29 July 2002

PHYSICS LETTERS A

Physics Letters A 300 (2002) 107–114

www.elsevier.com/locate/pla

Dissipation in a 2-dimensional Hilbert space: various forms of complete positivity ☆

R.A. Bertlmann *, W. Grimus

Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria

Received 6 February 2002; received in revised form 11 June 2002; accepted 13 June 2002

Communicated by P.R. Holland

Abstract

We consider the time evolution of the density matrix ρ in a 2-dimensional complex Hilbert space. We allow for dissipation by adding to the von Neumann equation a term $D[\rho]$, which is of Lindblad type in order to assure complete positivity of the time evolution. We present five equivalent forms of $D[\rho]$. In particular, we connect the familiar dissipation matrix L with a geometric version of $D[\rho]$, where L consists of a positive sum of projectors onto planes in $\mathbb{R}^3$. We also study the minimal number of Lindblad terms needed to describe the most general case of $D[\rho]$. All proofs are worked out comprehensively, as they present at the same time a practical procedure how to determine explicitly the different forms of $D[\rho]$. Finally, we perform a general discussion of the asymptotic behaviour $t \to \infty$ of the density matrix and we relate the two types of asymptotic behaviour with our geometric version of $D[\rho]$. © 2002 Published by Elsevier Science B.V.

PACS: 03.65.-w; 03.65.Yz

Keywords: Dissipation; Complete positivity; Decoherence

1. Introduction

Particle physics is not only a field where fundamental interactions are explored, but it has also become a testing ground for possible deviations from the quantum-mechanical time evolution. The time evolution of the density matrix ρ is given by the master equation

$$\frac{d\rho}{dt} = -iH\rho + i\rho H^\dagger - D[\rho], \tag{1}$$

where $D[\rho]$ is a dissipative term which adds to the quantum-mechanical term on the right-hand side of Eq. (1). Such a term can emerge if the system under consideration is not fully closed but interacts weakly with the environment. In general the nature of such an interaction is unknown, but experimental data from suitable systems can be used to place bounds on the parameters of such hypothesized interactions. In the general case, the Weisskopf–Wigner approximation [1] allows to incorporate also unstable particles, by using non-Hermitian Hamiltonians $H = M - i\Gamma/2$, where M and $\Gamma \geqslant 0$ are Hermitian.

It is not only necessary that the time evolution (1) respects $\rho(t) \geqslant 0\ \forall t$, but the stronger requirement of *complete positivity* [2] seems to be a natural and phys-

☆ This research was performed within the FWF Project No. P14143-PHY of the Austrian Science Foundation.

* Corresponding author.

E-mail address: bertlman@ap.univie.ac.at (R.A. Bertlmann).

0375-9601/02/$ – see front matter © 2002 Published by Elsevier Science B.V.
PII: S0375-9601(02)00816-2

ical concept (for the general structure of completely positive maps, see Refs. [3,4]). This concept is defined in the following way. Let us assume that the system under consideration is described by elements of the finite complex Hilbert space $\mathcal{H}$ ($\dim\mathcal{H} \equiv d < \infty$) with time evolution $\rho(t) \equiv \gamma_t(\rho)$ with $\rho(0) = \rho$. Considering in addition the finite-dimensional Hilbert space $\mathbb{C}^n$, one can extend the time evolution γ_t on $\mathcal{H}$ to a time evolution $\gamma_{n;t}$ on $\mathcal{H} \otimes \mathbb{C}^n$ by defining $\gamma_{n;t} = \gamma_t \otimes \mathbb{C}$. If the time evolution γ_t derived from Eq. (1) with the dissipative term $D[\rho]$ has the property that $\gamma_{n;t}(\rho) \geqslant 0$ is valid for all times t, all $n = 0, 1, 2, \ldots$ and all density matrices on the space $\mathcal{H} \otimes \mathbb{C}^n$, then γ_t is called completely positive.

Complete positivity of γ_t determines the general structure of $D[\rho]$ [5,6] (see also Refs. [7–10]). It has been shown by Lindblad [5] (see also Ref. [11]) that γ_t of Eq. (1) is completely positive if and only if $D[\rho]$ has the structure

$$\begin{aligned} D[\rho] &= \frac{1}{2}\sum_{j=1}^{r}\left(A_j^\dagger A_j\rho + \rho A_j^\dagger A_j - 2A_j\rho A_j^\dagger\right) \\ &= \frac{1}{2}\sum_{j=1}^{r}\left(\left[A_j^\dagger, A_j\rho\right] + \left[\rho A_j^\dagger, A_j\right]\right), \end{aligned} \tag{2}$$

where the operators A_j act on $\mathcal{H}$.

