1 Lie Groups

Definition (4.1 1) A *Lie Group* G is a set that is

- a group
- a differential manifold with the property that

$$\begin{aligned} \mu : G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned}$$

  and

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

  are smooth.

Definition (4.1 2) A *Lie Subgroup* of G is a subset H of G such that (i) H is a subgroup of G and (ii) H is a submanifold of G and (iii) topological group with respect to subspace topology.

Definition (4.1 3) The *left and right translations* are diffeomorphisms of G defined by

$$\begin{aligned} r_g : G &\to G, \\ g' &\mapsto g'g \end{aligned} \qquad \begin{aligned} l_g : G &\to G \\ g' &\mapsto gg' . \end{aligned}$$

l_g and r_g satisfy

$$l_{g_1} \circ l_{g_2} = l_{g_1 g_2} \quad \textit{and} \quad r_{g_1} \circ r_{g_2} = r_{g_2 g_1} .$$

$g \mapsto l_g$ and $g \mapsto r_g$ are an isomorphism and an anti-isomorphism (bijection, $\phi(ab) = \phi(b)\phi(a)$ and same for inverse), respectively.

A homomorphism of Lie groups is a smooth group homomorphism.

1.1 Examples

1. $\mathbb{R}^n$ with $+$
2. $S^1 := \{x \in \mathbb{C} \,|\, |x| = 1\}$: Circle in complex plane is group under multiplication but also manifold (circle).

3. real general linear group $GL(n,\mathbb{R}) := \{A \in M(n,\mathbb{R}) |\det A \neq 0\}$. Differential structure given by bijection with $\mathbb{R}^{n^2}$. Because det is continuous and $\{0\}$ is closed $\det^{-1}(0)$ is closed and the complement, $GL(n,\mathbb{R})$ is open. Every open subset of an n-dimensional manifold is an n-dimensional submanifold.

   Decomposes into two disjoint components with det $> / < 0$

   Dimension is n^2.

4. Similarly $GL(n,\mathbb{C}) := \{A \in M(n,\mathbb{C}) |\det A \neq 0\}$, dimension $2n^2$. But $GL(n,\mathbb{C})$ is connected while $GL(n,\mathbb{R})$ is not.

5. connected component of $GL(n,\mathbb{R})$: $GL^+(n,\mathbb{R}) := \{A \in M(n,\mathbb{R}) |\det A > 0\}$ this is subgroup of $GL(n,\mathbb{R})$ because

   - $\mathbb{1} \in GL^+(n,\mathbb{R})$
   - $\det AB = \det A \det B \Rightarrow (A, B \in GL^+(n,\mathbb{R}) \Rightarrow AB \in GL^+(n,\mathbb{R}))$
   - $\det A^{-1} = (\det A)^{-1} \Rightarrow (A \in GL^+(n,\mathbb{R}) \Rightarrow A^{-1} \in GL^+(n,\mathbb{R}))$

   $\mathbb{1} \in GL^+(n,\mathbb{R})$

6. $SL(n,\mathbb{R}) := \{A \in M(n,\mathbb{R}) |\det A = 1\}$

7. $O(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) | AA^T = \mathbb{1}\} \Rightarrow \det A = \pm 1$ is a compact Lie Group of $(n^2 - n)/2$ dimensions.

8. $SO(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) | AA^T = \mathbb{1}, \det A = 1\}$ also has dimension $(n^2 - n)/2$.

9. Generalization: $O(p,q,\mathbb{R})$ orthogonal with respect to metric with signature p, q. e.g. $O(3,1)$ Lorentz group. ($SO(p,q,\mathbb{R})$)

2 Lie Algebra of a Lie Group

Definition A Lie Algebra A is a vector space with an additional map

$$\begin{aligned} A \times A &\to A \\ X_1, X_2 &\mapsto [X_1, X_2] \end{aligned}$$

such that

$$\begin{aligned} [aX + bY, Z] &= a[X, Z] + b[Y, Z] \\ [X, Y] &= -[Y, X] \\ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] &= 0 \qquad \text{(Jacobi Identity)} \end{aligned}$$

Example: VFs on a Manifold.

