We will write x for (x^1, x^2, x^3).

[*Reminder:* The Galilean transformations read

$$t' = t \tag{1}$$
$$x' = R(x - vt) + c\,; \tag{2}$$

where v is the velocity vector, R an orthogonal Matrix, and c is a translation vector. (In components: $t' = t$, $x^i{}' = R^i{}_j(x^j - v^j t) + c^i$.]

1. (a) Show by a direct calculation that Newton's law,

$$\ddot{x}(t) = 0\,, \tag{3}$$

is invariant under Galilean transformations.

(b) Show that the Galilean transformations form a group.

2. The *harmonic oscillator equation*,

$$m\ddot{x}(t) = -kx(t) \qquad (k > 0)\,, \tag{4}$$

describes the motion of an object of mass m attached to a spring.

Is this equation Galilei-invariant? If not, how come? After all, (4) is the second Newton's law ($ma = F$), and therefore should be invariant under Galileo transformations?

Write down a system of equations for two objects connected by a spring satisfying Hook's law. Is this system invariant under Galileo transformations?

3. Recall that, in dimension three, a vector field X is a triplet of functions $X = (X^1, X^2, X^3)$. Show that for any vector field X we have

$$\nabla \times (\nabla \times X) = \mathrm{grad}(\mathrm{div} X) - \Delta X\ ,$$

where grad is the gradient of a function: $\mathrm{grad} f := (\partial_1 f, \partial_2 f, \partial_3 f)$; div is the divergence of a vector field $\mathrm{div} X := \partial_1 X^1 + \partial_2 X^2 + \partial_3 X^3$; and $\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator. Here $\partial_i := \frac{\partial}{\partial x^i}$. Finally $\nabla \times X$ is the curl of a vector field X:

$$\nabla \times X := \left(\frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial z}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial X^1}{\partial z} - \frac{\partial X^3}{\partial x}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial X^2}{\partial x} - \frac{\partial X^1}{\partial y}\right)\hat{\mathbf{e}}_3\ ,$$

where the $\hat{\mathbf{e}}_\mathbf{i}$'s are the canonical basis vectors.

4. A matrix $A = (A^i{}_j)$ is called orthogonal if for all vectors $z = (z^i)$ we have

$$\sum_i (A^i{}_j z^j)^2 = \sum_i (z^i)^2$$

(the summation convention is used on the j-index). Show that the following statements are equivalent:

(a) A is orthogonal,

(b) $A^\mathsf{T} A = I$, where A^T is the transpose of A, and I is the identity matrix,

(c) $\sum_i A^i{}_j A^i{}_k = \delta_{jk}$, where δ_{jk} vanishes for j distinct from k, and equals one otherwise,

(d) $AA^\mathsf{T} = I$,

(e) $\sum_i A^j{}_i A^k{}_i = \delta^{jk}$, where δ^{jk} vanishes for j distinct from k, and equals one otherwise.

5. [*Remark: Difficult; this is for self-study, and is unlikely to be discussed in class*] Die Schrödinger-Gleichung lautet

$$i\hbar\partial_t\psi = -\hbar^2/(2m)\Delta\psi \, .$$

Als nichtrelativistische Gleichung müsste die Schrödinger-Gleichung Galilei-invariant sein. Zeige, dass dies tatsächlich der Fall ist, aber nur wenn man voraussetzt, dass $\psi(t, x)$ nicht wie ein Skalar transformiert, sondern gemäß

$$\psi'(t', x') = e^{-imvx/\hbar + imv^2 t/(2\hbar)}\psi(t, x) \, ,$$

wobei wir hier der Einfachheit halber nur eine Raumdimension betrachten. (Also: $t' = t$, $x' = x - vt$, wo x und v eindimensional sind.) Hinweis: Man zeigt dann, dass

$$\left(i\hbar\partial_t + \hbar^2/(2m)\partial_x^2\right)\psi(t, x) = \left(i\hbar\partial_t + \hbar^2/(2m)\partial_x^2\right)\left(e^{miv/\hbar(x-vt/2)}\psi'(t'(t, x), x'(t, x))\right) = 0 \, ,$$

wenn $\psi'(t', x')$ die "gestrichene" Schrödinger-Gleichung erfüllt. Was wird geschehen wenn man noch dazu eine Translation addiert?

6. **[Summation convention]** For each of the following, either write out the equation with the summation signs included explicitly or explain why the equation is ambiguous or does not make sense. Provide a possible correct version, or versions, of the wrong or incoherent equations. (Recall that $\delta^a_b = 1$ if $a = b$ and is zero otherwise.)

(i) $X^a = L^a{}_b M^{bc} \hat{X}_c$	(v) $X^a = L^a{}_b \hat{X}^b + M^{ab} \hat{X}^b$
(ii) $X^a = L^b{}_c M^c{}_d \hat{X}^d$	(vi) $X^a = L^a{}_b \hat{X}^b + M^a{}_c \hat{X}^c$
(iii) $\delta^a_b = \delta^a_c \delta^c_d$	(vii) $X^a = L^a{}_c \hat{X}^c + M^b{}_c \hat{X}^c$
(iv) $\delta^a_b = \delta^a_b \delta^c_c$	(viii) $X^a = L^a{}_c \hat{X}^c + \sum_c M^{ac} \hat{X}^c$

(ix) Given two matrices $A = (A^\mu{}_\nu)$, $B = (B^\mu{}_\nu)$, verify that the product matrix

$$A \cdot B = ((A \cdot B)^\mu{}_\nu)$$

has entries

$$(A \cdot B)^\mu{}_\nu = A^\mu{}_\sigma B^\sigma{}_\nu \,.$$

Instead of writing $(A^\mu{}_\nu)$ for the matrix A, one often simply writes $A^\mu{}_\nu$.

(x) Verify that $A^\mu{}_\nu x_\mu$ corresponds to the matrix operation $A^\top x$, where $A^\top$ denotes the transpose of A.

Hence, if one considers $\eta = (\eta_{\mu\nu})$ to be a matrix, then $\eta_{\nu\mu}$ corresponds to the transpose of η.

(xi) What matrix operations are associated with $A^\mu{}_\nu\eta_{\mu\sigma}$, $A^\mu{}_\nu\eta_{\sigma\mu}$, $A^\mu{}_\nu B^{\nu\sigma}$ and $A^\mu{}_\nu B^{\sigma\nu}$?