Using the relation

$$A_j = B_j + s_j\mathbb{1} \quad \text{with } \operatorname{Tr} B_j = 0, \tag{3}$$

the dissipative term (2) can be reformulated as

$$\begin{aligned} &D[\rho] = -i[\Delta H, \rho] + D'[\rho] \quad \text{with} \\ &\Delta H = \frac{i}{2}\sum_{j=1}^{r}\left(s_j B_j^\dagger - s_j^* B_j\right), \end{aligned} \tag{4}$$

where $D'[\rho]$ is obtained from $D[\rho]$ by the replacement $A_j \to B_j$. This reformulation has the effect that part of $D[\rho]$ is shifted into the quantum-mechanical term of the time evolution (1), such that a new Hamiltonian $H' = H + \Delta H$ appears. Note that for Hermitian operators A_j we have $\Delta H = 0$. In the space of traceless operators on the Hilbert space $\mathcal{H}$ we can choose a basis $\{F_j \mid j = 1, \ldots, d^2 - 1\}$ with the property

$$\operatorname{Tr}\left(F_j^\dagger F_k\right) = \delta_{jk} \tag{5}$$

and expand the operators B_j as

$$B_j = \sum_{k=1}^{d^2-1} C_{kj} F_k. \tag{6}$$

Then we obtain the expression

$$\begin{aligned} D'[\rho] &= -\frac{1}{2}\sum_{j=1}^{r}\left(\left[B_j, \rho B_j^\dagger\right] + \left[B_j\rho, B_j^\dagger\right]\right) \\ &= -\frac{1}{2}\sum_{k,l=1}^{d^2-1} c_{kl}\left(\left[F_k, \rho F_l^\dagger\right] + \left[F_k\rho, F_l^\dagger\right]\right), \end{aligned} \tag{7}$$

where (c_{kl}) is a positive matrix defined by

$$c_{kl} = \sum_{j=1}^{r} C_{kj} C_{lj}^*. \tag{8}$$

The form (7) of the dissipative term has been derived by Gorini, Kossakowski and Sudarshan [6]. It is equivalent to the Lindblad form (2). Thus the time evolution with H and $D[\rho]$ is equivalent to the one with H' and $D'[\rho]$.

In particle physics, searches for deviations from the quantum-mechanical time evolution are going on in neutral meson–antimeson systems ($K^0\bar{K}^0$ and $B^0\bar{B}^0$) (for a list of papers see, e.g., Refs. [12–21]) and neutrino physics (for a list of papers see, e.g., Refs. [22–26]). The importance of complete positivity, in particular, in $K^0\bar{K}^0$ and analogous systems, has been stressed in Refs. [16,18]. For instance, only if γ_t is completely positive, the positivity of the time evolution $\gamma_t \otimes \gamma_t$ in the space $\mathcal{H} \otimes \mathcal{H}$ is guaranteed. This follows from the decomposition $\gamma_t \otimes \gamma_t = (\gamma_t \otimes \mathbb{1})(\mathbb{1} \otimes \gamma_t)$, where both factors are positive according to complete positivity. Forming the tensor product $\gamma_t \otimes \gamma_t$ is a method for implementing the 1-particle time evolution at the 2-particle level, which is an often used procedure, e.g., in the $K^0\bar{K}^0$ system. For an example where γ_t is only positive but not completely positive, with ensuing non-positivity of $\gamma_t \otimes \gamma_t$, see Ref. [18].

For simplicity we assume a Hermitian Hamiltonian H from now on, but this assumption is irrelevant for our discussion of complete positivity; it only concerns the investigation of the asymptotic limit of the time evolution, where loss of probability due to a non-Hermitian H as obtained in the Weisskopf–Wigner approximation [1] will lead to $\rho(t) \to 0$ for $t \to \infty$.

Apart from complete positivity of the time evolution, the assumptions on which the present work is based are the following:

1. We work in a 2-dimensional complex Hilbert space, which we can identify with $\mathbb{C}^2$;
2. We assume Hermitian Lindblad operators, i.e., $A_j = A_j^\dagger\ \forall j$.

Both assumptions are crucial for the following discussions. The first one is motivated by the applications in particle physics, whereas the second assumption guarantees that the entropy $S[\rho] = -\operatorname{Tr}(\rho \ln \rho)$ cannot decrease as a function of time [27]. In this framework we will discuss the different forms of the dissipative term $D[\rho]$ used in the literature and we will show their equivalence. We will put emphasis on the formulation of the time evolution (1) in $\mathbb{R}^3$, where we will represent $D[\rho]$ as a positive sum over projectors onto planes. We will also study in detail the matrix formulation of $D[\rho]$ as advocated by Benatti and Floreanini, e.g., in Refs. [16,18], and relate this formulation with the geometric version of $D[\rho]$ as a sum of projectors. Finally, we will investigate the limit $t \to \infty$ of the density matrix.