Each Lie Group has an associated Lie Algebra which encodes many properties of the group (e.g. dimension, compactness, if G is simply connected every rep. of Lie alg. gives rep. of Lie group).

Definition (4.3.1) A VF X on a Lie Group G is *left-invariant* if it is l_g-related to itself for all $g \in G$, i.e.

$$\begin{aligned} l_{g*}X = X \qquad \forall g \in G \\ \Leftrightarrow l_{g*}X_{g'} = X_{gg'} \qquad \forall g, g' \in G \end{aligned}$$

Definition (4.3.2) A VF X on a Lie Group G is *right-invariant* if it is r_g-related to itself for all $g \in G$, i.e.

$$\begin{aligned} r_{g*}X = X \qquad \forall g \in G \\ \Leftrightarrow r_{g*}X_{g'} = X_{g'g} \qquad \forall g, g' \in G \end{aligned}$$

The set of all left-invariant VFs is called $L(G)$ and is a VS.

Fact (eq 3.1.31) If VFs X_1 and X_2 on manifold $\mathcal{M}$ are h-related to VFs Y_1 and Y_2 on $\mathcal{N}$ (i.e. $h_*X_1 = Y_1$) then $[X_1, X_2]$ is h-related to $[Y_1, Y_2]$.

$\Rightarrow$ if X_1 and X_2 are left-invariant then $l_{g*}[X_1, X_2] = [l_{g*}X_1, l_{g*}X_2] = [X_1, X_2]$ is also left-invariant.

Therefore $L(G)$ is sub Lie algebra of the lie algebra of all VFs on G.

Question: Are there any left-invariant VFs?

Theorem (4.1) There exists an isomorphism $i : T_eG \to L(G)$, $A \mapsto L^A$.

i given by

$$L^A_g = l_{g*}A \quad \forall g \in G\,.$$

This is left invariant because

$$l_{g'*}L^a_g = l_{g'*} \circ l_{g*}A = l_{g'g*}A = L^A_{g'g}\,.$$

It is an isomorphism because:

- If $L^A = L^B$ then $L^A_e = L^B_e \Rightarrow A = B$ and i is therefore injective.
- If L is left-invariant then $L_g = l_{g*}L_e$ which is equal to $L^{L_e}_g$. Therefore i is surjective.

This means that $\dim L(G) = \dim T_eG = \dim G$.

Theorem (4.2) $f : G \to H$ smooth homomorphism between Lie-Groups then $\Rightarrow f_* : L(G) \to L(H)$ is homomorphism between Lie Algebras.

Proof omitted.

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \uparrow ? & & \uparrow ? \\ L(G) & \xrightarrow{f_*} & L(H) \end{array}$$

If $\{E_1, E_2, \dots E_n\}$ is basis of $L(G)$ then commutator must be linear combination of these:

$$[E_\alpha, E_\beta] = \sum_{\gamma=1}^{n} C^\gamma_{\alpha\beta} E_\gamma$$

$C^\gamma_{\alpha\beta}$ are called the *structure constants* of the Lie algebra.

2.1 Exponential Map

Definition An *integral curve* of a VF X is a map $\sigma : \mathbb{R} \to G$ such that

$$\sigma_* \left(\frac{d}{dt}\right)_t = X_{\sigma(t)} \,.$$

This means when applied to a coordinate function $x^i : G \to \mathbb{R}$

$$\sigma_* \left(\frac{d}{dt}\right)_t (x^i) = \left(\frac{d}{dt}\right)_t (x^i \circ \sigma) = \frac{d}{dt} x^i(\sigma(t))\bigg|_t = X_{\sigma(t)}(x^i) = X^i_{\sigma(t)}$$

Definition (4.4 1) We call $\exp_A : t \mapsto \exp tA$ the unique integral curve of the left invariant VF L^A satisfying $A = \exp_{A*}(\frac{d}{dt})\big|_0 (\Leftrightarrow \exp 0A = e)$. $(A \in T_e G)$

This is defined for all t because every left-invariant VF is complete. (Not enough time for proof, idea is to extend curve by using group multiplication.)

