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Testing Q^2 Duality with Non-Relativistic Potentials

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Abstract. The process of electron–positron annihilation into quark–antiquark is considered with non-relativistic potential account of final state interaction. Calculations are made for confining potentials, with discrete bound final states, and for corresponding non-confining potentials, with continua of finally free quark–antiquark pairs. When suitably averaged over energy the two ways of calculating agree closely, illustrating well the local duality concept.

1. Introduction

In the quark–parton model the production of hadrons in e^+e^- collisions is supposed to be dominated at high energies by the production of quasi-free quark–antiquark pairs

$$e^+e^- \to q\bar{q} \tag{1}$$

which fragment into hadrons. At low energies these quark–antiquark pairs form resonances, vector mesons,

$$e^+e^- \to V \tag{2}$$

which again decay into hadrons. Although the cross-sections of these two processes look completely different the hypothesis of duality [1] states that they become equal when averaged suitably over energy. They are dual descriptions of one and the same physical phenomenon:

$$\int s\cdot\sigma_v d\sqrt{s} = \int s\cdot\sigma_{q\bar{q}} d\sqrt{s} \tag{3}$$

There is a considerable literature on this idea [1–3]. We are concerned here with the way in which it is illustrated in non-relativistic potential models, such as came into vogue in connection with the J/ψ and associated excited states. There are several papers in which it is shown how duality emerges in JWKB approximation [4–6]. The present work is a sequel rather to that of Ishikawa and Sakurai [7], in which the Schrödinger equation was solved numerically, for various confining potentials, and the results compared with duality predictions.

We sharpen the illustration of duality by several variations from the procedure of Ishikawa and Sakurai:

1) We use non-relativistic kinematics consistently in both confined and unconfined calculations.

2) While they considered only how the average over discrete resonances built up the *asymptotic* form of the comparison amplitude, we wish to see a more local agreement. We include therefore the finite mass threshold effects in the unconfined amplitude (as done in J.J. Sakurai (1)).

3) We include also in the comparison amplitude the effect of the short range singular part of the potential. The basic idea underlying duality [see for example [4] or [3]], that the energy average is determined at small times and distances, does not justify the omission of this part.

4) We believe that we perform somewhat more systematically and accurately than they the numerical solution of Schrödinger equations. Our results, for the same potentials, are somewhat different; in particular the sequence of levels is much more regular.

These changes are mainly dictated by the fact that our intention is rather different from that of Ishikawa and Sakurai. We wish to illustrate duality in non-relativistic models, leaving aside completely the question of the adequacy of those models in the real situation, especially for high excitation.

2. Computations

In the Born approximation the cross-section for process (1), for a single flavour, is given by

$$s\cdot\sigma_{q\bar{q}} = \frac{4\pi\alpha^2}{3}\cdot 3Q^2\cdot\frac{3v}{2}\left(1-\frac{v^2}{3}\right)\cdot\theta(s-(2m_q)^2) \tag{4}$$

where Q is the quark charge in units of e, the factor

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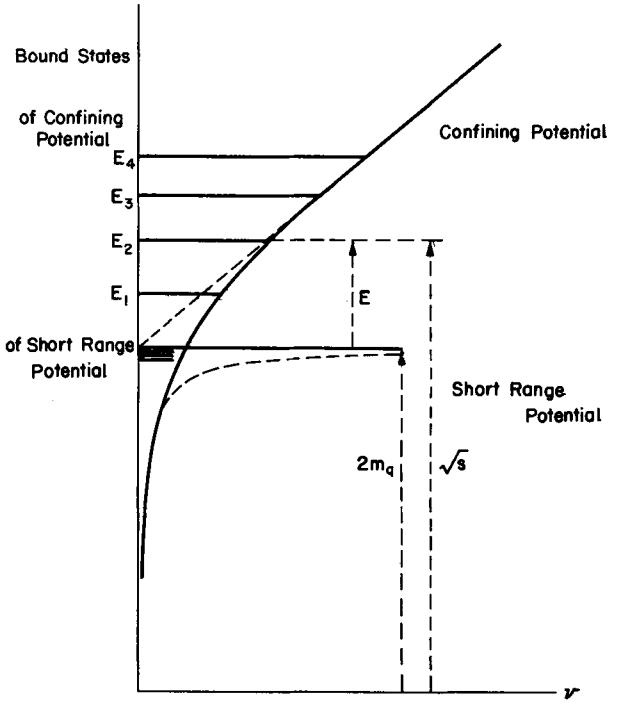

