Cauchy problems for the Einstein equations: an Introduction

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Contents

Part I

The Einstein equations

Chapter 1

The local evolution problem

1.1 The nature of the Einstein equations

The *vacuum Einstein equations with cosmological constant* Λ read

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0\,, \tag{1.1.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor,

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}\,, \tag{1.1.2}$$

while $R_{\alpha\beta}$ is the Ricci tensor of the Levi-Civita connection of g, and R the scalar curvature. We will sometimes refer to those equations as *the vacuum Einstein equations*, regardless of whether or not the cosmological constant vanishes. Taking the trace of (1.1.1) one obtains

$$R = \frac{2(n+1)}{n-1}\Lambda\,, \tag{1.1.3}$$

where, as elsewhere, $n+1$ is the dimension of spacetime. This leads to the following equivalent version of (1.1.1):

$$\mathrm{Ric} = \frac{2\Lambda}{n-1} g\,. \tag{1.1.4}$$

Thus the Ricci tensor of the metric is proportional to the metric. Pseudo-Lorentzian manifolds the metric of which satisfies Equation (1.1.4) are called *Einstein manifolds* in the mathematical literature; see, e.g., [59].

Given a manifold $\mathscr{M}$, Equation (1.1.1) or, equivalently, Equation (1.1.4) forms a system of partial differential equations for the metric. Indeed, recall that for the Levi-Civita connection we have

$$\Gamma^\alpha{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}(\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma})\,, \tag{1.1.5}$$

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}\,, \tag{1.1.6}$$

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}\,. \tag{1.1.7}$$

We see that the Ricci tensor is an object built out of the Christoffel symbols and their first derivatives, while the Christoffel symbols are built out of the metric and its first derivatives. These equations further show that the Ricci tensor is linear in the second derivatives of the metric, with coefficients which are rational functions of the $g_{\alpha\beta}$'s, and quadratic in the first derivatives of g, again with coefficients rational in g. Equations linear in the highest order derivatives are called *quasi-linear*, hence the vacuum Einstein equations constitute a second order system of quasi-linear partial differential equations for the metric g.

In the discussion above we have assumed that the manifold $\mathscr{M}$ has been given. Such a point of view might seem to be too restrictive, and sometimes it is argued that the Einstein equations should be interpreted as equations both for the metric and the manifold. The sense of such a statement is far from being clear, one possibility of understanding that is that the manifold arises as a result of the evolution of the metric g. We are going to discuss in detail the evolution point of view below, let us, however, anticipate and mention the following: there exists a natural class of spacetimes, called *maximal globally hyperbolic* (see Appendix A.22, p. 233, for a definition), which are obtained by the vacuum evolution of initial data, and which have topology $\mathbb{R} \times \mathscr{S}$, where $\mathscr{S}$ is the n-dimensional manifold on which the initial data have been prescribed. Thus, these spacetimes (as defined precisely in Theorem 2.1.1 below) have topology and differentiable structure which are determined by the initial data. As will be discussed in more detail in Chapter 2, the spacetimes so constructed are sometimes *extendible*. Now, there do not seem to exist conditions which would guarantee uniqueness of extensions of the maximal globally hyperbolic solutions, while examples of non-unique extensions are known. Therefore it does not seem useful to consider the Einstein equations as equations determining the manifold beyond the maximal globally hyperbolic region. We conclude that in the evolutionary point of view the manifold can be also thought as being given *a priori*, namely $\mathscr{M} = \mathbb{R} \times \mathscr{S}$. We stress, however, that the decomposition $\mathscr{M} = \mathbb{R} \times \mathscr{S}$ has no intrinsic meaning in general, in that there is no natural time coordinate which can always be constructed by evolutionary methods and which leads to such a decomposition.

Now, there exist standard classes of partial differential equations which are known to have good properties. They are determined by looking at the algebraic properties of those terms in the equations which contain derivatives of highest order, in our case of order two. Inspection of (1.1.1) shows (see (1.1.22) below) that this equation does not fall in any of the standard classes, such as hyperbolic, parabolic, or elliptic. In retrospect this is not surprising, because equations in those classes typically lead to unique solutions. On the other hand, given any solution g of the Einstein equations (1.1.4) and any diffeomorphism Φ, the pull-back metric Φ^*g is also a solution of (1.1.4), so whatever uniqueness there might be will hold only *up to diffeomorphisms*. An alternative way of describing this, often found in the physics literature, is the following: suppose that we have a matrix $g_{\mu\nu}(x)$ of functions satisfying (1.1.1) in some coordinate system x^μ. If we perform a coordinate change $x^\mu \to y^\alpha(x^\mu)$, then the matrix of functions $\bar{g}_{\alpha\beta}(y)$ defined as

$$g_{\mu\nu}(x) \to \bar{g}_{\alpha\beta}(y) = g_{\mu\nu}(x(y))\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta} \tag{1.1.8}$$

will also solve (1.1.1), if the x-derivatives there are replaced by y-derivatives. This property is known under the name of *diffeomorphism invariance*, or *coordinate invariance*, of the Einstein equations. Physicists say that "the diffeomorphism group is the gauge group of Einstein's theory of gravitation".

Somewhat surprisingly, Choquet-Bruhat [205] proved in 1952 that there exists a set of *hyperbolic* equations underlying the Einstein equations. This proceeds by the introduction of so-called *wave coordinates*, also called *harmonic coordinates*, to which we turn our attention in the next section. Before doing that, let us pass to the derivation of a somewhat more explicit *and useful* form of the Einstein equations. In index notation, the definition of the Riemann tensor takes the form

$$\nabla_\mu\nabla_\nu X^\alpha - \nabla_\nu\nabla_\mu X^\alpha = R^\alpha{}_{\beta\mu\nu}X^\beta\,. \tag{1.1.9}$$

A contraction over α and μ gives

$$\nabla_\alpha\nabla_\nu X^\alpha - \nabla_\nu\nabla_\alpha X^\alpha = R_{\beta\nu}X^\beta\,. \tag{1.1.10}$$

Suppose that X is the gradient of a function ϕ, $X = \nabla\phi$, then we have

$$\nabla_\alpha X^\beta = \nabla_\alpha\nabla^\beta\phi = \nabla^\beta\nabla_\alpha\phi\,,$$

because of the symmetry of second partial derivatives. Further

$$\nabla_\alpha X^\alpha = \Box_g\phi\,,$$

where we use the symbol

$$\Box_g \equiv \nabla_\mu\nabla^\mu$$

to denote the wave operator associated with a Lorentzian metric g; *e.g.*, for a scalar field we have

$$\Box_g\phi \equiv \nabla_\mu\nabla^\mu\phi = \frac{1}{\sqrt{-\det g_{\alpha\beta}}}\partial_\mu(\sqrt{-\det g_{\rho\sigma}}g^{\mu\nu}\partial_\nu\phi)\,. \tag{1.1.11}$$

For gradient vector fields (1.1.10) can be rewritten as

$$\nabla_\alpha\nabla^\alpha\nabla_\nu\phi - \nabla_\nu\nabla_\alpha\nabla^\alpha\phi = R_{\beta\nu}\nabla^\beta\phi\,,$$

or, equivalently,

$$\Box_g d\phi - d(\Box_g\phi) = \mathrm{Ric}(\nabla\phi,\cdot)\,, \tag{1.1.12}$$

where d denotes exterior differentiation. Consider Equation (1.1.12) with ϕ replaced by y^A, where y^A is any collection of functions,

$$\Box_g dy^A = d\lambda^A + \mathrm{Ric}(\nabla y^A,\cdot)\,, \tag{1.1.13}$$

$$\lambda^A \equiv \Box_g y^A\,. \tag{1.1.14}$$

(The y^A's will be shortly assumed to form a coordinate system satisfying some convenient conditions, but this is irrelevant at this stage.) Set

$$g^{AB} \equiv g(dy^A, dy^B)\,; \tag{1.1.15}$$

this is consistent with the usual notation for the inverse metric when the y^A's form a coordinate system. For simplicity we have written g instead of $g^\sharp$ for the metric on T^*M. By the product rule we have

$$\begin{aligned}
\Box_g g^{AB} &= \nabla_\mu \nabla^\mu (g(dy^A, dy^B)) \\
&= \nabla_\mu (g(\nabla^\mu dy^A, dy^B) + g(dy^A, \nabla^\mu dy^B)) \\
&= g(\Box_g dy^A, dy^B) + g(dy^A, \Box_g dy^B) + 2g(\nabla_\mu dy^A, \nabla^\mu dy^B) \\
&= g(d\lambda^A, dy^B) + g(dy^A, d\lambda^B) + 2g(\nabla_\mu dy^A, \nabla^\mu dy^B) \\
&\quad +2\mathrm{Ric}(\nabla y^A, \nabla y^B)\,. \qquad (1.1.16)
\end{aligned}$$

Let us *suppose* that the functions y^A solve the homogeneous wave equation:

$$\lambda^A = \Box_g y^A = 0\,. \qquad (1.1.17)$$

The Einstein equation (1.1.4) inserted in (1.1.16) implies then

$$\begin{aligned}
E^{AB} &\equiv \Box_g g^{AB} - 2g(\nabla_\mu dy^A, \nabla^\mu dy^B) - \frac{4\Lambda}{n-1} g^{AB} \qquad (1.1.18a) \\
&= 0\,. \qquad (1.1.18b)
\end{aligned}$$

Now,

$$\begin{aligned}
\nabla_\mu (dy^A) &= \nabla_\mu (\partial_\nu y^A \, dx^\nu) \\
&= (\partial_\mu \partial_\nu y^A - \Gamma^\sigma{}_{\mu\nu} \partial_\sigma y^A) dx^\nu\,. \qquad (1.1.19)
\end{aligned}$$

Suppose that the dy^A's are linearly independent and form a basis of $T^*\mathscr{M}$, then (1.1.18b) is *equivalent* to the vacuum Einstein equation. Further we can choose the y^A's as coordinates, at least on some open subset of $\mathscr{M}$; in this case we have

$$\partial_A y^B = \delta^B_A\,, \qquad \partial_A \partial_C y^B = 0\,,$$

so that (1.1.19) reads

$$\nabla_B dy^A = -\Gamma^A{}_{BC} dy^C\,.$$

This, together with (1.1.18b), leads to

$$\Box_g g^{AB} - 2g^{CD} g^{EF} \Gamma^A_{CE} \Gamma^B_{DF} - \frac{4\Lambda}{n-1} g^{AB} = 0\,. \qquad (1.1.20)$$

Here the Γ^A_{BC}'s should be calculated in terms of the g_{AB}'s and their derivatives as in the usual equation for the Christoffel symbols (1.1.5), and the wave operator $\Box_g$ is understood as acting on scalars. We have thus shown that *in "wave coordinates", as defined by the condition $\lambda^A = 0$, the Einstein equation forms a second-order quasi-linear wave-type system of equations* (1.1.20) for the metric functions g^{AB}. This gives a strong hint that the Einstein equations possess a *hyperbolic*, evolutionary character; this fact will be fully justified in what follows.

