, A(\vec{a},\, \lambda)\, B(\vec{b},\, \lambda)\;. \tag{10.8}$$

Let us next consider a certain combination of such expectation values for measurements in different directions, but in the same state, i.e. for the same λ

$$\begin{aligned} E(\vec{a},\, \vec{b}) \,-\, E(\vec{a},\, \vec{b}') \;&=\; \int d\lambda\, \rho(\lambda)\, \Big(A(\vec{a},\, \lambda)\, B(\vec{b},\, \lambda) \,-\, A(\vec{a},\, \lambda)\, B(\vec{b}',\, \lambda) \Big) \\ &=\; \int d\lambda\, \rho(\lambda)\, \underbrace{A(\vec{a},\, \lambda)\, B(\vec{b},\, \lambda)}_{|\;|\leq 1} \Big(1 \,\pm\, A(\vec{a}',\, \lambda)\, B(\vec{b}',\, \lambda) \Big) \\ &-\; \int d\lambda\, \rho(\lambda)\, \underbrace{A(\vec{a},\, \lambda)\, B(\vec{b}',\, \lambda)}_{|\;|\leq 1} \Big(1 \,\pm\, A(\vec{a}',\, \lambda)\, B(\vec{b},\, \lambda) \Big)\;, \end{aligned} \tag{10.9}$$

where we just added and subtracted the same term to rewrite the equation. By then noting that the absolute values of the products of A and B on the right side must be smaller or equal to 1 we can make the following estimation:

$$\begin{aligned} \Big| E(\vec{a},\, \vec{b}) \,-\, E(\vec{a},\, \vec{b}') \Big| \;&\leq\; \Big| \int d\lambda\, \rho(\lambda)\, \Big(1 \,-\, A(\vec{a}',\, \lambda)\, B(\vec{b}',\, \lambda) \Big) \Big| \\ &+\; \Big| \int d\lambda\, \rho(\lambda)\, \Big(1 \,-\, A(\vec{a}',\, \lambda)\, B(\vec{b},\, \lambda) \Big) \Big|\;. \end{aligned} \tag{10.10}$$

Using Eq. (10.7) we get

$$\Big| E(\vec{a},\, \vec{b}) \,-\, E(\vec{a},\, \vec{b}') \Big| \leq 2 \,-\, \Big| E(\vec{a}',\, \vec{b}') \,+\, E(\vec{a}',\, \vec{b}) \Big|\;, \tag{10.11}$$

which we rewrite in terms of the so-called *Bell parameter function* $S(\vec{a},\, \vec{a}',\, \vec{b},\, \vec{b}')$ as

$$|\, S(\vec{a},\, \vec{a}',\, \vec{b},\, \vec{b}')\,| = \Big| E(\vec{a},\, \vec{b}) \,-\, E(\vec{a},\, \vec{b}') \Big| \,+\, \Big| E(\vec{a}',\, \vec{b}') \,+\, E(\vec{a}',\, \vec{b}) \Big| \,\leq\, 2\,. \tag{10.12}$$

Equation (10.12) is called the ***CHSH form of the Bell inequality***, named after Clauser, Horne, Shimony and Holt, who derived it separately from Bell. This inequality must be satisfied by any *local realistic theory*! (Clearly, satisfying Eq. (10.8).

The interesting question, of course, is whether or not quantum mechanics satisfies the inequality. Let us do the calculation. We first have to calculate the expectation value $E(\vec{a},\, \vec{b})$ for the state $|\,\psi^-\,\rangle$

$$\begin{aligned} E(\vec{a},\, \vec{b}) \;&=\; \langle\, \psi^- \,|\, \vec{\sigma}\vec{a} \otimes \vec{\sigma}\vec{b} \,|\, \psi^- \,\rangle \;= \\ &=\; \langle\, \psi^- \,| \begin{pmatrix} a_{\mathrm{z}} & a_{\mathrm{x}} \,-\, i a_{\mathrm{y}} \\ a_{\mathrm{x}} \,+\, i a_{\mathrm{y}} & -a_{\mathrm{z}} \end{pmatrix} \otimes \begin{pmatrix} b_{\mathrm{z}} & b_{\mathrm{x}} \,-\, i b_{\mathrm{y}} \\ b_{\mathrm{x}} \,+\, i b_{\mathrm{y}} & -b_{\mathrm{z}} \end{pmatrix} |\, \psi^- \,\rangle \;= \\ &=\; \frac{1}{2}\,(\,-\,2 a_{\mathrm{x}} b_{\mathrm{x}} \,-\, 2 a_{\mathrm{y}} b_{\mathrm{y}} \,-\, 2 a_{\mathrm{z}} b_{\mathrm{z}}\,) \;=\; -\,\vec{a}\,\vec{b}\;, \end{aligned} \tag{10.13}$$