Fig. 1. Picture of a confining and short range (Coulomb) potential with confining bound states and short range Coulomb bound states

three in front accounts for the three-quark colours, m_q is the quark mass and

$$v=\left(1-\frac{(2m_q)^2}{s}\right)^{1/2} \tag{5}$$

is the velocity of the quarks in the center-of-mass frame. We need here only the non-relativistic versions

$$(2m_q)^2\sigma_{q\bar{q}}=\frac{4\pi\alpha^2}{3}\cdot 3Q^2\cdot\frac{3v}{2}\cdot\theta(E) \tag{6}$$

$$v=\sqrt{\frac{E}{m_q}} \tag{7}$$

where

$$E=\sqrt{s}-2m_q. \tag{8}$$

In Eq. (4) final state interaction is ignored completely. But it is only the long range confining part of the potential that duality allows us to suppress, and not a short range singular part. We write the potential as a sum

$$V(r)=V_s(r)+V_c(r) \tag{9}$$

where the smooth confining part is such that

$$V_c(0)=0$$

and the singular short range part has the property

$$V_s(\infty)=0.$$

When the action of V_s is allowed for we have to add to Eq. (6) contributions from any bound states that V_s may have (see Fig. 1):

$$6\pi^2\sum_n\delta(E-E_{sn})\Gamma^{e\bar{e}}_{sn}. \tag{10}$$

The leptonic widths are given by the Van Royen–Weisskopf formula [8]

$$\Gamma^{e\bar{e}}_{sn}=16\pi\alpha^2Q^2|\psi_{sn}(0)|^2/(2m_q)^2$$

in terms of the magnitudes of the bound-state wave functions at the origin. In addition, for positive (E), Eq. (6) will be modified by some enhancement factor $K(E)$ because of the action of V_s.

When the full potential V is allowed for we have only terms like Eq. (10), involving, however, the real energies E_i (rather than the fictitious E_{sn}) and corresponding widths Γ_i and wave functions ψ_i. Dropping common factors the duality requirement is then

$$\sum_i 4\pi|\psi_i(0)|^2\delta(E-E_i)=\sum_n 4\pi|\psi_{sn}(0)|^2\delta(E-E_{sn})$$
$$+(m_q^2/\pi)\cdot\int dE'\,v(E')K(E')\theta(E')\delta(E-E') \tag{11}$$

when the delta functions are replaced by suitably broad distributions, or both sides averaged somehow over energy E. In our numerical calculations, for ease of comparison, we use for computing the energy levels and wave functions at the origin the same three very different potentials as in [7]. The first potential, somehow motivated by QCD considerations, is the Columb + linear potential [9]:

$$V=-\frac{\alpha_s}{r}+fr,\ \alpha_s=0.25,\ f=0.1795\ \text{GeV}^2,$$
$$m_c=1.6\ \text{GeV}. \tag{12}$$

The second potential is the Coulomb + cubic potential [10]:

$$V=-\frac{\alpha_s}{r}+fr^3,\quad \alpha_s=0.25,\quad f=0.0069\ \text{GeV}^4,$$
$$m_c=1.6\ \text{GeV}. \tag{13}$$

For the third potential the logarithmic potential [11] is taken which provides a level spacing independent of the quark masses and confines the quarks much more weakly than the above two potentials:

$$V=c\ln\frac{r}{r_0},\quad c=0.75\ \text{GeV},\quad r_0=1\ \text{GeV}^{-1},$$
$$m_c=1.6\ \text{GeV}. \tag{14}$$

For all three potentials we calculate the energy spectra and wave functions at origin by solving the Schrödinger equation with the computer. Our results are shown in Fig. 2. In comparing our energy spectrum with that of Fig. 1 of [7], it is important to keep in mind that Ishikawa and Sakurai have actually added a constant term to each of the three potentials to adjust the first bound state to coincide with ψ/J (3.1).