2. Equivalent forms of the dissipative term

We start with the original form of the dissipative term (2) and take into account that we confine ourselves to Hermitian Lindblad operators A_j. Thus $D[\rho]$ simplifies to

$$\text{Form A:}\quad D[\rho] = \frac{1}{2}\sum_{j=1}^{r}\left(A_j^2\rho + \rho A_j^2 - 2A_j\rho A_j\right) = \frac{1}{2}\sum_{j=1}^{r}\left[A_j,[A_j,\rho]\right]. \tag{9}$$

The number of Lindblad terms in Eq. (9) is denoted by r.

Next we note that the dissipative term can be rewritten in terms of projectors P_j and their orthogonal complements $P_j^\perp = \mathbb{1} - P_j$ as

$$\text{Form B:}\quad D[\rho] = \frac{1}{2}\sum_{j=1}^{r}\lambda_j\left(P_j\rho P_j^\perp + P_j^\perp\rho P_j\right). \tag{10}$$

The projectors P_j are non-trivial projectors in $\mathbb{C}^2$, which can be parameterized as

$$P_j = \frac{1}{2}\left(\mathbb{1} + \vec{n}_j\cdot\vec{\sigma}\right), \tag{11}$$

where the $\vec{n}_j$ are real unit vectors and $\vec{\sigma}$ denotes the vector of Pauli matrices. The quantities λ_j are real, positive numbers.

Proposition 1. *Forms A and B of $D[\rho]$ are equivalent.*

Proof. A general Hermitian 2×2 matrix A_j can be represented by

$$A_j = \frac{1}{2}\left(a_j\mathbb{1} + \sqrt{\lambda_j}\,\vec{n}_j\cdot\vec{\sigma}\right), \tag{12}$$

where a_j and λ_j are real numbers ($\lambda_j \geqslant 0$) and $\vec{n}_j$ is a unit vector. Note that the part of A_j proportional to the unit matrix does not contribute to $D[\rho]$, as evident from Eq. (9). Therefore, λ_j must not be zero in order to have a non-trivial effect of A_j. Consequently, we are allowed to replace A_j by $\sqrt{\lambda_j}\,P_j$ in $D[\rho]$. Using Eq. (9) and $P_j^2 = P_j$, we derive

$$D[\rho] = \frac{1}{2}\sum_{j=1}^{r}\lambda_j(P_j\rho + \rho P_j - 2P_j\rho P_j), \tag{13}$$

which can be rewritten in Form B of Eq. (10). □

The time evolution (1) is easily reformulated as a differential equation for a real 3-vector $\vec{\rho}$ by using

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{\rho}\cdot\vec{\sigma}) \quad\text{and}\quad H = \frac{1}{2}\left(\mathbb{1} + \vec{h}\cdot\vec{\sigma}\right). \tag{14}$$

Since for simplicity we have assumed that H is Hermitian, the 3-vector $\vec{h}$ is real as well (for an extension to non-Hermitian Hamiltonians see, e.g., Ref. [28]). The new version of Eq. (1) is given by

$$\frac{d\vec{\rho}}{dt} = \vec{h}\times\vec{\rho} - L\vec{\rho}. \tag{15}$$

With Form A or B of the dissipative term and Eq. (12) or (11), we arrive at a geometric version of $D[\rho]$:

$$\text{Form C:} \quad L\vec{\rho} = \frac{1}{2}\sum_{j=1}^{r} \lambda_j \big(\vec{\rho} - \vec{n}_j \vec{n}_j \cdot \vec{\rho}\big), \tag{16}$$

where the matrix L is a positive linear combination of projectors

$$\mathcal{P}\big(\vec{n}_j\big) = \mathbb{1}_3 - \vec{n}_j \vec{n}_j^T. \tag{17}$$

The projector (17) projects onto the plane orthogonal to $\vec{n}_j$. Clearly, Form C is equivalent to the the previous forms of the dissipative term.

The dissipation matrix L in Form C can be reformulated as [16]

$$\text{Form D:} \quad L_{\alpha\beta} = \frac{1}{2}(\Lambda \delta_{\alpha\beta} - \mathbf{q}_\alpha \cdot \mathbf{q}_\beta) \tag{18}$$

with

$$\mathbf{q}_\alpha \in \mathbb{R}^r \quad (\alpha = 1, 2, 3) \quad \text{and} \quad \Lambda = \sum_{\alpha=1}^{3} |\mathbf{q}_\alpha|^2. \tag{19}$$

Proposition 2. *Forms C and D are equivalent.*

Proof. The proof of this statement amounts to a mere rewriting of the elements of L by

$$\sum_{j=1}^{r} \lambda_j \big(\delta_{\alpha\beta} - \big(\vec{n}_j\big)_\alpha \big(\vec{n}_j\big)_\beta\big)$$
$$= \sum_{j=1}^{r} \lambda_j \delta_{\alpha\beta} - \sum_{j=1}^{r} \sqrt{\lambda_j}\, \big(\vec{n}_j\big)_\alpha \sqrt{\lambda_j}\, \big(\vec{n}_j\big)_\beta. \tag{20}$$

