Connections in a Bundle

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1 Reminders

1.1 Definitions

Definition 3.11 The *exterior derivative* of a function $f \in X^\infty(\mathcal{M})$ is the one-form $\mathrm{d}f$ defined by

$$\langle \mathrm{d}f, X\rangle := Xf = L_X f$$

for all vector felds X on $\mathcal{M}$. In local coordinates:

$$(\mathrm{d}f)_p = \sum_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu}\right)_p f(\mathrm{d}x^\mu)_p.$$

Definition 3.15 If ω is an n-form on $\mathcal{M}$ with $1 \leq n < \dim\mathcal{M}$ then the *exterior derivative* of ω is the $(n+1)$-form $\mathrm{d}\omega$ defined by

$$\begin{aligned}\mathrm{d}\omega(X_1, \ldots, X_{n+1}) := &\sum_{i=1}^{n+1}(-1)^{i+1}X_i(\omega(X_1, \ldots, \not{X}_i, \ldots, X_{n+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, \not{X}_i, \ldots, \not{X}_j, \ldots, X_{n+1})\end{aligned}$$

fr all vector fields $X_1, X_2, \ldots, X_{n+1}$.

If ω is a one-form, the 2-form $\mathrm{d}\omega$ acting on any pair of vector felds X, Y is

$$\mathrm{d}\omega(X, Y) = X(\langle\omega, Y\rangle) - Y(\langle\omega, X\rangle) - \langle\omega, [X, Y]\rangle.$$

Note that the notation $X(\langle\omega, Y\rangle)$ means the effect of acting with the vector field X on the function $\langle\omega, Y\rangle$ in $C^\infty(\mathcal{M})$.

Definition 4.10 Let G be a Lie group that has a right action $g \to \delta_g$ on a differentiable manifold $\mathcal{M}$. Then the vector field X^A on $\mathcal{M}$ *induced* by the action of the one-parameter subgroup $\exp tA$, $A \in T_eG$, is defined as

$$X_p^A(f) := \frac{\mathrm{d}}{\mathrm{d}t} f(p \exp tA)\Big|_{t=0}$$

where $f \in C^\infty(\mathcal{M})$, and $\delta_g(p)$ has been abbreviated to pg.

1.2 The pull back of a one form

Let $\mathcal{M}$, $\mathcal{N}$ be manifolds with local coordinates $\{x^1, x^2, \ldots, x^n\}$, $\{y^1, y^2, \ldots, y^m\}$ and $h : \mathcal{M} \to \mathcal{N}$. The local coordinate representation of the one-form ω in the manifold $\mathcal{N}$ is given by

$$\omega_{h(p)} = \sum_{\nu=1}^{n} \omega_\nu(h(p)) \, (\mathrm{d}y^\nu)_{h(p)} \qquad \text{for all} p \in \mathcal{M}.$$

The components of the pull-back of ω (then in $\mathcal{M}$) are given by

$$(h^*\omega)_\mu(p) \quad = \quad \left\langle h^*\omega, \frac{\partial}{\partial x^\mu} \right\rangle_p := \left\langle \omega, h_* \left(\frac{\partial}{\partial x^\mu} \right)_p \right\rangle_{h(p)} .$$

The push-forward of $(\partial/\partial x^\mu)_p$ at the point p can be expressed in terms of the Jacobian matrix of the map h:

$$(h^*\omega)_p = \sum_{\nu=1}^{n} \omega_\nu(h(p)) \sum_{\mu=1}^{m} \frac{\partial h^\nu}{\partial x^\mu}(p) \, (\mathrm{d}x^\mu)_p \, .$$

1.3 The Lie algebra of $GL(n, \mathbb{R})$

Consider the connected component $GL^+(n, \mathbb{R})$ of the general linear group $GL(n, \mathbb{R})$ (open subset of the linear space $M(n, \mathbb{R})$).

The tangent space at any point $g \in G$ can be identified with $M(n, \mathbb{R})$ which can therefore in turn be associated with the Lie algebra of $GL^+(n, \mathbb{R})$.

Coordinates on $G = GL^+(n, \mathbb{R})$ can be the matrix elements:

$$x^{ij}(g) := g^{ij}.$$

Let $A \in T_eG \cong M(n, \mathbb{R})$. Consider the with A associated left invariant vector field $L^A_g = l_{g*}(A)$:

$$L^A_g = \sum_{i,j=1}^{n} (L^A x^{ij})_g \left(\frac{\partial}{\partial x^{ij}} \right)_g$$

where

$$\left(L^A x^{ij}\right)_g = \frac{\mathrm{d}}{\mathrm{d}t} \left(x^{ij}(g \exp tA)\right)_{t=0}$$

(just the definition 4.10, but x^{ij} is the coordinate function). Since we are dealing with a matrix group, exp tA $= e^{tA}$ where e^A is the normal matrix exponential function. Thus we can caluclate the components $L^A_g x^{ij}$ of the left invariant vector field L^A:

$$L^A_g x^{ij} = \frac{\mathrm{d}}{\mathrm{d}t} x^{ij} \left(g \cdot e^{tA}\right) \Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \left(g \cdot e^{tA}\right)^{ij} \Big|_{t=0} = \sum_{k=1}^{n} g^{ik} \cdot \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \left(e^{tA}\right)^{kj} \Big|_{t=0}}_{=A^{kj}} = (g \cdot A)^{ij} \, .$$

Thus, the vector field has the form

$$L^A_g = \sum_{i,j}^{n} (g \cdot A)^{ij} \left(\frac{\partial}{\partial x^{ij}} \right)_g .$$

This representation gives

$$\left[L^{A'}, L^A\right] = L^{[A', A]}$$

where $[A', A]$ is the usual matrix commutator: hence the Lie algebra structure induced on $T_e GL^+(n,\mathbb{R}) \cong M(n,\mathbb{R})$ is just the commutator of the matrices.

