QUPON Tutorial Lecture 2005

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Entanglement and Decoherence in High Energy Physics

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Motivation

Composite quantum system in pure or mixed state

nonlocal — contextual features J.S. Bell

entanglement

basis for quantum communication, information and computing

Aim: understand features of entanglement

phenomenological $\longrightarrow$ conceptual $\longrightarrow$ mathematical aspects

elementary particles — massive, internal symmetries, decays

K-mesons strangeness

B-mesons beauty

Stability of quantum system

understand decoherence — entanglement loss

Detection of entanglement

criterion for separability / entanglement $\longrightarrow$ generalized Bell inequality

Part I

Bell inequality for strange mesons

Contents

- Strangeness — K-mesons
- QM of K-mesons
- Analogy: kaon — photon system
- Bell inequality for “quasi-spin” — CP violation

Strange mesons

Strange mesons selected by Nature to demonstrate fundamental principles of QM such as :

- superposition principle
- oscillation and decay property
- quasi-spin property

K-meson — bound state of quarks $(q\bar{q})$:

u ... up flavor

d ... down

s ... strange

mass: 494 - 497 MeV $J^P = 0^-$ pseudoscalar

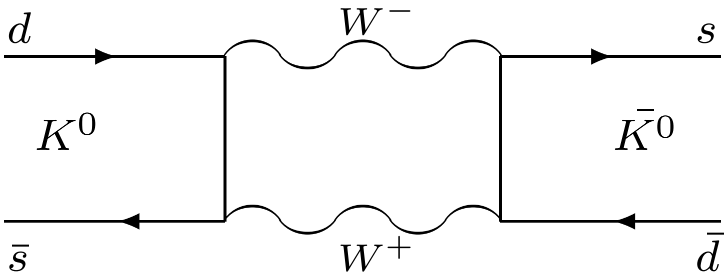
strong interactions: S conservation weak interactions: S violation

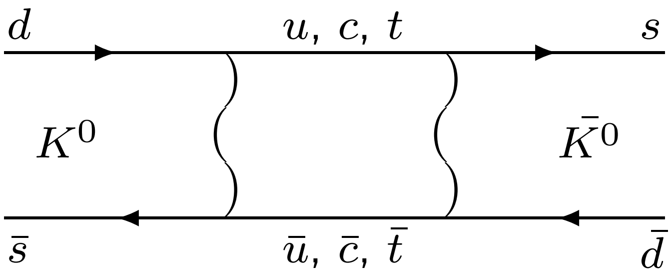
Properties of K-mesons

K-meson	quarks	S	I_3
K^+	$u\bar{s}$	$+1$	$+\frac{1}{2}$
K^-	$\bar{u}s$	-1	$-\frac{1}{2}$
K^0	$d\bar{s}$	$+1$	$-\frac{1}{2}$
$\bar{K}^0$	$\bar{d}s$	-1	$+\frac{1}{2}$

K^0 … particle $\bar{K}^0$ … antiparticle S … strangeness I … isospin

$K^0 \longleftrightarrow \bar{K}^0$ oscillation $|\Delta S| = 2$





QM of K-mesons

Strangeness eigenstates

$$S\,|K^0\rangle \;=\; +|K^0\rangle \qquad\qquad S\,|\bar{K}^0\rangle \;=\; -|\bar{K}^0\rangle$$

CP-transformation P... parity C... charge conjugation

$$CP\,|K^0\rangle \;=\; -|\bar{K}^0\rangle \qquad\qquad CP\,|\bar{K}^0\rangle \;=\; -|K^0\rangle$$

CP eigenstates

$$\begin{aligned} CP\,|K_1^0\rangle &= +|K_1^0\rangle & \qquad |K_1^0\rangle &= \tfrac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right) \\ CP\,|K_2^0\rangle &= -|K_2^0\rangle & \qquad |K_2^0\rangle &= \tfrac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right) \end{aligned}$$

strong interactions: CP conservation

weak interactions: small CP violation almost conserved

K-meson decays

K–decay $K_S \approx K_1^0 \longrightarrow 2\pi \qquad \tau_S \approx 10^{-10}$ sec short-lived

$K_L \approx K_2^0 \longrightarrow 3\pi \qquad \tau_L \approx 5 \cdot 10^{-8}$ sec long-lived

physical masses m_S, m_L with $\Delta m = m_L - m_S = 3.5 \cdot 10^{-6}$ eV small

CP-violation $K_L \longrightarrow 2\pi$ small

short-lived, long-lived states $|\varepsilon| \approx 10^{-3}$ CP-violating parameter

$$|K_S\rangle = \frac{1}{N}\left(p\,|K^0\rangle - q\,|\bar{K}^0\rangle\right) \qquad p = 1 + \varepsilon, \quad q = 1 - \varepsilon$$

$$|K_L\rangle = \frac{1}{N}\left(p\,|K^0\rangle + q\,|\bar{K}^0\rangle\right) \qquad N^2 = |p|^2 + |q|^2$$

decaying states evolve in time $\longrightarrow$ Weisskopf-Wigner approximation

$$|K_{\substack{S\\L}}(t)\rangle = e^{-i\lambda_{\substack{S\\L}} t}|K_{\substack{S\\L}}\rangle \quad \text{with} \quad \lambda_{\substack{S\\L}} = m_{\substack{S\\L}} - \frac{i}{2}\Gamma_{\substack{S\\L}} \quad \text{and} \quad \Gamma_{\substack{S\\L}} \sim \tau_{\substack{S\\L}}^{-1}$$

$\Longrightarrow$ time evolution for K^0 and $\bar{K}^0$

$K^0 \longleftrightarrow \bar{K}^0$ oscillation

K^0 beam produced at $t = 0$

probability for finding K^0 or $\bar{K}^0$ at $t > 0$

$$\left|\langle K^0|K^0(t)\rangle\right|^2 = \frac{1}{4}\left\{ e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2\, e^{-\Gamma t}\cos(\Delta m t) \right\}$$

$$\left|\langle \bar{K}^0|K^0(t)\rangle\right|^2 = \frac{1}{4}\left\{ e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2\, e^{-\Gamma t}\cos(\Delta m t) \right\}$$

with $\Delta m = m_L - m_S$ and $\Gamma = \frac{1}{2}(\Gamma_L + \Gamma_S)$ $\Gamma_{\substack{S\\L}} \sim \tau_{\substack{S\\L}}^{-1}$