Definition (4.4 2) The exponential map $\exp : T_e G \to G$ is defined by

$$\exp A := \exp tA\big|_{t=1} \,.$$

It is a local diffeomorphism around e (in a neighbourhood around e it is bijective and it and its inverse are smooth).

$\exp tA$ is a one parameter subgroup of G, i.e. it fulfils

$$\exp((t_1 + t_2)A) = (\exp t_1 A)(\exp t_2 A) \,.$$

In fact every one-parameter subgroup is of this form.

Theorem (4.4) If $\chi : \mathbb{R} \to G$ is one parameter subgroup then $\chi(t) = \exp tA$ with $A := \chi_* \left(\frac{d}{dt}\right)_0$.

Proof If $\chi : \mathbb{R} \to G$ is a one-parameter subgroup then $\chi(t_1 + t_2) = \chi(t_1)\chi(t_2)$ ($\Rightarrow \chi(0) = e$). This means $\chi \circ l_s = l_{\chi(s)} \circ \chi \quad \forall s \in \mathbb{R}$ (l_s is add. with s in $\mathbb{R}$). Therefore

$$\chi_* \left(\frac{d}{dt}\right)_s = \chi_* l_{s*} \left(\frac{d}{dt}\right)_0 = l_{\chi(s)*}\chi_* \left(\frac{d}{dt}\right)_0 = l_{\chi(s)*}(A) = L^A_{\chi(s)}$$

meaning $t \mapsto \chi(t)$ is integral curve for L^A. But these are unique and therefore $\chi(t) = \exp tA$. (unique because VF (tangent vector) and one starting point given).

Theorem (corollary) If $f : G \to H$ homomorphism between Lie groups G and H then

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ L(G) & \xrightarrow{f_*} & L(H) \end{array} \tag{4.2.39}$$

commutes, i.e. $\exp_H(f_* A) = f(\exp_G A) \quad \forall A \in T_e G$.

Proof Def. $\chi : \mathbb{R} \to H$ by $\chi(t) := f(\exp_G tA)$. Then

$$\begin{aligned} \chi(t_1 + t_2) &= f(\exp_G (t_1 + t_2)A) = f(\exp_G t_1 A \exp_G t_2 A) \\ &= f(\exp_G t_1 A) f(\exp_G t_2 A) = \chi(t_1)\chi(t_2) \end{aligned}$$

meaning χ is a one-parameter subgroup of H. This implies (by theorem 4.4) that it is given by

$$\chi(t) = \exp_H tB \quad \text{with} \quad B := \chi_* \left(\frac{d}{dt}\right)_0 \in T_e H\,. \tag{4.2.41}$$

Applying B to a function $k \in C^\infty(H)$ gives

$$B(k) = \left(\frac{d}{dt}\right)_0 (k \circ \chi) = \frac{d}{dt} k \circ f \circ \exp_G tA \bigg|_{t=0} = L^A_e (k \circ f)$$

Last step because $\exp_G tA$ is integral curve of L^A (see def. of integral curve). $L^A_e = A$ so $B(k) = A(k \circ f) = (f_* A)(k)$ or $B = f_*(A)$.

Inserting this into equation 4.2.41 gives

$$f(\exp_G tA) = \chi(t) = \exp_H t f_*(A)$$

which proves the theorem for $t = 1$.

Theorem (corollary) If $\mathrm{Ad}_g(g') := gg'g^{-1} \quad \forall g \in G$ then

$$\exp(\mathrm{Ad}_{g*}B) = g\exp(B)g^{-1} \tag{4.2.44}$$

Proof $\mathrm{Ad}_g(e) = e$ so Ad_{g*} maps T_eG to T_eG.For each $g \in G$, Ad_g is a homomorphism of G, therefore applying the above theorem gives

$$\exp \mathrm{Ad}_{g*}B = \mathrm{Ad}_g(\exp B) = g\exp(B)g^{-1}$$

The map $g \mapsto Ad_{g*}$ gives a representation of the Lie-group onto the Lie algebra called the *adjoint representation*.

