HOW HEAVY ARE THE QUARKS ? LIMITS FROM THE FEYNMAN–HELLMANN THEOREM

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In describing $q\bar{q}$ systems we assume the validity of a nonrelativistic treatment and the flavour independence of the quark forces. Then we find by using the Feynman–Hellmann theorem the following bounds for the masses of the quarks: $0.23 \leqslant M_u\ (= M_d) \leqslant 0.40$ GeV, $0.50 \leqslant M_s \leqslant 0.75$ GeV, $1.73 \leqslant M_c \leqslant 2.00$ GeV, $5.03 \leqslant M_b \leqslant 5.55$ GeV.

1. Introduction. Since long the quark model has been a very successful concept in particle physics in order to predict the spectra of the elementary particles. After the discovery of the J/ψ and the Υ families it was stimulated once again by a flood of papers offering essentially nonrelativistic models to explain the spectra of those heavy $q\bar{q}$ systems [1]. Trying to find a unique description for both the low-lying states like π, K, ρ, Φ, etc., and the heavy ones people obtained very promising results whether they used a relativistic description [2] or a nonrelativistic one treating relativistic corrections as perturbative effects [3]. Even a strictly nonrelativistic model, as shown by one of us (S.O.), can fit all data remarkably well [4].

In this paper we are still concerned with a nonrelativistic treatment of $q\bar{q}$ systems which is doubtless legal if the quarks are heavy enough. We want to remain as general as possible and our results do not rely on a specific potential model. We follow a procedure developed by Martin and one of us (R.A.B.) [5] which makes use of the Feynman–Hellmann theorem in order to derive mass inequalities between the quarks and their bound states. In taking the masses of the bound states from experiment one can predict the allowed range for the masses of the quarks. It has been done for the heavy quarks c and b yielding excellent results [5]. This encourages us to investigate further systems including the light quarks u, d, s, to see how far one can go in a nonrelativistic treatment.

2. Bounds on $M_{q_2} - M_{q_1}$. A system $q_2\bar{q}_1$ will be described nonrelativistically by a hamiltonian

$$H = p^2/m_{12} + V(r), \tag{1}$$

where

$$m_{12} = 2M_{q_1}M_{q_2}/(M_{q_1} + M_{q_2}) \tag{2}$$

is twice the reduced mass of the system ($M_{q_1} \leqslant M_{q_2}$).

The quark masses well defined in the nonrelativistic theory have the meaning of constituent masses. On this level we neglect the isosplitting of u and d ($M_u \approx M_d$), e.g. ρ is a pure $u\bar{u}$ state. Furthermore we remove the spin effects which are mass dependent by the center of mass combination

$$M = \tfrac{3}{4}M(\text{triplet}) + \tfrac{1}{4}M(\text{singlet}). \tag{3}$$

The potential $V(r)$ is assumed to be strictly flavour independent for which there is good evidence in the re-

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gion c to b [6,7], but which is certainly a strong requirement in the extension to the light quark u.

The basic idea to obtain the mass bounds is to use the Feynman–Hellmann theorem [8] with respect to m:

$$dE/dm = -T(m)/m\,, \tag{4}$$

where $T(m)$ is the kinetic energy. We integrate between two systems, say $q_1\bar{q}_1$ and $q_2\bar{q}_1$ ($M_{q_1} \leqslant M_{q_2}$):

$$E_{11} - E_{21} = \int_{m_{11}}^{m_{21}} \frac{T(m)}{m}\,dm\,, \tag{5}$$

and minimize or maximize the integral appropriately since we can control the mass behaviour of $T(m)$. For the kinetic energy of ground states the following two theorems hold [5]:

Theorem 1. $mT(m)$ is increasing in m for any potential. This property is saturated by the square well.

Theorem 2. $T(m)/m$ decreases in m for potentials satisfying the conditions

$$(d/dr)(V + rV') > 0\,, \quad V' > 0\,. \tag{6}$$

Saturation is obtained by the pure Coulomb case.