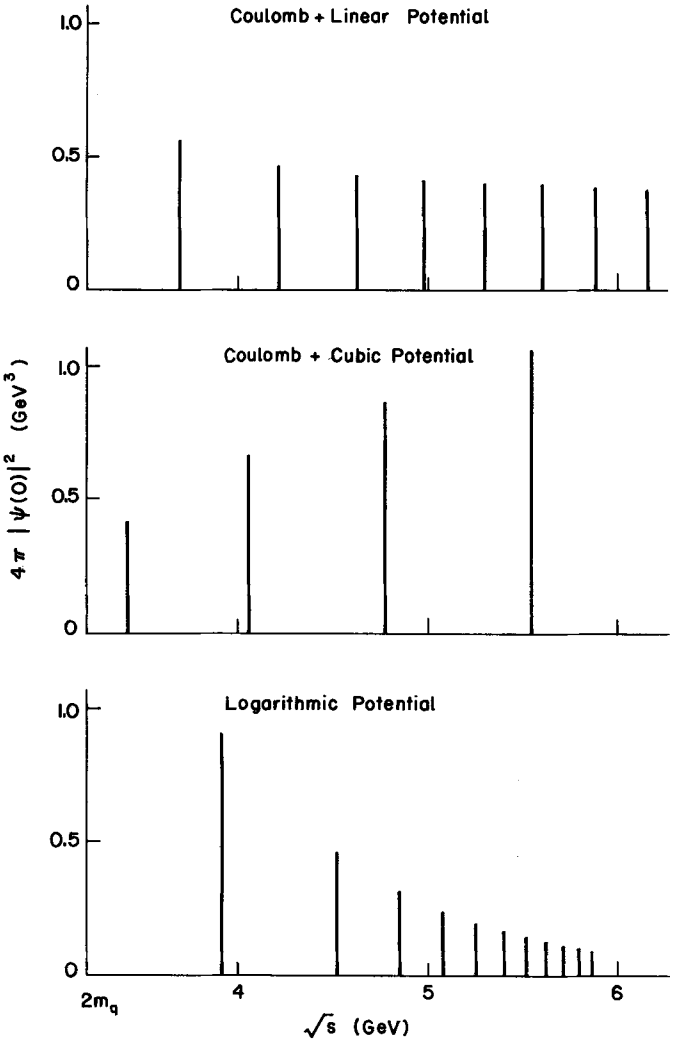

Fig. 2. $4\pi|\psi(0)|^2$ and the energy splitting as calculated by the computer for the Coulomb + linear potential Eq. (12), the Coulomb + Cubic Potential Eq. (13) and the logarithmic potential Eq. (14)

Table 1. Comparison of the Schrödinger solutions $4\pi|\psi_i(0)|^2$ for the Coulomb + linear potential Eq. (12) with the corresponding values given by the duality Eq. (11)

i	GeV E_i	GeV^3 $4\pi\|\psi_i(0)\|^2$ Schrödinger solution	GeV^3 $4\pi\|\psi_i(0)\|^2$ According to duality Eq. (11)
1	0.488	0.561	0.633
2	1.013	0.463	0.472
3	1.422	0.430	0.432
4	1.778	0.412	0.412
5	2.101	0.400	0.400
6	2.400	0.392	0.392
7	2.681	0.385	0.385

Table 2. Comparison of the Schrödinger solutions $4\pi|\psi_i(0)|^2$ for the Coulomb + cubic potential Eq. (13) with the corresponding values given by the duality Eq. (11)

i	GeV E_i	GeV^3 $4\pi\|\psi_i(0)\|^2$ Schrödinger solution	GeV^3 $4\pi\|\psi_i(0)\|^2$ According to duality Eq. (11)
1	0.216	0.419	0.439
2	0.855	0.663	0.660
3	1.569	0.868	0.867
4	2.336	1.055	1.054
5	3.143	1.231	1.230
6	3.984	1.398	1.397
7	4.854	1.558	1.558