A natural basis for $M(n,\mathbb{R})$ is the set of matrices E_{ij} defined as

$$(E_{ij})_{kl} := \delta_{ik}\delta_{jl}$$

and the associated left-invariant vector fields are

$$L_g^{ij} = \sum_{k=1}^{n} g^{ki} \left(\frac{\partial}{\partial x^{kj}} \right)_g .$$

1.4 The Cartan-Maurer form

Definition 4.6 The *Cartan-Maurer* form Ξ is the $L(G)$ (= left invariant vector fields on G) valued one-form on G that associates with any $v \in T_g G$ the left-invariant vector field on G whose value at $g \in G$ is precisely the given tangent vector v.

Specifically, if $\langle \Xi, v \rangle$ denotes this left-invariant vector field then

$$\langle \Xi, v \rangle (g') := l_{g'*}(l_{g^{-1}*} v)$$

for all $v \in T_g G$.

- On the left-invariant vector fields L^A, the expression becomes

$$\langle \Xi, L_g^A \rangle (g') = L_{g'}^A .$$

- Since $L(G) \cong T_e G$, we may write

$$\langle \Xi, L_g^A \rangle = A.$$

- Consider $G = GL(n,\mathbb{R})$ with $T_e G \cong M(n,\mathbb{R})$. The Cartan-Maurer form has to fulfill

$$\langle \Xi^{ij}, L_g^A \rangle = A^{ij}.$$

  Hence Ξ^{ij} is given by

$$\Xi_g^{ij} = \sum_{k=1}^{n} (g^{-1})^{ik} (\mathrm{d}x^{kj})_g$$

  which can be shown easily:

$$\begin{aligned} \langle \Xi^{ij}, L_g^A \rangle &= \sum_{k,l,m=1}^{n} (g^{-1})^{ik} (gA)^{lm} \underbrace{\left(\frac{\partial}{\partial x^{lm}} \right)_g \left(\mathrm{d}x^{kj} \right)_g}_{=\delta_l^k \delta_m^j} \\ &= \sum_{k,n=1}^{n} \underbrace{(g^{-1})^{ik} g^{kn}}_{\delta^{in}} A^{nj} = A^{ij} \end{aligned}$$

- Consider now a map $\Omega : \mathcal{M} \to G$ where $\mathcal{M}$ is some differentiable manifold and G is a group of matrices. Ω could be thought of as a gauge function. Then $\Omega^* \Xi$ is a $L(g)$-valued one-form on $\mathcal{M}$. We calculate the components of $\Omega^*\Xi$:

$$\begin{aligned}
\left\langle (\Omega^*\Xi)^{ij}_p, \left(\frac{\partial}{\partial x^\mu}\right)_p \right\rangle &= \left\langle \Xi^{ij}, \Omega_*\left(\frac{\partial}{\partial x^\mu}\right)\right\rangle_{\Omega(p)} \\
&= \left\langle \sum_{k=1}^n \left(\Omega^{-1}(p)\right)^{ik} \left(\mathrm{d}x^{kj}\right)_{\Omega(p)}, \Omega_*\left(\frac{\partial}{\partial x^\mu}\right)\right\rangle_{\Omega(p)} \\
&= \sum_{k=1}^n \left(\Omega^{-1}(p)\right)^{ik} \Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p \left(x^{kj}\right) \\
&= \sum_{k=1}^n \left(\Omega^{-1}(p)\right)^{ik} \frac{\partial}{\partial x^\mu} \underbrace{x^{kj}(\Omega(p))}_{=\Omega^{kj}(p)}
\end{aligned}$$

Hence we get

$$(\Omega^*\Xi)^{ij}_p = \sum_{\mu=1}^m \sum_{k=1}^n \left(\Omega^{-1}(p)\right)^{ik} \frac{\partial}{\partial x^\mu}\Omega^{kj}(p)(\mathrm{d}x^\mu)_p$$

which is often written rather symbolically as

$$\Omega^*\Xi = \Omega^{-1}\mathrm{d}\Omega.$$

2 Connections in a Principal Bundle

2.1 Introduction

Consider a principal bundle $G \to P \to \mathcal{M}$ ($\mathcal{M} \cong P/G$). We want to compare points in neighbouring fibres and need therefore vectors that point from one fibre to another.

We know already that to each $A \in L(G)$ (left invariant vector fields on G) there corresponds an induced vector field X^A on P (in an isomorphic way) which represents the Lie algebra of G homomorphically, i.e. $[X^A, X^B] = X^{[A,B]}$ for all $A, B \in L(G)$. The vector $X^A_p \in T_pP$ is tangent to the fibre at $p \in P$. This gives raise to the following definition.

Definition Let $G \to P \to \mathcal{M}$ be a principal bundle and $p \in P$. The *vertical subspace* V_pP of a tangent space T_pP at p is defined to be

$$V_p := \{\tau \in T_pP | \pi_*\tau = 0\}$$

where $\pi : P \to \mathcal{M}$ is the projection in the bundle.