Then we define

$$\mathbf{q}_\alpha = \begin{pmatrix} \sqrt{\lambda_1}\, (\vec{n}_1)_\alpha \\ \vdots \\ \sqrt{\lambda_r}\, (\vec{n}_r)_\alpha \end{pmatrix} \quad \text{and} \quad \Lambda = \sum_{j=1}^{r} \lambda_j. \tag{21}$$

The sum over the square of the lengths of the three vectors $\mathbf{q}_\alpha$ is performed via

$$\sum_{\alpha=1}^{3} |\mathbf{q}_\alpha|^2 = \sum_{\alpha=1}^{3} \sum_{j=1}^{r} \lambda_j \big(\vec{n}_j\big)_\alpha^2$$
$$= \sum_{j=1}^{r} \lambda_j \left(\sum_{\alpha=1}^{3} \big(\vec{n}_j\big)_\alpha^2 \right) = \sum_{j=1}^{r} \lambda_j. \tag{22}$$

Here, we have used that the $\vec{n}_j$ are 3-dimensional unit vectors. Thus we have obtained Form D of L from Form C. This procedure can be reversed: given three vectors $\mathbf{q}_\alpha \in \mathbb{R}^r$, we can use Eq. (21) to construct r unit vectors $\vec{n}_j$ and positive numbers λ_j. In this way, we gain Form C from Form D. □

The question arises how many terms are necessary in $D[\rho]$ in the most general case. This question is answered by the following theorem.

Theorem 1. *The most general case is covered by three Lindblad terms in $D[\rho]$, i.e., if $D[\rho]$ is given by a sum over more than three terms, then it can be rewritten as a sum of at most three terms. If $D[\rho]$ is formulated with the minimal number of terms, then, using Forms B or C, there are three distinct minimal cases referring to one, two or three linearly independent vectors $\vec{n}_j$ with $\lambda_j > 0$; we will denote these cases by an* index $\ell \in \{1, 2, 3\}$.[1]

Proof. Let us first assume that $r > 3$. Since there are only three vectors $\mathbf{q}_\alpha$, we can find a rotation R acting on $\mathbb{R}^r$ such that

$$R\mathbf{q}_\alpha = \begin{pmatrix} \mathbf{Q}_\alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{with } \mathbf{Q}_\alpha \in \mathbb{R}^3. \tag{23}$$

In the most general case the set of vectors $Q = \{\mathbf{Q}_\alpha \mid \alpha = 1, 2, 3\}$ is linearly independent and can be parameterized by

$$\mathbf{Q}_\alpha = \begin{pmatrix} q_\alpha^1 \\ q_\alpha^2 \\ q_\alpha^3 \end{pmatrix} = \begin{pmatrix} \sqrt{\mu_1}\, (\vec{m}_1)_\alpha \\ \sqrt{\mu_2}\, (\vec{m}_2)_\alpha \\ \sqrt{\mu_3}\, (\vec{m}_3)_\alpha \end{pmatrix}. \tag{24}$$

The real and positive numbers μ_j are chosen in such a way that the vector $\vec{m}_1$, extracted from the first elements of the vectors $\mathbf{Q}_\alpha$, is a unit vector; the same is done for the second and third elements.[2] In this case we have an index $\ell = 3$. If Q spans a 2-dimensional space, we choose the rotation R such that the third elements of all $\mathbf{Q}_\alpha$ are zero; if Q spans a 1-dimensional space, the second and third elements are

[1] Obviously, in the minimal formulation of $D[\rho]$ we have $r \equiv \ell$.

[2] This procedure is analogous to the one to prove that Form C follows from Form D.

taken to be zero. These two cases refer to index $\ell = 2$ and 1, respectively. Form D of the dissipative term tells us that $D[\rho]$ is independent of any rotation R. Consequently, we arrive at

$$\sum_{j=1}^{r} \lambda_j \left(\delta_{\alpha\beta} - (\vec{n}_j)_\alpha (\vec{n}_j)_\beta\right) = \sum_{j=1}^{\ell} \mu_j \left(\delta_{\alpha\beta} - (\vec{m}_j)_\alpha (\vec{m}_j)_\beta\right), \tag{25}$$

which proves the theorem for $r > 3$. For $r \leqslant 3$ we follow analogous steps performed after Eq. (23), with the rotation R acting now on a 3 or 2-dimensional space. For $r = 1$ the procedure is trivial. □

On the other hand, to find the minimal number of operators A_j needed for the most general dissipative term (2), we could start from the Gorini–Kossakowski–Sudarshan expression (7). Transforming back to the Lindblad structure (2) by using the relations

$$c_{kl} = \left(U\hat{c}U^\dagger\right)_{kl} \quad \text{and} \quad B_j = \sum_{k=1}^{d^2-1} U_{kj}\sqrt{\hat{c}_{jj}}\, F_k, \tag{26}$$

where $\hat{c} \geqslant 0$ is diagonal and U a unitary matrix, it can be seen that, in the general case, we obtain $d^2 - 1$ terms in $D[\rho]$ of Eq. (2); if some of the elements $\hat{c}_{jj}$ are zero, we will have less than $d^2 - 1$ terms. Applying this to $d = 2$, we find at most three terms, which agrees with the result of the explicit calculations leading to Theorem 1.