REMARK 1.1.1 Using the explicit formula for the Ricci tensor and the Christoffel symbols, and *without imposing any coordinate conditions*, one has

$$\begin{aligned} R_{\nu\rho}[g] &= \frac{1}{2}\left\{\frac{\partial}{\partial x^\delta}\left(g^{\delta\eta}\left[-\frac{\partial g_{\rho\nu}}{\partial x^\eta}+\frac{\partial g_{\nu\eta}}{\partial x^\rho}+\frac{\partial g_{\rho\eta}}{\partial x^\nu}\right]\right)-\frac{\partial}{\partial x^\rho}\left(g^{\delta\eta}\frac{\partial g_{\delta\eta}}{\partial x^\nu}\right)\right\} \\ &+\frac{1}{4}\left\{g^{\lambda\pi}\left(\frac{\partial g_{\delta\pi}}{\partial x^\lambda}+\frac{\partial g_{\lambda\pi}}{\partial x^\delta}-\frac{\partial g_{\lambda\delta}}{\partial x^\pi}\right)g^{\delta\eta}\left(\frac{\partial g_{\nu\eta}}{\partial x^\rho}+\frac{\partial g_{\rho\eta}}{\partial x^\nu}-\frac{\partial g_{\rho\nu}}{\partial x^\eta}\right)\right. \\ &\left.-g^{\lambda\eta}\left(\frac{\partial g_{\delta\eta}}{\partial x^\rho}+\frac{\partial g_{\rho\eta}}{\partial x^\delta}-\frac{\partial g_{\rho\delta}}{\partial x^\eta}\right)g^{\delta\pi}\left(\frac{\partial g_{\nu\pi}}{\partial x^\lambda}+\frac{\partial g_{\lambda\pi}}{\partial x^\nu}-\frac{\partial g_{\lambda\nu}}{\partial x^\pi}\right)\right\}. \end{aligned} \tag{1.1.21}$$

This is clearly not very enlightening, and fortunately almost never needed.

It should be kept in mind that the coefficients $g^{\delta\eta}$ of the matrix $(g^{\delta\eta})$ inverse to $(g_{\mu\nu})$ take the form $g^{\delta\eta} = (\det(g_{\mu\nu}))^{-1} p^{\delta\eta}$, with $p^{\delta\eta}$'s being homogeneous polynomials, of degree one less than the dimension of the manifold, in the $g_{\mu\nu}$'s. In particular the Ricci tensor is an analytic function of the metric and its first and second derivatives away from the set $\det(g_{\mu\nu}) = 0$.

Let us denote by

$$T^*\mathscr{M} \otimes S^2\mathscr{M} \ni (k,h) \mapsto \sigma(k)_{\mu\nu}[h] \in S^2\mathscr{M}$$

the symbol of the Ricci tensor. Here the Ricci tensor is understood as a quasi-linear PDE operator, and we denote by $S^2\mathscr{M}$ the bundle of two-covariant symmetric tensors. By definition, the map $\sigma(k)$ is obtained by keeping in $R_{\mu\nu}$ only those terms which involve second derivatives of the metric, and replacing each term $\partial_\alpha\partial_\beta g_{\mu\nu}$ by

$$\partial_\alpha\partial_\beta g_{\mu\nu} \to k_\alpha k_\beta h_{\mu\nu}\,,$$

with $h \in S^2\mathscr{M}$: From (1.1.21) we find

$$\sigma(k)_{\nu\rho}[h] = \frac{1}{2}\left\{k_\delta g^{\delta\eta}\left[-h_{\rho\nu}k_\eta + h_{\nu\eta}k_\rho + h_{\rho\eta}k_\nu\right] - k_\rho g^{\delta\eta}h_{\delta\eta}k_\nu\right\}. \tag{1.1.22}$$

Now, the type of a PDE operator is determined by the properties of the kernel of the symbol. For example, one says that an operator is *elliptic* if for every $k \neq 0$ its symbol is invertible as a linear map. (See e.g. [166] for a definition of *hyperbolic* linear operators in terms of the algebraic properties of the symbol.)

A calculation shows (cf., e.g., [215]):

1. For every covector η the tensor

$$h_{\mu\nu} = k_\mu\eta_\nu + \eta_\mu k_\nu$$

is in the kernel of $\sigma(k)$. Such tensors arise from covariance of the Ricci tensor under diffeomorphisms, and exhaust the kernel when $k_\alpha k^\alpha \neq 0$.

2. If $k \neq 0$ is null, in dimension $(n+1)$ the kernel has dimension $n(n+1)/2$ and is spanned by tensors of the form

$$h_{\mu\nu} = \ell_{\mu\nu} + k_\mu\eta_\nu + \eta_\mu k_\nu\,,$$

with $k^\mu\ell_{\mu\nu} = 0$ and $g^{\mu\nu}\ell_{\mu\nu} = 0$.

It follows, e.g., that the equation $R_{\mu\nu} = 0$ is certainly *not* elliptic, whether g is Riemannian, Lorentzian, or else. In fact, one also finds [215] that there is no known notion of hyperbolicity which applies directly to the Ricci tensor. □

REMARK 1.1.2 Our derivation so far of the "harmonically reduced equations" has the advantage of giving an explicit form of the lower order terms in the equation in a compact form. An alternative, more standard, derivation of those equations starts from (1.1.21) and proceeds as follows: The explicit formula for the "harmonicity functions" $\lambda^\mu := \Box x^\mu$ reads

$$\begin{aligned}\lambda^\mu &= \Box_g x^\mu = g^{\delta\eta}\nabla_\delta\partial_\eta x^\mu = g^{\delta\eta}(\partial_\delta\partial_\eta x^\mu - \Gamma^\sigma_{\delta\eta}\partial_\sigma x^\mu) = -g^{\delta\eta}\Gamma^\mu_{\delta\eta} \\ &= -\frac{1}{2}g^{\delta\eta}g^{\mu\sigma}(\partial_\delta g_{\sigma\eta} + \partial_\eta g_{\sigma\delta} - \partial_\sigma g_{\delta\eta}) \\ &= -g^{\delta\eta}g^{\mu\sigma}(\partial_\delta g_{\sigma\eta} - \frac{1}{2}\partial_\sigma g_{\delta\eta})\,. \end{aligned} \tag{1.1.23}$$

If we write "l.o.t." for terms which do not contain second derivatives of the metric, from (1.1.21) we find

$$\begin{aligned}R_{\nu\rho}[g] &= \frac{1}{2}\Big\{ - g^{\delta\eta}\frac{\partial^2 g_{\rho\nu}}{\partial x^\delta\partial x^\eta} + \underbrace{g^{\delta\eta}\frac{\partial^2 g_{\nu\eta}}{\partial x^\delta\partial x^\rho}}_{-g_{\nu\mu}\partial_\rho\lambda^\mu + \frac{1}{2}g^{\delta\eta}\frac{\partial^2 g_{\delta\eta}}{\partial x^\rho\partial x^\nu}} + \underbrace{g^{\delta\eta}\frac{\partial^2 g_{\rho\eta}}{\partial x^\delta\partial x^\nu}}_{-g_{\rho\mu}\partial_\nu\lambda^\mu + \frac{1}{2}g^{\delta\eta}\frac{\partial^2 g_{\delta\eta}}{\partial x^\rho\partial x^\nu}} \\ &\quad - g^{\delta\eta}\frac{\partial^2 g_{\delta\eta}}{\partial x^\rho\partial x^\nu}\Big\} + \text{l.o.t.} \\ &= -\frac{1}{2}\left(\Box_g g_{\nu\rho} + g_{\nu\mu}\partial_\rho\lambda^\mu + g_{\rho\mu}\partial_\nu\lambda^\mu\right) + \text{l.o.t.}\end{aligned} \tag{1.1.24}$$

This can be seen to coincide with the principal part of (1.1.16).

It turns out that (1.1.18b) allows one also to *construct* solutions of Einstein equations [205], this will be done in the following sections.

INCIDENTALLY: Before analyzing the existence question, it is natural to ask the following: given a solution of the Einstein equations, can one always find local coordinate systems y^A satisfying the wave condition (1.1.17)? The answer is yes, the standard way of obtaining such functions proceeds as follows: Let $\mathscr{S}$ be any spacelike hypersurface in $\mathscr{M}$; by definition, the restriction of the metric g to $T\mathscr{S}$ is positive non-degenerate. Let $\mathscr{O} \subset \mathscr{S}$ be any open subset of $\mathscr{S}$, and let X be any smooth vector field on $\mathscr{M}$, defined along $\mathscr{O}$, which is transverse to $\mathscr{S}$; by definition, this means that for each $p \in \mathscr{O}$ the tangent space $T_p\mathscr{M}$ is the direct sum of $T_p\mathscr{S}$ and of the linear space $\mathbb{R}X(p)$ spanned by $X(p)$. (Any timelike vector X would do — *e.g.*, the unit normal to $\mathscr{S}$ — but transversality is sufficient for our purposes here.) The following result is well known (cf., e.g., [376, Theorem 8.6] or [321, Theorem 7.2.2]):

THEOREM 1.1.4 *Let $\mathscr{S}$ be a smooth spacelike hypersurface in a smooth spacetime $(\mathscr{M}, g)$. For any smooth functions f, h on $\mathscr{O} \subset \mathscr{S}$ there exists a unique smooth solution ϕ defined on $\mathscr{D}(\mathscr{O})$ of the problem*

$$\Box_g\phi = 0\,, \qquad \phi|_{\mathscr{O}} = f\,, \quad X(\phi)|_{\mathscr{O}} = h\,.$$

Once a hypersurface $\mathscr{S}$ has been chosen, *local wave coordinates adapted to* $\mathscr{S}$ may be constructed as follows: Let $\mathscr{O}$ be any coordinate patch on $\mathscr{S}$ with coordinate functions x^i, $i = 1, \ldots, n$, and let e^0 be the field of unit future pointing normals to $\mathscr{O}$. On $\mathscr{D}(\mathscr{O})$ define the y^A's to be the unique solutions of the problem

$$\Box_g y^A = 0\,,$$

$$y^0|_{\mathscr{O}} = 0\,, \quad e^0(y^0)|_{\mathscr{O}} = 1\,, \tag{1.1.25}$$

$$y^i|_{\mathscr{O}} = x^i\,, \quad e^0(y^i)|_{\mathscr{O}} = 0\,, \quad i = 1, \ldots, n\,. \tag{1.1.26}$$

It follows from (1.1.25)-(1.1.26) that $\partial y^A/\partial x^\mu$ is nowhere vanishing on $\mathscr{O}$. By continuity, $\partial y^A/\partial x^\mu$ will be non-zero in a neighborhood of the initial data surface, and the inverse function theorem shows that there exists a neighborhood $\mathscr{U} \subset \mathscr{D}(\mathscr{O})$ of $\mathscr{O}$ which is coordinatized by the y^A's.

We note that there is a considerable freedom in the construction of the y^i's as above because of the freedom of choice of the x^i's, but the function y^0 is defined uniquely by $\mathscr{S}$ and (1.1.25). □

1.2 Linearised gravity

To get some insight into the problem at hand, we consider first the Einstein equations linearised at Minkowski spacetime. Our presentation follows [45].