K^0 beam oscillates with frequency $\Delta m / 2\pi$

oscillation visible at $t \sim \tau_S$, $\Delta m\, \tau_S = 0.47$

K_0 beam at $t = 0$ $\Longrightarrow$ $\bar{K}^0$ equal to K^0 far away from source

through K_L

“Quasi-spin” of K-mesons

	K_0	$\sim \Uparrow$	$\bar{K}^0$	$\sim \Downarrow$	2-dim Hilbertspace
strangeness	$+1$	up	-1	down	

operators in “quasi-spin” space: $S \sim \sigma_3 \quad CP \sim -\sigma_1 \quad \cancel{C}P \sim \sigma_2$

Hamiltonian $H = M - \frac{i}{2}\Gamma = \frac{1}{2}\left(a \cdot \mathbb{1} + \vec{h} \cdot \vec{\sigma}\right)$ with $|K_{\substack{S \\ L}}\rangle$ eigenstates

Analogy

K-meson	spin-$\frac{1}{2}$	photon
$\lvert K^0\rangle$	$\lvert\Uparrow\rangle_z$	$\lvert V\rangle$
$\lvert\bar{K}^0\rangle$	$\lvert\Downarrow\rangle_z$	$\lvert H\rangle$
$\lvert K_S\rangle$	$\lvert\Rightarrow\rangle_y$	$\lvert L\rangle = \frac{1}{\sqrt{2}}(\lvert V\rangle - i\lvert H\rangle)$
$\lvert K_L\rangle$	$\lvert\Leftarrow\rangle_y$	$\lvert R\rangle = \frac{1}{\sqrt{2}}(\lvert V\rangle + i\lvert H\rangle)$

Entanglement

Schrödinger 1935: Trilogy "On the present situation in QM"

"... *entangled states* ..." — "... *verschränkte Zustände* ..."

quantum system with physically distant parts

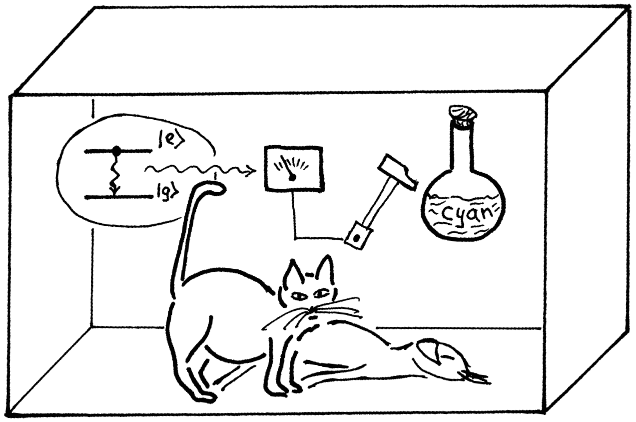
E. Schrödinger: *"The whole is in a definite state, the parts taken individually are not."*

Illustration of entanglement: Schrödinger's cat

box, decaying atom, hammer, cyanide, cat

entangled state ⟷ separable state

convex combination of product states

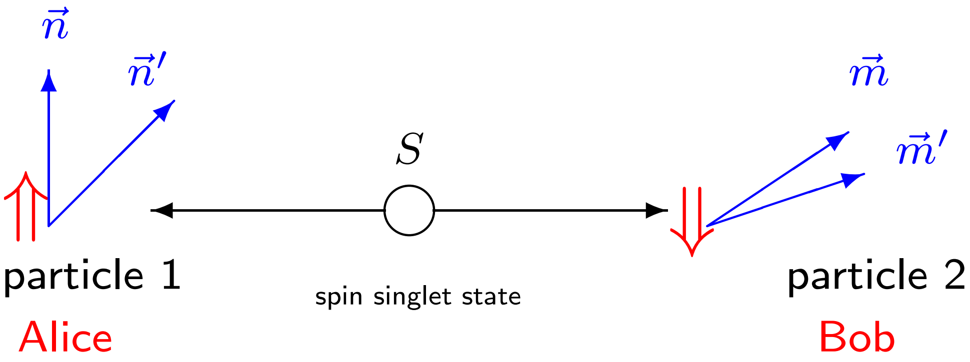


$$|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle|alive\rangle + |g\rangle|dead\rangle)$$

Bell's Theorem

Bell's Theorem 1964

J.S. Bell: *"... in a certain experimental situation all LRT (local realistic theories) are incompatible with QM."*



EPR-type experiment

Bell inequality

LRT satisfy BI — choice:

- fix quasi-spin — freedom in time
- freedom in quasi-spin — fix time

choose: fix time — vary quasi-spin of K-meson rotation in quasi-spin space

for BI we need 3 different "angles" – quasi-spins: $|K_S\rangle\,,\ |\bar{K}^0\rangle\,,\ |K_1^0\rangle$ choice

$\Longrightarrow$ Bell inequality of Wigner-type

$$P(K_S, \bar{K}^0) \ \leq \ P(K_S, K_1^0) + P(K_1^0, \bar{K}^0)$$

contains unphysical CP-even state $|K_1^0\rangle$ P... probability

But!

BI $\Longrightarrow$ inequality on physical CP-parameter — experimentally testable **!**

how does it come **?**

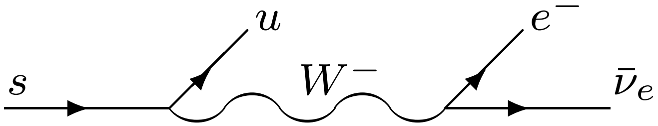
Experiment

BI $\Longrightarrow$ optimal **Inequality** for complex weights p, q of $|K_S\rangle$, $|\bar{K}^0\rangle$, $|K_1^0\rangle$

$$|\,p\,| \;\leq\; |\,q\,|$$ experimentally testable **!**

Experiment

decay of K-mesons



semileptonic decay of strange mesons

$$K^0(d\bar{s}) \longrightarrow \pi^-(d\bar{u}) \quad l^+ \quad \nu_l$$

$$\bar{K}^0(\bar{d}s) \longrightarrow \pi^+(\bar{d}u) \quad l^- \quad \bar{\nu}_l$$

quark level

$$\bar{s} \longrightarrow \bar{u} \quad l^+ \quad \nu_l$$

$$s \longrightarrow u \quad l^- \quad \bar{\nu}_l$$

$\Longrightarrow$ l^+ tags K^0 in K_L state

l^- tags $\bar{K}^0$ $l = \mu, e$

$|p|^2 \ldots$ probability for K^0 in K_L

$|q|^2 \ldots$ probability for $\bar{K}^0$ in K_L

charge asymmetry

$$\delta = \frac{\Gamma(K_L \to \pi^- l^+ \nu_l) - \Gamma(K_L \to \pi^+ l^- \bar{\nu}_l)}{\Gamma(K_L \to \pi^- l^+ \nu_l) + \Gamma(K_L \to \pi^+ l^- \bar{\nu}_l)} \quad = \quad \frac{|p|^2 - |q|^2}{|p|^2 + |q|^2}$$