2.2 The Lie Algebra of $GL(n,\mathbb{R})$

Consider $GL(n,\mathbb{R})^+$. It is a subset of $M(n,\mathbb{R})$ and a natural system of coordinates are the matrix elements given by $x^i{}_j : GL(n,\mathbb{R})^+ \to \mathbb{R}; \quad x^i{}_j(g) := g^i{}_j$. Therefore the tangent space at every point is $M(n,\mathbb{R})$.

We want to find the explicit form of the lie algebra. The coordinate representation of the left invariant vector fields (i.e. the lie algebra) is

$$L^A_g = L^A_g(x^i{}_j)\left(\frac{\partial}{\partial x^i{}_j}\right)_g .$$

The components of the vector field can be written as

$$L^A_g(x^i{}_j) = (l_{g*}A)(x^i{}_j) = (l_{g*}(\exp tA)_*\left(\frac{d}{dt}\right)_0)(x^i{}_j)$$
$$= ((g\exp tA)_*\left(\frac{d}{dt}\right)_0)(x^i{}_j) = \frac{d}{dt}\left(x^i{}_j(g\exp tA)\right)\Big|_{t=0} .$$

For matrices we can consider the curve $t \mapsto e^{tA}$, which is a one-parameter subgroup of $GL(n,\mathbb{R})^+$ ($e^{t_1A}e^{t_2A} = e^{(t_1+t_2)A}$) and whose derivative at $t = 0$ is A. This means

$$e^{tA} = \exp tA \qquad \forall t \in \mathbb{R} \quad \forall A \in T_eG \cong M(n,\mathbb{R})\,.$$

Inserting this into the expression for the components of the vector field L^A gives

$$L^A_g(x^i{}_j) = \frac{d}{dt}x^i{}_j(ge^{tA})\Big|_{t=0} = \frac{d}{dt}g^i{}_k(e^{tA})^k{}_j\Big|_{t=0}$$
$$= g^i{}_k\frac{d}{dt}(e^{tA})^k{}_j\Big|_{t=0} = g^i{}_kA^k{}_j = (gA)^i{}_j$$

So the left-invariant VF L_g^A has the local coordinate representation

$$L_g^A = (gA)^i{}_j \left(\frac{\partial}{\partial x^i{}_j} \right)_g .$$

To understand the Lie algebra we also need the coordinate representation of the Lie bracket. Calculating the Lie bracket of the VFs L^A and L^B gives

$$\begin{aligned}
[L^A, L^B]_g &= (gA)^i{}_j (\partial_i{}^j)_g (gB)^{i'}{}_{j'} \partial_{i'}{}^{j'} - (gB)^i{}_j (\partial_i{}^j)_g (gA)^{i'}{}_{j'} \partial_{i'}{}^{j'} \\
&= g^i{}_k A^k{}_j (\partial_i{}^j g^{i'}{}_l|_g) B^l{}_{j'} (\partial_{i'j'})_g + g^i{}_k A^k{}_j g^{i'}{}_l B^l{}_{j'} (\partial_i{}^j \partial_{i'}{}^{j'})_g \\
&-g^i{}_k B^k{}_j \underbrace{(\partial_i{}^j g^{i'}{}_l|_g)}_{\delta_i{}^{i'} \delta^j{}_l} A^l{}_{j'} (\partial_{i'}{}^{j'})_g - g^i{}_k B^k{}_j g^{i'}{}_l A^l{}_{j'} (\partial_i{}^j \partial_{i'}{}^{j'})_g \\
&= g^i{}_k A^k{}_j B^j{}_{j'} \partial_i{}^{j'} - g^i{}_k B^k{}_j A^j{}_{j'} \partial_i{}^{j'} = g^i{}_k [A, B]^k{}_{j'} \partial_i{}^{j'} \\
&= (g[A, B])^i{}_{j'} \partial_i{}^{j'} .
\end{aligned}$$

This means that

$$[L^A, L^B] = L^{[A,B]}$$

i.e. the matrix commutator gives the Lie bracket.

2.3 Left-Invariant Forms

Analogous to left/right-invariant VFs, define left/right-invariant n-forms.