Consider a metric g which, in the natural coordinates on $\mathbb{R}^{n+1}$, takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{1.2.1}$$

where η denotes the Minkowski metric. Suppose that there exists a small constant ϵ such that we have

$$|h_{\mu\nu}| \,,\ |\partial_\sigma h_{\mu\nu}| \,,\ |\partial_\sigma \partial_\rho h_{\mu\nu}| = O(\epsilon) \,. \tag{1.2.2}$$

If we use the metric η to raise and lower indices one has

$$R_{\beta\delta} \;=\; \frac{1}{2}\left[\partial_\alpha\{\partial_\beta h^\alpha{}_\delta + \partial_\delta h^\alpha{}_\beta - \partial^\alpha h_{\beta\delta}\} - \partial_\delta\partial_\beta h^\alpha{}_\alpha\right] + O(\epsilon^2) \,. \tag{1.2.3}$$

Coordinate transformations $x^\mu \mapsto x^\mu + \zeta^\mu$, with

$$|\zeta_\mu| \,, \quad |\partial_\sigma \zeta_\mu| \,, \quad |\partial_\sigma\partial_\rho \zeta_\mu| \,, \quad |\partial_\sigma\partial_\rho\partial_\nu \zeta_\mu| = O(\epsilon) \,, \tag{1.2.4}$$

preserve (1.2.1)-(1.2.2), and lead to the "gauge-freedom"

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu + O(\epsilon^2) \,. \tag{1.2.5}$$

In what follows we ignore all $O(\epsilon^2)$-terms in the equations above. Then vacuum linearised gravity becomes a theory of a tensor field $h_{\mu\nu}$ with the gauge-freedom (1.2.5) and satisfying the equations

$$0 = \partial_\alpha\{\partial_\beta h^\alpha{}_\delta + \partial_\delta h^\alpha{}_\beta - \partial^\alpha h_{\beta\delta}\} - \partial_\delta\partial_\beta h^\alpha{}_\alpha \,. \tag{1.2.6}$$

1.2.1 The Cauchy problem for linearised gravity

For definiteness we assume that the space-dimension $n = 3$, the general case proceeds as below with only trivial modifications.

Solving the following wave equation

$$\Box \zeta_\alpha = -\partial_\beta h^\beta{}_\alpha + \frac{1}{2}\partial_\alpha h^\beta{}_\beta \,,$$

where $\Box \equiv \Box_\eta$ is the wave-operator of the Minkowski metric, and performing (1.2.5) leads to a new tensor $h_{\mu\nu}$, still denoted by the same symbol, such that

$$\partial_\beta h^\beta{}_\alpha = \frac{1}{2}\partial_\alpha h^\beta{}_\beta \,, \tag{1.2.7}$$

together with the usual wave equation for h:

$$\Box h_{\beta\delta} = 0\,. \tag{1.2.8}$$

Solutions of this last equation are in one-to-one correspondence with their Cauchy data at $t = 0$. However, those data are not arbitrary, which can be seen as follows: Equations (1.2.7)-(1.2.8) imply

$$\Box(\partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta) = 0\,. \tag{1.2.9}$$

This implies that (1.2.7) will hold if and only if

$$\left(\partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta\right)\Big|_{t=0} = 0 = \partial_0\left(\partial_\beta h^\beta{}_\alpha - \frac{1}{2}\partial_\alpha h^\beta{}_\beta\right)\Big|_{t=0}\,. \tag{1.2.10}$$

Equivalently, taking (1.2.8) into account,

$$\partial_0(h_{00} + h^i{}_i)|_{t=0} = 2\partial_i h^i{}_0|_{t=0}\,, \tag{1.2.11}$$

$$\partial_0 h_{0i}|_{t=0} = (\partial_j h^j{}_i + \tfrac{1}{2}\partial_i(h_{00} - h^j{}_j))|_{t=0}\,, \tag{1.2.12}$$

$$\Delta h^i{}_i|_{t=0} = \partial_i\partial_j h^{ij}|_{t=0}\,, \tag{1.2.13}$$

$$\partial_j(\partial_0 h^j{}_i - \partial_0 h^k{}_k\delta^j_i)|_{t=0} = (\Delta h_{0i} - \partial_i\partial_j h^j{}_0)|_{t=0}\,. \tag{1.2.14}$$

The last two equations are the linearisations, at the Minkowski metric, of the "scalar and vector constraint equations" that will be encountered in Section 1.4.

There remains the freedom of choosing $\zeta_\alpha|_{t=0}$ and $\partial_t\zeta_\alpha|_{t=0}$. It turns out to be convenient to require

$$\begin{aligned}(\partial_0 h^k{}_k - 2\partial_k h^k{}_0 - 2\Delta\zeta_0)|_{t=0} &= 0\,,\\ (h_{00} + 2\partial_0\zeta_0)|_{t=0} &= 0\,,\\ (h_{0i} + \partial_i\zeta_0 + \partial_0\zeta_i)|_{t=0} &= 0\,,\\ D_i(h^i{}_j - \tfrac{1}{3}h^k{}_k\delta^i_j + D^i\zeta_j + D_j\zeta^i - \tfrac{2}{3}D^k\zeta_k\delta^i_j)|_{t=0} &= 0\,,\end{aligned} \tag{1.2.15}$$

where $D_i \equiv D^i \equiv \partial_i$ in background Cartesian coordinates. Indeed, given any $h_{\mu\nu}$ and $\partial_0 h_{\mu\nu}|_{t=0}$, the first equation can be solved for $\zeta_0|_{t=0}$ if one assumes that

$$(\partial_0 h^k{}_k - 2\partial_k h^k{}_0)|_{t=0} \tag{1.2.16}$$

belongs to a suitable weighted Sobolev or Hölder space, the precise requirements being irrelevant for the conceptual overview here. The second equation in (1.2.15) defines $\partial_0\zeta_0|_{t=0}$; the third defines $\partial_0\zeta_i|_{t=0}$; finally, the last equation is an elliptic equation for the vector field $\zeta_i|_{t=0}$ which can be solved [113] if one again assumes that

$$\partial_i(h^i{}_j - \frac{1}{3}h^k{}_k\delta^i_j)|_{t=0} \tag{1.2.17}$$

belongs to a weighted Sobolev or Hölder space. (We note, however, that if some components of h_{ij} behave as $1/r$, then ζ will behave like $\ln r$ in general, which is likely to introduce $\ln r/r$ terms in the gauge-transformed metric, a feature which one sometimes wishes to avoid.) After performing this gauge-transformation, we end up with a tensor field $h_{\mu\nu}$ which satisfies

$$\partial_0 h^k{}_k|_{t=0} = h_{00}|_{t=0} = h_{0i}|_{t=0} = \partial_i(h^i{}_j - \frac{1}{3}h^k{}_k\delta^i_j)|_{t=0} = 0\,. \qquad (1.2.18)$$

Inserting this into (1.2.11)-(1.2.14) we find

$$\partial_0 h_{00}|_{t=0} = 0\,, \qquad (1.2.19)$$
$$\partial_0 h_{0i}|_{t=0} = -\tfrac{1}{6}\partial_i h^j{}_j|_{t=0}\,, \qquad (1.2.20)$$
$$\Delta h^i{}_i|_{t=0} = 0\,, \qquad (1.2.21)$$
$$\partial_j(\partial_0 h^j{}_i - \partial_0 h^k{}_k\delta^j_i)|_{t=0} = 0\,. \qquad (1.2.22)$$

Now, so far we have been considering vacuum fields with initial data on $\mathbb{R}^n$. However, any domain $\Omega \subset \mathbb{R}^n$ would have worked provided that the Laplace equation could be solved on Ω. On the other hand, the hypothesis that $\Omega = \mathbb{R}^n$ becomes important when considering (1.2.21). Indeed, in this case the further requirement that $h^i{}_i$ goes to zero as r tends to infinity together with the maximum principle gives

$$h^i{}_i|_{t=0} = 0\,. \qquad (1.2.23)$$

We conclude (compare [20]) that at any given time $t = t_0$ every linearised gravitational initial data set $(h_{\mu\nu}, \partial_t h_{\mu\nu})|_{t=t_0}$ can be gauge-transformed to the so-called *TT*-gauge. Here "*TT*" stands for "transverse traceless". In this gauge, using the notation

$$k_{ij} := \frac{1}{2}\partial_0 h_{ij}|_{t=t_0}\,,$$

it holds that

$$h^k{}_k|_{t=t_0} = \partial_i h^i{}_j|_{t=t_0} = k^k{}_k = \partial_i k^i{}_j = 0\,. \qquad (1.2.24)$$

We say that both h and k are transverse and traceless. From what has been said and from uniqueness of solutions of the wave equation we also see that in this gauge we will have for all t

$$h_{00} = h_{0i} = h^k{}_k = \partial_i h^i{}_j = 0\,, \qquad (1.2.25)$$

which further implies that (1.2.24) is preserved by evolution.

Summarising, we have proved:

THEOREM 1.2.1 *Linearised gravitational fields on $\mathbb{R}\times\mathbb{R}^n$, with initial data tending to zero sufficiently fast as one recedes to infinity, can be gauge-transformed to fields satisfying*

$$\Box_\eta h_{ij} = 0 = \partial_i h^i{}_j = h^i{}_i = h_{00} = h_{0i}\,. \qquad (1.2.26)$$

The initial data at $\{t = t_0\}$ are given by two symmetric tensor fields $(h_{ij}(t_0,\cdot), k_{ij}(\cdot))$ satisfying (1.2.24). These constraint equations are preserved by evolution.

The situation is different if dealing with the linearised equations with sources confined to a ball $B(R)$. Then the fields are vacuum on the complement of $B(R)$ and we can repeat the construction above in the vacuum region. But (1.2.23) will not hold in general, and the trace of h_{ij} might be non-trivial. Since $h^i{}_i$ is harmonic on $\mathbb{R}^n \setminus B(R)$, it will have an expansion in terms of inverse powers of r, starting with $1/r$-terms associated with the total mass of the configuration. One can still shield the solutions inside cones using the methods of Carlotto and Schoen [84], discussed in Section 3.6, but this requires much more sophisticated techniques.

1.2.2 The Weyl tensor formulation

It can be shown that the vacuum Einstein equations imply the following equation for the Weyl tensor [218],

$$\nabla_\mu C^\mu{}_{\alpha\beta\gamma} = 0\,. \tag{1.2.27}$$

This equation implies a symmetrizable-hyperbolic system of equations in dimension $1+3$ (cf., e.g., [208]), which can be used to obtain solutions of the Einstein equations.

When linearised at the Minkowski metric (or, more generally, at a Weyl-flat meric), the equations for the metric perturbations and for the Weyl tensor perturbations decouple, so that one can consider (1.2.27) in its own, with ∇ the covariant-derivative operator of the Minkowski metric, as an equation for a tensor field $C^\mu{}_{\alpha\beta\gamma}$ with the algebraic symmetries of the Weyl tensor.