Conclusion

$\xrightarrow{|p| \leq |q|}$ Bell inequality for δ

$\delta \ \leq \ 0$ Experiment: $\delta_{exp} \ = \ (3.27 \pm 0.12) \cdot 10^{-3}$

BI violated **!**

- consider 2 BI's $\delta \ \leq \ 0$ and $\delta \ \geq \ 0$ $\bar{K}^0 \longrightarrow K^0$, $p \longleftrightarrow q$

$\Longrightarrow$ $\delta \ = \ 0$ CP conservation

in contradiction to experiment **!**

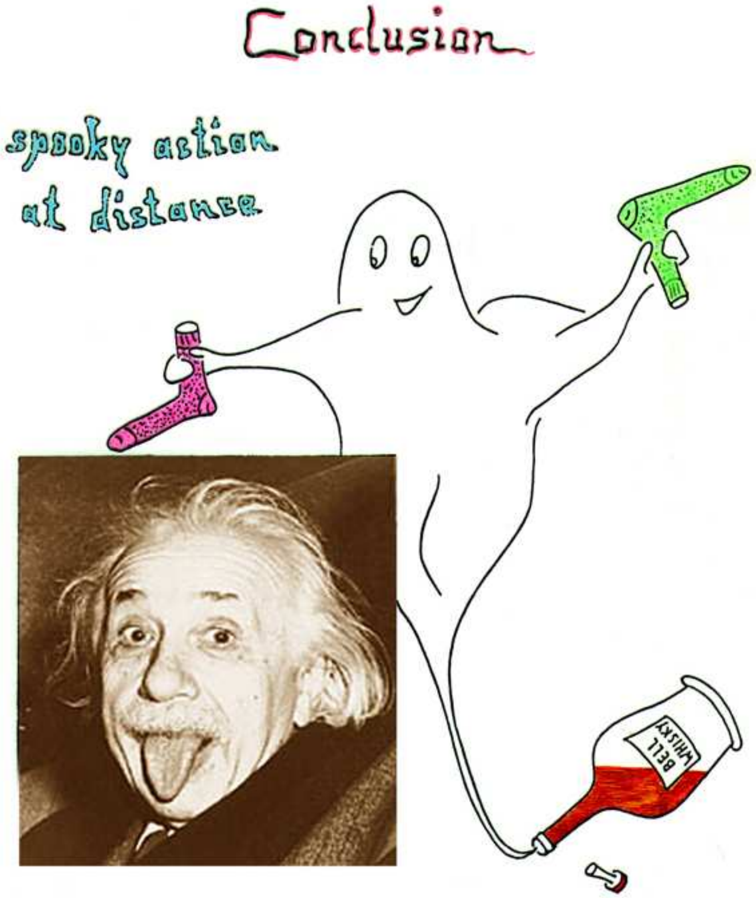
Conclusion

LRT are only compatible with strict CP conservation in $K^0\bar{K}^0$ mixing **!**

$\delta \ \neq \ 0$ $\Longleftrightarrow$ $K^0\bar{K}^0$ entanglement

CP violation nonlocal — contextual

Spooky



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Part II

Decoherence of entangled beauty

Contents

- Beauty — B-mesons
- $B^0\bar{B}^0$ production in B-factory
- Decoherence of entangled beauty
- Decoherence — entanglement loss

Beauty mesons

Neutral B-meson $m_B = 5.3$ GeV $J^P = 0^-$

B^0 bound state of quarks $(d\bar{b})$ $\bar{B}^0$ $(\bar{d}b)$

b ... beauty or bottom $+1$ -1

QM formalism analogous to K-meson $s \longrightarrow b$

- $B^0 \longleftrightarrow \bar{B}^0$ oscillation
- B-meson decay $\longrightarrow$ B_H ... heavy, B_L ... light states

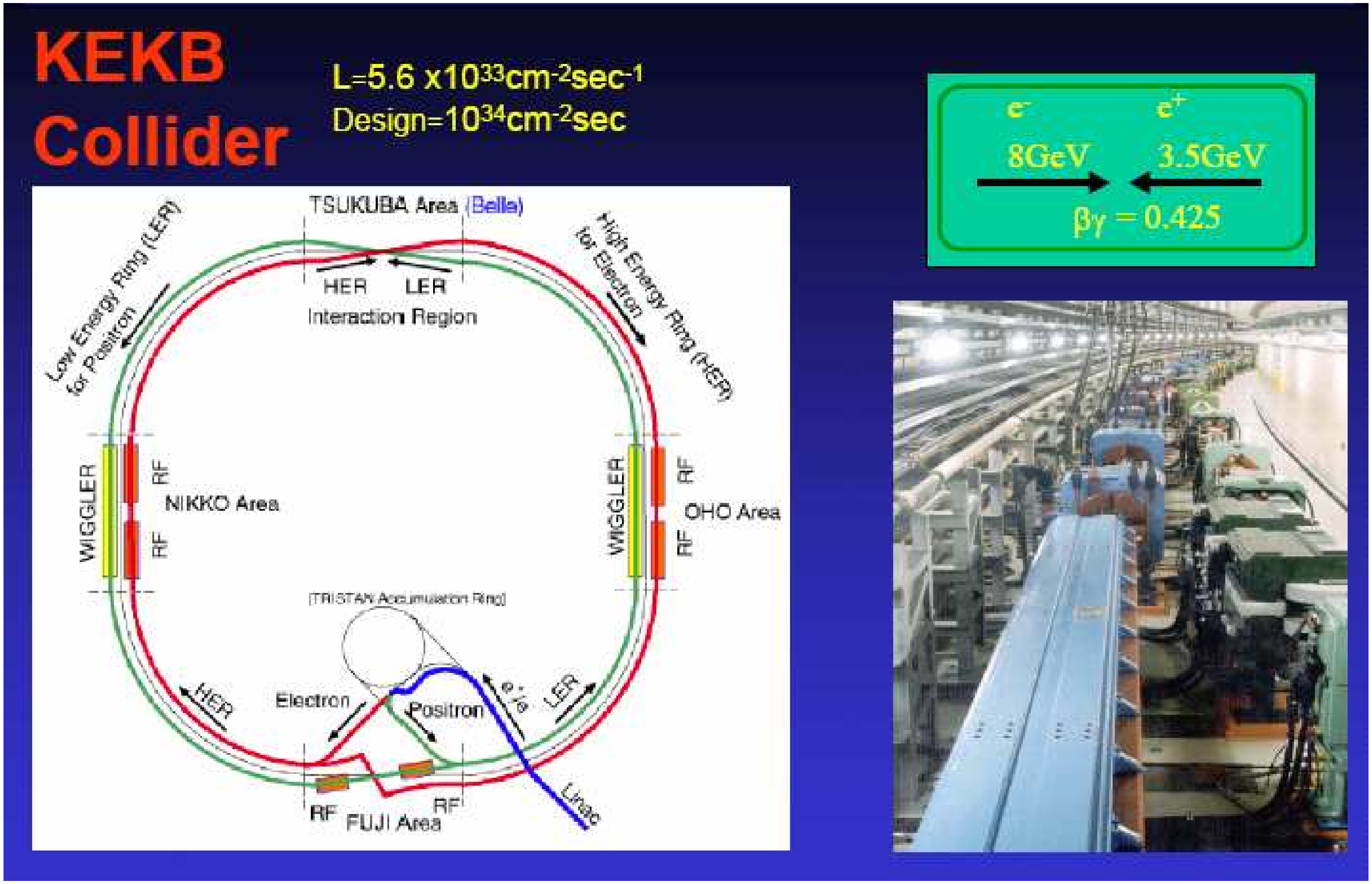
  time evolution $|B_H(t)\rangle, |B_L(t)\rangle$ or $|B_0(t)\rangle, |\bar{B}_0(t)\rangle$