7. Let $V = \mathbb{R}^4$, $t_\mu = (1, \vec{0}^\top)$, where $\vec{a}^\top$ denotes the transpose of a vector $\vec{a}$. Let $h^{\mu\nu}$ be of the form

$$h^{\mu\nu} = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & \mathbf{1} \end{pmatrix},$$

where $\mathbf{1}$ is the 3×3 identity matrix. Moreover, let

$$A^\mu{}_\nu = \begin{pmatrix} 1 & \vec{0}^\top \\ -\vec{v} & \mathbf{1} \end{pmatrix}.$$

Show that

(a)

$$A^\mu{}_\nu A^\lambda{}_\rho h^{\nu\rho} = h^{\mu\lambda}, \tag{5}$$

(b)

$$A^\mu{}_\nu t_\mu = t_\nu .$$

Explain how the matrix $A^{\mu\nu}$ describes Galilean boosts, and why (5) corresponds to the limit $c \to \infty$ in the equation defining Lorentz boosts.

8. Write the Lorentz transformation

$$\begin{aligned} t' &= \gamma \left(t - \frac{v}{c^2}\, x \right), \\ x' &= \gamma\, (x - v\, t)\ , \end{aligned}$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

as 2×2–matrix $L(v)$:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = L(v) \begin{pmatrix} t \\ x \end{pmatrix} .$$

Verify that $L(v)$ satisfies the defining relation for Lorentz matrices. Show that

$$L(v_1)\, L(v_2) = L\left(\frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\right) .$$

Compute the relativistic addition of n equal velocities. How many times does one need to add $c/2$ to end up with a velocity of $0.99c$ and $0.999c$, respectively?

[*Hint: Show that* $(1, c)^{\mathsf{T}}$ *is an eigenvector of L. What is the associated eigenvalue?*]

9. Let u^μ, v^ν be two vectors of the Minkowski space satisfying $u^\mu\, u_\mu = v^\mu\, v_\mu = -1$ and $u^\mu\, v_\mu < 0$. We remind the convention $u_\mu = \eta_{\mu\nu} u^\nu$, and $v_\mu = \eta_{\mu\nu} v^\nu$. Consider the linear mapping:

$$L^\mu{}_\nu = \delta^\mu{}_\nu - 2v^\mu\, u_\nu + (1 - u^\alpha\, v_\alpha)^{-1}\, (u^\mu + v^\mu)\,(u_\nu + v_\nu)\,. \tag{6}$$

Prove that:

(a) $L^\mu{}_\nu\, u^\nu = v^\mu$,

(b) $L^\mu{}_\nu\, L^\lambda{}_\rho\, \eta_{\mu\lambda} = \eta_{\nu\rho}$.

[*Hint: calculations are simpler if you introduce* $w^\mu := u^\mu + v^\mu$, $\phi := 1 - u^\alpha v_\alpha$, *rewrite* $L^\mu{}_\nu$ *in terms of those, calculate* $w^\mu u_\mu$, $w^\mu v_\mu$, *deduce* $w^\mu w_\mu$, *and continue from there.*]

What is the interpretation of $L^\mu{}_\nu$? [*Hint: Show that each map* (25) *is an identity on the space orthogonal to Span*$\{u, v\}$.]

Do the matrices (25) form a group?

[*Reminder:* Given two vectors $X \equiv (X^\alpha)$ and $Y \equiv (Y^\beta)$, their scalar product is calculated as

$$\eta(X, Y) := \eta_{\mu\nu} X^\mu Y^\nu \equiv -X^0 Y^0 + X^1 Y^1 + X^2 Y^2 + X^3 Y^3 .$$

A vector $X \neq 0$ is called timelike if $\eta(X, X) < 0$, spacelike if $\eta(X, X) > 0$, null or lightlike if $\eta(X, X) = 0$.]

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10. Let $\{e_0, e_1, e_2, e_3\}$ be a (pseudo)-orhonormal basis of Minkowski spacetime. Consider the vectors:

$$v_0 = \begin{pmatrix} 1 \\ -1/2 \\ 0 \\ 0 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} .$$

Which of these vectors is space/time/lightlike? Find a fourth vector v_3, othogonal to these three vectors. Is v_3 space/time/lightlike?

11. A clock C is at rest at the spatial origin of an inertial frame S. A second clock C' is at rest at the spatial origin of an inertial frame S' moving with constant speed v relative to S. The clocks read $t = t' = 0$ when the two spatial origins coincide. When C' reads t'_2 it receives a radio signal from C sent out when C reads t_1. Draw a space-time diagram describing this process. Determine the space-time coordinates (ct_2, x_2) in S of the point (event) at which C' receives the radio signal. Hence show that

$$t_1 = t'_2 \sqrt{\frac{1 - v/c}{1 + v/c}} .$$

Is there a relationship with the Doppler effect?

12. Find four linearly independent: (a) spacelike vectors, (b) timelike vectors, and (c) null vectors.

13. Let v and w be two timelike and linearly independent vectors. Prove that the line $\{v + \lambda w \,|\, \lambda \in \mathbb{R}\}$ intersects the light cone at the origin in exactly two points.

14. [*For self-study, unlikely to be covered in class.*] Prove explicitly that the property of the Lorentz transformations

$$L^{\mathsf{T}} \eta L = \eta$$

forms a system of ten independant equations in the coefficients of L (from which it should follow that the group of Lorentz transformation is a 6-parameter group?).

15. We say that a vector k is null, or lightlike, if $\eta(k, k) = 0$ but $k \neq 0$.

Let k be null. Show that the space $k^{\perp}$ of vectors orthogonal to k does not contain any timelike vectors, and that all null vectors in $k^{\perp}$ are multiples of k.

16. The coordinates (ct, x, y, z) and (ct', x', y', z') in two inertial frames S and S' respectively are related by

$$\begin{aligned} t' &= (\cosh\lambda)t - (\sinh\lambda)c^{-1}x \\ x' &= -(\sinh\lambda)c\,t + (\cosh\lambda)x \\ y' &= y \\ z' &= z\,, \end{aligned}$$

for a real number λ.

(a) Show that this defines a Lorentz transformation. If the origin in S' has speed V in S, what is V in terms of λ?

(b) A particle has 3-velocity $(a, b, 0)$ as measured in S and $(a', b', 0)$ as measured in S'. Find the relation between these 3-velocities in terms of λ.

(c) A light ray γ in S lies in the plane $z = 0$ and makes an angle α with the positive x-axis. Show that γ lies in $z' = 0$ in S'. Show that, if γ makes an angle α' with the positive x'-axis then $\tan(\alpha')/\tan(\alpha)$ is a function of V and $\cos\alpha$, which should be found.