3. Complete-positivity conditions on the dissipation matrix L

Benatti and Floreanini [16–18] parameterized the dissipation matrix L by 6 real constants and expressed complete positivity in the form of inequalities satisfied by these parameters. Thus they have the version

$$\text{Form E:} \quad L = 2\begin{pmatrix} a & b & c \\ b & \alpha & \beta \\ c & \beta & \gamma \end{pmatrix}, \tag{27}$$

together with

$$2R \equiv \alpha + \gamma - a \geqslant 0,$$
$$2S \equiv a + \gamma - \alpha \geqslant 0,$$
$$2T \equiv a + \alpha - \gamma \geqslant 0; \tag{28a}$$
$$RS \geqslant b^2, \quad RT \geqslant c^2, \quad ST \geqslant \beta^2; \tag{28b}$$
$$RST \geqslant 2bc\beta + R\beta^2 + Sc^2 + Tb^2. \tag{28c}$$

We will show now that Form E is equivalent to the forms presented in the previous section.

First we want to put Eqs. (28a), (28b), (28c) into a simpler equivalent form.

Lemma 1. *Given L, there exists a symmetric matrix M such that*

$$L = \frac{1}{2}(\mathrm{Tr}\, M\mathbb{1}_3 - M). \tag{29}$$

In terms of M, Eqs. (28a), (28b), (28c) *are given by*

(i) $M_{\alpha\alpha} \geqslant 0 \quad (\alpha = 1, 2, 3)$,
(ii) $M_{\alpha\alpha}M_{\beta\beta} \geqslant M_{\alpha\beta}^2 \quad \forall \alpha \neq \beta$,
(iii) $\det M \geqslant 0$, (30)

respectively.

Proof. For $\alpha \neq \beta$ we have $M_{\alpha\beta} = -2L_{\alpha\beta}$. The diagonal elements of M are determined by $M_{\alpha\alpha} = -L_{\alpha\alpha} + L_{\beta\beta} + L_{\gamma\gamma}$, where $\alpha \neq \beta \neq \gamma \neq \alpha$. The second part of the lemma is proved by plugging Eq. (29) into Eqs. (28a), (28b), (28c). □

The next lemma will allow us to connect M possessing properties (30) with Form D of the dissipative term.

Lemma 2. *A real and symmetric* 3×3 *matrix M has the properties of Eq.* (30) *if and only if there exist vectors* $\mathbf{q}_\alpha \in \mathbb{R}^3$ $(\alpha = 1, 2, 3)$ *such that*

$$M_{\alpha\beta} = \mathbf{q}_\alpha \cdot \mathbf{q}_\beta. \tag{31}$$

Proof. ($\Leftarrow$) This direction of the proof is quickly dealt with. Since $M_{\alpha\alpha} = \mathbf{q}_\alpha^2 \geqslant 0$, property (i) is valid. Furthermore, for $\alpha \neq \beta$, with the Cauchy–Schwarz inequality we derive $M_{\alpha\beta}^2 = (\mathbf{q}_\alpha \cdot \mathbf{q}_\beta)^2 \leqslant \mathbf{q}_\alpha^2\, \mathbf{q}_\beta^2 = M_{\alpha\alpha}M_{\beta\beta}$ and property (ii) holds as well. Defining a 3×3 matrix $q = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, we have $\det M = \det(q^T q) = (\det q)^2 \geqslant 0$ and thus property (iii). This completes the first half of the proof.

($\Rightarrow$) This direction of the proof is more involved. If all $M_{\alpha\alpha}$ were zero, then according to property (ii) of Eq. (30) we would have $M = 0$. Therefore, at least one of the diagonal elements of M is non-zero. Without loss of generality we assume $M_{11} > 0$. First we consider the case

$$M_{11}M_{22} - M_{12}^2 > 0. \tag{32}$$

Denoting the elements of $\mathbf{q}_\alpha$ by q_α^j, we can define a vector $\mathbf{q}_1$ by $q_1^1 = \sqrt{M_{11}}$, $q_1^2 = q_1^3 = 0$. From $M_{12} = \mathbf{q}_1 \cdot \mathbf{q}_2$, it follows after a sign choice that $q_2^1 = M_{12}/\sqrt{M_{11}}$. Taking into account that $\mathbf{q}_2 \cdot \mathbf{q}_2 = M_{22}$ and defining $q_2^3 = 0$, the first two vectors are given by

$$\mathbf{q}_1 = \begin{pmatrix} \sqrt{M_{11}} \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{q}_2 = \begin{pmatrix} M_{12}/\sqrt{M_{11}} \\ \sqrt{M_{22} - M_{12}^2/M_{11}} \\ 0 \end{pmatrix}. \tag{33}$$