  according to Weisskopf–Wigner approximation

B-meson in contrast to K-meson

- $\Delta\Gamma = \Gamma_H - \Gamma_L \approx 0$ small $\Gamma_{B^0}^{-1} = \tau_{B^0} = 1.5 \cdot 10^{-12}$ sec
- $\Delta m = m_H - m_L = 3 \cdot 10^{-4}$ eV large

KEK-B



KEKB Collider

L=5.6 x10^{33}cm^{-2}sec^{-1}

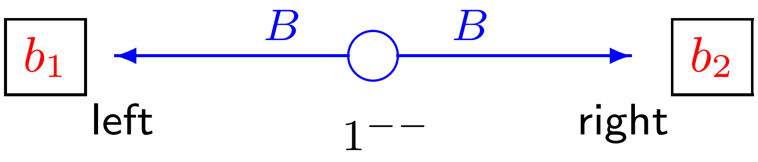
Design=10^{34}cm^{-2}sec

e^- 8GeV → ← e^+ 3.5GeV

$\beta\gamma = 0.425$

$$e^+e^- \longrightarrow \Upsilon(4S) \longrightarrow B^0\bar{B}^0$$

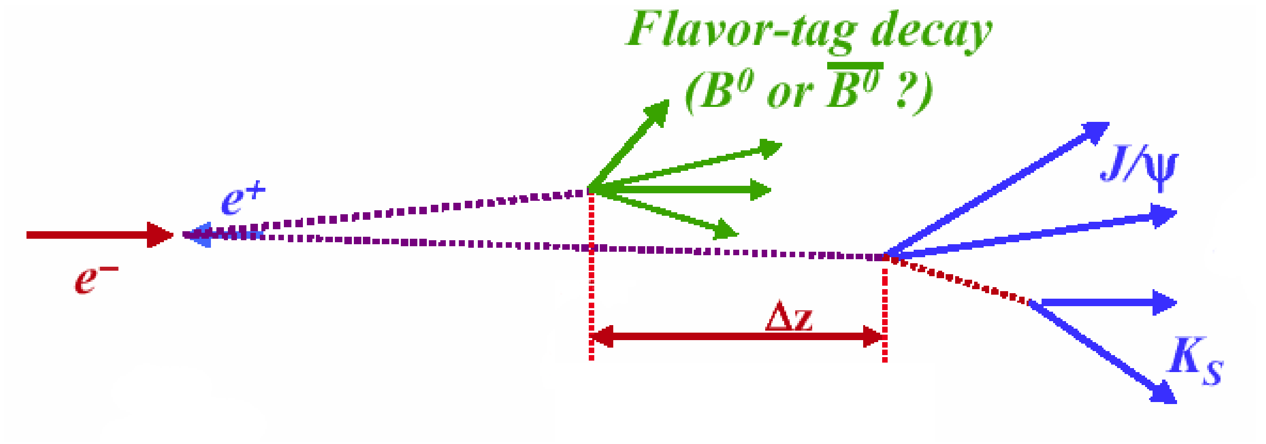
Entangled beauty



entangled beauty system propagates left ⟷ right entangled state QM

$$|\psi(t_l, t_r)\rangle \;=\; \frac{N_{HL}}{\sqrt{2}} \left\{ |B_H(t_l)\rangle_l \otimes |B_L(t_r)\rangle_r \;-\; |B_L(t_l)\rangle_l \otimes |B_H(t_r)\rangle_r \right\}$$

Experiment KEKB



Question: How to measure possible decoherence in entangled beauty **?**

Decoherence parameter

Probability to detect beauty $|b_1\rangle_l$ on left side $\longleftrightarrow$ $|b_2\rangle_r$ on right side

$$\Big|\langle b_1|\langle b_2|\,\psi(t_l,t_r)\rangle\Big|^2 \quad =$$

$$\frac{|N_{HL}|^2}{2}\Big\{ \big|\langle b_1|B_H(t_l)\rangle_l\big|^2 \big|\langle b_2|B_L(t_r)\rangle_r\big|^2 \;+\; \big|\langle b_1|B_L(t_l)\rangle_l\big|^2 \big|\langle b_2|B_H(t_r)\rangle_r\big|^2$$

$$-\;2\;\underbrace{(1-\zeta)}_{\text{modification}}\;\Re e\Big[\langle b_1|B_H(t_l)\rangle_l^*\,\langle b_2|B_L(t_r)\rangle_r^*\,\langle b_1|B_L(t_l)\rangle_l\,\langle b_2|B_H(t_r)\rangle_r\Big]\Big\}$$