17. Let p be the explosion of the Supernova 1987A, and consider the following events:

- q_1 = birth of Albert Einstein in Ulm
- q_2 = death of Jacques Albrespic in Tours
- q_3 = death of Richard Feynman in Los Angeles
- q_4 = birth of Lucy (*Australopithecus afarensis*)

Can p have caused q_i? or the other way round?

[*Hint: dates can be found on Wikipedia.*]

18. A beam of neutrinos is sent from CERN to the Gran Sasso National Laboratory for detection. The neutrinos travel with superluminal velocity $w > c$ measured at the rest frame of the Earth's crust, which we assume to be inertial. Another inertial observer travels in the same direction with subluminal velocity $v < c$. Assuming that the speed of light is c for all inertial observers, how fast does the moving observer have to travel in order to observe the detection in Gran Sasso happening *before* the beam is produced in CERN, i.e. to observe the neutrinos propagating backwards from Italy to Switzerland? Derive the general formula. What speed do you obtain if $w = (1 + 2 \times 10^{-5})c$? How does this value of w relate to the speed v of protons in the SPS and LHC in Geneva?

19. Consider two particles moving in an inertial frame in perpendicular directions with the same velocity v. Evaluate the relative velocity between these particles, both in Newtonian physics and in special relativity.

20. The cousins Yksi und Kaksi, two Emperor Pinguins, hatched at the exact same moment. While Yksi lived peacefuly in Antarctica, Kaksi was captured and taken to a zoo in Signapore. After 30 years Kaksi was allowed to retire from his zoo and return to Antractica. To his surprise, he discovered that Yksi was older than he was. By how much? You can ignore the travel time to Singapore and back. Should you take the motion of the Earth around the Sun into account?

[*Reminder:* Recall that the *proper-time parameterisation* of a worldline $\tau \mapsto x^\mu(\tau)$ is defined by the condition

$$\eta(\dot{x}, \dot{x}) = -1\,,$$

where $\dot{x} \equiv dx/d\tau \equiv (dx^\mu/d\tau)$. The *four-acceleration* is then defined as $a = (a^\mu) := \left(\dfrac{d^2x^\mu}{d\tau^2}\right)$.]

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21. **[Time dilation]** Let X be an inertial observer with four-velocity u, and let X' be an inertial observer with four-velocity w. Calculate the intersection of the surface of simultaneity $t' = \text{const}$ of X' with the time-axis of X; call the corresponding time coordinate t. Show that $t = \gamma^{-1} t'$.

    Calculate, next, the intersection of the surface of simultaneity $t = \text{const}$ of X with the time-axis of X'; call the corresponding time coordinate t'. Show that $t' = \gamma^{-1} t$.

    Conclude that every observer "sees" the time of the other as flowing slower than his (since $\gamma^{-1} < 1$).

22. (a) Relative to an observer X (with coordinates $\{t, \vec{x}\}$) the worldline of a particle is described by

    $$(-1,1) \ni \lambda \mapsto x^\mu(\lambda) = \begin{pmatrix} \sqrt{5}\lambda \\ 1 + \lambda^2 \sin\lambda \\ \lambda^2 \cos\lambda \\ 0 \end{pmatrix} .$$

    Compute the three-velocity $\vec{v}(t)$ of the particle. What is its four-velocity. (Why has the assumption $\lambda \in (-1,1)$ been imposed rather than e.g. $\lambda \in \mathbb{R}$?)

    (b) A particle moves on a circular path with radius r; let ω be its angular frequency. (Some care is needed here. Why?) Determine the worldline of the particle. Parameterize the worldline by the proper time. Compute the four-velocity as well as the four-acceleration of the particle.

[*Reminder:* The wave vector k of a photon]

The wave vector k of a photon is defined as $k^\mu = (\omega, \vec{k})$, where $\omega = |\vec{k}| \neq 0$ is the frequency of the photon, and $\vec{k}$ its direction of propagation. Given an inertial observer O moving with four-velocity u, the frequency seen by O equals $-\eta(u,k)$.

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23. **[Doppler effect I]**

    (a) Consider an inertial frame attached to Earth. Check that the worldline

    $$(t(\tau), x(\tau)) = \left(g^{-1}\sinh(g\tau), g^{-1}(\cosh(g\tau) - 1)\right) ,$$

    describes a spaceship with constant acceleration g passing through the origin at $\tau = 0$ with zero initial velocity, where τ is the proper time of the spaceship. What is the asymptotic behaviour of the proper time $\tau \to \infty$ as a function of the time coordinate $t(\tau)$ of the spaceship?

    (b) Express the four velocity of the spaceship in the inertial frame of Earth. Deduce the four velocity of the Earth in an instantaneous inertial frame attached to the spacecraft.

(c) Consider a photon emitted from Earth towards the spaceship. Express the wave vector of the photon both in the instantaneous inertial frame of the spaceship and in the inertial frame of the earth. Deduce that the frequencies of the photon such as measured on the spaceship ω_S and on Earth ω_E are related by

$$\omega_E = \omega_S \sqrt{\frac{1+v(\tau)}{1-v(\tau)}},$$

where $v(\tau)$ is the velocity of the spacecraft in the earth frame.

(d) When, in Earth’s time, and the astronaut’s time, will the astronaut start seeing the blue oceans of the Earth ($\lambda_b = 450\,nm$) as red ($\lambda_r = 700\,nm$) if a equals the earth acceleration? How far will then the spaceship be?

(e) Same question with $g = 5g_{\oplus}$ (high-$g_{\oplus}$ rollercoaster)? $g = 9g_{\oplus}$ (military pilot training)? $g = 100g_{\oplus}$ (brief exposure in a crash)?