A class of potentials satisfying the above conditions is a superposition of power potentials:

$$V(r) = \int_{-1}^{\infty} \epsilon(\lambda)\rho(\lambda)r^{\lambda}\,d\lambda\,, \tag{7}$$

with $\epsilon(\lambda) = \lambda/|\lambda|$ and $\rho(\lambda) \geqslant 0$. This includes all simple favourite potentials used for quarkonia.

To obtain a lower bound on the quark mass difference $M_{q_2} - M_{q_1}$ we now apply theorem 1 to minimize the integral (5):

$$E_{11} - E_{21} \geqslant m_{11}\,T(m_{11})\left(\frac{1}{m_{11}} - \frac{1}{m_{21}}\right) = \frac{T(M_{q_1})}{2M_{q_2}}(M_{q_2} - M_{q_1})\,. \tag{8}$$

Adding quark masses and respecting

$$M(q_2\bar{q}_1) = M_{q_2} + M_{q_1} + E_{21}\,, \quad M(q_1\bar{q}_1) = 2M_{q_1} + E_{11}\,, \tag{9}$$

gives the result

$$\frac{M(q_2\bar{q}_1) - M(q_1\bar{q}_1)}{1 - T(M_{q_1})/2M_{q_2}} \leqslant M_{q_2} - M_{q_1}\,. \tag{10}$$

Maximizing on the other hand the integral (5) by theorem 2 we get an upper bound

$$M_{q_2} - M_{q_1} \leqslant \frac{M(q_2\bar{q}_1) - M(q_1\bar{q}_1)}{1 - T(M_{q_1})/(M_{q_1} + M_{q_2})}\,. \tag{11}$$

Additional bounds we are going to use are collected in the appendix.

The lower bounds hold completely generally, the upper bounds for potentials satisfying conditions (7) or (6), or superficially speaking for monotonous potentials not more singular than $-1/r$.

Now, we have as experimental information not only the mass of the system $M(q_i\bar{q}_j)$ but also its kinetic energy. For $T(M_q)$ holds a very simple formula [5]

$$T(M_q) = \langle T\rangle^{1s}_{q\bar{q}} \approx \tfrac{3}{4}(E_{1p} - E_{1s})(1 + \tfrac{7}{9}c)\,, \tag{12}$$

with

$$c = \left(\frac{2E_{1p} - E_{2s} - E_{1s}}{E_{2s} - E_{1s}}\right)^2\,.$$

For the energy differences we just take the center of mass differences of the corresponding states and get

$$T(M_c) = 0.36\ \text{GeV}\,, \quad T(M_u) = 0.46\ \text{GeV}. \tag{13,14}$$

Whereas $T(M_c)$ can be determined quite accurately the value of $T(M_u)$ has an error of about 2–3% due to experimental errors in the mass input.

3. Bounds on M_q. We have now a large set of inequalities which have to be fulfilled for all possible quark mass differences. These inequalities define surfaces in a four-dimensional space separating the allowed regions from the forbidden ones. All surfaces intersect and restrict in this way the range of the masses of the quarks.

We begin to solve the inequalities by allowing for a variation

$$\begin{aligned} 0.23 &\leqslant M_u\ (=M_d) \leqslant 0.4\ \text{GeV}\,,\\ 0.3 &\leqslant M_s \leqslant 0.75\ \text{GeV}\,,\\ 1.0 &\leqslant M_c \leqslant 2.0\ \text{GeV}\,,\\ 4.0 &\leqslant M_b \leqslant 6.0\ \text{GeV}\,, \end{aligned} \tag{15}$$

in the correction term proportional to T. Then we nearly pass through the whole particle spectrum and select the optimal bounds

$$M_s - M_u \geqslant 0.27 \text{ GeV} \quad \text{from } K^*-\rho, \quad \text{eq. (10)}, \qquad (16)$$

$$M_c - M_u \geqslant 1.50 \text{ GeV} \quad \text{from } D^*-\rho, \quad \text{eq. (10)}, \qquad (17)$$

$$M_b - M_c \geqslant 3.30 \text{ GeV} \quad \text{from } \Upsilon-J/\psi, \quad \text{eq. (A2)}. \qquad (18)$$