With the relation of Eq. (32) we find that $\mathbf{q}_2$ is a well-defined real 3-vector. Next we use $M_{13} = \mathbf{q}_1 \cdot \mathbf{q}_3$ and $M_{23} = \mathbf{q}_2 \cdot \mathbf{q}_3$ and obtain

$$q_3^1 = \frac{M_{13}}{\sqrt{M_{11}}} \quad \text{and} \quad q_3^2 = \frac{M_{23} - M_{12}M_{13}/M_{11}}{\sqrt{M_{22} - M_{12}^2/M_{11}}}. \tag{34}$$

It remains to take into account $\mathbf{q}_3 \cdot \mathbf{q}_3 = M_{33}$. After some algebra we arrive at

$$\begin{aligned}\left(q_3^3\right)^2 &= M_{33} - \left(q_3^1\right)^2 - \left(q_3^2\right)^2 \\ &= \frac{\det M}{M_{11}M_{22} - M_{12}^2} \geqslant 0.\end{aligned} \tag{35}$$

The positivity follows from property (iii) of Eq. (30). Thus, q_3^3 is well-defined and we have proven relation (31), provided condition (32) holds.

It remains to check the same for the special case

$$M_{11}M_{22} = M_{12}^2, \tag{36}$$

which was excluded by Eq. (32). From Eq. (36) we obtain $M_{12} = \eta\sqrt{M_{11}M_{22}}$ with $\eta = \pm 1$ and

$$\det M = -\left(\sqrt{M_{11}}\,M_{23} - \eta\sqrt{M_{22}}\,M_{13}\right)^2. \tag{37}$$

Since $\det M \geqslant 0$ (see Eq. (30)), it follows that

$$M_{23} = \eta M_{13}\sqrt{M_{22}/M_{11}}. \tag{38}$$

With this relation and taking into account condition (36), it is easy to check that

$$\mathbf{q}_1 = \sqrt{M_{11}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{q}_2 = \eta\sqrt{M_{22}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{q}_3 = \begin{pmatrix} M_{13}/\sqrt{M_{11}} \\ \sqrt{M_{33} - M_{13}^2/M_{11}} \\ 0 \end{pmatrix} \tag{39}$$

represents a consistent choice of vectors, which fulfills Eq. (31). This completes the proof. □

With Lemmata 1 and 2 we now readily see that a matrix L fulfills the conditions of Eqs. (28a), (28b), (28c) if and only if L is given by $L_{\alpha\beta} = (1/2)(\Lambda\delta_{\alpha\beta} - \mathbf{q}_\alpha \cdot \mathbf{q}_\beta)$, where $\mathbf{q}_\alpha \in \mathbb{R}^3$ and $\Lambda = \sum_{\alpha=1}^{3} |\mathbf{q}_\alpha|^2$.

Proposition 3. *Forms D and E of the dissipation matrix L are equivalent.*

Proof. According to Lemma 1, from the matrix L we construct a matrix M with properties (30). Lemma 2 tells us that such an M is represented by a matrix of scalar products (see Eq. (31)) and vice versa. Therefore, our statement is true. □

4. The asymptotic limit $t \to \infty$

Now we consider the asymptotic limit of the density matrix $\rho(t)$ with time evolution (1) [20]. For this purpose we use the results of Theorem 1, where we have also defined the index ℓ of L. For a general study of the large time behaviour of the density matrix starting with expression (7), see Ref. [29].

Theorem 2. *For index $\ell = 2$ or $\ell = 3$, the asymptotic limit of the density matrix is given by*

$$\lim_{t\to\infty} \rho(t) = \frac{1}{2}\mathbb{1}. \tag{40}$$

Proof. This statement is most easily proved by using Form C, Eq. (16), of the dissipative term and Eq. (15) of the time evolution. Denoting by $\mathcal{A}$ the operator on the right-hand side of Eq. (15), we have

$$\mathcal{A}\mathbf{x} = \vec{h} \times \mathbf{x} - \frac{1}{2}\sum_{j=1}^{\ell} \lambda_j \mathcal{P}(\vec{n}_j)\mathbf{x} \tag{41}$$

for arbitrary complex 3-vectors $\mathbf{x}$. If we can show that every eigenvalue c of $\mathcal{A}$ fulfills $\operatorname{Re} c < 0$, the theorem is proved. Let $\mathbf{x}$ be a normalized eigenvector with eigenvalue c. Then, with $\mathbf{x}^\dagger \mathbf{x} = 1$ we have

$$c = \mathbf{x}^\dagger \mathcal{A}\mathbf{x} = -2i\vec{h} \cdot (\operatorname{Re}\mathbf{x} \times \operatorname{Im}\mathbf{x}) - \frac{1}{2}\sum_{j=1}^{\ell} \lambda_j \mathbf{x}^\dagger \mathcal{P}(\vec{n}_j)\mathbf{x} \tag{42}$$

and, consequently,

$$\operatorname{Re} c = -\frac{1}{2}\sum_{j=1}^{\ell} \lambda_j \mathbf{x}^\dagger \mathcal{P}(\vec{n}_j)\mathbf{x}. \tag{43}$$