24. **[Doppler effect II]** Let X be an observer and Y a source of photons. Let α be the angle between the direction of motion of the observer X and the photons (as observed from Y, the source of the photons.) Prove that there exists a unique angle α_{noD} of the relative velocity v_{XY} of X and Y, so the Doppler effect disappears (that is to say that $\omega_X = \omega_Y$). Prove finally that, for small velocities, the following relation is true:

$$\alpha_{\mathrm{noD}} = \frac{\pi}{2} - \frac{v_{XY}}{2} + O\left(v_{XY}^3\right) .$$

25. **[Doppler effect III]** A rigid ring of radius $R = 1$ m spins with constant angular frequency $\omega = 2.1 \cdot 10^8$ Hz around its axis of symmetry. Every infinitesimal element of the ring emits electromagnetic radiation of wavelength 450 nm (i.e. monochromatic blue light) as measured in the comoving frame of that element. What is the color of the ring perceived by (a) an observer at the center of symmetry the ring, stationary with respect to that center, (b) by an observer situated somewhere on the ring, moving together with the ring? [*Hint: calculate the emitted and observed frequencies in the rest frame of the center of the ring.*]

26. **[Alice through the moving mirror—Doppler effect IV]** A plane mirror moves in the direction of its normal with uniform velocity v, towards Alice, in Alice’s rest frame S, and facing her (so in Alice’s rest frame the three-velocity of the mirror is $(-v, 0, 0)^{\mathsf{T}}$ with $v > 0$). A ray of light of frequency ω_1, which we will call ingoing, strikes the mirror at an angle of incidence θ, and is reflected with frequency ω_2 at an angle of reflection φ. The purpose is to prove that

$$\frac{\tan\frac{\theta}{2}}{\tan\frac{\varphi}{2}} = \frac{c+v}{c-v}\,, \quad \frac{\omega_2}{\omega_1} = \frac{\sin\theta}{\sin\varphi} = \frac{c\cos\theta + v}{c\cos\varphi - v} = \frac{c + v\cos\theta}{c - v\cos\varphi}\,. \tag{7}$$

Because of the geometry of the problem, we can assume that we are in three-dimensional Minkowski space-time.

(a) Assuming that the wave vector of the incoming light ray with respect to the frame S is

$$k_1 = \omega_1 (1, \cos\theta, \sin\theta)^{\mathsf{T}} ,$$

find the wave vector k_1' in the frame of the mirror. Using the standard law of reflection in the mirror's frame, deduce the wave vector k_2' of the outgoing light ray in the frame S'. Find the outgoing light ray in the frame S, say k_2, by transforming back k_2' into the frame S.

(b) Introducing φ by $k_2 = \omega_2 \,(1, -\cos\varphi, \sin\varphi)^{\mathsf{T}}$, deduce that

$$\frac{\omega_2}{\omega_1} = \frac{\sin\theta}{\sin\varphi}, \;\; \cos\varphi = \frac{2v + v^2\cos\theta + \cos\theta}{1 + v^2 + 2v\cos\theta}, \;\; \sin\varphi = \frac{\sin\theta}{\gamma^2(1 + v^2 + 2v\cos\theta)}.$$

(c) Use the identity

$$\tan\frac{\varphi}{2} = \frac{\sin\varphi}{1 + \cos\varphi} ,$$

to derive the first equation in (7).

27. **[Fizeau experiment]** Consider the Fizeau experiment as presented in the lecture. Use both the non-relativistic and the relativistic laws of addition of velocities, derive the respective formulae for the difference of time of arrival of the photons at the interference point. Determine the associated phase shift.

28. Consider the two worldlines

$$x_X(t) = \left(t, -\frac{1}{2}\sin t, 0, 0\right)^{\mathsf{T}} , \quad x_Y(t) = \left(t, \frac{3}{4}\arctan t, 0, 0\right)^{\mathsf{T}} , \tag{8}$$

given in some inertial coordinate system $\{t, \vec{x}\}$.

(a) Compute the relative velocity $|\vec{v}| = |\vec{v}_{XY}|$ between the particles at time $t = 0$.

(b) Using the formula for the relative γ factor show that

$$|\vec{v}| = 2\frac{|3 + 2(1 + t^2)\cos t|}{8(1 + t^2) + 3\cos t} . \tag{9}$$

Can this be interpreted as a relative velocity of the observers for $t \neq 0$? Are there ambiguities here? For instance, for $t \approx 1.90172$, the formula yields $|\vec{v}| = 0$ which suggests that we can conclude that the particles X and Y represented by the worldlines are at rest with respect to each other at that time. Is this unambiguous?

29. Derive the conditions on the constant vectors $\vec{E}_0$, $\vec{B}_0$ and k^μ so that the fields

$$\vec{E} = \vec{E}_0 e^{ik_\mu x^\mu} \text{ and } \vec{B} = \vec{B}_0 e^{ik_\mu x^\mu}$$

satisfy the source-free Maxwell equations.

30. An astronomer observes four stars $S_1, \ldots S_4$ and makes a note of their angular distances θ_{ij} between S_i und S_j for all i, j. Show that the quantity

$$\frac{(1-\cos\theta_{12})(1-\cos\theta_{34})}{(1-\cos\theta_{13})(1-\cos\theta_{24})}$$

is independent of the state of motion of the astronomer.

31. (a) Show that a photon cannot spontaneously disintegrate into an electron-positron pair.

    (b) Recall that the Zero Momentum (ZM) frame is defined as the inertial frame in which the space-momentum vector of the system vanishes. Find the velocity of the ZM frame of two photons of frequencies ν_1 and ν_2 that travel in the positive and negative x-directions respectively.

32. Recall, from the lectures, the special relativistic formula for the aberration of light:

$$\cos(\bar{\theta}) - 1 = \frac{(1+v)(\cos(\theta)-1)}{1-v\cos(\theta)}, \tag{10}$$

where θ is the angle, in the frame of O, between the velocity of the moving observer $\overline{O}$ and the direction of the photon. Show that

$$\begin{aligned} \cos(\bar{\theta}) &= \frac{\cos(\theta)-v}{1-v\cos(\theta)}, & (11)\\ \tan\left(\frac{\bar{\theta}}{2}\right) &= \pm\sqrt{\frac{1+v}{1-v}}\tan\left(\frac{\theta}{2}\right). & (12) \end{aligned}$$

Which sign is correct? At what velocity will a moving observer $\overline{O}$ looking at an angle $\pi/4$ away from his direction of motion see an object which lies behind him at an angle $3\pi/4$ in the frame of O.

33. Radiation energy from the Sun is received on Earth on the equator at the rate of 1.94 calories per minute per square centimeter. Assuming the distance of the Sun to be 150 000 000 km, find the total mass lost by the Sun per second, and the force exerted by solar radiation on a black disk of the same diameter as the Earth (use 12 800 km), at the location of the Earth.

34. How fast must a particle move before its *kinetic energy*, defined as the difference between the total energy and the rest energy, equals the rest energy?

35. Consider the photoproduction of pions π^0

$$\gamma + \mathsf{p} \to \pi^0 + \mathsf{p},$$

with the target proton p at rest. What is the minimal energy of the photon γ for this process to take place? (Assume that the mass of the proton is 0.94 GeV, and the mass of the pion 140 MeV.) Compare the resulting frequency to that of hard X rays ($\lambda_X = 10^{-2}$ nm). What will the result be if the target proton is moving in the same direction as the photon?