Let us show that a theory of such tensor fields on Minkowski spacetime is equivalent to linearised gravity as formulated above. For this, we suppose first that $R_{\mu\nu\rho\sigma}$ is a tensor field on on a star-shaped subset of $\mathbb{R}^d$, $d > 2$, having the algebraic symmetries of the Riemann tensor and satisfying the (linearised) Bianchi identity

$$\partial_{[\mu} R_{\nu\rho]\sigma\tau} = 0\,. \tag{1.2.28}$$

(We will see shortly that what follows applies to $C_{\mu\nu\rho\sigma}$, which has the right algebraic symmetries, and note that at this stage we are not assuming tracelessness, which holds for $C_{\mu\nu\rho\sigma}$ but not necessarily for $R_{\mu\nu\rho\sigma}$.) We will construct a tensor field $h_{\mu\nu}$, defined up to the usual gauge transformations, such that

$$R_{\mu\nu\rho\sigma} = 2\,\partial_{[\mu} h_{\nu][\rho,\sigma]}\,. \tag{1.2.29}$$

This is precisely the condition that $R_{\mu\nu\rho\sigma}$ equals to the linearisation, at the Minkowski metric η, of the map which assigns to $h_{\alpha\beta}$ the Riemann tensor of the metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$. Equivalently, the right-hand side of (1.2.29) multiplied by ϵ is, up to $O(\epsilon^2)$ terms, the Riemann tensor of the metric $\eta_{\mu\nu} + \epsilon h_{\mu\nu}$. We will say that $R_{\mu\nu\rho\sigma}$ is the linearised Riemann tensor associated with $h_{\mu\nu}$.

To prove (1.2.29), we start by noting that (1.2.28) implies

$$R_{\mu\nu\rho\sigma} = \partial_{[\mu} F_{\nu]\rho\sigma} \tag{1.2.30}$$

with $F_{\mu\nu\rho} = F_{\mu[\nu\rho]}$. But, since $R_{[\mu\nu\rho]\sigma} = 0$, there exists a tensor field $H_{\alpha\beta}$ such that

$$F_{[\mu\nu]\rho} = \partial_{[\mu} H_{\nu]\rho}\,. \tag{1.2.31}$$

Inserting the identity

$$F_{\nu\rho\sigma} = F_{[\sigma\nu]\rho} + F_{[\sigma\rho]\nu} - F_{[\rho\nu]\sigma} \tag{1.2.32}$$

into (1.2.31), and the resulting equation into (1.2.30), we find indeed (1.2.29) after setting

$$h_{\mu\nu} = H_{(\mu\nu)} \,.$$

The addition of a pure-trace tensor to $h_{\mu\nu}$ does not change the trace-free part of $R_{\mu\nu\rho\sigma}$. So, for a tensor $C_{\mu\nu\rho\sigma}$ with Weyl-symmetries satisfying $\partial_{[\mu}C_{\nu\rho]\sigma\tau} = 0$, there exists a potential $h_{\mu\nu}$ as in (1.2.29), which is trace-free.

We continue with an analysis of the kernel of the map sending $h_{\mu\nu}$ into $R_{\mu\nu\rho\sigma}$. Namely, when $R_{\mu\nu\rho\sigma} = 0$, from (1.2.29) we infer

$$h_{\mu[\nu,\rho]} = \partial_\mu A_{\nu\rho} \,, \tag{1.2.33}$$

for some tensor field satisfying $A_{\nu\rho} = A_{[\nu\rho]}$. But, since $\partial_{[\mu}A_{\nu\rho]} = 0$,

$$A_{\mu\nu} = \partial_{[\mu}B_{\nu]} \,. \tag{1.2.34}$$

Now defining $k_{\mu\nu} = h_{\mu\nu} + \partial_\mu B_\nu$, there results

$$k_{\mu[\nu,\rho]} = h_{\mu[\nu,\rho]} + \partial_\mu\partial_{[\rho}B_{\nu]} = 0 \,, \tag{1.2.35}$$

so that $k_{\mu\nu} = \partial_\mu E_\nu$, whence $h_{\mu\nu} = \partial_\mu(E_\nu - B_\nu)$. Finally, using the symmetry of $h_{\mu\nu}$, it follows that

$$h_{\mu\nu} = \partial_{(\mu}\Lambda_{\nu)} \tag{1.2.36}$$

with $\Lambda_\mu = E_\mu - B_\mu$.

We continue with the proof of equivalence of (1.2.27) to the equations arising in the metric formulation of the theory. Here the key observation is that, in spacetime dimension four, (1.2.30) is equivalent to [111, Proposition 4.3]

$$\partial_{[\alpha}C_{\beta\gamma]\mu\nu} = 0 \,. \tag{1.2.37}$$

As just pointed out, this implies existence of a symmetric trace-free tensor field $h_{\mu\nu}$ such that

$$C_{\mu\nu\rho\sigma} = 2\,\partial_{[\mu}h_{\nu][\rho,\sigma]} \,. \tag{1.2.38}$$

Now, the right-hand side of (1.2.38) is the linearised Riemann tensor associated with the linearised metric perturbation $h_{\mu\nu}$. Since the left-hand side of (1.2.38) has vanishing traces, we conclude that the linearised Ricci tensor associated with $h_{\mu\nu}$ vanishes. Equivalently, $h_{\mu\nu}$ satisfies the linearised Einstein equations.

Let E_{ij} denote the “electric part” and B_{ij} the “magnetic part” of the Weyl tensor:

$$E_{ij} := C_{0i0j} \,, \quad B_{ij} := \star C_{0i0j} \,, \tag{1.2.39}$$

with

$$\star C_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}C_{\mu\nu\gamma\delta} \,.$$

Then both E_{ij} and B_{ij} are symmetric and traceless. Indeed, symmetry and tracelessness of E_{ij}, as well as tracelessness of B_{ij} are obvious from the symmetries of the Weyl tensor. The symmetry of B_{ij} follows from the less-obvious double-dual symmetry of the Weyl tensor (see (A.12.49), Appendix A.12.5)

$$\epsilon_{\alpha\beta}{}^{\mu\nu}C_{\mu\nu\gamma\delta} = \epsilon_{\gamma\delta}{}^{\mu\nu}C_{\mu\nu\alpha\beta} \, .$$

Using this notation, and in spacetime dimension four, (1.2.30) split into two evolution equations for E_{ij} and B_{ij},

$$\partial_t E_{ij} = -\epsilon_i{}^{k\ell}\partial_k B_{\ell j} \, , \quad \partial_t B_{ij} = \epsilon_i{}^{k\ell}\partial_k E_{\ell j} \, , \tag{1.2.40}$$

and two constraint equations

$$D^i E_{ij} = 0 = D^i B_{ij} \, . \tag{1.2.41}$$

These equations are strongly reminiscent of the sourceless Maxwell equations. The above structure remains true for the full non-linear equations, cf., e.g., [218].

It is of interest to enquire about the relation of the last constraints with the ones satisfied by $h_{\mu\nu}$. It turns out that the vanishing of the divergence of E_{ij} is closely related to the linearised scalar constraint equation (1.2.13), while the symmetry of B_{ij} relates to the vector constraint equation (1.2.14). This can be seen as follows:

To understand the nature of the divergence constraint $D^i E_{ij} = 0$, let us denote by r_{ijkl} the linearised Riemann tensor of the three-dimensional metric $\delta_{ij} + h_{ij}$, with the associated linearised Ricci tensor $r_{ij} = r^k{}_{ikj}$. We have just seen that $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$ for solutions of $\partial_\alpha C^\alpha{}_{\beta\gamma\delta} = 0$, which gives for such solutions

$$0 = R_{ij} = R^\alpha{}_{i\alpha j} = C^\alpha{}_{i\alpha j} = -C_{0i0j} + r_{ij} = -E_{ij} + r_{ij} \, . \tag{1.2.42}$$

Here we have used the fact that the three-dimensional Riemann tensor differs from the four-dimensional one by terms quadratic in the extrinsic curvature (cf. (1.4.17) below), hence both tensors coincide when linearised at Minkowski spacetime. The vanishing of the divergence of the Einstein tensor implies

$$D^i r_{ij} = \frac{1}{2} D_j r \, ,$$

which together with (1.2.42) shows that the constraint equation $D^i E_{ij} = 0$ is, for asymptotically flat solutions, equivalent to the linearised scalar constraint $r = 0$.

Let us show that symmetry of B_{ij} is equivalent to the vector constraint equation. For this let

$$k_{ij} = \frac{1}{2}(\partial_0 h_{ij} - \partial_i h_{0j} - \partial_j h_{0i})$$

denote the linearised extrinsic curvature tensor of the slices $t = \text{const}$. By a direct calculation, or by linearising the relevant embedding equations, we find

$$R_{0ij\ell} = \partial_\ell k_{ij} - \partial_j k_{i\ell} \, . \tag{1.2.43}$$

Again for solutions of $\partial_\alpha C^\alpha{}_{\beta\gamma\delta} = 0$ it holds that

$$\begin{aligned}\epsilon^{n\ell m} B_{\ell m} &= \frac{1}{2}\epsilon^{n\ell m}\epsilon_{mrs} C_{0\ell}{}^{rs} = \frac{1}{2}\epsilon^{n\ell m}\epsilon_{mrs} R_{0\ell}{}^{rs} = 2\delta^{[n}_r \delta^{\ell]}_s D^s k_\ell{}^r \\ &= D^\ell (k_\ell{}^n - k^m{}_m \delta^n_\ell)\,, \qquad (1.2.44)\end{aligned}$$

which is the linearised version of the vector constraint equation, as claimed.

1.3 Existence local in time and space in wave coordinates

Let us return to (1.1.16). Assume again that the y^A's form a local coordinate system, but do not assume for the moment that the y^A's solve the wave equation. In that case (1.1.16) together with the definition (1.1.18a) of E^{AB} lead to

$$R^{AB} = \frac{1}{2}(E^{AB} - g^{AC}\partial_C \lambda^B - g^{BC}\partial_C\lambda^A) + \frac{2\Lambda}{n-1} g^{AB}\,. \qquad (1.3.1)$$

For the purpose of the calculations that follow, it turns out to be convenient to treat the upper indices on the λ's as vector indices, and change the partial derivatives in (1.3.1) to vector-covariant ones:

$$\begin{aligned}&E^{AB} - g^{AC}\partial_C\lambda^B - g^{BC}\partial_C\lambda^A = \\ &\quad \underbrace{E^{AB} + g^{AC}\Gamma^B{}_{CD}\lambda^D + g^{BC}\Gamma^A{}_{CD}\lambda^D}_{=:\hat E^{AB}} \\ &\quad - g^{AC}(\partial_C\lambda^B + \Gamma^B{}_{CD}\lambda^D) - g^{BC}(\partial_C\lambda^A + \Gamma^A{}_{CD}\lambda^D)\,. \qquad (1.3.2)\end{aligned}$$

One can then rewrite (1.3.1) as

$$R^{AB} = \frac{1}{2}(\hat E^{AB} - \nabla^A\lambda^B - \nabla^B\lambda^A) + \frac{2\Lambda}{n-1} g^{AB}\,. \qquad (1.3.3)$$

The idea, due to Yvonne Choquet-Bruhat [205], is to use the hyperbolic character of the equation

$$\hat E^{AB} = 0 \qquad (1.3.4)$$

to construct a metric g. If we manage to make sure that λ^A vanishes as well, it will then follow from (1.3.1) that g will also solve the Einstein equation.

In any case, we need the following result, which again is standard (cf., e.g., [242, 264, 306, 376, 399]; see Appendix A.22 for a short review of causality theory, in particular for the definition of global hyperbolicity):

THEOREM 1.3.1 *For any initial data*

$$g^{AB}(y^i, 0) \in H^{k+1}\,, \quad \partial_0 g^{AB}(y^i,0) \in H^k\,, \qquad k > n/2\,, \qquad (1.3.5)$$

prescribed on an open subset $\mathscr{O} \subset \{0\}\times\mathbb{R}^n \subset \mathbb{R}\times\mathbb{R}^n$ there exists an open neighborhood $\mathscr{U} \subset \mathbb{R}\times\mathbb{R}^n$ of $\mathscr{O}$ and a unique solution g^{AB} of (1.3.4) defined on $\mathscr{U}$. The set $\mathscr{U}$ can be chosen so that $(\mathscr{U}, g)$ globally hyperbolic with Cauchy surface $\mathscr{O}$.