Definition 6.1 A *connection* in a principal bundle $G \to P \to \mathcal{M}$ is a smooth assignment to each point $p \in P$ of a subspace H_pP of T_pP such that

$$\begin{array}{lll}
(a) & T_pP \simeq V_pP \oplus H_pP & \text{for all } p \in P \\
(b) & \delta_{g*}(H_pP) = H_{pg}P & \text{for all } g \in G, p \in P
\end{array}$$

where $\delta_g(p) := pg$ denotes the right action of G on P.

- Any tangent vector $\tau \in T_pP$ can be decomposed uniquely into a sum of *vertical* and *horizontal* components lying in V_pP and H_pP, $\tau = \mathrm{ver}(\tau) + \mathrm{hor}(\tau)$. These components will be denoted by $\mathrm{ver}(\tau)$ and $\mathrm{hor}(\tau)$ respectively.

- Consider the isomorphic map $\iota : L(G) \to \mathrm{VFlds}(P)$, $A \mapsto X^A$. A connection can be associated with a certain $L(G)$-valued one-form ω on P in the following way:

$$\omega_p(\tau) := \iota^{-1}(\mathrm{ver}(\tau)).$$

  Note that

  1. $\omega_p(X^A) = A$ for all $p \in P, A \in L(G)$
  2. $\delta_g^*\omega = \mathrm{Ad}_{g^{-1}}(\omega)$, i.e., $(\delta_g^*\omega)_p(\tau) = \mathrm{Ad}_{g^{-1}}(\omega_p(\tau))$, for all $\tau \in T_pP$
     where $\mathrm{Ad}_g(g') = gg'g^{-1}$ (adjoint map)
     (Remember theorem 4.10: $X^{\mathrm{Ad}_{g*}(A)} = \delta_{g^{-1}*}(X^A)$).
  3. $\tau \in H_pP \Leftrightarrow \omega_p(\tau) = 0$.

2.2 Local representatives of a connection

Theorem 6.1 Let $\sigma : U \subset \mathcal{M} \to P$ be a local section of a principal bundle $G \to P \to \mathcal{M}$ which is equipped with a connection one-form ω. Define the local σ-representative of ω to be the $L(G)$ valued one-form ω^U on the open set $U \subset \mathcal{M}$ given by $\omega^U = \sigma^*\omega$. Let $h : U \times G \to \pi^{-1}(U) \subset P$ be the local trivialisation of P induced by σ according to $h(x, g) := \sigma(x)g$.

Then if $(\alpha, \beta) \in T_{(x,g)}(U \times G) \simeq T_xU \oplus T_gG$, the local representative $h^*\omega$ of ω on $U \times G$ can be written in terms of the local 'Yang-Mills' field ω^U as

$$(h^*\omega)_{(x,g)}(\alpha, \beta) = \mathrm{Ad}_{g^{-1}}\left(\omega^U_x(\alpha)\right) + \Xi_g(\beta)$$

where Ξ is the Cartan-Maurer $L(G)$-valued one-form on G.

Proof Factor the map $h : U \times G \to P$ as

$$\begin{array}{ccccc} U \times G & \stackrel{\sigma\times\mathrm{id}}{\longrightarrow} & P \times G & \stackrel{\delta}{\longrightarrow} & P \\ (x, g) & \mapsto & (\sigma(x), g) & \mapsto & \sigma(x)g \end{array}$$

Then,

$$\begin{aligned} (h^*\omega)_{(x,g)}(\alpha, \beta) &= ((\sigma \times \mathrm{id})^*\delta^*\omega)_{(x,g)}(\alpha, \beta) \\ &= (\delta^*\omega)_{(\sigma(x),g)}(\sigma_*\alpha, \beta) = \omega_{\sigma(x)g}((\delta \circ i_g)_*\sigma_*\alpha + (\delta \circ j_{\sigma(x)})_*\beta) \end{aligned}$$

where $i_g : P \to P \times G$, $p \mapsto (p, g)$, and $j : G \to P \times G$, $g \mapsto (g, p)$, so that

$$\begin{aligned} \delta \circ i_g(p) &= \delta(p, g) = pg, \text{ i.e., } \delta \circ i_g = \delta_g : P \to P \\ \delta \circ j_p(g) &= \delta(p, g) = pg, \text{ i.e., } \delta \circ j_p = P_p : G \to P \end{aligned}$$

Therefore (using the definition of the pull-back of a one-form in the first summand)

$$\begin{aligned} (h^*\omega)_{(x,y)}(\alpha, \beta) &= \omega_{\sigma(x)g}((\delta \circ i_g)_*\sigma_*\alpha) + \omega_{\sigma(x)g}((\delta \circ j_{\sigma(x)})_*\beta) \\ &= (\delta_g^*\omega_{\sigma(x)g})(\sigma_*\alpha) + \omega_{\sigma(x)g}(P_{\sigma(x)*}\beta). \end{aligned}$$

- We have already discussed: $\delta_g^*\omega_{\sigma(x)g} = \mathrm{Ad}_{g^{-1}}(\omega_{\sigma(x)})$
- For some $A \in L(G)$ it is $\beta = L_g^A$. Therefore $\Xi_g(\beta) = \langle \Xi_g, \beta \rangle = A$
- This A is the second summand: We have $P_{\sigma(x)*}(L_g^A) = X^A_{\sigma(x)g}$ and $\omega(X^A) = A$.