36. **[Energy efficiency of the production of particles]** We consider the production of pions by collision of protons in a particle accelerator. The effective energy E_{out} of the production of particles is the sum of the energies of the particles $\sum_j m_j$ (up to a c^2-factor in International Units) produced by the collision. The minimal kinetic energy, in the rest frame of the accelerator, necessary to produce the new particles from the collision of the original particles is denoted by E_{in}. The coefficient of energy efficiency of the particle generation is defined as $\kappa = E_{\text{out}}/E_{\text{in}}$. Compute κ for

$$\mathsf{p} + \mathsf{p} \to \mathsf{p} + \mathsf{p} + \pi^0 \,,$$

where p is a proton and π^0 is a pion when

(a) one proton is at rest in the rest frame of the accelerator;

(b) both protons are colliding with opposite velocities as measured in the rest frame of the accelerator.

Does κ depend on the frame in which it is computed?

37. The energy-density ρ and the *Poynting vector* $\vec{p}$ for the Maxwell field are defined as

$$\rho = \frac{1}{8\pi}\left((\vec{E})^2 + (\vec{B})^2\right), \quad \vec{p} = \frac{1}{4\pi}\vec{E} \times \vec{B}\,.$$

Calculate $(\rho, \vec{p})$ for a plane wave:

$$\vec{E} = \cos(\omega t - \vec{k}\vec{x})\,\vec{e}\,, \quad \vec{B} = \omega^{-1}\cos(\omega t - \vec{k}\vec{x})\,\vec{k} \times \vec{e}\,,$$

with constant vectors $\vec{e}$ and $\vec{k} \neq \vec{0}$, such that $\vec{e} \perp \vec{k}$, and with $\omega = |\vec{k}|$.

38. **[Changes of coordinates]** Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let X be the vector field which in the above coordinate system equals $(1, 1, 0, 0)^{\mathsf{T}}$, and let α be a one-form which in the above coordinate system equals $(1, 1, 0, 0)$.

Find the metric coefficients $\tilde{g}_{ab}$, and the components of X and α, in each of the following coordinate systems.

(a) $\tilde{x}^0 = t - z$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = z$,

(b) $\tilde{x}^0 = \tau, \;\; \tilde{x}^1 = \phi, \;\; \tilde{x}^2 = y, \;\; \tilde{x}^3 = z,$

(c) $\tilde{x}^0 = t, \;\; \tilde{x}^1 = r, \;\; \tilde{x}^2 = \theta, \;\; \tilde{x}^3 = \phi,$

where, in the first case, (r, θ) are plane polar coordinates in the (x, y) plane: $x = r\cos\theta$, $y = r\sin\theta$; in the second, (τ, ϕ) are 'Rindler coordinates', defined by $t = \tau\cosh\phi$, $x = \tau\sinh\phi$; and, in the third, (r, θ, ϕ) are spherical polar coordinates. In each case, state which region of Minkowski space the coordinate system covers. [*Hint: A quick method for the metric is to write it as* $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ *and substitute, for example,*

$$\mathrm{d}x = \mathrm{d}(r\cos\theta) = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta\,,$$

and so on. Here we have used the definition $\mathrm{d}f = \partial_\mu f\,\mathrm{d}x^\mu$ *of the differential* $\mathrm{d}f$ *of a function* f*. Of course you should convince yourself that this is legitimate.*]

39. **[Raising and lowering of indices]** Let $g_{\mu\nu}$ be a symmetric tensor field with non-zero determinant, and let $g^{\alpha\beta}$ denote a tensor field such that $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$, where δ^α_γ equals 1 if both indices are equal, and zero otherwise. (When thinking in terms of matrices, then the matrix $(g^{\alpha\beta})$ is inverse to the matrix $(g_{\alpha\beta})$.) Define

$$B_\alpha := g_{\alpha\beta}A^\beta\,, \quad C^\gamma := g^{\gamma\sigma}B_\sigma\,. \tag{13}$$

Show that

$$C^\gamma = A^\gamma\,. \tag{14}$$

The first operation in (13) is called "lowering an index with the metric"; the second "raising an index with the metric". What does (14) say about these operations?

From now on we shall simply write

$$A_\alpha := g_{\alpha\beta}A^\beta\,, \quad B^\gamma := g^{\gamma\sigma}B_\sigma\,.$$

Show that

$$A_\alpha B^\alpha = A^\alpha B_\alpha\,.$$

40. The *alternating tensor* $\epsilon_{\alpha\beta\gamma\delta}$ in four space-time dimensions is defined by the requirement that it changes sign under the permutation of any two indices (such tensors are called *totally antisymmetric*), and

$$\epsilon_{0123} = 1\,.$$

Does this indeed define $\epsilon_{\alpha\beta\gamma\delta}$ uniquely? [*Hint: What is the value of* $\epsilon_{\alpha\beta\gamma\delta}$ *when some indices coincide?*]

Define $\epsilon^{\alpha\beta\gamma\delta}$ by raising the indices using *some* symmetric two-contravariant tensor $\eta^{\mu\nu}$, with inverse tensor $\eta_{\mu\nu}$, possibly, but not necessarily, equal to the Minkowski metric:

$$\epsilon^{\alpha\beta\gamma\delta} = \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\gamma\rho}\eta^{\delta\sigma}\epsilon_{\mu\nu\rho\sigma}\,.$$

Show that $\epsilon^{\alpha\beta\gamma\delta}$ is totally antisymmetic. Explain why

$$\epsilon^{0123} = \det \eta^{\alpha\beta} \ .$$

[*Hint: how would the corresponding equation look like in two-dimensions? and in three? If in doubt, look up the definition of the determinant on e.g. Wikipedia.*] Similarly show that

$$\Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta}\Lambda^{\gamma'}{}_{\gamma}\Lambda^{\delta'}{}_{\delta}\epsilon_{\alpha'\beta'\gamma'\delta'} = \det\Lambda\epsilon_{\alpha\beta\gamma\delta} \ .$$

How can this be generalised to other dimensions? or to the Euclidean metric? How many totally antisymmetric tensors with five indices are there in dimension n, $1 \le n \le 7$?