REMARK 1.3.2 On bounded sets, C^k functions belong to the Sobolev spaces H^k, so the reader unfamiliar with Sobolev spaces can interpret this theorem as saying that for $C^{k+1}\times C^k$ initial data there exist local solutions of the equations. The results in [272–275, 398] and references therein allow one to reduce the differentiability threshold above. □

We note that $\mathscr{U}$ above is *not* uniquely defined without imposing further conditions. Various further results and notions will be useful before we address this issue in Chapter 2.

It remains to find out how to ensure the harmonicity conditions (1.1.17). The key observation of Yvonne Choquet-Bruhat is that (1.3.4) and the Bianchi identities imply a wave equation for λ^A's. In order to see that, recall that it follows from the Bianchi identities that the Ricci tensor of the metric g necessarily satisfies a divergence identity:

$$\nabla_A\Big(R^{AB}-\frac{R}{2}g^{AB}\Big)=0\,.$$

Assuming that (1.3.4) holds, (1.3.3) implies the algebraic identity

$$R=-\nabla_C\lambda^C+2\Lambda\frac{n+1}{n-1}\,.$$

Since the metric is covariantly constant, the divergence identity leads to

$$\begin{aligned}0 &= -\nabla_A\Big(\nabla^A\lambda^B+\nabla^B\lambda^A-\nabla_C\lambda^C g^{AB}\Big)\\ &= -\Big(\Box\lambda^B+\nabla_A\nabla^B\lambda^A-\nabla^B\nabla_C\lambda^C\Big)\\ &= -\Big(\Box\lambda^B+R^B{}_A\lambda^A\Big)\,. \qquad (1.3.6)\end{aligned}$$

This shows that λ^A necessarily satisfies the second order hyperbolic system of equations

$$\Box\lambda^B+R^B{}_A\lambda^A=0\,. \qquad (1.3.7)$$

Now, it is a standard fact in the theory of hyperbolic equations that we will have

$$\lambda^A\equiv 0$$

on the domain of dependence $\mathscr{D}(\mathscr{O})$ provided that both λ^A and its derivatives vanish at $\mathscr{O}$.

REMARK 1.3.3 Actually the vanishing of $\lambda:=(\lambda^A)$ as above is a completely standard result only if the metric is $C^{1,1}$; this is proved by a simpler version of the argument that we are about to present. But the result remains true under the weaker conditions of Theorem 1.3.1, which can be seen as follows. Consider initial data as in (1.3.5), with some $k\in\mathbb{R}$ satisfying $k>n/2$. Then the derivatives of the metric are in L^∞,

$$|\partial g|\le C\,,$$

for some constant C. In the argument below we will use the letter C for a generic constant which might change from line to line. Let $\mathscr{S}_t$ be a foliation by spacelike hypersurfaces of a conditionally compact domain of dependence $\mathscr{D}(\mathscr{S}_0)$, where $\mathscr{S}_0$

is a subset of the initial data surface $\mathscr{S}$. When λ vanishes at $\mathscr{S}_0$, a standard energy calculation for (1.3.7) gives the inequality

$$\begin{aligned}
\|\lambda\|^2_{H^1(\mathscr{S}_t)} &\le C\int_0^t \|\big((1+|\mathrm{Ric}|)|\lambda|+(1+|\partial g|)|\partial\lambda|\big)|\partial\lambda|\|_{L^1(\mathscr{S}_s)}ds \\
&\le C\int_0^t \big(\|(1+|\mathrm{Ric}|)\lambda\|_{L^2(\mathscr{S}_s)}\|\partial\lambda\|_{L^2(\mathscr{S}_s)}+\|\lambda\|^2_{H^1(\mathscr{S}_t)}\big)ds \\
&\le C\int_0^t \big(\|(1+|\mathrm{Ric}|)\lambda\|_{L^2(\mathscr{S}_s)}\|\lambda\|_{H^1(\mathscr{S}_s)}+\|\lambda\|^2_{H^1(\mathscr{S}_s)}\big)ds\,. \quad (1.3.8)
\end{aligned}$$

We want to use this inequality to show that λ vanishes everywhere; the idea is to estimate the integrand by a function of $\|\lambda\|^2_{H^1(\mathscr{S}_s)}$, the vanishing of λ will follow then from the Gronwall lemma. Such an estimate is clear from (1.3.8) if $|\mathrm{Ric}|$ is in L^∞, which proves the claim for metrics in $C^{1,1}$, but is not obviously apparent for less regular metrics. Now, the construction of g in the course of the proof of Theorem 1.3.1 provides a metric such that $\partial g|_{\mathscr{S}_s}\in H^k$ and $\mathrm{Ric}|_{\mathscr{S}_s}\in H^{k-1}$. By Sobolev embedding for $n>2$ we have [25]

$$\|\lambda\|_{L^p(\mathscr{S}_s)}\le C\|\lambda\|_{H^1(\mathscr{S}_s)}\,,$$

where $p=2n/(n-2)$. We can thus use Hölder's inequality to obtain

$$\||\mathrm{Ric}|\lambda\|_{L^2(\mathscr{S}_s)}\le\|\mathrm{Ric}\|_{L^n(\mathscr{S}_s)}\|\lambda\|_{L^p(\mathscr{S}_s)}\le C\|\mathrm{Ric}\|_{L^n(\mathscr{S}_s)}\|\lambda\|_{H^1(\mathscr{S}_s)}\,.$$

Equation (1.3.8) gives thus

$$\|\lambda\|^2_{H^1(\mathscr{S}_t)} \le C\int_0^t \big(1+\|\mathrm{Ric}\|_{L^n(\mathscr{S}_s)}\big)\|\lambda\|^2_{H^1(\mathscr{S}_s)}ds\,,$$

which is the desired inequality provided that $\|\mathrm{Ric}\|_{L^n(\mathscr{S}_s)}$ is finite. But, again by Sobolev,

$$\|\mathrm{Ric}\|_{L^p(\mathscr{S}_s)}\le C\|\mathrm{Ric}\|_{H^{k-1}(\mathscr{S}_s)}\ \text{ provided that }\ \frac{1}{p}\ge\frac12-\frac{k-1}{n}\,,$$

and we see that $\mathrm{Ric}\in L^n(\mathscr{S}_s)$ will hold for $k>n/2$, as assumed in Theorem 1.3.1. □

Remark 1.3.4 There exists a simple generalization of the wave coordinates condition $\Box_g x^\mu=0$ to

$$\Box_g y^A=\mathring{\lambda}^A(y^B,x^\mu,g_{\alpha\beta})\,. \quad (1.3.9)$$

Instead of solving the equation $\hat E^{AB}=0$ one then solves

$$\hat E^{AB}=\nabla^A\mathring{\lambda}^B+\nabla^B\mathring{\lambda}^A\,. \quad (1.3.10)$$

There exists a variation of Theorem 1.3.1 that applies when (1.3.10) is used: Equation (1.3.3) can then be rewritten as

$$R^{AB}=\frac12(\underbrace{\hat E^{AB}-\nabla^A\mathring{\lambda}^B-\nabla^B\mathring{\lambda}^A}_{=0})-\nabla^A(\lambda^B-\mathring{\lambda}^B)-\nabla^B(\lambda^A-\mathring{\lambda}^A)+\frac{2\Lambda}{n-1}g^{AB}\,. \quad (1.3.11)$$

This allows one to repeat the calculation (1.3.6), with λ^A there replaced by $\lambda^A-\mathring{\lambda}^A$.

There remains the easy task to adapt the calculations that follow, done in the case $\mathring{\lambda}^A=0$, to the modified condition (1.3.9), leading to initial data satisfying the right conditions. □

Remark 1.3.5 We can further generalize to include matter fields. Consider, for example, a set of fields ψ^I, $I = 1, \ldots, \mathring{N}$, for some $\mathring{N} \in \mathbb{N}$, satisfying a system of equations of the form

$$\Box_g \psi^I = F^I(\psi^J, \partial\psi^J, g, \partial g)\,. \tag{1.3.12}$$

We assume that there exists an associated energy-momentum tensor

$$T_{\mu\nu}(\psi^J, \partial\psi^J, g, \partial g)$$

which is identically divergence-free when (1.3.12) hold:

$$\nabla_\mu T^{\mu\nu} = 0\,.$$

Allowing (1.3.9), instead of solving the equation $\hat{E}^{AB} = 0$ one solves

$$\hat{E}^{AB} = \nabla^A \mathring{\lambda}^B + \nabla^B \mathring{\lambda}^A + 16\pi \frac{G}{c^4}\Big(T^{AB} - \frac{1}{n-1} g^{CD} T_{CD} g^{AB}\Big)\,. \tag{1.3.13}$$

Theorem 1.3.1 applies to this equation as well. Equation (1.3.3) can then be rewritten as

$$\begin{aligned} R^{AB} \;=\;& \frac{1}{2}\Big(\underbrace{\hat{E}^{AB} - \nabla^A \mathring{\lambda}^B - \nabla^B \mathring{\lambda}^A - 16\pi \frac{G}{c^4}\Big(T^{AB} - \frac{1}{n-1} g^{CD} T_{CD} g^{AB}\Big)}_{=0}\Big) \\ & - \nabla^A(\lambda^B - \mathring{\lambda}^B) - \nabla^B(\lambda^A - \mathring{\lambda}^A) \\ & + \frac{2\Lambda}{n-1} g^{AB} + 8\pi \frac{G}{c^4}\left(T^{AB} - \frac{1}{n-1} g^{CD} T_{CD} g^{AB}\right)\,. \end{aligned} \tag{1.3.14}$$

When T^{AB} has identically vanishing divergence, one can again repeat the calculation (1.3.6), with λ^A there replaced by $\lambda^A - \mathring{\lambda}^A$. As before, the right initial data will lead to a solution with $\lambda^A = \mathring{\lambda}^A$, and hence to the desired solution of the Einstein equations with sources. □

We return to the vanishing of λ^A and its derivatives on $\mathscr{S}$. It is convenient to assume that y^0 is the coordinate along the $\mathbb{R}$ factor of $\mathbb{R} \times \mathbb{R}^n$, so that set $\mathscr{O}$ carrying the initial data is a subset of $\{y^0 = 0\}$; this can always be done. We have

$$\begin{aligned} \Box y^A &= \frac{1}{\sqrt{|\det g|}} \partial_B \Big(\sqrt{|\det g|} g^{BC} \partial_C y^A\Big) \\ &= \frac{1}{\sqrt{|\det g|}} \partial_B \Big(\sqrt{|\det g|} g^{BA}\Big)\,. \end{aligned}$$

So $\Box y^A$ will vanish at the initial data surface if and only if certain time derivatives of the metric are prescribed in terms of the space ones:

$$\partial_0 \Big(\sqrt{|\det g|} g^{0A}\Big) = -\partial_i \Big(\sqrt{|\det g|} g^{iA}\Big)\,. \tag{1.3.15}$$

This implies that the initial data (1.3.5) for the equation (1.3.4) cannot be chosen arbitrarily if we want both (1.3.4) and the Einstein equation to be simultaneously satisfied.