Thus we have

$$(h^*\omega)_{(x,g)}(\alpha, \beta) = \mathrm{Ad}_{g^{-1}}(\omega_{\sigma(z)}(\sigma_*\alpha)) * \Xi_g(\beta) = \underbrace{\mathrm{Ad}_{g^{-1}}(\omega^U_x(\alpha))}_{\text{Yang−Mills field on } \mathcal{M}} + \Xi_g(\beta)$$

for all $(\alpha, \beta) \in T_xU \oplus T_gG$, as desired. □

2.3 Local gauge transformations

Definition In general, a *gauge transformation* in the principal bundle $G \to P \to \mathcal{M}$ is defined to be any principal automorphism of the bundle.

Theorem 6.2 Let ω be a connection on the principal bundle $G \to P \to \mathcal{M}$ and let $\sigma_1 : U_1 \to P$ and $\sigma_2 : U_2 \to P$ be two local trivialisations on open sets $U_1, U_2 \subset \mathcal{M}$ such that $U_1 \cap U_2 \neq \emptyset$. Let $A_\mu^{(1)} = \sigma_1^* \omega$ and $A_\mu^{(2)} = \sigma_2^* \omega$ denote the local representatives of ω with respect to σ_1 and σ_2 respectively. Let $\Omega : U_1 \cap U_2 \to G$ be the unique local gauge function defined by

$$\sigma_2(x) = \sigma_1(x)\Omega(x) = \delta_{\Omega(x)}(\sigma_1(x)).$$

Then the local representatives are related on $U_1 \cap U_2$ by

$$A_\mu^{(2)}(x) = \mathrm{Ad}_{\Omega(x)^{-1}}(A_\mu^{(1)}(x)) + (\Omega^* \Xi)_\mu(x).$$

Proof Consider $A_\mu^{(2)}(x) := (\sigma_2^* \omega)_x(\partial_\mu)$. Now we factorise $\sigma_2 : U_1 \cap U_2 \to P$ as

$$\begin{array}{ccccc} U_1 \cap U_2 & \xrightarrow{\sigma_1 \times \Omega} & P \times G & \xrightarrow{\delta} & P \\ x & \mapsto & (\sigma_1(x), \Omega(x)) & \mapsto & \sigma_1(x)\Omega(x). \end{array}$$

Thus we write

$$\begin{aligned} A_\mu^{(2)}(x) &= ((\sigma_1 \times \Omega)^* \delta^* \omega)_x(\partial_\mu) \\ &= (\delta^* \omega)_{(\sigma_1(x), \Omega(x))}(\sigma_{1*}(\partial_\mu)_x, \Omega_*(\partial_\mu)_x) = \omega_{\sigma_1(x)\Omega(x)}(\delta_*(\sigma_{1*}(\partial_\mu)_x, \Omega_*(\partial_\mu)_x) \\ &= \omega_{\sigma_1(x)\Omega(x)}(\delta_{\Omega(x)*}\sigma_{1*}(\partial_\mu)_x + P_{\sigma_1(x)*}\Omega_*(\partial_\mu)_x) \\ &= \omega_{\sigma_1(x)\Omega(x)}(\delta_{\Omega(x)*}\sigma_{1*}(\partial_\mu)_x) + \omega_{\sigma_1(x)\Omega(x)}(P_{\sigma_1(x)*}\Omega_*(\partial_\mu)_x) \\ &= \delta_{\Omega(x)}^{\;*}\omega_{\sigma_1(x)}(\sigma_{1*}(\partial_\mu)_x) + \omega_{\sigma_1(x)\Omega(x)}(P_{\sigma_1(x)*}\Omega_*(\partial_\mu)_x) \end{aligned}$$

Now, we use the same arguments as in the previous proof. E.g. there is an $A \in T_e G$ such that $\Omega_*(\partial_\mu)_x = L^A_{\Omega(x)}$. We obtain

$$\begin{aligned} A_\mu^{(2)}(x) &= \mathrm{Ad}_{\Omega(x)^{-1}}(\omega_{\sigma_1(x)}(\sigma_{1*}(\partial_\mu)_x)) + \langle \Xi_{\Omega(x)}, \Omega_*(\partial_\mu)_x \rangle \\ &= \mathrm{Ad}_{\Omega(x)^{-1}}(A_\mu^{(1)}(x)) + (\Omega^* \Xi)_\mu(x). \quad \square \end{aligned}$$

Matrix groups Now, we assume G to be a matrix group. The group action will be the matrix multiplication. Thus we can calculate the adjoint map:

$$\mathrm{Ad}_{\Omega(x)^{-1}}(A_\mu^{(1)}(x)) = \Omega(x)^{-1} A_\mu^{(1)}(x) \Omega(x).$$

We also discussed already the pull-back of the Cartan-Maurer form on a matrix group with a map $\Omega : \to G$:

$$(\Omega^* \Xi)_\mu(x) = \sum_{k=1}^{n} \left(\Omega^{-1}(p)\right)^{ik} \frac{\partial}{\partial x^\mu} \Omega^{kj}(x) = \Omega^{-1}(p) \partial_\mu \Omega(x)$$

Altogether, one obtains

$$A_\mu^{(2)}(x) = \Omega(x)^{-1} A_\mu^{(1)}(x) \Omega(x) + \Omega^{-1}(p) \partial_\mu \Omega(x).$$

2.4 Example: Connections in the frame bundle

The base space is an m-dimensional manifold $\mathcal{M}$. The total space $\mathbf{B}(\mathcal{M})$ is the space of all frames b (= ordered set $(b_1, b_2, \ldots, b_m)$ of basis vectors of $T_x\mathcal{M}$, $x \in \mathcal{M}$) at all points in $\mathcal{M}$. The projection map $\pi : \mathbf{B}(\mathcal{M}) \to \mathcal{M}$ takes a frame into the point to which it is attached.