[*A word of caution: we will see in the remainder of this course that ϵ is not quite a tensor, but a tensor density. It is, however, consistent to consider it as a tensor when considering only orthochronous orientation preserving Lorentz transformations.*]

41. Let a bracket over indices denote complete antisymmetrisation, and a parenthesis over indices denote complete symmetrisation: for example,

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) \ , \quad A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) \ , \quad \delta^{[\alpha}_{\mu}\delta^{\gamma]}_{\rho} = \frac{1}{2}(\delta^{\alpha}_{\mu}\delta^{\gamma}_{\rho} - \delta^{\gamma}_{\mu}\delta^{\alpha}_{\rho}) \ ,$$

$$A_{[\mu\nu\rho]} = \frac{1}{6}(A_{\mu\nu\rho} + A_{\nu\rho\mu} + A_{\rho\mu\nu} - A_{\nu\mu\rho} - A_{\mu\rho\nu} - A_{\rho\nu\mu}) \ ,$$

and similarly for four or more indices (with combinatorial prefactors $1/n!$). Show that

(a) $A_{(\mu\nu)} = A_{(\nu\mu)}$, $A_{[\mu\nu]} = -A_{[\nu\mu]}$,

(b) $A_{\mu\nu} = A_{[\mu\nu]} + A_{(\mu\nu)}$,

(c) if $C_{\alpha\beta}$ is symmetric, i.e. $C_{\alpha\beta} = C_{\beta\alpha}$, and if $D^{\alpha\beta}$ is antisymmetric, i.e. $D^{\alpha\beta} = -D^{\beta\alpha}$, then $C_{\alpha\beta}D^{\alpha\beta} = 0$,

(d) $A^{[\mu\nu]}B_{\mu\nu} = A^{\mu\nu}B_{[\mu\nu]} = A^{[\mu\nu]}B_{[\mu\nu]}$,

(e) $A^{(\mu\nu)}B_{\mu\nu} = A^{\mu\nu}B_{(\mu\nu)} = A^{(\mu\nu)}B_{(\mu\nu)}$,

(f) $A^{[\mu\nu\rho]}B_{\mu\nu\rho} = A^{\mu\nu\rho}B_{[\mu\nu\rho]}$,

(g) $\delta^{[\alpha}_{\mu}\delta^{\gamma]}_{\rho} = \delta^{\alpha}_{[\mu}\delta^{\gamma}_{\rho]} = \delta^{[\alpha}_{[\mu}\delta^{\gamma]}_{\rho]}$,

(h) $\delta^{[\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma]}_{\rho} = \delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho]} = \delta^{[\alpha}_{[\mu}\delta^{\beta}_{\nu}\delta^{\gamma]}_{\rho]}$,

(i) $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\rho} = -6\delta^{\delta}_{\rho}$,

(j) $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\nu\rho} = -4\delta^{\gamma}_{[\nu}\delta^{\delta}_{\rho]}$.

42. Assuming the tensorial transformation law of $F^{\mu\nu}$ (this means that, under Lorentz transformations $x^{\mu} \mapsto \Lambda^{\mu}{}_{\nu}x^{\nu}$, we have $F^{\mu\nu} \mapsto \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$) , derive the explicit formulae for the transformation laws of the electric and magnetic fields under a boost along the first coordinate axis.

43. Let $\vec{E}\cdot\vec{B}=0$, and suppose that $|\vec{E}|^2 \neq |\vec{B}|^2$. Show that there exists a Lorentz frame in which either $\vec{E}$ or $\vec{B}$ vanishes. [*Hint: apply a boost with $\vec{v}$ proportional to $\vec{E}\times\vec{B}$.*]

44. Let $*F_{\alpha\beta} := \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$. Show that the contraction $F_{\alpha\beta}F^{\alpha\beta}$ is invariant (more precisely, behaves as a scalar) under Lorentz transformations, while $F^{\alpha\beta}(*F_{\alpha\beta})$ either remains invariant, or changes sign. Express those contractions in terms of $\vec{E}$ and $\vec{B}$.

45. [*For self-study, unlikely to be covered in class.*]

    (a) Assuming that $\Lambda^\alpha{}_\beta$ is a Lorentz matrix, show that the matrix $A^\alpha{}_\nu := \eta^{\alpha\beta}\Lambda^\mu{}_\beta\eta_{\mu\nu}$ is inverse to $\Lambda^\alpha{}_\beta$.

    (b) Recall that we required that $F^{\mu\nu}$ transforms as a *two-contravariant tensor* under Lorentz transformations: if $\bar{x}^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$, then

    $$\overline{F}^{\mu\nu}(\bar{x}) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta}(x)\ ,$$

    and that $F_{\mu\nu}$ has been defined as

    $$F_{\mu\nu} := \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}\ .$$

    Use (a) to show that $F_{\alpha\beta}$ transforms as

    $$\overline{F}_{\mu\nu}(\bar{x}) = (\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu F_{\alpha\beta}(x)$$

    (this is called the *transformation law of a two-covariant tensor*).

46. Define a $*$-operation on anti-symmetric tensors as

    $$*F_{\alpha\beta} := \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}F_{\gamma\delta}\ , \qquad *F^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\ ,$$

    where

    $$\epsilon_{\alpha\beta}{}^{\gamma\delta} := \eta_{\alpha\mu}\eta_{\beta\nu}\epsilon^{\mu\nu\gamma\delta}\ .$$

    (a) Show that

    $$\partial_\mu(*F^{\mu\nu}) = 0 \quad\Longleftrightarrow\quad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad\Longleftrightarrow\quad \partial_{[\alpha}F_{\beta\gamma]} = 0\,.$$

    (b) Show that the double star of an anti-symmetric tensor is the negative of this tensor.

    (c) Show that if $F_{\mu\nu}$ and $G_{\mu\nu}$ are anti-symmetric, then $*F^{\alpha\beta}G_{\alpha\beta} = F^{\alpha\beta} *G_{\alpha\beta}$. Conclude that $*F^{\alpha\beta} *F_{\alpha\beta} = -F^{\alpha\beta}F_{\alpha\beta}$. Can you think of a simpler proof of the last equality?

47. Consider a charged particle which moves along *a straight line in Minkowski spacetime* in an electromagnetic field. Show that its velocity $\vec{v}$ satisfies $\vec{E}+\vec{v}\times\vec{B}=0$ and $\vec{E}\cdot\vec{v}=0$. Find all possible solutions for $\vec{v}$ in terms of $\vec{E}$ and $\vec{B}$. What can you say about $F^{\alpha\beta}F_{\alpha\beta}$ and $*F^{\alpha\beta}F_{\alpha\beta}$?

48. The tensor field
$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\eta_{\mu\nu}\right) \qquad (15)$$
is called the *energy-momentum tensor* of the electromagnetic field. Express T_{00}, T_{0i}, and T_{ij} in terms of E^i and B^i.

