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Reinhold A. Bertlmann

Generalized Bell Inequality and Entanglement Witness

Institute for Theoretical Physics
University of Vienna

Motivation

Composite quantum system: bipartite – multipartite in mixed state

Quantum correlations**?** not reproducible by classical means

nonlocal – contextual features J.S.Bell

Entanglement

easy for pure states $\longrightarrow$ Bell Inequality $\longrightarrow$ entanglement criterion

difficult for mixed states $\longrightarrow$ entanglement criterion

separable states — entangled states

classical correlations — quantum correlations

given a composite mixed system

quantum state ρ separable or entangled **?**

$\longrightarrow$ Generalized Bell Inequality

Contents

- Composite quantum system: Alice & Bob in mixed state
- Usual Bell inequalities (BI) fail as entanglement criterion
- Generalized Bell inequality (GBI) $\longleftrightarrow$ entanglement witness
- Theorem: entangled state distance $\longleftrightarrow$ violation of GBI
- Example: 2 spins $\vec{\sigma}_A$ and $\vec{\sigma}_B$ Alice & Bob
- Geometry in spin space

States – observables

consider tensor product of Hilbert spaces

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad \text{Alice and Bob}$$

Observable: A hermitian operator, matrix

State: ρ density matrix

ρ, A elements of $\mathcal{H}$

scalar product — expectation value of A

$$(\rho \mid A) = \mathrm{tr}(\rho A) \quad \longrightarrow \quad \langle A \rangle_\psi$$

norm $\qquad \|A\|_2 = \left(\mathrm{tr} A^2\right)^{\frac{1}{2}}$

Separability

Set of separable states defined by

$$S = \Big\{ \rho = \sum_i p_i \, \rho_A^i \otimes \rho_B^i \;\Big|\; 0 \leq p_i \leq 1, \; \sum_i p_i = 1 \Big\}$$

Entangled state — if not separable $\omega \in S^c$ complement of S $\quad S \cup S^c = \mathcal{H}$

Separability criterion $\qquad$ (2×2 and 2×3 dimensions)

1. Theorem: positive partial transposition $\qquad$ Peres, Horodecki

$$(\mathbb{1}_A \otimes T_B)\, \rho \geq 0 \quad \Longleftrightarrow \quad \rho \text{ separable}$$

2. Theorem: reduction $\qquad$ Horodecki

$$\mathbb{1}_A \otimes \rho_B - \rho \geq 0 \quad \Longleftrightarrow \quad \rho \text{ separable}$$

↑

reduced density matrix $\rho_B = \mathrm{tr}_\mathrm{A}\, \rho$

Usual BI's

consider usual Bell inequalities

CHSH inequality $\qquad A_{\text{CHSH}} \ldots$ CHSH operator

$(\rho \mid A_{\text{CHSH}}) \leq 2$ with $A_{\text{CHSH}} = \vec{n}\cdot\vec{\sigma}_A \otimes (\vec{m} - \vec{m}')\cdot\vec{\sigma}_B + \vec{n}'\cdot\vec{\sigma}_A \otimes (\vec{m} + \vec{m}')\cdot\vec{\sigma}_B$

$\uparrow$

$\rho_{\text{local}} \supset \rho_{\text{separ}}$

QM

$(\rho^- \mid A_{\text{CHSH}}) = 2\sqrt{2} \qquad \rho^- = |\psi^-\rangle\langle\psi^-|$ Bell singlet

with $|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\Uparrow\rangle_A \otimes |\Downarrow\rangle_B - |\Downarrow\rangle_A \otimes |\Downarrow\rangle_B\right)$

Werner states

shift CHSH-operator

$$(\rho \,|\, 2\,\mathbb{1} - A_{\text{CHSH}}) \geq 0 \qquad \text{and} \qquad (\rho^- \,|\, 2\,\mathbb{1} - A_{\text{CHSH}}) < 0$$

↑

inequality valid for all entangled states **?**

YES for pure entangled states

NO for mixed entangled states

Example: Werner state

$$\rho_W = p\,\rho^- + (1-p)\frac{1}{4}\mathbb{1} \qquad \text{for} \qquad \frac{1}{3} < p \leq \frac{1}{\sqrt{2}}$$

satisfies CHSH Bell inequality **!**

usual BI's not good to find entangled states

GBI – entanglement witness

Generalized Bell inequality   GBI   detects entanglement of state

- $\exists$ observable A such that

$$(\rho \mid A) \;\geq\; 0 \quad \forall\, \rho \in S \qquad\qquad (\omega \mid A) \;<\; 0 \quad \text{for some} \quad \omega \in S^c$$

given entangled state ω $\exists$ (always) some operator A satisfying GBI

$A \in \mathcal{A}_w$ entanglement witness

$A \in \mathcal{A}_t$ tangent functional

if $\exists\, \rho_0 \in S$ such that $(\rho_0 \mid A) = 0$ then $\rho_0 \in \partial S$