Since the projectors $\mathcal{P}(\vec{n}_j)$ are positive operators and $\lambda_j > 0$, we have $\operatorname{Re} c \leqslant 0$. If $\operatorname{Re} c = 0$, it is necessary that $\mathbf{x}^\dagger \mathcal{P}(\vec{n}_j)\mathbf{x} = 1 - |\vec{n}_j \cdot \mathbf{x}|^2 = 0\ \forall j = 1, \ldots, \ell$. Therefore, the eigenvector $\mathbf{x}$ is proportional to $\vec{n}_j\ \forall j = 1, \ldots, \ell$. This is a contradiction for $\ell = 2$ or 3 independent vectors $\vec{n}_j$ and, indeed, every eigenvalue c has a negative non-zero real part. □

Theorem 3. *For $\ell = 1$ ($\vec{n}_1 \equiv \vec{n}$, $P_1 \equiv P$, $\lambda_1 \equiv \lambda$) we have either $\vec{h} \parallel \vec{n}$ or, equivalently, $[H, P] = 0$, in which case we obtain*

$$\lim_{t\to\infty} \rho(t) = P\rho(0)P + P^\perp \rho(0) P^\perp; \tag{44}$$

or $\vec{h} \nparallel \vec{n}$, i.e., $[H, P] \neq 0$, then the asymptotic limit of $\rho(t)$ is the same as in Theorem 2.

Proof. We follow the same strategy as in the proof of the previous theorem. Thus either $\operatorname{Re} c < 0$ or the eigenvector $\mathbf{x}$ is proportional to $\vec{n}$, in which case $\operatorname{Re} c = 0$. Assuming $\mathbf{x} = \vec{n}$ without loss of generality, we have now $\mathcal{A}\mathbf{x} = \vec{h} \times \vec{n} = c\vec{n}$, where the eigenvalue c is imaginary. This equation is only soluble for $\vec{h} \propto \vec{n}$, whence it follows that $c = 0$. The relation $\vec{h} \propto \vec{n}$ is equivalent to $[H, P] = 0$. In this case we can write the Hamiltonian as $H = hP + h'P^\perp$. Decomposing an arbitrary density matrix ρ as

$$\rho = \rho_0 + \rho_1 \quad \text{with}$$
$$\rho_0 = P\rho P + P^\perp \rho P^\perp \quad \text{and}$$
$$\rho_1 = P\rho P^\perp + P^\perp \rho P, \tag{45}$$

we find

$$\frac{d\rho_0}{dt} = 0 \quad \text{and}$$

$$\frac{d\rho_1}{dt} = \left[-i(h - h') - \lambda/2\right] P\rho P^\perp + \left[i(h - h') - \lambda/2\right] P^\perp \rho P. \tag{46}$$

Therefore, we arrive at

$$\rho_0(t) = \rho_0(0) \quad \text{and}$$
$$\rho_1(t) = e^{[-i(h-h')-\lambda/2]t} P\rho(0)P^\perp + e^{[i(h-h')-\lambda/2]t} P^\perp \rho(0) P. \tag{47}$$

Since $\lim_{t\to\infty} \rho_1(t) = 0$, the theorem is proven. □

The different limits of $\rho(t)$ discussed in Theorems 2 and 3 have been noticed in Refs. [20,23,26]. In the case of $\ell = 1$ and $[H, P] = 0$, the limit (44) of the density matrix has the form $\rho = \mu P + (1 - \mu)P^\perp$ with $0 \leqslant \mu \leqslant 1$; all density matrices obeying $d\rho/dt = 0$ have this form for $[H, P] = 0$. For $\ell = 1$ and $[H, P] \neq 0$, and for $\ell = 2, 3$, the unique density matrix which is time-independent is proportional to the unit matrix.

5. Summary and conclusions

In this Letter we have considered the quantum-mechanical time evolution of a 2×2 density matrix $\rho(t)$, where the von Neumann equation is modified by a dissipative term $D[\rho]$, which, therefore, must be of the Lindblad type or, equivalently, of the Gorini–Kossakowski–Sudarshan type in order to assure a completely positive time evolution. We have, furthermore, assumed that the Lindblad operators A_j are all Hermitian which ensues that the entropy is non-decreasing with time. Our starting point conforms with many applications of the open-systems approach in particle physics.