49. Describe the gauge transformations which preserve the Lorenz gauge condition.

50. [*For self-study, unlikely to be covered in class.*] Let $T_{\mu\nu}$ be given by (15). Show that
$$T_{\mu\rho}T^\rho{}_\nu = \frac{1}{4}T_{\alpha\beta}T^{\alpha\beta}\eta_{\mu\nu}\ .$$
[*Hint: An efficient proof uses the Cayley-Hamilton theorem.*]

51. Let $A_{\mu\nu}$ be an invertible matrix, with inverse $B^{\mu\nu}$. Show that
$$\frac{\partial(\det A_{\mu\nu})}{\partial A_{\rho\sigma}} = (\det A_{\mu\nu})B^{\sigma\rho}\ .$$
Deduce that
$$\frac{\partial\sqrt{-\det g_{\mu\nu}}}{\partial g_{\alpha\beta}} = \frac{1}{2}\sqrt{-\det g_{\mu\nu}}\,g^{\alpha\beta}\,, \qquad \frac{\partial\sqrt{-\det g_{\mu\nu}}}{\partial g^{\alpha\beta}} = -\frac{1}{2}\sqrt{-\det g_{\mu\nu}}\,g_{\alpha\beta}\,,$$
[*Hint: You can find $\partial g_{\alpha\beta}/\partial g^{\mu\nu}$ by differentiating the defining identity $g^{\pi\rho}g_{\rho\sigma} = \delta^\pi_\sigma$.*].
What is $\frac{\partial\sqrt{|\det g_{\mu\nu}|}}{\partial g_{\alpha\beta}}$?
Further show that
$$\partial_\alpha\sqrt{\det A_{\mu\nu}} = \frac{1}{2}\sqrt{\det A_{\mu\nu}}B^{\rho\sigma}\partial_\alpha A_{\sigma\rho}\,,$$
as well as
$$\nabla_\alpha X^\alpha = \frac{1}{\sqrt{-\det g_{\mu\nu}}}\partial_\alpha(\sqrt{-\det g_{\mu\nu}}X^\alpha)\,. \qquad (16)$$

52. Write-down the Euler-Lagrange equations, the canonical energy-momentum tensor
$$t^\mu{}_\nu = \frac{\partial L}{\partial\phi^A{}_{,\mu}}\phi^A{}_{,\nu} - L\delta^\mu_\nu\,,$$
and the symmetric energy-momentum tensor
$$T_{\mu\nu} = 2\frac{\partial L}{\partial g^{\mu\nu}}\,,$$
for the following Lagrageans:

(a) $L = \frac{1}{2}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + m^2\phi^2\right)\sqrt{-\det g_{\mu\nu}}$
(massive scalar field);

(b) $L = \frac{1}{16\pi}\left(g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}\right)\sqrt{-\det g_{\mu\nu}}$,
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (Maxwell field); and

(c) $L = \frac{1}{16\pi}\left(e^{-a\phi}g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu} + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\sqrt{-\det g_{\mu\nu}}$,
where as before $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $a \in \mathbb{R}$ (dilaton-Maxwell system). (Here ϕ is *not* A_0, the fields ϕ and A_μ are independent.)

In the calculation of the Euler-Lagrange equations and of the canonical energy-momentum tensor you can assume that $g_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric.

Can you write the field equations in an explicitly tensorial form for a general metric? [*Hint: Use* (16).]

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[*Reminder:* The Christoffel symbols of the Levi-Civita connection are defined by the formula

$$\Gamma^a{}_{bc} \equiv \Gamma^a_{bc} := \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \,.$$

A vector with components $(X^1, \dots, X^n)$ will often be written as $X^1\partial_1 + \dots + X^n\partial_n \equiv X^a\partial_a$. Thus, ∂_1 is the same as the vector with components $(1, 0, \dots 0)$, etc. Similarly, a covector $(\alpha_1, \dots, \alpha_n)$ will be written as $\alpha_1 dx^1 + \dots + \alpha_n dx^n \equiv \alpha_a dx^a$.]

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53. **[Christoffel symbols]** Calculate the Christoffel symbols for (a) the Euclidean metric on $\mathbb{R}^2$ in polar coordinates: $g = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2$, and (b) the unit round metric on S^2: $h = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$.

    [*Hint: Calculate first the Christoffel symbols for a metric of the form* $\mathrm{d}x^2 + e^{2f(x)}\mathrm{d}y^2$.]

54. Using the results of the previous exercise, calculate the Riemann tensor, the Ricci tensor and the Ricci scalar of the unit round metric on S^2, namely $g = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$.

55. **[Christoffel symbols]** Establish the transformation law

$$\Gamma^a_{bc} = \tilde\Gamma^d_{ef}\frac{\partial x^a}{\partial \tilde x^d}\frac{\partial \tilde x^e}{\partial x^b}\frac{\partial \tilde x^f}{\partial x^c} + \frac{\partial x^a}{\partial \tilde x^d}\frac{\partial^2 \tilde x^d}{\partial x^b \partial x^c}$$

by direct calculation. Explain why this implies that the Christoffel symbols do *not* define a tensor.

56. **[Euler-Lagrange equations for geodesics]**

    (a) We consider curves $s \mapsto x(s)$, set $\dot x := dx/ds$, Show that the Euler-Lagrange equations

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot x^a}\right) = \frac{\partial L}{\partial x^a} \tag{17}$$

associated with the Lagrange function

$$L(x^c, \dot{x}^c) = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b \tag{18}$$

can be written as

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0 \; . \tag{19}$$

(b) Show that if the metric does not explicitly depend upon a coordinate, say x^1, then $g(\dot{x}, \partial_1)$ is constant along every geodesic.

(c) Use equation (17) to calculate again the associated Christoffel symbols for the two-dimensional metrics g and h of Problem 51.

57. Write down the geodesic equation for proper-time parameterised timelike geodesics in the “post-Newtonian metric”:

$$g_{00} = -\left(1 - \frac{2GM}{r}\right) , \quad g_{0i} = 0 \; , \quad g_{ij} = \left(1 + \frac{2GM}{r}\right) \delta_{ij} \; , \tag{20}$$

with $i, j \in \{1, 2, 3\}$. (This is the Newtonian approximation, for $GM/r \ll 1$, of the metric tensor of a spherically symmetric body of mass M.) In your calculations neglect all terms quadratic in GM.

58. Write down the geodesic equation for affinely parameteried geodesics in the Schwarzschild metric:

$$g = -V dt^2 + V^{-1} dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) \, , \quad V = 1 - \frac{2m}{r} \, ,$$

where m is a constant. Use the result to determine the Christoffel symbols. Can you solve some of the equations? Can you think of some obvious solutions of the whole system?