Table 3. Comparison of the Schrödinger solutions for the logarithmic potential Eq. (14) with the values given by the duality Eq. (11)

i	E_i in GeV	$4\pi\|\psi_i(0)\|^2$ in GeV^3	$4\pi\|\psi_i(0)\|^2$ According to Eq. (11)
1	0.715	0.907	0.877
2	1.317	0.463	0.505
3	1.649	0.315	0.325
4	1.878	0.241	0.244
5	2.054	0.196	0.197
6	2.196	0.165	0.165
7	2.316	0.143	0.143
8	2.419	0.126	0.126
9	2.510	0.113	0.113
10	2.591	0.103	0.103

We need also bound states and enhancement factors K for the short range parts V_s. With the potentials Eqs. (12) and (13) it is natural to identify V_s with the Coulomb parts. Then

$$E_{sn} = -m_q\alpha_s^2/4n^2$$
$$4\pi|\psi_{sn}(0)|^2 = m_q^3\alpha_s^3/2n^3$$
$$K(E) = 2\pi\eta/(e^{2\pi\eta}-1), \quad \eta = -\alpha_s/2v.$$

For the logarithmic potential there is no such natural splitting. We take

$$V_s = c\ln\left(\tanh\frac{r}{r_0}\right) \tag{15}$$

$$V_c = c\ln\frac{r}{r_0} - c\ln\left(\tanh\frac{r}{r_0}\right). \tag{16}$$

This potential has no bound states [12]. The radial Schrödinger equation is integrated numerically for positive E to find K. The latter emerges from the asymptotic form of the radial wave functions for unit slope at the origin

$$(Kpk)^{-1/2}\sin\left(\int_0^r p(r')dr' + \delta\right) \tag{17}$$

where

$$p(r) = (k^2 - m_q V_s(r))^{1/2}, \quad k = (m_q E)^{1/2}.$$

Table 4

Δ	$\sum_i 4\pi\lvert\psi_i(0)\rvert^2 B(E-E_i)$		$\sum_i 4\pi\lvert\psi_i(0)\rvert^2 G(E-E_i)$	
GeV	Schrödinger solution	Duality solution according to Eqs. (11) and (20)	Schrödinger solution	Duality solution according to Eqs. (11) and (21)
1.013	0.398	0.431	0.417	0.464
1.422	0.397	0.419	0.467	0.498
	values for Coulomb + linear potential Eq. (12)			
0.855	0.511	0.510	0.452	0.451
1.569	0.546	0.546	0.511	0.511
	values for Coulomb + cubic potential Eq. (13)			

First, following [7], we just integrate both sides of Eq. (18) over bins associated with each level E_i, extending between

$$\tfrac{1}{2}(E_i + E_{i-1}) \quad \text{and} \quad \tfrac{1}{2}(E_i + E_{i+1})$$

except that the lower bound of the first bin is taken at zero. Duality gives

$$4\pi|\psi_i(0)|^2 = (m_q^2/\pi)\cdot \int_{\Delta E} dE\, v(E) K(E) \tag{18}$$

The results of calculating $|\psi_i(0)|^2$ from this formula and directly from the Schrödinger equation are given in Tables 1, 2 and 3. The agreement between the two values is not very good for the first state, mainly because of an inappropriate averaging procedure. But for the second state the discrepancy is already only 1.9% for the Coulomb plus linear potential (0.5% for the Coulomb plus cubic and 9.1% for the logarithmic), and rapidly diminishes to a few per mile for the higher states considered. As in [4] we would emphasize the importance of the enhancement factor K in Eq. (18). It contributes a factor of about 1.57 for the second level of Table 1, dropping to about 1.33 for the last considered. The corresponding numbers for Tables 2 and 3 are 1.63–1.24 and 1.78–1.29, respectively.