There is a natural free right-action of $GL(m, \mathbb{R})$ on $\mathbf{B}(\mathcal{M})$ given by

$$(b_1, b_2, \ldots, b_m)g := \left(\sum_{j_1=1}^{m} b_{j_1} g_{j_1 1}, \sum_{j_2=1}^{m} b_{j_2} g_{j_2 2}, \ldots, \sum_{j_m=1}^{m} b_{j_m} g_{j_m m} \right) \quad \Leftrightarrow \quad \delta_g(b) = b \cdot g$$

for all $g \in GL(m, \mathbb{R})$.

Let $U \subset \mathcal{M}$ be a coordinate neighbourhood with coordinate functions $(x_1, x_2, \ldots, x_m)$. Then any base $b = (b_1, b_2, \ldots, b_m)$ for the vector space $T_x\mathcal{M}$, $x \in U$ can be expanded uniquely as

$$b_i = \sum_{j=1}^{m} b_i^j \left(\frac{\partial}{\partial x^j} \right)_x, \quad i = 1, 2, \ldots, m$$

for some non singular matrix $b_i^j \in GL(m, \mathbb{R})$. Any local coordinate chart (U, ϕ) on $\mathcal{M}$ provides a local section

$$\sigma : U \to \mathbf{B}(\mathcal{M}), \qquad x \mapsto \left(\left(\frac{\partial}{\partial x^1} \right)_x, \ldots, \left(\frac{\partial}{\partial x^m} \right)_x \right).$$

Let ω be a ($L(GL(m, \mathbb{R}))$ valued) connection one-form on $\mathbf{B}(\mathcal{M})$ and let

$$\Gamma := \sigma^* \omega, \quad \Gamma_\mu(x) = (\sigma^* \omega)_x(\partial_\mu)$$

be the local σ-representative of ω. We now want to calculate the local σ'-representative Γ' of ω associated with another coordinate chart (U', ϕ') such that $U \cap U' \neq \emptyset$ where

$$\sigma' : U' \to \mathbf{B}(\mathcal{M}), \qquad x \mapsto \left(\left(\frac{\partial}{\partial x'^1} \right)_x, \ldots, \left(\frac{\partial}{\partial x'^m} \right)_x \right).$$

The coordinate transformation for all $x \in U \cap U'$ is given by

$$(\partial_{\mu'})_x = \sum_{\nu=1}^{m} J_\mu^\nu(x) (\partial_\nu)_x, \quad J_\mu^\nu(x) := \frac{\partial x^\nu}{\partial x'^\mu}(x) \quad \text{(Jacobian)}$$

Then

$$\begin{aligned}
\Gamma'_\mu(x) \quad &= \quad (\sigma'^* \omega)_x \frac{\partial}{\partial x'^\mu} = \sum_{\alpha=1}^{m} J_\mu^\alpha(x) (\sigma'^* \omega)_x \frac{\partial}{\partial x^\alpha} \\
&\overset{\text{Theorem 6.2}}{=} \quad \sum_{\alpha=1}^{m} J_\mu^\alpha(x) \left(\mathrm{Ad}_{J(x)^{-1}} \left((\sigma^* \omega)_x \frac{\partial}{\partial x^\alpha} \right) + (J^* \Xi)_\alpha(x) \right) \\
&= \quad \sum_{\alpha=1}^{m} J_\mu^\alpha(x) \left(J^{-1}(x) \Gamma_\alpha(x) J(x) + J^{-1}(x) \partial_\alpha J(x) \right).
\end{aligned}$$

The Lie algebra of $GL(m, \mathbb{R})$ is $M(m, \mathbb{R})$. We can take a basis of this space $\{G_\chi^\lambda | \chi, \lambda = 1, 2, \ldots, m\}$ and express the entries of the matrix-valued one-form Γ_μ in virtue of this basis:

$$(\Gamma_\mu)_\delta^\epsilon = \sum_{\lambda, \chi=1}^{m} \Gamma_{\mu\lambda}{}^{\chi} (G_\chi^\lambda)_\delta^\varepsilon$$

If one chooses the basis $(G^\lambda_\chi)^\epsilon_\delta := \delta^\epsilon_\chi \delta^\lambda_\delta$ one obtains