59. **[Lie bracket]** Recall that vector fields can be identified with homogeneous linear first order partial differential operators $X = X^a \partial_a$ acting on functions as $X(f) = X^a \partial_a f$.

The Lie-bracket $[X, Y]$ of two vector fields X and Y is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \; .$$

Show that $[X, Y]$ also is a vector field, i.e. a homogeneous linear first order differential operator, with components

$$[X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a \; . \tag{21}$$

Check, by a direct coordinate calculation, that the right-hand-side of (21) transforms as a vector field under changes of coordinates.

[*For self-study, unlikely to be covered in class*] Prove the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \; .$$

60. **[Schwarzschild: Orders of magnitude]** Recall that the Schwarzschild metric g takes the form

$$g = -\left(1 - \frac{2MG}{rc^2}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{2MG}{rc^2}} + r^2\left(d\theta^2 + \sin\theta^2 d\varphi^2\right) , \tag{22}$$

where M is the total mass of the gravitating object, G is Newton’s constant and c is the speed of light. Calculate the deviation of g_{00}/c^2 from minus one, where $x^0 = t$, as well as the deviation of $\partial_i g_{00}/c^2$ from zero, a) at the surface of the sun when M is the mass of the sun, b) at the orbit of the earth when M is the mass of the sun, c) at the surface of the earth when M is the mass of the earth, d) at the orbit of the moon when M is the mass of the earth, and e) at the surface of the moon when M is the mass of the moon.

Clearly, something wrong is happening with (22) at $r = 2MG/c^2$; this is called the *Schwarzschild radius*. Calculate the Schwarzschild radius of a) the sun, b) the earth, and c) the moon. Calculate the corresponding mass densities, compare the result to the density of a neutron.

61. **[An exact gravitational wave]** Let η_{ab} and η^{ab} be the covariant and contravariant metric tensors on Minkowski space $\mathcal{M}$, with standard Lorentzian coordinates x^a so that

$$(\eta_{ab}) = -\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{23}$$

Let n_a be a constant covector on $\mathcal{M}$ satisfying $\eta^{ab} n_a n_b = 0$. Define a new metric on $\mathcal{M}$ by

$$g_{ab} = \eta_{ab} + f n_a n_b ,$$

where f is a function on $\mathcal{M}$ such that $\eta^{ab} n_a \partial_b f = 0$, where $\partial_a = \partial/\partial x^a$.

(a) Show that the Christoffel symbols of g_{ab} are given by

$$\Gamma^a{}_{bc} = \tfrac{1}{2}\eta^{ad}\left(n_d n_b \partial_c f + n_d n_c \partial_b f - n_b n_c \partial_d f\right) .$$

[*Hint: Look for g^{ab} of the form $\eta^{ab} + h n^a n^b$, where $n^a = \eta^{ab} n_b$.*]

(b) The Ricci tensor is defined as

$$-R_{ac} = \partial_d \Gamma^d{}_{ac} - \partial_a \Gamma^d{}_{dc} + \Gamma^d{}_{de}\Gamma^e{}_{ac} - \Gamma^d{}_{ae}\Gamma^e{}_{dc} . \tag{24}$$

Show that

$$R_{ab} \text{ is proportional to } n_a n_b \eta^{cd} \partial_c \partial_d f , \tag{25}$$

[*Hint: Check that $\Gamma^d{}_{de}$ vanishes. Show, next, that any contraction of n^a with the Christoffel symbols vanishes, and conclude that the last term in* (24) *gives no contribution either.*]

(c) Show that Einstein’s vacuum field equations,

$$R_{ab} = 0 ,$$

have solutions as above with $f = \alpha \sin(k_a x^a)$, with $\alpha \in \mathbb{R}$, and with k_a having constant components in the coordinate system where η_{ab} takes the form (23), provided that k_a satisfies $\eta^{ab} k_a k_b = \eta^{ab} k_a n_b = 0$. Deduce that such a k_a is proportional to n_a.

62. **[Symmetries of the curvature tensor]**

(a) What does it mean for a connection ∇ on a space-time with metric g_{ab} to be (a) a *metric connection*, (b) *torsion free*?

Assume henceforth that ∇ is torsion free.

In what follows, it is often useful to use the preceding results to do the next ones.

(c) Given an arbitrary smooth covector field A_a, and a smooth antisymmetric tensor field F_{ab}, show that

$$H_{ab} := \nabla_a A_b - \nabla_b A_a \quad \text{and} \quad \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab}$$

are both independent of the choice of the connection.

(d) Hence or otherwise show that

$$\nabla_a H_{bc} + \nabla_b H_{ca} + \nabla_c H_{ab} = 0 .$$

Recall that the curvature tensor is defined as

$$\nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c = R^c{}_{dab} V^d .$$

(e) Show that

$$\nabla_a \nabla_b A_c - \nabla_b \nabla_a A_c = -R^d{}_{cab} A_d .$$

(f) Hence show that $R^d{}_{abc} + R^d{}_{bca} + R^d{}_{cab} = 0$ for a torsion-free connection.

(g) Show further that, for a tensor T_{ab},

$$\nabla_a \nabla_b T_{cd} - \nabla_b \nabla_a T_{cd} = -R^e{}_{cab} T_{ed} - R^e{}_{dab} T_{ce} .$$

(h) Hence show that $R_{abcd} = -R_{bacd}$ if ∇ is metric and torsion-free.

(b) Show that the symmetry $R_{abcd} = R_{cdab}$ follows from $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$ and $R_{a[bcd]} = 0$.

63. **[Counting components]**

(a) In four dimensions, a tensor satisfies $T_{abcde} = T_{[abcde]}$. Show that $T_{abcde} = 0$.

(b) A tensor T_{ab} is *symmetric* if $T_{ab} = T_{(ab)}$. In n-dimensional space, it has n^2 components, but only $\frac{1}{2}n(n+1)$ of these can be specified independently—for example the components T_{ab} for $a \leq b$. How many independent components do the following tensors have (in n dimensions)?

i. F_{ab} with $F_{ab} = F_{[ab]}$.

ii. A tensor of type $(0, k)$ such that $T_{ab\dots c} = T_{[ab\dots c]}$ (distinguish the cases $k \leq n$ and $k > n$, bearing in mind the result of question (1)).

iii. R_{abcd} with $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$.

iv. R_{abcd} with $R_{abcd} = R_{[ab]cd} = R_{ab[cd]} = R_{cdab}$.

Show that, in four dimensions, a tensor with the symmetries of the Riemann tensor has 20 independent components.