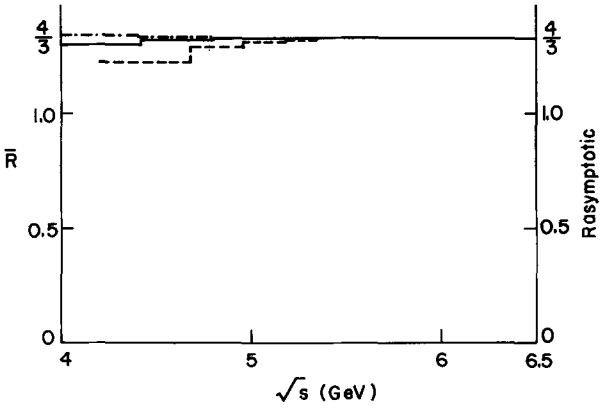

Fig. 3. The resonance by resonance averaged colliding beam ratio R [Eq. (19)] for
——— Coulomb + linear potential Eq. (12)
–·–·–· Coulomb + cubic potential Eq. (13)
– – – – – – logarithmic potential Eq. (14)

For comparison with [7] we show in Fig. 3, to be compared with Fig. 2 of [7], how the "colliding beam ratio"

$$\bar{R}_i = \frac{4}{3}\cdot\frac{4\pi|\psi_i(0)|^2}{(m_q^2/\pi)\cdot\int_{\Delta E_i} dE\, v(E) K(E)} \tag{19}$$

varies from level to level. The factor 4/3 is needed here to have the same normalization as Ishikawa and Sakurai [7], who include colour and fractional (2/3) charge factors in their discrete state calculation, but only a single charged muon in the smooth comparison amplitude. That our histograms stay more close to the asymptotic value (4/3) is presumably for the reasons outlined in the Introduction. Finally as a more sophisticated version of the duality idea we replaced the delta functions in Eq. (11) by distributions of finite width [3]. We tried both the Lorentz–Breit–Wigner shape

$$\delta(E-E') \to B(E-E') = (\Delta/\pi)\cdot[(E-E')^2 + \Delta^2]^{-1} \tag{20}$$

and the Gaussian

$$\delta(E-E') \to G(E-E') = [\sqrt{2\pi}\tau\cdot\exp((E-E')^2/2\tau^2)]^{-1} \tag{21}$$

with

$$\tau = \Delta/1.18.$$

In Eqs. (20) and (21) we took for the half width at half maximum $\Delta = 1.013$ GeV and $\Delta = 1.422$ GeV for the Coulomb + linear potential Eq. (12) and $\Delta = 0.855$ and $\Delta = 1.569$ GeV for the Coulomb + cubic potential Eq. (13) (the values of the second and third energy levels) and integrated the distributions up to $E' = (E_{12} + E_{13})/2$. With $E = 0$.

The numerical computations shown in Table 4 indicate that the Lorentz-Breit-Wigner and the Gaussian shape are equally good. The bigger Δ the better is the agreement with duality Eqs. (11), (20)

and (21). The contribution of the Coulomb bound-states is of the order of some per cent. They improve the results in the case of the Coulomb + cubic potential but make them worse in the of Coulomb + linear potential since duality underestimates the exact results in the former but overestimates in the latter.

3. Conclusion

We have found that charmonium potential models illustrate nicely the duality ideas, and that in a context where everything is calculable non perturbatively. Coarse graining with respect to energy largely removes from the cross-section dependence on the confining part of the potential. We have seen this for three very different potentials. Of course these findings are not independent of what has been seen already in terms of the JWKB approximation—at least when one looks above the first few levels. When v and K in Eq. (18) vary little from one level to the next, a good approximation to the right-hand side gives

$$4\pi|\psi_i(0)|^2 = \pi^{-1} m_q^{3/2} E_i^{1/2} K(E_i) \Delta E_i$$

as derived by JWKB approximation in [4] and without the enhancement factor K (for potentials smooth at short range) in [5] and [6].

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