where we used the representation of Eq. (10.5) and Eq. (7.27). Furthermore we work with normalized vectors $|\vec{a}| = |\vec{b}| = 1$ and obtain

$$E(\vec{a}, \vec{b}) \,=\, -\,\vec{a}\,\vec{b} \,=\, -\,\cos(\alpha\,-\,\beta)\;, \tag{10.14}$$

where $(\alpha\,-\,\beta)$ is the angle between the directions $\vec{a}$ and $\vec{b}$. Choosing the directions in the plane perpendicular to the propagation axis in steps of $45°$[1], i.e. such that

$$\alpha - \beta \,=\, \alpha' \,-\, \beta' \,=\, \alpha' \,-\, \beta \,=\, \frac{\pi}{4}\;,\;\; \alpha\,-\,\beta' \,=\, 3\,\frac{\pi}{4}\;, \tag{10.15}$$

we get the Bell parameter function as predicted by quantum mechanics

$$\begin{aligned} S_{\mathrm{QM}} \;&=\; |-\,\cos(\alpha\,-\,\beta) \,+\, \cos(\alpha\,-\,\beta')| \,+\, |-\,\cos(\alpha'\,-\,\beta') \,-\, \cos(\alpha'\,-\,\beta)| \,= \\ &=\; |-\,\frac{\sqrt{2}}{2} \,-\, \frac{\sqrt{2}}{2}| \,+\, |-\,\frac{\sqrt{2}}{2} \,-\, \frac{\sqrt{2}}{2}| \,=\, 2\sqrt{2} \,>\, 2\,. \end{aligned} \tag{10.16}$$

This is the maximal possible violation of the Bell inequality, showing that all local realistic theories are *incompatible* with quantum mechanics which in turn must be a nonlocal and/or nonrealistic theory.

Experiments: In the past many experiments by several have been carried out, testing Bell inequalities. The first ones were Clauser and Freedman [23] and Fry and Thompson [24] in the mid 70ties. They used a radiative cascade of calcium or mercury to create the desired entangled states and had static analyzers. The next step forward happened in the early 80ties, when A. Aspect and his group [25] included an optical switch mechanism in the set-up of their instruments in order to achieve a spacelike separation of the event intervals. The experimental outcome of all experiments was in accordance with quantum mechanics.

The best experiment up to now concerning tests of Bell inequalities has been performed by A. Zeilinger and his group [26] in the late 90ties. They used a revolutionary new source, a BBO crystal, for producing the entangled states and were able to fabricate a truly random and ultrafast electro-optical switch mechanism for the analyzers, see Fig. 10.2. In this way Alice couldn't get any information from Bob with velocities less than the speed of light, which means that the strict Einstein locality conditions have been fulfilled. The experimental value for the Bell parameter function in this experiment was determined to be

$$S_{\mathrm{exp}} \,=\, 2,73\,\pm\,0,02\;, \tag{10.17}$$

in perfect agreement with quantum mechanics, which implies that LRT for describing Nature are ruled out, i.e. Nature contains a kind of nonlocality in the sense described above.

[1] Funny enough, these angles were called *awkward Irish angels* by John Bell himself.