decoherence parameter

$$0 \le \zeta \le 1$$

↗ ↖

pure QM total decoherence Furry–Schrödinger

spontan. factorization of wave function

Aim : determine range of ζ by experimental data

Asymmetry

How to detect ζ **?**

consider as detected particles

like-beauty events (B^0, B^0) and $(\bar{B}^0, \bar{B}^0)$

unlike-beauty events $(B^0, \bar{B}^0)$ and $(\bar{B}^0, B^0)$

Probabilities

$$\begin{aligned} P_\zeta(B^0, t_l; B^0, t_r) &= \frac{|p|^2}{|q|^2}\,\frac{1}{8}\left(P^{QM}_{\text{like}}(t_l, t_r) + 2\,\zeta\, e^{-\Gamma(t_l+t_r)} \cos\Delta m \Delta t\right) \\ P_\zeta(B^0, t_l; \bar{B}^0, t_r) &= \frac{1}{8}\left(P^{QM}_{\text{unlike}}(t_l, t_r) - 2\,\zeta\, e^{-\Gamma(t_l+t_r)} \cos\Delta m \Delta t\right) \end{aligned}$$

Asymmetry directly sensitive to interference term

$$A(t_l, t_r) = \frac{P_{\text{unlike}}(t_l, t_r) - P_{\text{like}}(t_l, t_r)}{P_{\text{unlike}}(t_l, t_r) + P_{\text{like}}(t_l, t_r)}$$

$$A_\zeta(t_l, t_r) = (1-\zeta)\, A^{\text{QM}}(t_l, t_r) \qquad \text{with} \quad A^{\text{QM}}(t_l, t_r) = \frac{\cos\Delta m\,\Delta t}{\cosh(\frac{1}{2}\Delta\Gamma\Delta t)}$$

Belle experiment at KEKB $\quad A_{exper} \Longrightarrow \zeta_{exper} \quad$ HEPHY Vienna group

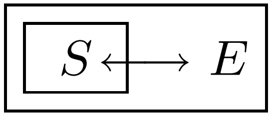
Open quantum system

Decoherence in open quantum system

system S interacts with environment E

dissipation: energy flow

decoherence: mixing of states



Hamilton for beauty meson decay, oscillation

$$\left(H \;=\; M - \frac{i}{2}\,\Gamma\right) |B_{\substack{H\\L}}\rangle \;=\; \lambda_{\substack{H\\L}} |B_{\substack{H\\L}}\rangle \quad \text{with} \quad \lambda_{\substack{H\\L}} = m_{\substack{H\\L}} - \tfrac{i}{2}\Gamma_{\substack{H\\L}}$$

composite system — 2-particle case

$$|e_1\rangle \;=\; |B_H\rangle_l \otimes |B_L\rangle_r \qquad |e_2\rangle \;=\; |B_L\rangle_l \otimes |B_H\rangle_r$$

total Hamiltonian

$$H \;=\; H_l \otimes \mathbb{1}_r \;+\; \mathbb{1}_l \otimes H_r$$

Quantum master equation

Quantum master equation

$$\frac{d\rho}{dt} = -iH\rho + i\rho H^{\dagger} - D[\rho]$$

Model for decoherence dissipator — projectors to eigenstates of H

$$D[\rho] = \lambda\left(P_i \rho P_j + P_j \rho P_i\right) \quad \text{with} \quad P_j = |e_j\rangle\langle e_j| \quad (j = 1, 2)$$

start with entangled state Bell singlet state

$$\rho^{-} = |\psi^{-}\rangle\langle\psi^{-}| \quad \text{where} \quad |\psi^{-}\rangle = \frac{1}{\sqrt{2}}\left(|e_1\rangle - |e_2\rangle\right)$$

time evolution given by master equation

⟹ time dependent density matrix

$$\rho(t) = \frac{1}{2} e^{-\Gamma t}\Big(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| - \underset{\nearrow}{e^{-\lambda t}}\left(|e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|\right)\Big)$$

decoherence ⟹ mixed state

Decoherence parameter relation

calculate probabilities: $P_\lambda(B^0 t_l, B^0 t_r)$, $P_\lambda(B^0 t_l, \bar{B}^0 t_r)$, ...

Asymmetry

$$A(t_l, t_r) \; = \; \frac{P_{\rm unlike}(t_l, t_r) \; - \; P_{\rm like}(t_l, t_r)}{P_{\rm unlike}(t_l, t_r) \; + \; P_{\rm like}(t_l, t_r)}$$

$\Longrightarrow$

$$A^\lambda(t_l, t_r) \; = \; \underset{\substack{\uparrow \\ (1-\zeta)}}{e^{-\lambda \min(t_l, t_r)}} A^{QM}(t_l, t_r) \qquad \text{with} \quad A^{\rm QM}(t_l, t_r) \; = \; \frac{\cos \Delta m \, \Delta t}{\cosh(\frac{1}{2} \Delta\Gamma \Delta t)}$$

in $\zeta-$ formalism

comparison $\zeta \longleftrightarrow \lambda$ decoherence parameters

$\Longrightarrow$ **Relation**

$$\zeta(t_l, t_r) \; = \; 1 \; - \; e^{-\lambda \min(t_l, t_r)}$$

Entanglement measure

at $t = 0$ totally entangled $B^0 \bar{B}^0$ system

for $t > 0$ decoherence $\Longrightarrow$ entanglement loss

Measure for entanglement via entropy of system

von Neumann entropy — good for pure quantum states

entanglement of formation — good for mixed quantum states

Entropy of quantum system

- measures degree of uncertainty in a state of a quantum system

von Neumann entropy

$$S\big(\rho(t)\big) \;=\; -\,\mathrm{Tr}\,\{\,\rho(t)\;\log_2 \rho(t)\,\}$$

mixed density matrix — ensemble of pure states

$$\rho \;=\; \sum_i p_i\,\rho_i \qquad \text{with} \quad \rho_i \;=\; |\psi_i\rangle\langle\psi_i| \qquad p_i \geq 0 \quad \sum_i p_i = 1$$

EOF – concurrence

Entanglement of formation for mixed states EOF Bennett et al.