Respecting the previous bound $M_u \geqslant 0.23$ GeV increases the lower limits to

$$M_s \geqslant 0.27 + 0.23 = 0.5 \text{ GeV}, \qquad (19)$$

$$M_c \geqslant 1.50 + 0.23 = 1.73 \text{ GeV}, \qquad (20)$$

$$M_b \geqslant 3.30 + 1.17 = 5.03 \text{ GeV}. \qquad (21)$$

These new limits determine the upper bounds for the quark mass differences for which we get as best values

$$M_s - M_u \leqslant 0.45 \text{ GeV} \quad \text{from } K^*-\rho, \quad \text{eq. (11)}, \qquad (22)$$

$$M_c - M_u \leqslant 1.60 \text{ GeV} \quad \text{from } D^*-\rho, \quad \text{eq. (11)}, \qquad (23)$$

$$M_b - M_c \leqslant 3.55 \text{ GeV} \quad \text{from } \Upsilon-J/\psi, \quad \text{eq. (A5)}. \qquad (24)$$

A comparison with (15) finally results in

$$\begin{aligned} 0.23 &\leqslant M_u \; (= M_d) \leqslant 0.40 \text{ GeV}, \\ 0.50 &\leqslant M_s \leqslant 0.75 \text{ GeV}, \\ 1.73 &\leqslant M_c \leqslant 2.00 \text{ GeV}, \\ 5.03 &\leqslant M_b \leqslant 5.55 \text{ GeV}. \end{aligned} \qquad (25)$$

These values correspond to the best solutions of our inequalities for which we used all available data. As a surprise the allowed range for M_s, M_c and M_b turns out to be higher than expected thus making a nonrelativistic approach to systems containing s, c and b reliable enough. This is somewhat in contrast to a quark mass determination from the leptonic width given by the Van Royen–Weisskopf formula which leads to smaller mass values [1]. However this formula is subject to considerable corrections (Barbieri et al. [3], [9]).

But can we trust more our mass determination from particle spectra only? We must not forget the two rather strong assumptions we made at the beginning, first the validity of a nonrelativistic motion and secondly the flavour independence of the potential. But besides that our results are model independent. Systems containing u certainly must be handled with great care since their internal motion is quite high. Relativistic corrections affect our result, they tend to decrease it by about 20%. A more detailed discussion and comparison with specific models will be presented elsewhere [10].

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Appendix. Integrating eq. (4) between $(q_2\bar{q}_1)$ and $(q_2\bar{q}_2)$ and minimizing subsequently by theorem 1 gives the lower bound

$$\frac{M(q_2\bar{q}_2) - M(q_2\bar{q}_1)}{1 - T(M_{q_1})/2M_{q_2}} \leqslant M_{q_2} - M_{q_1}, \qquad (A1)$$

and integration from $(q_1\bar{q}_1)$ to $(q_2\bar{q}_2)$ yields

$$\frac{M(q_2\bar{q}_2) - M(q_1\bar{q}_1)}{2[1 - T(M_{q_1})/2M_{q_2}]} \leqslant M_{q_2} - M_{q_1}. \qquad (A2)$$

Including furthermore a third quark with $M_{q_1} \leqslant M_{q_2} \leqslant M_{q_3}$ results in

$$\frac{M(q_3\bar{q}_2) - M(q_2\bar{q}_1)}{1 - T(M_{q_1})/2M_{q_3}} \leqslant M_{q_3} - M_{q_1}, \qquad (A3)$$

and

$$\frac{M(q_3\bar{q}_1) - M(q_2\bar{q}_1)}{1 - [T(M_{q_1})/2M_{q_3}]\, M_{q_1}/M_{q_2}} \leqslant M_{q_3} - M_{q_2}. \qquad (A4)$$

Maximizing the integral (5) between $(q_1\bar{q}_1)$ and $(q_2\bar{q}_2)$ by theorem 2 gives the upper bound

$$M_{q_2} - M_{q_1} \leqslant \frac{M(q_2\bar{q}_2) - M(q_1\bar{q}_1)}{2[1 - T(M_{q_1})/2M_{q_1}]}. \qquad (A5)$$

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