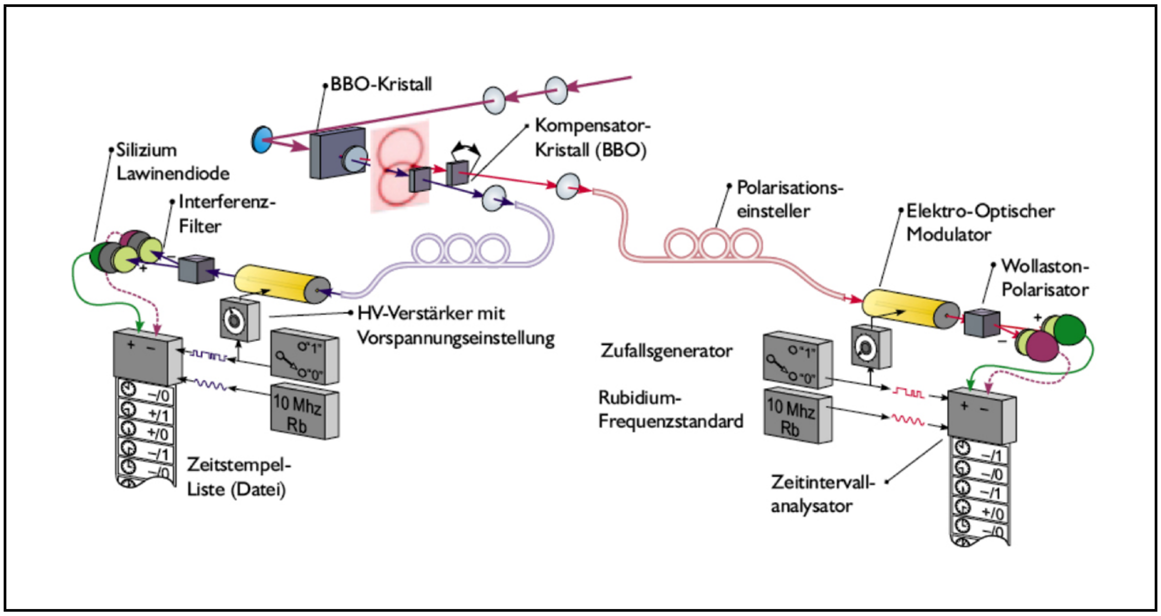

Figure 10.2: Experimental set-up of the Bell-type experiment of G. Weihs, T. Jennewein, C. Simon, H. Weinfurter and A. Zeilinger [26]: A BBO crystal is pumped by a laser and produces vertically V and horizontally H polarized light on two separate cones which overlap. In the overlap regions the photons (not "knowing" whether they are V or H) are entangled and are caught by two fibres which lead to the electro-optical modulators of the two observer stations Alice and Bob, where the data registration is carried out. The modulators rotate the polarization of the photons at random and ultrafast that no signal can travel from Alice to Bob with velocities less than the speed of light. In this way the group obtained a violation of Bell's inequality $S_{\mathrm{exp}} = 2,73 \pm 0,02$ under strict Einstein locality conditions. The picture is taken from the PhD Thesis of Gregor Weihs, see http://www.uibk.ac.at/exphys/photonik/people/weihs.html

10.4 Quantum Teleportation

Quantum teleportation, invented by Charles Bennett and coauthors in 1993 [27], is one of the most exciting issues in quantum information and quantum communication. It's a special information transfer from Alice to Bob and often called "*beaming*" in the popular literature, but actually it's not such a thing, there is *no* transfer of matter or energy involved. Alice's particle has not been physically moved to Bob, only the particle's quantum state has been transferred. Let's see how it works.

10.4.1 Qubits

The information stored in a classical bit with value 0 or 1 is encoded in a physical two level system, which definitely is in one of two possible states. It can be realized by spin $\frac{1}{2}$ particles being in either spin $\uparrow$ or $\downarrow$ state, or by photons having polarization state H or V, to name just a few examples.

Quantum mechanically, however, superpositions of such definite states, e.g. $\uparrow$ and $\downarrow$, are possible, i.e.

$$\begin{aligned} |\,\psi\,\rangle &= a\,|\uparrow\rangle + b\,|\downarrow\rangle \\ &= a\,|\,H\,\rangle + b\,|\,V\,\rangle \\ &= a\,|\,0\,\rangle + b\,|\,1\,\rangle\ , \end{aligned} \tag{10.18}$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. Such a state is called a ***qubit***. When measured, the qubit reveals 1 bit of information, either 0 or 1 with probabilities $|a|^2$ and $|b|^2$ respectively, which immediately follows when applying the corresponding projectors

$$|\,0\,\rangle\langle\,0\,|\,\psi\,\rangle \;=\; a\,|\,0\,\rangle \tag{10.19}$$

$$|\,1\,\rangle\langle\,1\,|\,\psi\,\rangle \;=\; b\,|\,1\,\rangle\ . \tag{10.20}$$

We see that a quantum mechanical system contains 1 qubit of information as long as it is not measured, which is more information than encoded in a classical bit. This fact is, for example, utilized in quantum communication or in developing a quantum computer. Another interesting application is quantum teleportation, what we want to discuss now in more detail.