$$E(\rho) \;=\; \min \sum_i p_i \, S(\rho_i^l)$$

average entanglement of pure states, minimized over decompositions

least expected entanglement of ensemble of pure states

with reduced density matrix of subsystem $\rho^{\,l}(t) \;=\; \mathrm{Tr}_r\{\,\rho(t)\}$

how to calculate EOF **?**

Theorem

Wootters & Hill

- $E(\rho) \;\equiv\; \mathcal{E}\left(C(\rho)\right) \;=\; H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2}\right)$ $0 \leq E \leq 1$

with $H(x) \;=\; -\,x \log_2 x \;-\; (1-x)\log_2(1-x)$ binary entropy function

$C \ldots$ Concurrence $0 \leq C \leq 1$ entanglement measure

Entanglement loss – decoherence

Concurrence formula

$$C\,\big(\,\rho_N(t)\,\big) \;=\; \max\,\big\{\,0, \underset{\uparrow}{\alpha_1} - \underset{\uparrow}{\alpha_2}\,\big\} \;=\; e^{-\lambda t} \qquad \text{our model}$$

eigenvalues of ρ

recall relation $\zeta \longleftrightarrow \lambda$ decoherence parameters

$$\zeta(t) \;=\; 1 - e^{-\lambda t}$$

Loss of entanglement Bertlmann-Durstberger-Hiesmayr

$$1 \,-\, C\,\big(\,\rho_N(t)\,\big) \quad = \quad \zeta(t)$$

$$1 \,-\, \underset{\uparrow}{E}\,\big(\,\rho_N(t)\,\big) \quad \doteq \quad \frac{1}{\ln 2}\,\underset{\uparrow}{\zeta(t)} \;\doteq\; \frac{\lambda}{\ln 2}\;t$$

Proposition

- loss of entanglement $=$ decoherence parameter

measuring ζ or λ $\Longrightarrow$ $1 - E$ entanglement loss quantitative **!**

Analysis : HEPHY Vienna group for BELLE experiment at KEK

Literature – decoherence

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Part III

Generalized Bell inequality
Entanglement witness

Contents

- Composite quantum system: Alice & Bob in mixed state
- Usual Bell inequalities (BI) fail as entanglement criterion
- Generalized Bell inequality (GBI) $\longleftrightarrow$ entanglement witness
- Theorem: entangled state distance $\longleftrightarrow$ violation of GBI
- Example: 2 spins $\vec{\sigma}_A$ and $\vec{\sigma}_B$ Alice & Bob
- Geometry in spin space

Separability

Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ Alice & Bob

Set of separable states defined by

$$S \;=\; \Big\{ \rho \,=\, \sum_i p_i \, \rho_A^i \otimes \rho_B^i \;\Big|\; 0 \leq p_i \leq 1 , \; \sum_i p_i = 1 \Big\}$$

Entangled state — if not separable $\omega \in S^c$ complement of S $S \cup S^c = \mathcal{H}$

Separability criterion

(2×2 and 2×3 dimensions)

1. Theorem: positive partial transposition Peres, Horodecki

$$(\mathbb{1}_A \otimes T_B)\, \rho \;\geq\; 0 \qquad \Longleftrightarrow \qquad \rho \;\text{separable} \qquad T_B(\sigma_B^i)_{kl} = (\sigma_B^i)_{lk}$$

2. Theorem: reduction Horodecki

$$\mathbb{1}_A \otimes \rho_B - \rho \;\geq\; 0 \qquad \Longleftrightarrow \qquad \rho \;\text{separable}$$

↑

reduced density matrix $\rho_B \;=\; \mathrm{tr}_A \rho$

Usual BI's

consider usual Bell inequalities

CHSH inequality $A_{\text{CHSH}} \ldots$ CHSH operator Clauser-Horne-Shimony-Holt

$$(\rho \mid A_{\text{CHSH}}) \;\leq\; 2 \quad \text{with} \quad A_{\text{CHSH}} \;=\; \vec{n}\cdot\vec{\sigma}_A \otimes (\vec{m} - \vec{m}')\cdot\vec{\sigma}_B + \vec{n}'\cdot\vec{\sigma}_A \otimes (\vec{m} + \vec{m}')\cdot\vec{\sigma}_B$$

$\uparrow$

$\rho_{\text{local}} \supset \rho_{\text{separ}}$

QM

$$(\rho^- \mid A_{\text{CHSH}}) \;=\; 2\sqrt{2} \qquad \rho^- = |\psi^-\rangle\langle\psi^-|$$

Bell singlet

$$\text{with} \quad |\psi^-\rangle \;=\; \frac{1}{\sqrt{2}}\left(\,|\Uparrow\rangle_A \otimes |\Downarrow\rangle_B \;-\; |\Downarrow\rangle_A \otimes |\Downarrow\rangle_B\right)$$

Werner states

shift CHSH-operator

$$(\rho \,|\, 2 \cdot \mathbb{1} - A_{\mathrm{CHSH}}) \;\geq\; 0 \qquad \text{and} \qquad (\rho^{-} \,|\, 2 \cdot \mathbb{1} - A_{\mathrm{CHSH}}) \;<\; 0$$

↑

inequality valid for all entangled states **?**

YES for pure entangled states

NO for mixed entangled states

Example Werner state

$$\rho_W \;=\; p\,\rho^{-} \;+\; (1-p)\,\frac{1}{4}\,\mathbb{1} \qquad \text{for} \qquad \frac{1}{3} < p \leq \frac{1}{\sqrt{2}}$$

satisfies CHSH Bell inequality **!**

usual BI's not good to find entangled states **!**

GBI – entanglement witness

Generalized Bell inequality GBI detects entanglement of state

- $\exists$ observable A such that

$$(\rho \mid A) \geq 0 \quad \forall \rho \in S \qquad (\omega \mid A) < 0 \quad \text{for some} \quad \omega \in S^c$$

given entangled state ω $\exists$ (always) some operator A satisfying GBI

$A \in \mathcal{A}_w$ entanglement witness

$A \in \mathcal{A}_t$ tangent functional

if $\exists\, \rho_0 \in S$ such that $(\rho_0 \mid A) = 0$ then $\rho_0 \in \partial S$

$\mathcal{A}_t$ characterizes convex set S of separable states

entanglement measure — distance $D(\omega)$

$$E(\omega) \longrightarrow D(\omega) = \min_{\rho \in S} \|\rho - \omega\|_2 \leq \sqrt{2}$$

distance $D(\omega)$ of entangled state $\omega \in S^c$ to set S of separable states

GBI – theorem

consider GBI

$$\xrightarrow[\text{GBI}]{} \qquad (\rho \mid A) \; - \; (\omega \mid A) \; \geq \; 0 \qquad\qquad \forall\, \rho \in S$$

maximal violation of GBI

$$B(\omega) \; = \max_{\|A-\alpha\|_2 \leq 1} \left[\, \min_{\rho \in S}(\rho \mid A) \; - \; (\omega \mid A)\,\right]$$