$$\begin{aligned}
{\Gamma'}_{\mu\delta}{}^{\epsilon}(x) &= (\Gamma'_\mu(x))^\varepsilon_\delta = \sum_{\alpha=1}^{m} J^\alpha_\mu(x)\left(J^{-1}(x)\Gamma_\alpha(x)J(x) + J^{-1}(x)\partial_\alpha J(x)\right)^\epsilon_\delta \\
&= \sum_{\alpha,\rho,\chi=1}^{m} J^\alpha_\mu(x)(J^{-1})^\epsilon_\chi(x)\Gamma_{\alpha\rho}{}^{\chi}(x)J^\rho_\delta(x) + \sum_{\alpha,\lambda=1}^{m} J^\alpha_\mu(x)(J^{-1})^\epsilon_\lambda(x)\partial_\alpha J^\lambda_\delta(x) \\
&= \sum_{\alpha,\rho,\chi=1}^{m} \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\epsilon}{\partial x^\chi}\frac{\partial x^\rho}{\partial x'^\delta}\Gamma_{\alpha\rho}{}^{\chi} + \sum_{\alpha,\lambda=1}^{m} \frac{\partial x'^\epsilon}{\partial x^\lambda}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x^\lambda}{\partial x^\alpha \partial x'^\delta} \\
&= \sum_{\alpha,\rho,\chi=1}^{m} \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\delta}\frac{\partial x'^\epsilon}{\partial x^\chi}\Gamma_{\alpha\rho}{}^{\chi} + \sum_{\lambda=1}^{m} \frac{\partial x'^\epsilon}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\delta},
\end{aligned}$$

the transformation law for the Christoffel symbols.

3 The curvature two-form

Definition 6.8

1. If ω is any k-form on a principal bundle space $P(\xi)$, the *exterior covariant derivative* of ω is the horizontal (k+1)-form $\mathrm{D}\omega$ defined by

$$D\omega := \mathrm{d}\omega \circ \mathrm{hor}$$

   i.e.,

$$\mathrm{D}\omega(X_1, x_2, \ldots, X_{k+1} = \mathrm{d}\omega(\mathrm{hor}X_1, \mathrm{hor}X_2, \ldots, \mathrm{hor}X_{k+1})$$

   for any set $\{X_1, X_2, \ldots, X_{k+1}\}$ of vector fields on $P(\xi)$.

2. If ω is a connection one-form on $P(\xi)$, the *curvature two-form* of ω is defined as $G := \mathrm{D}\omega$.

Theorem 6.4 If $G = \mathrm{D}\omega$ is the curvature 2-form of the connection ω, then on an arbitrary pair of vector fields X and Y on $P(\xi)$ we have, for all $p \in P(\xi)$,

$$G_p(X, Y) = \mathrm{d}\omega_p(X, Y) + [\omega_p(X), \omega_p(Y)]$$

where $[\omega_p(X), \omega_p(Y)]$ denotes the Lie bracket in $L(G)$ between the Lie algebra elements $\omega_p(X)$ and $\omega_p(Y)$.

Proof Since both sides of the assertion are linear in X and Y, it suffices to prove the relation for the three choices: (i) X, Y are horizontal; (ii) X, Y are vertical; (iii) X is horizontal, Y is vertical.

(i) Remember: $\omega_p(\tau)$ yields the vector in $L(G)$ that induces via $A \mapsto X^A$ the vertical part of T_pP. Thus $\omega_p(\tau) = \omega_p(\mathrm{ver}(\tau) + \mathrm{hor}(\tau)) = \omega(\mathrm{ver}(\tau))$.

In this case: $[\omega_p(X), \omega_p(Y)] = [0, 0] = 0$ and per definition $\mathrm{D}\omega_p(X, Y) = \mathrm{d}\omega_p(\mathrm{hor}(X), \mathrm{hor}(Y)) = \mathrm{d}\omega_p(X, Y)$.

(ii) If X and Y are vertical vector fields then there exist $A, B \in L(G)$ which induce X, Y, i.e. $X_p = X_p^A, Y_p = X_p^B$. We calculate the right hand side of the assertion:

$$\begin{aligned}
&\mathrm{d}\omega_p(X,Y) + [\omega_p(X), \omega_p(Y)] \\
&\quad = \mathrm{d}\omega_p(X^A, X^B) + [\omega_p(X^A), \omega_p(X^B)] \\
&\quad = X_p^A(\omega_p(X^B)) - X_p^B(\omega_p(X^A)) - \omega_p(\underbrace{[X^A, X^B]}_{=X^{[A,B]}}) + [\omega_p(X^A)\omega_p(X^B)] \\
&\quad = \underbrace{X_p^A(B)}_{=0} - \underbrace{X_p^B(A)}_{=0} - [A,B] + [A,B] = 0
\end{aligned}$$

Now, we calculate the left hand side:

$$G_p(X,Y) = \mathrm{d}\omega(\mathrm{hor}(X), \mathrm{hor}(Y)) = \mathrm{d}\omega(0,0) = 0$$

(iii) X is horizontal, Y is vertical. The left hand side is easy:

$$G_p(X,Y) = \mathrm{d}\omega(\mathrm{hor}(X), \mathrm{hor}(Y)) = \mathrm{d}\omega(X,0) = 0$$

Next, we calculate

$$[\omega_p(X), \omega_p(Y)] = [0, \omega_p(Y)] = 0.$$

It remains $\mathrm{d}\omega_p(X,Y)$. Again, we use that there is an $A \in L(G)$ such that $Y_p = X_p^A$. This yields

$$\mathrm{d}\omega_p(X,Y) = \underbrace{X(\omega_p(X_p^A))}_{=X(A)=0} - X_p^A(\underbrace{\omega_p(X)}_{=0}) - \underbrace{\omega_p([X, X^A])}_{=0 \text{ since } [X,X^A] \text{ is horizontal}} = 0. \quad \square$$

3.1 Gauge field tensor

Let $\{E_1, E_2, \ldots, E_n\}$ be a basis of $L(G)$, let $\{\partial_1, \partial_2, \ldots, \partial_m\}$ be a basis of $T_p\mathcal{M}$ and let $\{d^1, d^2, \ldots, d^m\}$ be the dual basis, thus a basis of $T_p\mathcal{M}^*$. We consider

$$\begin{aligned}
\sigma^*\omega(X) &=: A(X) = A^\alpha(X)E_\alpha = A^\alpha_\beta d^\beta(X) E_\alpha \\
&= A^\alpha_\beta X^\gamma \underbrace{d^\beta(\partial_\gamma)}_{\delta^\beta_\gamma} E_\alpha = A^\alpha_\beta X^\beta E_\alpha \Rightarrow \quad A^\alpha(X) = A^\alpha_\beta X^\beta.
\end{aligned}$$

Note that $A^\alpha(\partial_\mu) = A^\alpha_\mu$.