10.4.2 Quantum Teleportation Set-Up

The idea of quantum teleportation is the transmission of an (even unknown) quantum state – 1 qubit – from one system at Alice's location onto another system at Bob's location. This transmission occurs instantaneously[2] by EPR correlations. In order for Bob to get any meaningful results he needs additional information from Alice, that is transmitted by a classical channel. This is also the reason why special relativity is not violated, since the classical information can not travel faster than the speed of light.

Let us now try to phrase this procedure in terms of quantum mechanics. Suppose Alice wants to teleport the state $|\,\psi\,\rangle$ of the incoming particle 1, given by

$$|\,\psi\,\rangle \,=\, a\,|\,H\,\rangle_1 + b\,|\,V\,\rangle_1 \qquad 1 \text{ qubit}\ . \tag{10.21}$$

An EPR source creates a pair of entangled particles, shared by Alice and Bob, which are in the Bell state

$$\left|\,\psi^-\,\right\rangle_{23} \,=\, \frac{1}{\sqrt{2}}\,(\,|\,H\,\rangle_2\,|\,V\,\rangle_3 \,-\, |\,V\,\rangle_2\,|\,H\,\rangle_3\,)\ . \tag{10.22}$$

Alice, on her side, performs a Bell-measurement, i.e. she entangles particles 1 and 2,

[2]Depending on the choice of interpretation of the quantum state, i.e. the wave function, one might come to different conclusions as to when the transmission of the state actually takes place. The transmission of the classical information, encoded in the qubit, however, is agreed upon by everybody, since this has to be supplemented by classical communication.

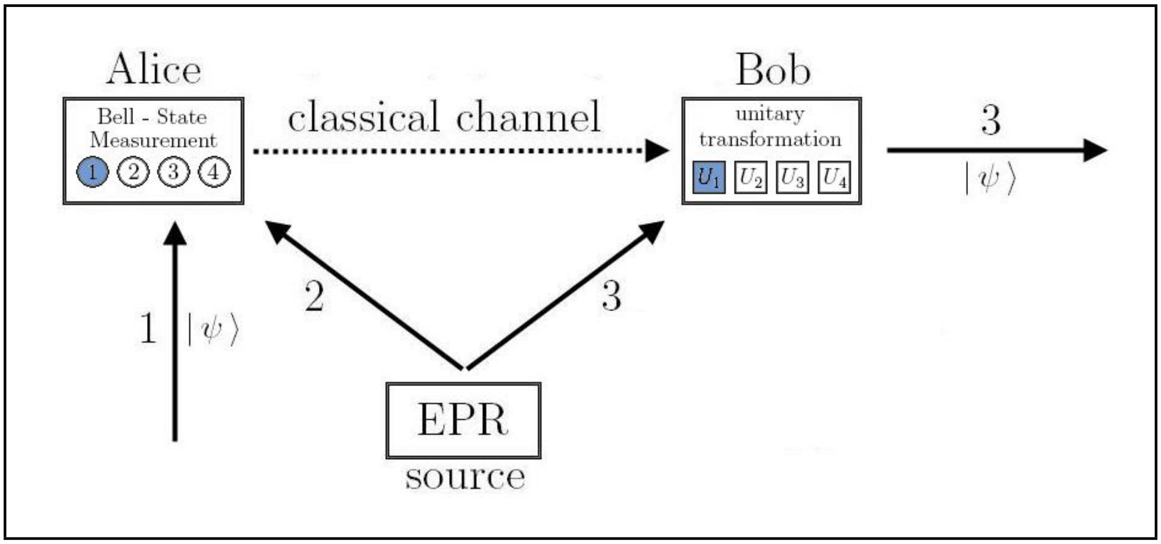

Figure 10.3: Quantum Teleportation Scheme: The state $|\psi\rangle$, carried at first by particle 1 at Alice's location is transferred to particle 3 at Bob's location via the EPR – entangled – pair of particles 2 & 3. This is done by entangling particles 1 and 2 via a Bell measurement and supplying information in addition to the EPR correlations via a classical channel.