It should be emphasized that there is considerable freedom in choosing the wave coordinates, which is reflected in the freedom to adjust the initial values of g^{0A}'s. A popular choice is to require that on the initial hypersurface $\{y^0 = 0\}$ we have

$$g^{00} = -1\,, \qquad g^{0i} = 0\,, \tag{1.3.16}$$

and this choice simplifies the algebra considerably.

We will show in Proposition 1.5.2 below that for any spacetime $(\mathscr{M}, g)$ and for any spacelike hypersurface $\mathscr{S}$ one can find local coordinates so that (1.3.16) holds. This makes precise the sense in which there is no loss of generality when making this choice.

Equation (1.3.15) determines then the time derivatives $\partial_0 g^{0A}|_{\{y^0=0\}}$ needed in Theorem 1.3.1, once $g_{ij}|_{\{y^0=0\}}$ and $\partial_0 g_{ij}|_{\{y^0=0\}}$ are given. So, from this point of view, the essential initial data for the evolution problem become the space metric

$$h := g_{ij} dy^i dy^j\,,$$

together with its time derivatives.

Incidentally: To show that the harmonic coordinates do not impose any restrictions on the metric functions $g_{\mu\nu}|_{t=0}$, we need to relax (1.3.16). For this it is convenient to introduce the Arnowitt-Deser-Miser (ADM) notation (see Appendix A.19),

$$h_{ij} := g_{ij}\,, \quad N^{-2} := -g^{00}\,, \quad N^i := -\frac{g^{0i}}{g^{00}}\,, \tag{1.3.17}$$

hence

$$g^{0i} = N^{-2} N^i\,, \quad g_{0i} = h_{ij} N^j\,, \quad g^{ij} = h^{ij} - N^{-2} N^i N^j\,, \tag{1.3.18}$$

$$g = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{1.3.19}$$

$$g_{00} = -(N^2 - h_{ij} N^i N^j)\,, \quad \det g = -N^2 \det h\,, \tag{1.3.20}$$

where h^{ij} is the metric inverse to the Riemannian metric h_{ij}. So, using (1.3.15), when all the functions $g_{\mu\nu}$ are prescribed at $t = 0$ we can calculate

$$\partial_t\Big(\sqrt{|\det g|} g^{00}\Big) = \partial_t\Big(\sqrt{|\det h|} N^{-1}\Big) \text{ and } \partial_t\Big(\sqrt{|\det g|} g^{0i}\Big) = \partial_t\Big(\sqrt{|\det h|} N^{-1} N^i\Big) \tag{1.3.21}$$

at $t = 0$ in terms of the $g_{\mu\nu}$'s and their space derivatives. We have already seen how to determine $\partial_t g_{ij} = \partial_t h_{ij}$ at $t = 0$ in terms of K_{ij} and the remaining metric functions. One can then algebraically determine all the derivatives $\partial_t g_{\mu\nu}|_{t=0}$ using (1.3.21) and (1.3.15). □

It turns out that further constraints arise from the requirement of the vanishing of the derivatives of λ. Supposing that (1.3.15) holds on $\{y^0 = 0\}$ — equivalently, supposing that λ vanishes on $\{y^0 = 0\}$, we then have

$$\partial_i \lambda^A = 0$$

on $\{y^0 = 0\}$, where the index i is used to denote tangential derivatives. In order that all derivatives vanish initially it remains to ensure that some transverse derivative does. A transverse direction is provided by the field N of unit timelike normals to $\{y^0 = 0\}$ and, as we are about to show, the vanishing of $\nabla_N \lambda$ can be expressed as

$$\Big(G_{\mu\nu} + \Lambda g_{\mu\nu}\Big) N^\mu = 0\,. \tag{1.3.22}$$

For this, it is most convenient to use an ON frame $e_a = e_a{}^\mu\partial_\mu$, with $e_0 = N$, so that $g_{ab} := g(e_a, e_b) = \eta_{ab}$, where η is the Minkowski metric. It follows from the equation $E_{AB} = 0$ and (1.3.3) that

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{1}{2}\Big(\nabla_\mu\lambda_\nu + \nabla_\nu\lambda_\mu - \nabla^\alpha\lambda_\alpha g_{\mu\nu}\Big)\,,$$

which gives, using frame components $\lambda_a := g_{\mu\nu}\lambda^\mu e_a{}^\mu$:

$$\begin{aligned}-2\Big(G_{\mu\nu} + \Lambda g_{\mu\nu}\Big)N^\mu N^\nu &= 2\nabla_0\lambda_0 - \nabla^\alpha\lambda_\alpha \underbrace{g_{00}}_{=-1} \\ &= 2\nabla_0\lambda_0 + (-\nabla_0\lambda_0 + \underbrace{\nabla_i\lambda_i}_{=0}) \\ &= \nabla_0\lambda_0\,. \end{aligned} \tag{1.3.23}$$

Equation (1.3.23) shows that the vanishing of $\nabla_0\lambda_0$ is equivalent to the vanishing of the 0 frame-component of (1.3.22). Finally

$$\begin{aligned}-2\Big(G_{i0} + \Lambda g_{i0}\Big) &= \underbrace{\nabla_i\lambda_0}_{=0} + \nabla_0\lambda_i - \nabla^\alpha\lambda_\alpha \underbrace{g_{i0}}_{=0} \\ &= \nabla_0\lambda_i\,, \end{aligned} \tag{1.3.24}$$

as desired.

Equations (1.3.22) are called the *general relativistic constraint equations*. We will shortly see that (1.3.15) has quite a different character from (1.3.22); the former will be referred to as a *gauge equation*.

Summarizing, we have proved:

THEOREM 1.3.7 *Under the hypotheses of Theorem 1.3.1, suppose that the initial data (1.3.5) satisfy (1.3.15), (1.3.16) as well as the constraint equations (1.3.22). Then the metric given by Theorem 1.3.1 on the globally hyperbolic set $\mathscr{U}$ satisfies the vacuum Einstein equations.*

In conclusion, in the wave gauge $\lambda^A = 0$ the Cauchy data for the vacuum Einstein equations consist of

1. An open subset $\mathscr{O}$ of $\mathbb{R}^n$,
2. together with matrix-valued functions g^{AB}, $\partial_0 g^{AB}$ prescribed there, so that g^{AB} is symmetric with signature $(-,+,\cdots,+)$ at each point.
3. The constraint equations (1.3.22) hold, and
4. the algebraic gauge equation (1.3.15) holds.

INCIDENTALLY: Let us derive some alternative explicit forms of the Einstein equations. For once, so far we have been using the notation y^A for the wave coordinates. Let us revert to a standard notation, x^μ, for the local coordinates. In this notation, (1.1.20) can be rewritten as

$$E^{\alpha\beta} = \Box_g g^{\alpha\beta} - 2g^{\gamma\delta}g^{\epsilon\phi}\Gamma^\alpha_{\gamma\epsilon}\Gamma^\beta_{\delta\phi} - \frac{4\Lambda}{n-1}g^{\alpha\beta} \tag{1.3.25}$$

(recall that we want this to be zero in vacuum). Set

$$\varphi := \sqrt{|\det g_{\pi\rho}|}\,, \qquad \mathfrak{g}^{\alpha\beta} := \varphi g^{\alpha\beta}\,. \tag{1.3.26}$$

In terms of $\mathfrak{g}$, the wave conditions take the particularly simple form

$$\partial_\alpha \mathfrak{g}^{\alpha\beta} = 0\,. \tag{1.3.27}$$

It is therefore convenient to rewrite Einstein equations as a system of wave equations for $\mathfrak{g}^{\alpha\beta}$. In order to do that, we calculate as follows:

$$\begin{aligned}
\partial_\mu\varphi &= \partial_\mu\left(\sqrt{|\det g_{\alpha\beta}|}\right) = \frac{1}{2}\sqrt{|\det g_{\pi\rho}|}g^{\alpha\beta}\partial_\mu g_{\alpha\beta} = -\frac{1}{2}\sqrt{|\det g_{\pi\rho}|}g_{\alpha\beta}\partial_\mu g^{\alpha\beta}\\
&= -\frac{1}{2}\varphi g_{\alpha\beta}\partial_\mu g^{\alpha\beta}\,,\\
\Box_g\varphi &= \nabla^\mu\partial_\mu\varphi = -\frac{1}{2}\nabla^\mu\left(\varphi g_{\alpha\beta}\partial_\mu g^{\alpha\beta}\right)\\
&= -\frac{1}{2}\Big(\nabla^\mu\varphi\underbrace{g_{\alpha\beta}\partial_\mu g^{\alpha\beta}}_{-2\partial_\mu\varphi/\varphi}+\varphi g^{\mu\nu}\partial_\nu g_{\alpha\beta}\partial_\mu g^{\alpha\beta}+\varphi g_{\alpha\beta}\underbrace{\Box_g g^{\alpha\beta}}_{=E^{\alpha\beta}+\ldots}\Big)\\
&= \varphi^{-1}\nabla^\mu\varphi\partial_\mu\varphi-\frac{\varphi}{2}\Big(g^{\mu\nu}\partial_\nu g_{\alpha\beta}\partial_\mu g^{\alpha\beta}+g_{\alpha\beta}\big(E^{\alpha\beta}+2g^{\gamma\delta}g^{\epsilon\phi}\Gamma^\alpha_{\gamma\epsilon}\Gamma^\beta_{\delta\phi}+\frac{4\Lambda}{n-1}g^{\alpha\beta}\big)\Big)\,,\\
\Box_g\mathfrak{g}^{\alpha\beta} &= \varphi\Box_g g^{\alpha\beta}+2\nabla^\mu\varphi\partial_\mu g^{\alpha\beta}+\Box_g\varphi g^{\alpha\beta}\,.
\end{aligned}$$

Thus, in harmonic coordinates,

$$\begin{aligned}
\Box_g\mathfrak{g}^{\alpha\beta} &= \varphi\big(E^{\alpha\beta}+2g^{\gamma\delta}g^{\epsilon\phi}\Gamma^\alpha_{\gamma\epsilon}\Gamma^\beta_{\delta\phi}+\frac{4\Lambda}{n-1}g^{\alpha\beta}\big)+2\nabla^\mu\varphi\partial_\mu g^{\alpha\beta}+\Big[\varphi^{-1}\nabla^\mu\varphi\partial_\mu\varphi\\
&\quad -\frac{\varphi}{2}\Big(g^{\mu\nu}\partial_\nu g_{\rho\sigma}\partial_\mu g^{\rho\sigma}+g_{\rho\sigma}\big(E^{\rho\sigma}+2g^{\gamma\delta}g^{\epsilon\phi}\Gamma^\rho_{\gamma\epsilon}\Gamma^\sigma_{\delta\phi}+\frac{4\Lambda}{n-1}g^{\rho\sigma}\big)\Big)\Big]g^{\alpha\beta}\,;
\end{aligned} \tag{1.3.28}$$

also note that the Λ terms can be grouped together to $-2\Lambda\mathfrak{g}^{\alpha\beta}$.