We have discussed five equivalent forms of $D[\rho]$ which all have their merits depending on the problem considered. We have put particular emphasis on the time evolution in the form of Eq. (15), where the density matrix and the Hamiltonian are represented by real 3-vectors, and Form C, Eq. (16), of the dissipative term, where the dissipative term is a positive linear combination of projectors onto 2-dimensional planes in $\mathbb{R}^3$.

We have studied the question of the minimal number ℓ of Lindblad terms needed in order to reproduce a given $D[\rho]$ and formulated the result in Theorem 1; the proof of this theorem represents at the same time a procedure how to determine ℓ in practice. An other procedure would be to start with the Gorini–Kossakowski–Sudarshan expression from which it also can be seen that $D[\rho]$ can be generally decomposed into 3 terms.

We have also connected the approach where the dissipative term is given by a matrix L specified by the conditions (28a), (28b), (28c) with the geometric picture of $D[\rho]$ as given by Form C, Eq. (16). Again, the proof which shows the equivalence between the two approaches, given by Lemmata 1 and 2, indicates a practical way to obtain the projectors $\mathcal{P}(\vec{n}_j)$ (17) associated with $D[\rho]$.

Finally, we have presented a general discussion of the limit $t \to \infty$ of $\rho(t)$, where the usefulness of Form C of $D[\rho]$ was exemplified.

References

[1] V.F. Weisskopf, E.P. Wigner, Z. Phys. 63 (1930) 54;
V.F. Weisskopf, E.P. Wigner, Z. Phys. 65 (1930) 18.

[2] W.F. Stinespring, Proc. Am. Math. Soc. 6 (1955) 211.

[3] M.-D. Choi, Linear Algebra Appl. 10 (1975) 285.

[4] A. Fujiwara, P. Algoet, Phys. Rev. A 59 (1999) 3290.

[5] G. Lindblad, Comm. Math. Phys. 48 (1976) 119.

[6] V. Gorini, A. Kossakowski, E.C.G. Sudarshan, J. Math. Phys. 17 (1976) 821.

[7] V. Gorini, A. Frigerio, M. Verri, A. Kossakowski, E.C.G. Sudarshan, Rep. Math. Phys. 13 (1978) 149.

[8] V. Gorini, A. Kossakowski, J. Math. Phys. 17 (1976) 1298.

[9] R. Alicki, K. Lendi, Quantum Dynamical Semigroups and Applications, Lecture Notes in Physics, Vol. 286, Springer, Heidelberg, 1987.

[10] A.S. Holevo, Statistical Structure of Quantum Theory, Lecture Notes in Physics, Vol. m67, Springer, Heidelberg, 2001.

[11] S.L. Adler, Phys. Lett. A 265 (2000) 58.

[12] J. Ellis, J.S. Hagelin, D.V. Nanopoulos, M. Srednicki, Nucl. Phys. B 241 (1984) 381.

[13] J. Ellis, J.L. Lopez, N.E. Mavromatos, D.V. Nanopoulos, Phys. Rev. D 53 (1996) 3846.

[14] T. Banks, L. Susskind, M.E. Peskin, Nucl. Phys. B 244 (1984) 125.

[15] P. Huet, M.E. Peskin, Nucl. Phys. B 434 (1995) 3;
P. Huet, M.E. Peskin, Nucl. Phys. B 488 (1997) 335.

[16] F. Benatti, R. Floreanini, Phys. Lett. B 389 (1996) 100.

[17] F. Benatti, R. Floreanini, Mod. Phys. Lett. A 12 (1997) 1465.

[18] F. Benatti, R. Floreanini, hep-ph/9806450.

[19] A.A. Andrianov, J. Taron, R. Tarrach, Phys. Lett. B 507 (2001) 200.

[20] R.A. Bertlmann, W. Grimus, Phys. Rev. D 64 (2001) 056004.

[21] F. Benatti, R. Floreanini, R. Romano, Nucl. Phys. B 602 (2001) 541.

[22] K. Enqvist, K. Kainulainen, J. Maalampi, Nucl. Phys. B 373 (1992) 498.

[23] C.-S. Chang et al., Phys. Rev. D 60 (1999) 033006.

[24] E. Lisi, A. Marrone, D. Montanino, Phys. Rev. Lett. 85 (2000) 1166.

[25] H.V. Klapdor-Kleingrothaus, H. Päs, U. Sarkar, Eur. Phys. J. A 8 (2000) 577.

[26] F. Benatti, R. Floreanini, JHEP 2 (2000) 32.

[27] F. Benatti, H. Narnhofer, Lett. Math. Phys. 15 (1988) 325.

[28] R.A. Bertlmann, B.C. Hiesmayr, Phys. Rev. A 63 (2001) 062112.

[29] K. Lendi, J. Phys. A: Math. Gen. 20 (1987) 15.