The next aim is to calculate $F := \sigma^* G = \sigma^* \mathrm{D}\omega$. First, we calculate as helping identity

$$\begin{aligned}
(A^b \wedge A^c)(X,Y) &= \sum_{\mu,\nu=}^{m} (A^b \wedge A^c)(\partial_\mu, \partial_\nu) \\
&= \sum_{\mu,\nu,\beta,\gamma} X^\mu Y^\nu A^b_\beta A^c_\gamma \underbrace{(d^\beta \wedge d^\gamma)(\partial_\mu, \partial_\nu)}_{=\delta^\beta_\mu \delta^\gamma_\nu - \delta^\gamma_\mu \delta^\beta_\nu} = \sum_{\beta,\gamma} (X^B A^b_\beta Y^\nu A^c_\nu - X^\nu A^c_\nu Y^\beta A^b_\beta) \\
&= A^b(X) A^c(Y) - A^c(X) A^b(Y).
\end{aligned}$$

Now, we calculate

$$\begin{aligned}
F(X,Y) &\overset{\text{Theorem 6.4}}{=} \mathrm{d}A(X,Y) + [A(X),A(Y)] \\
&= \mathrm{d}A(X,Y)^\alpha E_\alpha + \sum_{\beta,\gamma} A^\beta(X)A^\gamma(Y) \underbrace{[E_\beta,E_\gamma]}_{=\sum_\alpha C^\alpha_{\beta\gamma}E_\alpha} \qquad |C^\alpha_{\beta\gamma} = -C^\alpha_{\gamma\beta} \\
&= \mathrm{d}A(X,Y)^\alpha E_\alpha + \frac{1}{2}\sum_{\alpha,\beta,\gamma} \underbrace{(A^\beta(X)A^\gamma(Y) - A^\gamma(X)A^\beta(Y))}_{=(A^\beta\wedge A^\gamma)(X,Y)} C^\alpha_{\beta\gamma}E_\alpha \\
&= \underbrace{\left(\mathrm{d}A(X,Y)^\alpha + \frac{1}{2}\sum_{\beta,\gamma}(A^\beta\wedge A^\gamma)(X,Y)C^\alpha_{\beta\gamma}\right)}_{=F^\alpha(X,Y)} E_\alpha
\end{aligned}$$

We also can calculate the ccoridnate representation of F:

$$\begin{aligned}
F_{\mu\nu} &:= F(\partial_\mu,\partial_\nu) = \mathrm{d}A(\partial_\mu,\partial_\nu) + [A_\mu,A_\nu] \\
&= \partial_\mu(A(\partial_\nu)) - \partial_\nu(A\partial_\mu)) - A(\underbrace{[\partial_\mu,\partial_\nu]}_{=0}) + [A_\mu,A_\nu] \\
\Rightarrow \quad F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu,A_\nu].
\end{aligned}$$

If $\sigma_1 : U_1 \to P$ and $\sigma_2 : U_2 \to P$ are a pair of local sections with $U_1 \cap U_2 \neq \emptyset$, there exists some local gauge function $\Omega : U_1 \cap U_2 \to G$ such that $\sigma_2(x) = \sigma_1(x)\Omega(x)$. Correspondingly, there are two local representatives for the curvature 2-form G - namely $F^{(1)} := \sigma_1^* G$ and $F^{(2)} := \sigma_2^* G$. Using an analysis very similar to that employed in the proof of theorem 6.1, it can be shown that these curvature representatives are related by

$$F^{(2)}_{\mu\nu}(x) = \Omega(x)^{-1}F^{(1)}_{\mu\nu}(x)\Omega(x)$$

for all $x \in U_1 \cap U_2$.

Another useful identity is the Bianchi Identity $DG = 0$:

$$\begin{aligned}
G(X,Y) &= \omega(X,Y) + [\omega(X),\omega(Y)] \\
\Rightarrow \quad \mathrm{D}G(X,Y,Z) &= X(G(Y,Z)) - Y(G(X,Z)) + Z(G(X,Y)) \\
&\quad -G([X,Y],Z) + G([X,Z],Y)) - G([Y,Z],X) \\
&\vdots \\
&= 0
\end{aligned}$$

4 Parallel Transport

4.1 Parallel transport in a principal bundle

Definition 6.2 Since $\pi_* : H_pP \to T_{\pi(p)}\mathcal{M}$ is an isomorphism, to each vector field X on $\mathcal{M}$ there exists a unique vector field, denoted $X^\uparrow$, on P such that, for all $p \in P$,

(a) $\pi_*(X^\uparrow_p) = X_{\pi(p)}$

(b) $\mathrm{ver}(X^\uparrow_p) = 0$.

This vector field is known as the *horizontal lift* of X.