Definition (4.5) An n-form ω is left-invariant if

$$l_g^* \omega = \omega \quad \forall g \in G \quad \Leftrightarrow \quad l_g^*(\omega_{g'}) = \omega_{g^{-1}g'} \quad \forall g, g' \in G .$$

Because pullbacks commute with the exterior derivative d this means

$$l_g^*(d\omega) = d(l_g^* \omega) = d\omega ,$$

i.e. if ω is left-invariant then $d\omega$ is too.

The set of all left-invariant one-forms is denoted by $L^*(G)$.

We know the structure constants for left-invariant VFs:

$$[E_\alpha, E_\beta] = C^\gamma_{\alpha\beta} E_\gamma . \tag{4.3.5}$$

Define a dual basis $\omega^1, \omega^2, \ldots, \omega^n$ for $L^*(G)$ by

$$\langle w^\alpha, E_\beta \rangle := \delta^\alpha_\beta .$$

The analogue of 4.3.5 for one-forms is the *Cartan-Maurer* equation

$$d\omega^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma}\, \omega^\beta \wedge \omega^\gamma = 0\,.$$

This contains the exterior derivative because while the lie bracket of two VFs gives another VF, the wedge product of two one-forms gives a two-form.

Definition (4.6) The *Cartan-Maurer form* Ξ is the $L(G)$ valued one-form ($\Xi : TG \to L(G)$) on G such that

$$\langle \Xi, v \rangle_g = v \quad \forall v \in T_g G \quad \forall g \in G\,.$$

Or equivalently

$$\langle \Xi, v \rangle_{g'} := l_{g'*}(l_{g^{-1}*} v) \quad \forall v \in T_g G \quad \forall g, g' \in G\,.$$

The Cartan-Maurer form is left-invariant.
Applying it to a left-invariant VF L^A gives $\langle \Xi, L^A_g \rangle_{g'} = L^A_{g'}$.

3 Infinitesimal Transformations

Definition (4.8) A *right-action* of a Lie-group G on a manifold M is a homomorphism $\delta : G \to \mathrm{Diff}(M); g \mapsto \delta_g$ i.e.

$$\delta_e(p) = p \qquad \delta_g(\delta_{g'}(p)) = \delta_{g'g}(p)\,.$$

such that the map $G \times M \ni (g,p) \mapsto \delta_g(p) \in M$ is smooth.

Often $\delta_g(p)$ is written as pg. The Homomorphism condition is then $(pg_1)g_2 = p(g_1 g_2)$.

Given such an action, every one-parameter subgroup of G gives a manifold-filling family of curves on M. These do not cross because

$$m\sigma(t) = m'\sigma(t') \Rightarrow m\sigma(t)\sigma(-t') = m' \underbrace{\sigma(t')\sigma(-t')}_{e} \Rightarrow m\sigma(t-t') = m'\,.$$

No self intersection because $m\sigma(t) = m\sigma(t') \Rightarrow m\sigma(t+\Delta t) = m\sigma(t'+\Delta t)$.

By taking the tangent vector to these curves this defines the *induced vector field*.

Definition (4.10) If a Lie-group G has a right action on a manifold M then the VF X^A on M induced by $t \mapsto \exp tA$ is defined as

$$X^A_p(f) := \left.\frac{d}{dt} f(p \exp tA)\right|_{t=0}$$

with $f \in C^\infty(M)$.

This means $\phi_t^A(p) := p \exp tA$ is a *flow* of X^A.
Define
$$M_p : G \to M \quad \forall p \in M \qquad M_p(g) := pg\,.$$
Using this
$$\begin{aligned}(M_{p*}L_g^A)(f) =& L_g^A(f \circ M_p) = (l_{g*}A)(f \circ M_p) = A(f \circ M_p \circ l_g) = A(f \circ M_{pg}) \\ =& \frac{d}{dt} f(M_{pg}(\exp tA))\bigg|_{t=0} = \frac{d}{dt} f(pg \exp tA)\bigg|_{t=0} = X_{pg}^A(f)\,.\end{aligned}$$
Therefore $M_{p*}L_g^A = X_{pg}^A$ and $M_{p*}A = X_p^A$ (alternate definition of induced VF).