which means that the state of particle 1 and 2 is projected onto one of the four Bell states

$$|\psi^{\pm}\rangle_{12} = \frac{1}{\sqrt{2}}\left(|H\rangle_1|V\rangle_2 \pm |V\rangle_1|H\rangle_2\right) \tag{10.23}$$

$$|\phi^{\pm}\rangle_{12} = \frac{1}{\sqrt{2}}\left(|H\rangle_1|H\rangle_2 \pm |V\rangle_1|V\rangle_2\right) . \tag{10.24}$$

Thus before Alice's measurement the state of all three particles is expressed by

$$\begin{aligned} |\psi\rangle_{123} &= \frac{a}{\sqrt{2}}\left(|H\rangle_1|H\rangle_2|V\rangle_3 - |H\rangle_1|V\rangle_2|H\rangle_3\right) \\ &+ \frac{b}{\sqrt{2}}\left(|V\rangle_1|H\rangle_2|V\rangle_3 - |V\rangle_1|V\rangle_2|H\rangle_3\right) . \end{aligned} \tag{10.25}$$

Next we utilize the fact, that the Bell states (Eqs. (10.23) - (10.24)) form a CONS for the 2-qubit states, which allows for any state of the form $|\ \rangle_1|\ \rangle_2$ to be expanded in terms of the Bell states. Rewriting the total state (10.25) in such an expansion yields

$$\begin{aligned} |\psi\rangle_{123} = \frac{1}{2}\{ \ & |\psi^-\rangle_{12} \ \left(-a\,|H\rangle_3 - b\,|V\rangle_3\right) + \\ + \ & |\psi^+\rangle_{12} \ \left(-a\,|H\rangle_3 + b\,|V\rangle_3\right) + \\ + \ & |\phi^-\rangle_{12} \ \left(+a\,|H\rangle_3 + b\,|V\rangle_3\right) + \\ + \ & |\phi^+\rangle_{12} \ \left(+a\,|H\rangle_3 - b\,|V\rangle_3\right) \} . \end{aligned} \tag{10.26}$$

Now Alice performs a Bell measurement, i.e. she projects the state of Eq. (10.26) onto one of the four Bell states, e.g. onto $|\,\psi^-\,\rangle_{12}$

$$|\,\psi^-\,\rangle_{12}\,\langle\,\psi^-\,|\,\psi\,\rangle_{123} \;=\; \frac{1}{2}\,|\,\psi^-\,\rangle_{12}\,(\,-\,a\,|\,H\,\rangle_3 \,-\, b\,|\,V\,\rangle_3\,) \;=\; -\,\frac{1}{2}\,|\,\psi^-\,\rangle_{12}\,|\,\psi\,\rangle_3\;, \quad (10.27)$$

then particle 3 on Bob's side is instantaneously in the state initially carried by particle 1, apart from some unitary transformation, i.e. the expansion from Eq. (10.26) can be rewritten as

$$\begin{aligned} |\,\psi\,\rangle_{123} \;=\; \frac{1}{2}\,\{ \quad & |\,\psi^-\,\rangle_{12} \quad U_1\,|\,\psi\,\rangle_3 \;+ \\ +\quad & |\,\psi^+\,\rangle_{12} \quad U_2\,|\,\psi\,\rangle_3 \;+ \\ +\quad & |\,\phi^-\,\rangle_{12} \quad U_3\,|\,\psi\,\rangle_3 \;+ \\ +\quad & |\,\phi^+\,\rangle_{12} \quad U_4\,|\,\psi\,\rangle_3 \;\}\;. \end{aligned} \quad (10.28)$$

However, since Bob does not know onto which of the four Bell states the total state was projected by Alice, because this happens randomly, he needs additional classical information from Alice (via mobile phone, e.g.) to decide, which of the four unitary operations U_1, U_2, U_3 or U_4 he has to perform. But once given this information, he will find particle 3 in exactly the same polarization state as particle 1 was before. The unitary transformations can be represented by the matrices

$$U_1 \;=\; -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\;,\quad U_2 \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\;,\quad U_3 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\;,\quad U_4 \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\;. \quad (10.29)$$

All these transformations can be experimentally easily realized.