$\mathcal{A}_t$ characterizes convex set S of separable states

Entanglement measure — defined by distance $D(\omega)$

$$E(\omega) \longrightarrow D(\omega) \;=\; \min_{\rho \in S} \|\rho - \omega\|_2 \;\leq\; \sqrt{2}$$

distance $D(\omega)$ of entangled state $\omega \in S^c$ to set S of separable states

GBI – theorem

consider GBI

$$\xrightarrow[\text{GBI}]{} \qquad (\rho \mid A) \; - \; (\omega \mid A) \; \geq \; 0 \qquad \forall\, \rho \in S$$

maximal violation of GBI

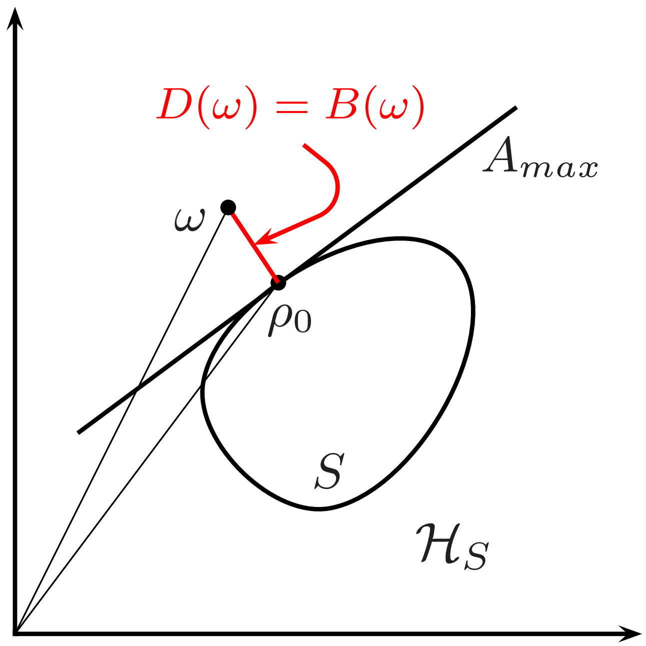
$$B(\omega) \; = \; \max_{\|A-\alpha\|_2 \leq 1} \left[\, \min_{\rho \in S}(\rho \mid A) \; - \; (\omega \mid A) \,\right]$$

Theorem Bertlmann-Narnhofer-Thirring

- $B(\omega) \; = \; D(\omega) \qquad \forall\, \omega \in S^c$
- min of $D \longrightarrow \rho_0$ max of $B \longrightarrow A_{max} \in \mathcal{A}_t$

$$A_{max} \; = \; \frac{\rho_0 - \omega \; - \; (\rho_0 \mid \rho_0 - \omega)\, \mathbb{1}}{\|\rho_0 - \omega\|_2}$$

Illustration of GBI – theorem



maximal violation of GBI

$B(\omega)$ is equal to distance $D(\omega)$

$\omega \longleftrightarrow$ set of separable states S

opt. oper. A_{max} is tangent to set S

Example Alice & Bob

2 spins $\vec{\sigma}_A$ and $\vec{\sigma}_B$ Alice & Bob

consider quantum states

$$\omega_\alpha = \frac{1}{4}\big(\mathbb{1} - \alpha\,\vec{\sigma}_A \otimes \vec{\sigma}_B\big) \qquad -\frac{1}{3} \leq \alpha \leq 1$$

possible range

separable state with minimum distance mixed product states

$$\rho_0 = \frac{1}{4}\big(\mathbb{1} - \frac{1}{3}\,\vec{\sigma}_A \otimes \vec{\sigma}_B\big) \in S$$

difference

$$\rho_0 - \omega_\alpha \;=\; \frac{1}{4}\big(\alpha - \frac{1}{3}\big)\,\vec{\sigma}_A \otimes \vec{\sigma}_B \qquad \text{for} \quad -\frac{1}{3} \leq \alpha \leq 1$$

Entanglement witness

Distance of ω to set S — measure of entanglement

$$D(\omega_\alpha) \;=\; \|\rho_0 - \omega_\alpha\|_2 \;=\; \frac{\sqrt{3}}{2}\big(\alpha - \frac{1}{3}\big) \qquad\qquad \|\vec{\sigma}_A \otimes \vec{\sigma}_B\|_2 = 2\sqrt{3}$$

Entanglement witness

$$A_{max} = \frac{\rho_0 - \omega_\alpha - (\rho_0 \mid \rho_0 - \omega_\alpha)\mathbb{1}}{\|\rho_0 - \omega_\alpha\|_2} = \frac{1}{2\sqrt{3}}(\mathbb{1} + \vec{\sigma}_A \otimes \vec{\sigma}_B)$$