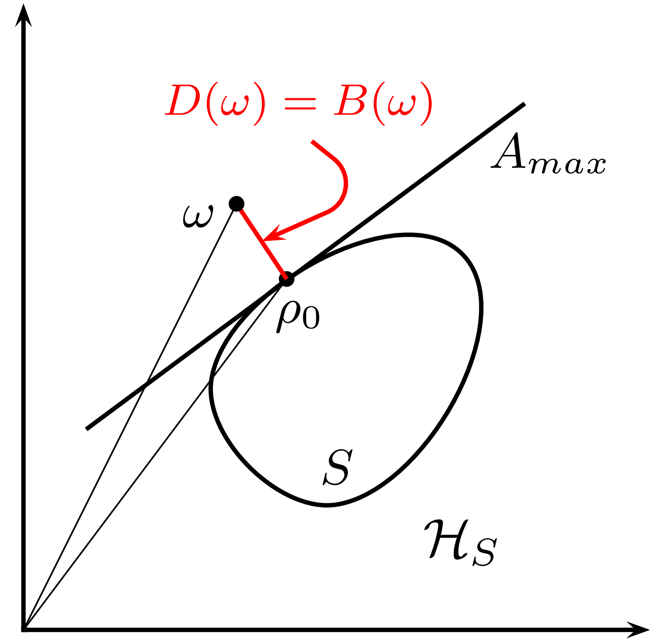
Theorem

Bertlmann-Narnhofer-Thirring

- $B(\omega) \; = \; D(\omega) \qquad\qquad \forall\, \omega \in S^c$
- min of $D \longrightarrow \rho_0$ max of $B \longrightarrow A_{max} \in \mathcal{A}_t$

$$A_{max} \; = \; \frac{\rho_0 - \omega \; - \; (\rho_0 \mid \rho_0 - \omega)\, \mathbb{1}}{\|\rho_0 - \omega\|_2}$$

Illustration of GBI – theorem



maximal violation of GBI

$B(\omega)$ is equal to distance $D(\omega)$

$\omega \longleftrightarrow$ set of separable states S

opt. oper. A_{max} is tangent to set S

Example Alice & Bob

Two spins $\vec{\sigma}_A$ and $\vec{\sigma}_B$ Alice & Bob

consider quantum states

$$\omega_\alpha = \frac{1}{4}\left(\mathbb{1} - \alpha\, \vec{\sigma}_A \otimes \vec{\sigma}_B\right) \qquad -\frac{1}{3} \leq \alpha \leq 1$$

possible range

separable state with minimum distance

mixed product states

$$\rho_0 = \frac{1}{4}\left(\mathbb{1} - \frac{1}{3}\, \vec{\sigma}_A \otimes \vec{\sigma}_B\right) \in S$$

difference

$$\rho_0 - \omega_\alpha = \frac{1}{4}\left(\alpha - \frac{1}{3}\right) \vec{\sigma}_A \otimes \vec{\sigma}_B \qquad \text{for} \quad -\frac{1}{3} \leq \alpha \leq 1$$

Entanglement witness

Distance

of ω to set S — measure of entanglement

$$D(\omega_\alpha) \;=\; \|\rho_0 - \omega_\alpha\|_2 \;=\; \frac{\sqrt{3}}{2}\,\big(\alpha - \frac{1}{3}\big) \qquad\qquad \|\vec{\sigma}_A \otimes \vec{\sigma}_B\|_2 \;=\; 2\sqrt{3}$$

Entanglement witness

$$A_{max} \;=\; \frac{\rho_0 - \omega_\alpha \;-\; (\rho_0 \mid \rho_0 - \omega_\alpha)\,\mathbb{1}}{\|\rho_0 - \omega_\alpha\|_2} \;=\; \frac{1}{2\sqrt{3}}\,(\mathbb{1} + \vec{\sigma}_A \otimes \vec{\sigma}_B)$$

Tangent functional

$$(\rho_0 \mid A_{max}) \;=\; \mathrm{tr}\left\{\frac{1}{4}\big(\mathbb{1} - \frac{1}{3}\,\vec{\sigma}_A \otimes \vec{\sigma}_B\big)\,\frac{1}{2\sqrt{3}}\,(\mathbb{1} + \vec{\sigma}_A \otimes \vec{\sigma}_B)\right\} \;=\; 0$$

GBI Alice & Bob

Separable state general

$$\rho \;=\; \frac{1}{4}\left(\mathbb{1} \;+\; \vec{n}\cdot\vec{\sigma}_A \otimes \mathbb{1}_B \;+\; \mathbb{1}_A \otimes \vec{m}\cdot\vec{\sigma}_B \;+\; \vec{n}\cdot\vec{\sigma}_A \otimes \vec{m}\cdot\vec{\sigma}_B\right)$$

GBI

$$(\rho \mid A_{max}) \;=\; \frac{1}{2\sqrt{3}}\,(1+\cos\delta) \;\geq\; 0 \qquad\qquad \vec{n}\cdot\vec{m} = \cos\delta$$

$$\longrightarrow \begin{cases} \frac{1}{\sqrt{3}} & \delta = 0 \\ 0 & \delta = 180^\circ \qquad \rho_0 \end{cases}$$

$$(\omega_\alpha \mid A_{max}) \;= \mathrm{tr}\left\{\,\frac{1}{4}\left(\mathbb{1} - \alpha\,\vec{\sigma}_A \otimes \vec{\sigma}_B\right)\,\frac{1}{2\sqrt{3}}\,\left(\mathbb{1} + \vec{\sigma}_A \otimes \vec{\sigma}_B\right)\right\}$$

$$= \frac{1}{2\sqrt{3}}\,(1-3\alpha) \;<\; 0 \qquad \text{for} \quad \frac{1}{3} < \alpha \leq 1$$

$$= \;-\;\frac{1}{\sqrt{3}} \qquad \text{for} \quad \alpha = 1 \qquad \omega_{\alpha=1} \;=\; \rho^{-}$$