Next, it might be convenient instead to write directly equations for $g_{\mu\nu}$ rather than $g^{\mu\nu}$ (compare (1.1.24)), or $\mathfrak{g}^{\mu\nu}$. For this, we use again $g_{\alpha\beta}g^{\beta\gamma}=\delta^\gamma_\alpha$. Keeping in mind that that $\Box_g$ is understood as an operator on *scalars*, we obtain

$$\begin{aligned}
\partial_\sigma g_{\alpha\beta} &= -g_{\alpha\gamma}g_{\beta\delta}\partial_\sigma g^{\gamma\delta}\,,\\
g^{\rho\sigma}\partial_\rho\partial_\sigma g_{\alpha\beta} &= -g^{\rho\sigma}\Big(\partial_\rho g_{\alpha\gamma}g_{\beta\delta}\partial_\sigma g^{\gamma\delta}+g_{\alpha\gamma}\partial_\rho g_{\beta\delta}\partial_\sigma g^{\gamma\delta}\\
&\quad +g_{\alpha\gamma}g_{\beta\delta}\partial_\rho\partial_\sigma g^{\gamma\delta}\Big)\\
&= -g^{\rho\sigma}\Big(\partial_\rho g_{\alpha\gamma}g_{\beta\delta}\partial_\sigma g^{\gamma\delta}+g_{\alpha\gamma}\partial_\rho g_{\beta\delta}\partial_\sigma g^{\gamma\delta}\Big)\\
&\quad -g_{\alpha\gamma}g_{\beta\delta}\underbrace{g^{\rho\sigma}\partial_\rho\partial_\sigma g^{\gamma\delta}}_{\Box_g g^{\gamma\delta}+g^{\rho\sigma}\Gamma^\lambda_{\rho\sigma}\partial_\lambda g^{\gamma\delta}}\\
\Box_g g_{\alpha\beta} &= g^{\rho\sigma}(\partial_\rho\partial_\sigma g_{\alpha\beta}-\Gamma^\lambda_{\rho\sigma}\partial_\lambda g_{\alpha\beta})\,.
\end{aligned}$$

One can use now the formula (1.3.25) expressing $\Box_g g^{\gamma\delta}$ in terms of $E^{\alpha\beta}$ to obtain an expression for $R_{\alpha\beta}$. In particular one finds

$$R_{\alpha\beta} = -\frac{1}{2}(\Box_g g_{\alpha\beta}+g_{\alpha\mu}\nabla_\beta\lambda^\mu+g_{\beta\mu}\nabla_\alpha\lambda^\mu)+\ldots\,, \tag{1.3.29}$$

where "..." stands for terms which do not involve second derivatives of the metric. □

1.4 The geometry of non-characteristic submanifolds

Let $\mathscr{S}$ be a hypersurface in a Lorentzian or Riemannian manifold $(\mathscr{M}, g)$, we want to analyse the geometry of such hypersurfaces. Set

$$h := g|_{T\mathscr{S}} . \tag{1.4.1}$$

More precisely,

$$\forall\ X, Y \in T\mathscr{S} \qquad h(X,Y) := g(X,Y) .$$

The tensor field h is called *the first fundamental form of* $\mathscr{S}$; when non-degenerate, it is also called *the metric induced by* g *on* $\mathscr{S}$. If $\mathscr{S}$ is considered as an abstract manifold with embedding $i : \mathscr{S} \to \mathscr{M}$, then h is simply the pull-back i^*g.

A hypersurface $\mathscr{S}$ will be said to be *spacelike* at $p \in \mathscr{S}$ if h is Riemannian at p, *timelike* at p if h is Lorentzian at p, and finally *null* or *isotropic* or *lightlike* at p if h is degenerate at p. $\mathscr{S}$ will be called *spacelike* if it is spacelike at all $p \in \mathscr{S}$, *etc.* An example of null hypersurface is given by $\dot{J}(p) \setminus \{p\}$ for any $p \in \mathscr{M}$, at least near p where $\dot{J}(p) \setminus \{p\}$ is differentiable.

When g is Riemannian, then h is always a Riemannian metric on $\mathscr{S}$, and then $T\mathscr{S}$ is in direct sum with $(T\mathscr{S})^\perp$. Whatever the signature of g, in this section we will always assume that this is the case:

$$T\mathscr{S} \cap (T\mathscr{S})^\perp = \{0\} \quad \Longrightarrow \quad T\mathscr{M} = T\mathscr{S} \oplus (T\mathscr{S})^\perp . \tag{1.4.2}$$

Recall that (1.4.2) fails precisely at those points $p \in \mathscr{S}$ at which h is degenerate. Hence, in this section we consider hypersurfaces which are either timelike throughout, or spacelike throughout. Depending upon the character of $\mathscr{S}$ we will then have

$$\epsilon := g(N,N) = \pm 1 , \tag{1.4.3}$$

where N is the field of unit normals to $\mathscr{S}$.

For $p \in \mathscr{S}$ let $P : T_p\mathscr{M} \to T_p\mathscr{M}$ be defined as

$$T_p\mathscr{M} \ni X \to P(X) = X - \epsilon g(X,N)N . \tag{1.4.4}$$

We note the following properties of P:

- P annihilates N:

$$P(N) = N - \epsilon g(N,N)N = N - \epsilon^2 N = 0 .$$

- P is a projection operator:

$$\begin{aligned} P(P(X)) &= P(X - \epsilon g(X,N)N) \\ &= P(X) - \epsilon g(X,N)P(N) = P(X) . \end{aligned}$$

- P restricted to $N^\perp$ is the identity:

$$g(X,N) = 0 \quad \Longrightarrow \quad P(X) = X .$$

- P is symmetric:

$$g(P(X),Y) = g(X,Y) - \epsilon g(X,N)g(Y,N) = g(X,P(Y)) .$$

The *Weingarten map* $B : T\mathscr{S} \to T\mathscr{S}$ is defined by the equation

$$T\mathscr{S} \ni X \quad \to \quad B(X) := P(\nabla_X N) \in T\mathscr{S} \subset T\mathscr{M} . \tag{1.4.5}$$

Here, and in other formulae involving differentiation, one should in principle choose an extension of N off $\mathscr{S}$; however, (1.4.5) involves only derivatives in directions tangent to $\mathscr{S}$, so that the result will not depend upon that extension.

In fact, the projector P is not needed in (1.4.5):

$$P(\nabla_X N) = \nabla_X N .$$

This follows from the calculation

$$0 = X(\underbrace{g(N,N)}_{\pm 1}) = 2g(\nabla_X N, N) ,$$

which shows that $\nabla_X N$ is orthogonal to N, hence tangent to $\mathscr{S}$.

The map B is closely related to the *second fundamental form* K of $\mathscr{S}$, also called the *extrinsic curvature tensor* in the physics literature:

$$T\mathscr{S} \ni X,Y \quad \to \quad K(X,Y) := g(P(\nabla_X N),Y) \tag{1.4.6a}$$
$$= g(\nabla_X N, Y) \tag{1.4.6b}$$
$$= g(B(X),Y) \tag{1.4.6c}$$
$$= h(B(X),Y) . \tag{1.4.6d}$$

EXAMPLE 1.4.1 As an example, consider a pseudo-Riemannian metric $g_{ij}dx^i dx^j$ on $\mathbb{R}^n$ with *constant coefficients*, let $c \in \mathbb{R}^*$ and consider a hypersurface $\mathscr{S}_c$ defined as

$$\mathscr{S}_c = \{g_{ij}x^i x^j = c\} . \tag{1.4.7}$$

Thus, $\mathscr{S}_c$ is a quadric, the nature of which depends upon the signature of g and the sign of c. A vector field X is tangent to $\mathscr{S}_c$ if and only if

$$X(g_{ij}x^i x^j) = 0 ,$$

Equivalently, $g_{ij}x^i X^j = 0$ for all vectors tangent to $\mathscr{S}_c$. Thus the one-form $g_{ij}x^i dx^j$ annihilates $T\mathscr{S}_c$. We conclude that

$$N_i = \pm \frac{1}{\sqrt{|c|}} g_{ij}x^j \quad \Longleftrightarrow \quad N = \pm \frac{1}{\sqrt{|c|}} x^i \partial_i ,$$

where the choice of sign is a matter of convention. When g is Riemannian, then the $\mathscr{S}_c$'s are spheres, and the plus sign is the usual choice. When g has signature $(-,+,\cdots,+)$ and $\mathscr{S}_c$ is spacelike (which will be the case if and only if $c < 0$), one usually requires N to be future-pointing, in which case the negative sign should be chosen on that component of $\mathscr{S}_c$ on which $x^0 > 0$.

Since the metric coefficients are constant, the Christoffel symbols vanish, and so for any two vector fields X and Y we have $\nabla_X Y^i = X(Y^i)$. In particular

$$\nabla_X N = \pm \frac{1}{\sqrt{|c|}} X(x^i)\partial_i = \pm \frac{1}{\sqrt{|c|}} X^i \partial_i = \pm \frac{1}{\sqrt{|c|}} X .$$

From the expression (1.4.6b) of K we find

$$T\mathscr{S} \ni X,Y \quad K(X,Y) = g(\nabla_X N, Y) = \pm\frac{1}{\sqrt{|c|}} g(X,Y) = \pm\frac{1}{\sqrt{|c|}} h(X,Y)\,. \quad (1.4.8)$$

Thus

$$K = \pm\frac{1}{\sqrt{|c|}} h\,, \quad (1.4.9)$$

with the plus sign for spheres in Euclidean space, and the minus sign for spacelike hyperboloids in Minkowski spacetime lying in the future of the origin. □

It is often convenient to have at our disposal formulae for the objects at hand using the index formalism. For this purpose let us consider a local ON frame $\{e_\mu\}$ such that $e_0 = N$ along $\mathscr{S}$. We then have

$$g_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$$

in the case of a spacelike hypersurface in a Lorentzian manifold.

Using the properties of P listed above,

$$\begin{aligned} K_{ij} &:= K(e_i, e_j) = h(B(e_i), e_j) = h(B^k{}_i e_k, e_j) \\ &= h_{kj} B^k{}_i\,, && (1.4.10) \\ B^k{}_i &:= \varphi^k(B(e_i))\,, && (1.4.11) \end{aligned}$$

where $\{\varphi^k\}$ is a basis of $T^*\mathscr{S}$ dual to the basis $\{P(e_i)\}$ of $T\mathscr{S}$. Equivalently,

$$B^k{}_i = h^{kj} K_{ji}\,,$$

and it is usual to write the right-hand side as $K^k{}_i$.

We continue by showing that K is symmetric: First, for X and Y tangent to $\mathscr{S}$,

$$\begin{aligned} K(X,Y) &= g(\nabla_X N, Y) \\ &= X(\underbrace{g(N,Y)}_{=0}) - g(N, \nabla_X Y)\,. && (1.4.12) \end{aligned}$$

Now, ∇ has no torsion, which implies

$$\nabla_X Y = \nabla_Y X + [X,Y]\,.$$

Further, the commutator of vector fields tangent to $\mathscr{S}$ is a vector field tangent to $\mathscr{S}$, which implies

$$\forall\ X,Y \in T\mathscr{S} \qquad g(N,[X,Y]) = 0\,.$$

Returning to (1.4.12), it follows that

$$K(X,Y) = -g(N, \nabla_Y X + [X,Y]) = -g(N, \nabla_Y X)\,,$$

and the equation

$$K(X,Y) = K(Y,X)$$

immediately follows from (1.4.12).