Tangent functional

$$(\rho_0 \mid A_{max}) = \mathrm{tr}\Big\{\frac{1}{4}\big(\mathbb{1} - \frac{1}{3}\,\vec{\sigma}_A \otimes \vec{\sigma}_B\big)\,\frac{1}{2\sqrt{3}}(\mathbb{1} + \vec{\sigma}_A \otimes \vec{\sigma}_B)\Big\} = 0$$

GBI Alice & Bob

Separable state general

$$\rho = \frac{1}{4}\big(\mathbb{1} + \vec{n}\cdot\vec{\sigma}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \vec{m}\cdot\vec{\sigma}_B + \vec{n}\cdot\vec{\sigma}_A \otimes \vec{m}\cdot\vec{\sigma}_B\big)$$

GBI

$$(\rho \mid A_{max}) \; = \frac{1}{2\sqrt{3}}\,(1+\cos\delta) \;\geq\; 0 \qquad\qquad \vec{n}\cdot\vec{m} = \cos\delta$$

$$\longrightarrow \begin{cases} \frac{1}{\sqrt{3}} & \delta = 0 \\ 0 & \delta = 180^\circ \qquad \rho_0 \end{cases}$$

$$(\omega_\alpha \mid A_{max}) \; = \mathrm{tr}\Big\{\frac{1}{4}\big(\mathbb{1} - \alpha\,\vec{\sigma}_A \otimes \vec{\sigma}_B\big)\frac{1}{2\sqrt{3}}\big(\mathbb{1} + \vec{\sigma}_A \otimes \vec{\sigma}_B\big)\Big\}$$

$$= \frac{1}{2\sqrt{3}}(1-3\alpha) \;<\; 0 \qquad \text{for} \quad \frac{1}{3} < \alpha \leq 1$$

$$= -\frac{1}{\sqrt{3}} \qquad \text{for} \quad \alpha = 1 \qquad \omega_{\alpha=1} = \rho^- \qquad \text{Bell singlet}$$

GBI maximal violation

maximal violation of GBI

$$B(\omega_\alpha) = \max_{\|A-\alpha\|_2 \leq 1} \left[\min_{\rho \in S} (\rho \mid A) - (\omega_\alpha \mid A) \right] = (\omega_\alpha \mid -A_{max})$$

$$\downarrow \qquad\qquad \downarrow$$

$$(\rho_0 \mid A_{max}) = 0 \qquad A_{max}$$

$$= \frac{\sqrt{3}}{2} (\alpha - \frac{1}{3}) \equiv D(\omega_\alpha)$$

distance in $\mathcal{H}$

nature of states

$-\frac{1}{3} \leq \alpha \leq \frac{1}{3}$ $\quad$ $\omega_\alpha \in S$ $\quad$ separable

$\frac{1}{3} < \alpha \leq 1$ $\quad$ mixed entangled $\quad$ Werner states

$\alpha = 1$ $\quad$ pure & maximally entangled $\quad$ Bell singlet ρ^-

Geometry of pure entangled states

4 Bell states

$$\psi^- \quad \rho^- = \frac{1}{4}\left(\mathbb{1} - \sigma_A^x \otimes \sigma_B^x - \sigma_A^y \otimes \sigma_B^y - \sigma_A^z \otimes \sigma_B^z\right) \quad P_0$$

$$\phi^- \quad \omega^- = \frac{1}{4}\left(\mathbb{1} - \sigma_A^x \otimes \sigma_B^x + \sigma_A^y \otimes \sigma_B^y + \sigma_A^z \otimes \sigma_B^z\right) \quad P_1$$

$$\phi^+ \quad \omega^+ = \frac{1}{4}\left(\mathbb{1} + \sigma_A^x \otimes \sigma_B^x - \sigma_A^y \otimes \sigma_B^y + \sigma_A^z \otimes \sigma_B^z\right) \quad P_2$$