Definition 6.3 Let α be a smooth curve that maps a closed interval $[a,b] \subset \mathbb{R}$ into $\mathcal{M}$ (i.e., α is the restriction to $[a,b]$ of a smooth curve defined on some open interval containing $[a,b,])$. A *horizontal lift* of α is a curve $\alpha^{\uparrow} : [a,b] \to P$ which is horizontal (i.e., $\mathrm{ver}[\alpha^{\uparrow}] = 0$) and such that $\pi(\alpha^{\uparrow}(t)) = \alpha(t)$ for all $t \in [a,b]$.

Theorem 6.3 For each point $p \in \pi^{-1}\{\alpha(a)\}$, there exists a unique horizontal lift of α such that $\alpha^{\uparrow}(a) = p$.

Definition 6.4 Let $\alpha : [a,b] \to \mathcal{M}$ be a curve in $\mathcal{M}$. The *parallel translation* along α is the map $\tau : \pi^{-1}(\{\alpha(a)\}) \to \pi^{-1}(\{\alpha(b)\})$ obtained by associating with each point $p \in \pi^{-1}(\{\alpha(a)\})$ the point $\alpha^{\uparrow}(b) \in \pi^{-1}(\{\alpha(b)\})$ where $\alpha^{\uparrow}$ is the unique horizontal lift of α that passes through p at $t = a$.

4.2 parallel transport in an associated bundle

Definition

(1) Let ω be a connection in the principal G-bundle $\xi = (P, \pi, \mathcal{M})$, and let $\xi[F] = P_F, \pi_F, \mathcal{M})$ be the bundle associated to ξ via the left action of G on F. The *vertical subspace* of the tangent space $T_y(P_F)$, $y \in P_F$ is defined as

$$V_Y(P_F) := \{\tau \in T_y(P_F) | \pi_{F*}\tau = 0\}.$$

(2) Let $k_v : P(\xi) \to P_F$, $v \in F$, be defined by $k_v(p) := [p, v]$. Then the *horizontal subspace* of the tangent space $T_{[p,v]}(P_F)$ is defined as

$$H_{[p,v]}(P_F) := k_{v*}(H_p P).$$

- Since $k_{g^{-1}v} \circ \delta_g = k_v$, the definition of $H[p,v](P_F)$ is independent ofthe choice of elements (p, v) in the equivalence class $y = [p, v] \in P_F$.
- Let $\alpha : [a,b] \to \mathcal{M}$ and let $[p, v]$ be any point $\pi_F^{-1}(\{\alpha(a)\}$. Let $\alpha^{\uparrow}$ be the unique horizontal lift of α to $P(\xi)$ such that $\alpha^{\uparrow}(a) = p$. Then the curve

$$\alpha_F^{\uparrow}(z) := k_v(\alpha^{\uparrow}(t)) = [\alpha^{\uparrow}(t), v]$$

is the horizontal lift of α to P_F that passes through $[p, v]$ at $t = a$. This leads to the concept of *parallel translation (or transportation)* in the associated bundle as the map $\tau_F : \pi_F^{-1}(\{\alpha(a)\}) \to \pi_F^{-1}(\{\alpha(b)\})$ obtained by taking each point $y \in \pi_F^{-1}(\{\alpha(a)\})$ into the point $\alpha_F^{\uparrow}(b)$, where $t \mapsto \alpha_F^{\uparrow}(t)$ is the horizontal lift of α to P_F that passes through y.

4.3 Covariant differentiation

Motivation: We seek for a derivative of a cross-section $\psi : \mathcal{M} \to P_V$ of a vector bundle. The problem is, that one cannot compare the values of ψ for any pair of neighbouring points in $\mathcal{M}$ without using a concrete bundle trivialisation because they lie in different fibres.

If the bundle is equipped with a connection one-form ω, one can use ω to 'pull-back' the second fibre over the first in order to subtract points in different fibres.

Definition 6.6 Let $\xi = (P, \pi, \mathcal{M})$ be a principal G-bundle and let V be a vector space that carries a linear representation of G. Let $\alpha : [0, \epsilon] \to \mathcal{M}$, $\epsilon > 0$, be a curve in $\mathcal{M}$ such that $\alpha(0) = x_0 \in \mathcal{M}$, and let $\psi : \mathcal{M} \to P_V$ be a cross-section of the associated vector bundle. The *covariant derivative* of ψ in the direction α at x_0, is

$$\nabla_a \psi := \lim_{t \to 0} \left(\frac{\tau_V^t \psi(\alpha(t)) - \psi(x_0)}{t} \right) \in \pi_V^{-1}(\{x_0\})$$

where τ_V^t is the (linear) parallel-transport map from the vector space $\pi_V^{-1}(\{\alpha(t)\})$ to the vector space $\pi_V^{-1}(\{x_0\})$.

Definition 6.7

- If $\nu \in T_x\mathcal{M}$, the *covariant derivative* of the section ψ of P_V *along* v is defined to be $\nabla_v \psi := \nabla_\alpha \psi$, where α is any curve in $\mathcal{M}$ that belongs to the equivalence class of v.
- If X is a vector field on $\mathcal{M}$, the *covariant derivative* along X is the linear operator $\nabla_X : \Gamma(P_V) \to \Gamma(P_V)$ on the set $\Gamma(P_V)$ of cross-sections of the vector bundle P_V defined by
$$(\nabla_X \psi)(x) := \nabla_{X_x} \psi.$$