Bell singlet

GBI maximal violation

maximal violation of GBI

$$B(\omega_\alpha) \;=\; \max_{\|A-\alpha\|_2 \leq 1} \left[\, \underset{\substack{\downarrow \\ (\rho_0 \,|\, A_{max}) = 0}}{\min_{\rho\,\in S}(\rho \mid A)} \;-\; (\omega_\alpha \mid \underset{\substack{\downarrow \\ A_{max}}}{A})\,\right] \;=\; (\omega_\alpha \mid -A_{max})$$

$$= \frac{\sqrt{3}}{2}\,(\alpha - \frac{1}{3}) \quad \equiv \quad D(\omega_\alpha)$$

distance in $\mathcal{H}$

nature of states

$-\frac{1}{3} \leq \alpha \leq \frac{1}{3}$ $\omega_\alpha \in S$ separable

$\frac{1}{3} < \alpha \leq 1$ mixed entangled Werner states

$\alpha = 1$ pure & maximally entangled Bell singlet ρ^-

Geometry of entangled states

4 Bell states

$$\begin{array}{llll}
\psi^- & \rho^- = \frac{1}{4}\left(\mathbb{1} - \sigma_A^x \otimes \sigma_B^x - \sigma_A^y \otimes \sigma_B^y - \sigma_A^z \otimes \sigma_B^z\right) & & P_0 \\
\phi^- & \omega^- = \frac{1}{4}\left(\mathbb{1} - \sigma_A^x \otimes \sigma_B^x + \sigma_A^y \otimes \sigma_B^y + \sigma_A^z \otimes \sigma_B^z\right) & & P_1 \\
\phi^+ & \omega^+ = \frac{1}{4}\left(\mathbb{1} + \sigma_A^x \otimes \sigma_B^x - \sigma_A^y \otimes \sigma_B^y + \sigma_A^z \otimes \sigma_B^z\right) & & P_2 \\
\psi^+ & \rho^+ = \frac{1}{4}\left(\mathbb{1} + \sigma_A^x \otimes \sigma_B^x + \sigma_A^y \otimes \sigma_B^y - \sigma_A^z \otimes \sigma_B^z\right) & & P_3
\end{array}$$

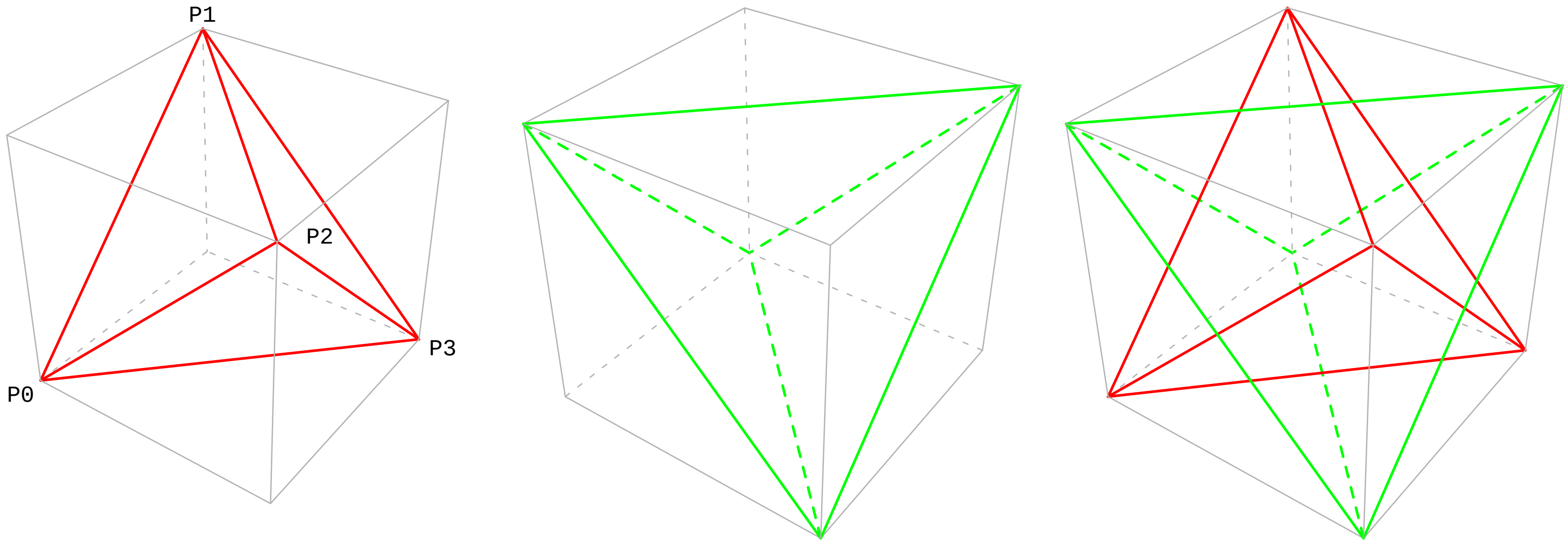
Density matrix in (c_1, c_2, c_3) parameter space

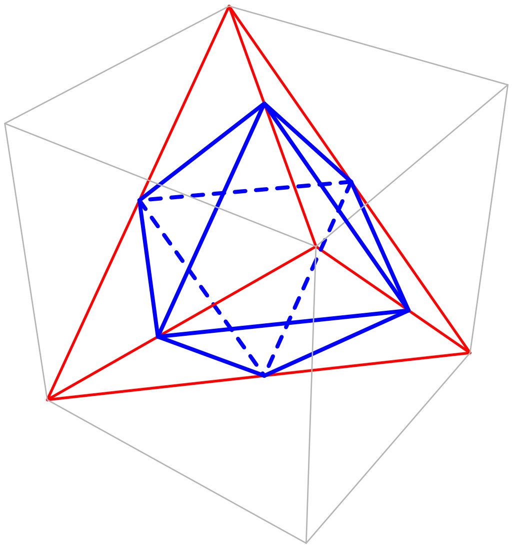
$$\omega_c = \frac{1}{4}\left(\mathbb{1} + \sum_{i=1}^{3} c_i \, \sigma_A^i \otimes \sigma_B^i\right)$$

positivity $\Longrightarrow$ tetrahedron (P_0, P_1, P_2, P_3)

partial transposition $\longrightarrow$ positivity $\longrightarrow$ separable   Thm Peres–Horodecki

Illustration of Geometry





intersection of tetrahedrons

$\Longrightarrow$ separable states

double-pyramide

(convex set)

Summary

- GBI criterion for entanglement

  entanglement measure — distance in $\mathcal{H}_S$

  $D(\omega) \; = \; B(\omega)$

  maximal violation of GBI

- usual BI criterion for nonlocality — contextuality

  basis for quantum communication and information

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