Theorem (4.8) Lie-group G has right action on manifolds M, M' with induced VFs X^A, X'^A and $f : M \to M'$ is *equivariant* ($\Leftrightarrow f(pg) = f(p)g \quad \forall p \in M, g \in G$) then
$$f_* X_p^A = X'^A_{f(p)}$$

Proof
$$(f \circ M_p)(g) = f(pg) = f(p)g = M'_{f(p)}(g)$$
$$f_* X_p^A = f_* M_{p*} A = (f \circ M_p)_* A = M'_{f(p)*} A = X'^A_{f(p)}$$
Special case: $M = G$ with action $\delta_g = r_g$ Then $M_g(g') = gg' = l_g(g')$
$$X_g^A = M_{g*}A = l_{g*}A = L_g^A\,.$$
So the left-invariant VFs are induced by right translation.
From definition of induced VF:
$$L_g^A(f) = \frac{d}{dt} f(g \exp tA)\bigg|_{t=0}\,.$$
This way of looking at the VFs L^A leads to

Theorem (4.9) For $A, B \in T_eG$
$$[L^A, L^B]_e = \frac{d}{dt} Ad_{\exp tA*} B\bigg|_{t=0}$$

Proof In general (eq. 3.2.20) if ϕ_t^X is a flow of X on M and Y some other VF then
$$[X, Y] = -\frac{d}{dt}\phi_{t*}^X Y\bigg|_{t=0} = \lim_{t \to 0} \frac{1}{t}\left(Y - \phi_{t*}^X Y\right)$$

Here $\phi_t^A = r_{\exp tA}$ and therefore

$$\begin{aligned}
[L^A, L^B]_e =& \lim_{t\to 0} \frac{1}{t} \left(L_e^B - r_{\exp tA*} L^B_{\exp -tA}\right) = \lim_{t\to 0} \frac{1}{t} \left(B - r_{\exp tA*} l_{\exp -tA*} B\right) \\
=& \lim_{t\to 0} \frac{1}{t} \left(B - Ad_{\exp -tA*} B\right) = \lim_{t\to 0} \frac{1}{t} \left(Ad_{\exp tA*} B - B\right) \\
=& \frac{d}{dt} Ad_{\exp tA*} B \bigg|_{t=0}
\end{aligned}$$

Theorem (4.11) If a Lie-group G has a right action on a manifold M then $A \mapsto X^A$ is a Lie-algebra homomorphism from $L(G)$ into Vfld(M) i.e.

$$[X^A, X^B] = X^{[AB]} := X^{[L^A, L^B]_e} \quad \forall A, B \in T_e G\,.$$

This means a representation of the Lie-group gives a representation of the Lie-algebra. (This also requires that $A \mapsto X^A$ is linear which is clear because $X_p^A = M_{p*} A$)

Proof First show $X^{Ad_{g*}A} = \delta_{g^{-1}*} X^A$:

$$X_p^{Ad_{g*}A} = M_{p*} Ad_{g*} A = (M_p \circ Ad_g)_* A$$

$$\begin{aligned}
(M_p \circ Ad_g)(g') = p(gg'g^{-1}) = (\delta_{g^{-1}} \circ M_{pg})(g') \\
X_p^{Ad_{g*}A} = (\delta_{g^{-1}} \circ M_{pg})_* A = \delta_{g^{-1}*} X^A_{pg}
\end{aligned}$$

Now use eq 3.2.20 again with the flow $\delta_{\exp tA}$ for X^A

$$\begin{aligned}
[X^A, X^B] &= \lim_{t\to 0} \frac{1}{t} \left(X^B - \delta_{\exp tA*} X^B\right) = \lim_{t\to 0} \frac{1}{t} \left(X^B - X^{Ad_{\exp -tA*} B}\right) \\
&= \lim_{t\to 0} \frac{1}{t} X^{B - Ad_{\exp -tA*} B} = \lim_{t\to 0} \frac{1}{t} X^{Ad_{\exp tA*} B - B} \\
&= X^{\lim_{t\to 0} (Ad_{\exp tA*} B - B)/t} = X^{[L^A, L^B]_e}\,.
\end{aligned}$$

The opposite direction (representation of algebra → representation of group) is possible if G is simply connected and M is compact ("Palais' theorem").