Remarks:

I) The state of particle 1, to be teleported, can be an unknown quantum state which even need not be well-defined (see the phenomenon of "*entanglement swapping*").

II) The initial state of particle 1 is destroyed by the teleportation process, i.e. one can not duplicate a given quantum state, which is expressed in the so called "*no cloning theorem*".

III) The teleportation process does not underly any distance restrictions, in principle, it can occur over any distances.

IV) Experiments on quantum teleportation have been repeatedly performed with great success, first of all, by Anton Zeilinger [28] and his group in Innsbruck in 1997, and since then by several other groups. A very appealing variant was performed by Rupert Ursin [29] in Vienna, see Fig. 10.4.

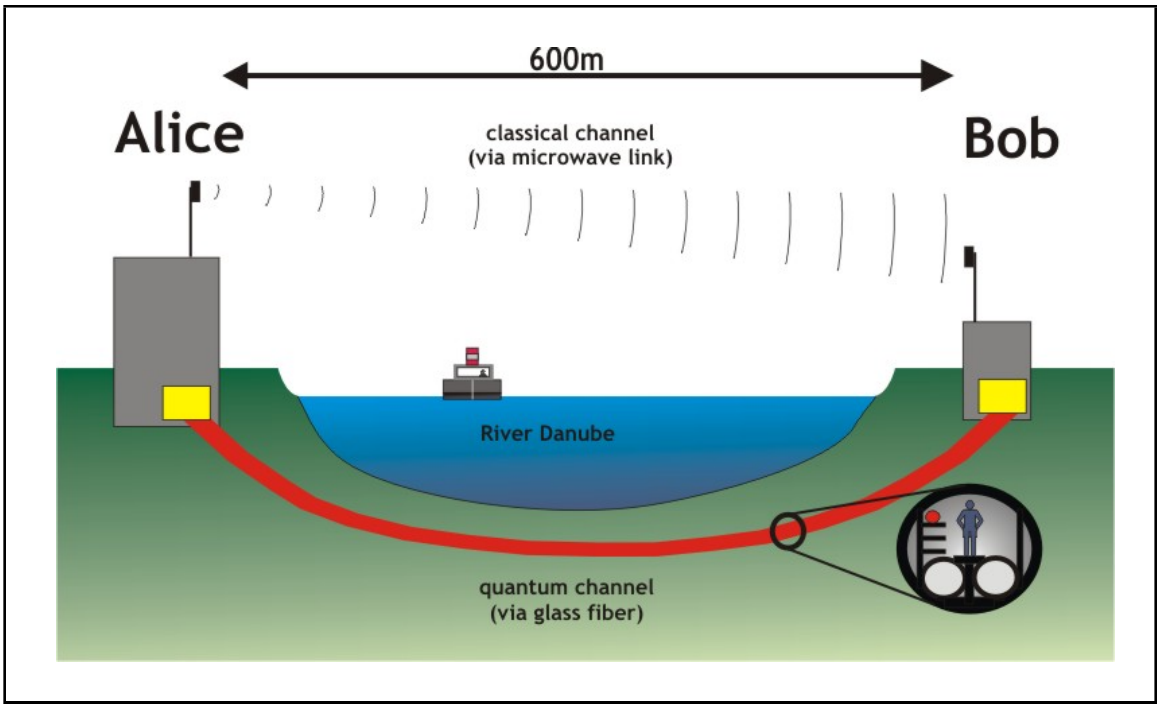

Figure 10.4: Quantum Teleportation under the Danube: The experiment was performed by R. Ursin of the Zeilinger group in 2004, where EPR pairs, created at Alice's location on one side of the Danube, were shared via a glass fiber to teleport quantum states across the river at a distance of ca. 600 m. Figure from: http://de.wikipedia.org/w/index.php?title=Bild:Experimental_overview.jpg