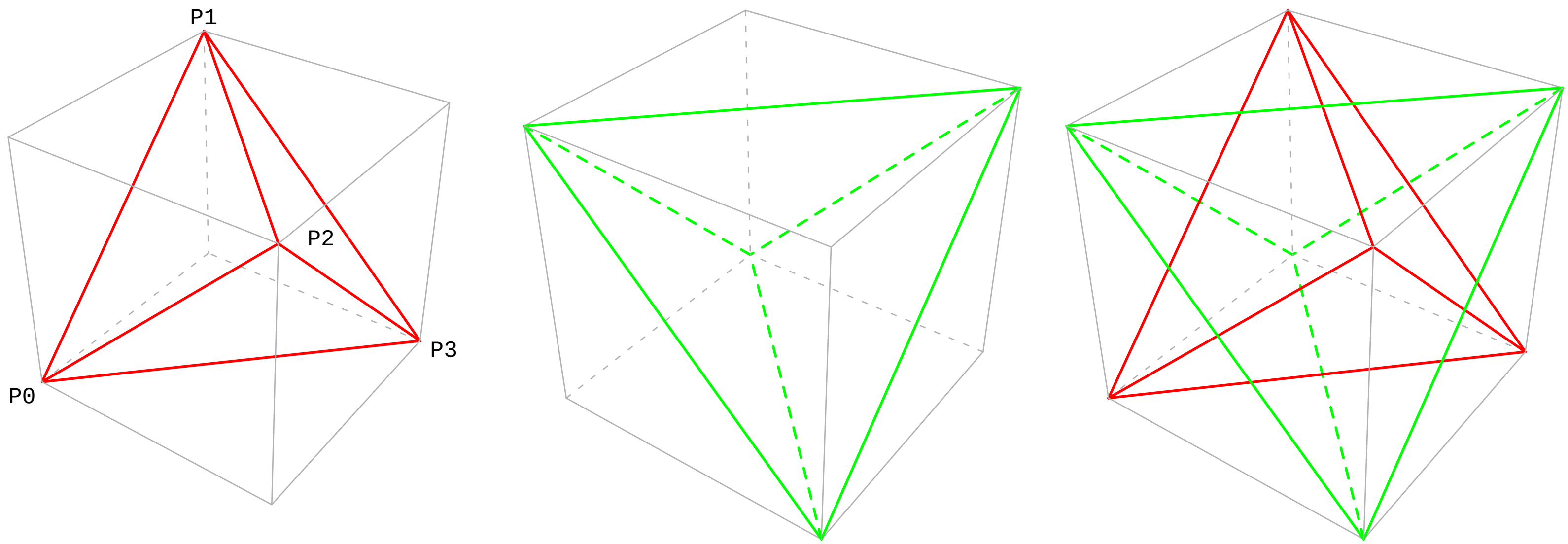
$$\psi^+ \quad \rho^+ = \frac{1}{4}\left(\mathbb{1} + \sigma_A^x \otimes \sigma_B^x + \sigma_A^y \otimes \sigma_B^y - \sigma_A^z \otimes \sigma_B^z\right) \quad P_3$$

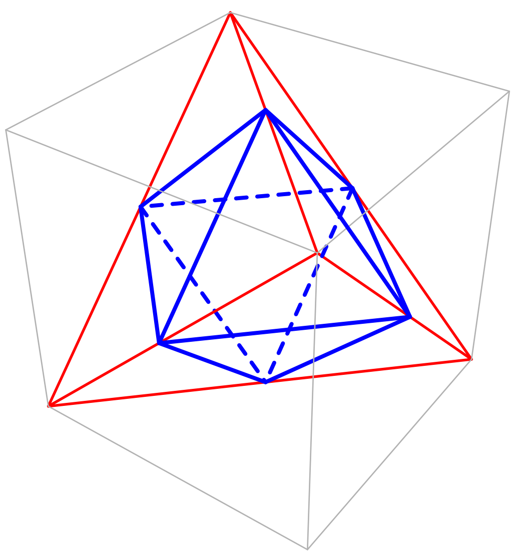
density matrix in (c_1, c_2, c_3) parameter space

$$\omega_c = \frac{1}{4}\left(\mathbb{1} + \sum_{i=1}^{3} c_i \, \sigma_A^i \otimes \sigma_B^i\right)$$

positivity $\Longrightarrow$ tetrahedron (P_0, P_1, P_2, P_3)

partial transposition $\longrightarrow$ positivity $\longrightarrow$ separable Thm Peres–Horodecki





intersection of tetrahedrons

⟹ separable states

double-pyramide

(convex set)

Summary

- GBI criterion for entanglement
  entanglement measure – distance in $\mathcal{H}_S$
  $D(\omega) = B(\omega)$
  maximal violation of GBI
- usual BI criterion for nonlocality – contextuality
  basis for quantum communication and information

Literature – GBI

- R.A. Bertlmann, H. Narnhofer W. Thirring: Phys.Rev.A 66, 032319 (2002)
- M. Horodecki, P. Horodecki, R. Horodecki: in Quantum Information, eds. G. Alber et al., Springer Tracts in Modern Physics Vol. 173, p. 151 (2001)
- K.G.H. Vollbrecht, R.F. Werner: quant-ph/0010095
- B. Terhal: Phys.Lett.A 271, 319 (2000) and quant-ph/0101032
- D. Bruß: quant-ph/0110078
- M. Lewenstein, D. Bruß, J.I. Cirac, B. Kraus, M. Kuś, J. Samsonowicz, A. Sanpera, R. Tarrach: quant-ph/0006064
- M. Lewenstein, B. Kraus, J.I. Cirac, P. Horodecki: Phys.Rev.A 62, 052310 (2000) and quant-ph/0005112