In adapted coordinates so that the vectors ∂_i are tangent to $\mathscr{S}$, from what has been said we find

$$K_{ij} = \frac{1}{2}(g(\nabla_i N, \partial_j) + g(\nabla_j N, \partial_i)) = \frac{1}{2}\mathscr{L}_N g_{ij}\,, \tag{1.4.13}$$

where $\mathscr{L}_N$ is the Lie derivative in the direction of N, see Appendix A.8. Formula (1.4.13) is very convenient for calculating K explicitly if moreover $N = \partial/\partial x^0$, since then $\mathscr{L}_N g_{ij} = \partial_0 g_{ij}$.

EXAMPLE 1.4.2 First, if $\mathscr{S} = \{t = 0\}$ in Minkowski spacetime, then $N = \partial_t$, the metric functions are t-independent and thus $K = 0$.

Next, consider a sphere $S^{n-1} = \{r = R\}$ embedded in Euclidean $\mathbb{R}^n$. In spherical coordinates the Euclidean metric δ takes the form

$$\delta = dr^2 + r^2 d\Omega^2 =: dr^2 + h\,,$$

with $\partial_r(d\Omega^2) = 0$ and $N = \partial_r$. The Lie derivative is again the coordinate derivative, and (1.4.13) gives

$$K = \frac{1}{2}\partial_r(r^2 d\Omega^2)|_{r=R} = \frac{1}{R}h\,,$$

as in (1.4.9). □

To continue, when X, Y are sections of $T\mathscr{S}$ we set

$$D_X Y := P(\nabla_X Y)\,. \tag{1.4.14}$$

First, we claim that D is a connection: Linearity with respect to addition in all variables, and with respect to multiplication of X by a function, is straightforward. It remains to check the Leibniz rule:

$$\begin{aligned} D_X(\alpha Y) &= P(\nabla_X(\alpha Y)) \\ &= P(X(\alpha)Y + \alpha\nabla_X Y) \\ &= X(\alpha)P(Y) + \alpha P(\nabla_X Y) \\ &= X(\alpha)Y + \alpha D_X Y\,. \end{aligned}$$

It follows that all the axioms of a covariant derivative on vector fields are fulfilled, as desired.

It turns out that D is actually the Levi-Civita connection of the metric h. Recall that the Levi-Civita connection is determined uniquely by the requirement of vanishing torsion, and that of metric-compatibility. Both results are straightforward:

$$D_X Y - D_Y X = P(\nabla_X Y - \nabla_Y X) = P([X, Y]) = [X, Y]\,; \tag{1.4.15}$$

in the last step we have again used the fact that the commutator of two vector fields tangent to $\mathscr{S}$ is a vector field tangent to $\mathscr{S}$. Equation (1.4.15) is precisely the condition for the vanishing of the torsion of D.

In order to establish metric-compatibility, we calculate for all vector fields X, Y, Z tangent to $\mathscr{S}$:

$$\begin{aligned}
X(h(Y,Z)) &= X(g(Y,Z)) \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
&= g(\nabla_X Y, \underbrace{P(Z)}_{=Z}) + g(\underbrace{P(Y)}_{=Y}, \nabla_X Z) \\
&= \underbrace{g(P(\nabla_X Y), Z) + g(Y, P(\nabla_X Z))}_{P \text{ is symmetric}} \\
&= g(D_X Y, Z) + g(Y, D_X Z) \\
&= h(D_X Y, Z) + h(Y, D_X Z)\,,
\end{aligned}$$

which is the condition for metric-compatibility of D.

Equation (1.4.14) turns out to be very convenient when trying to express the curvature of h in terms of that of g. To distinguish between both curvatures let us use the symbol ρ for the curvature tensor of h; by definition, for all vector fields tangential to $\mathscr{S}$,

$$\begin{aligned}
\rho(X,Y)Z &= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \\
&= P\Big(\nabla_X(P(\nabla_Y Z)) - \nabla_Y(P(\nabla_X Z)) - \nabla_{[X,Y]} Z\Big)\,.
\end{aligned}$$

Now, for any vector field W (not necessarily tangent to $\mathscr{S}$) we have

$$\begin{aligned}
P\Big(\nabla_X(P(W))\Big) &= P\Big(\nabla_X(W - \epsilon g(N,W)N)\Big) \\
&= P\Big(\nabla_X W - \underbrace{\epsilon X(g(N,W))N}_{P(N)=0} - \epsilon g(N,W)\nabla_X N\Big) \\
&= P\Big(\nabla_X W\Big) - \epsilon g(N,W) P\Big(\nabla_X N\Big) \\
&= P\Big(\nabla_X W\Big) - \epsilon g(N,W) B(X)\,.
\end{aligned}$$

Applying this equation to $W = \nabla_Y Z$ we obtain

$$\begin{aligned}
P\Big(\nabla_X(P(\nabla_Y Z))\Big) &= P(\nabla_X \nabla_Y Z) - \epsilon g(N, \nabla_Y Z) B(X) \\
&= P(\nabla_X \nabla_Y Z) + \epsilon K(Y,Z) B(X)\,,
\end{aligned}$$

and in the last step we have used (1.4.12). It now immediately follows that

$$\rho(X,Y)Z = P(R(X,Y)Z) + \epsilon\Big(K(Y,Z)B(X) - K(X,Z)B(Y)\Big)\,, \qquad (1.4.16)$$

an equation sometimes known as *Gauss' equation.*

In an adapted ON frame as discussed above, (1.4.16) reads

$$\boxed{\rho^i{}_{jk\ell} = R^i{}_{jk\ell} + \epsilon(K^i{}_k K_{j\ell} - K^i{}_\ell K_{jk})}\,. \qquad (1.4.17)$$

Here $K^i{}_k$ is the tensor field K_{ij} with an index raised using the contravariant form $h^\#$ of the metric h, compare (1.4.10).

Equation (1.4.17) is known as the *Gauss equation.*

We are ready now to derive the *general relativistic scalar constraint equation*: Let ρ_{ij} denote the Ricci tensor of the metric h, we then have

$$\begin{aligned}\rho_{j\ell} &:= \rho^i{}_{ji\ell} \\ &= \underbrace{R^i{}_{ji\ell}}_{=R^\mu{}_{j\mu\ell}-R^0{}_{j0\ell}} +\epsilon(K^i{}_iK_{j\ell}-K^i{}_\ell K_{ji}) \\ &= R_{j\ell}-R^0{}_{j0\ell}+\epsilon(\mathrm{tr}_hKK_{j\ell}-K^i{}_\ell K_{ji})\,.\end{aligned}$$

Defining $R(h)$ to be the scalar curvature of h, it follows that

$$\begin{aligned}R(h) &= \rho^j{}_j \\ &= \underbrace{R^j j}_{=R^\mu{}_\mu-R^0{}_0} - \underbrace{R^{0j}{}_{0j}}_{=R^{0\mu}{}_{0\mu}} +\epsilon(\mathrm{tr}_hKK^j{}_j-K^{ij}K_{ji}) \\ &= R(g)-2\underbrace{R^0{}_0}_{=\epsilon R_{00}} +\epsilon\Big((\mathrm{tr}_hK)^2-|K|^2_h\Big) \\ &= -16\pi\epsilon T_{00}+2\Lambda+\epsilon\Big((\mathrm{tr}_hK)^2-|K|^2_h\Big)\,,\end{aligned}$$

and we have used the Einstein equation,

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=8\pi\frac{G}{c^4}T_{\mu\nu}\,, \tag{1.4.18}$$

with $G=c=1$. Assuming that $\epsilon=-1$ we obtain the desired scalar constraint:

$$\boxed{R(h)=16\pi T_{\mu\nu}N^\mu N^\nu+2\Lambda+|K|^2_h-(\mathrm{tr}_hK)^2}\,. \tag{1.4.19}$$

(We emphasise that this equation is valid whatever the dimension of $\mathscr{S}$.) In particular in vacuum, with $\Lambda=0$, one obtains

$$R(h)=|K|^2_h-(\mathrm{tr}_hK)^2\,. \tag{1.4.20}$$

The *vector constraint equation* carries the remaining information contained in the equation $(G_{\mu\nu}+\Lambda g_{\mu\nu})N^\mu=8\pi T_{\mu\nu}N^\mu$. In order to understand that equation, let Y be tangent to $\mathscr{S}$, we then have

$$\begin{aligned}(G_{\mu\nu}+\Lambda g_{\mu\nu})N^\mu Y^\nu &= \Big(R_{\mu\nu}+(\Lambda-\frac{1}{2}R(g))g_{\mu\nu}\Big)N^\mu Y^\nu \\ &= \mathrm{Ric}(N,Y)+(\Lambda-\frac{1}{2}R(g))\underbrace{g(N,Y)}_{=0} \\ &= \mathrm{Ric}(N,Y)\,.\end{aligned} \tag{1.4.21}$$

We will relate this to some derivatives of K. By definition we have

$$(D_ZK)(X,Y)=Z(K(X,Y))-K(D_ZX,Y)-K(X,D_ZY)\,.$$

Now,

$$Z(K(X,Y)) = Z(g(\nabla_XN,Y))=g(\nabla_Z\nabla_XN,Y)+g(\nabla_XN,\nabla_ZY)\,.$$

Since $\nabla_X N$ is tangential, and P is symmetric, the last term can be rewritten as

$$\begin{aligned} g(\nabla_X N, \nabla_Z Y) &= g(P(\nabla_X N), \nabla_Z Y) = g(\nabla_X N, P(\nabla_Z Y)) \\ &= K(X, P(\nabla_Z Y)) = K(X, D_Z Y)\,. \end{aligned}$$

It follows that

$$\begin{aligned} &(D_Z K)(X,Y) - (D_X K)(Z,Y) \\ &\quad = g(\nabla_Z \nabla_X N, Y) - g(\nabla_X \nabla_Z N, Y) + K(X, D_Z Y) - K(Z, D_X Y) \\ &\qquad - K(D_Z X, Y) - K(X, D_Z Y) + K(D_X Z, Y) + K(Z, D_X Y) \\ &\quad = g(R(Z,X)N, Y) + \underbrace{g(\nabla_{[Z,X]} N, Y)}_{K([Z,X],Y)} - K(\underbrace{D_Z X - D_X Z}_{[Z,X]}, Y) \\ &\quad = g(R(Z,X)N, Y)\,. \end{aligned}$$

Thus,

$$(D_Z K)(X,Y) - (D_X K)(Z,Y) = g(R(Z,X)N, Y)\,. \tag{1.4.22}$$

This equation is known as the *Codazzi-Mainardi equation*, though it was known to Karl Mikhailovich Peterson before those authors.

In a frame in which the e_i's are tangent to the hypersurface $\mathscr{S}$, (1.4.22) can be rewritten as

$$D_k K_{ij} - D_i K_{kj} = R_{j\mu ki} N^\mu\,. \tag{1.4.23}$$

A contraction over i and j gives then

$$h^{ij}(D_k K_{ij} - D_i K_{kj}) = h^{ij} R_{j0ki} + \epsilon \underbrace{R_{00k0}}_{0} = g^{\mu\nu} R_{\mu 0 k \nu} = -R_{k0}\,.$$

Using the Einstein equation (1.4.18) together with (1.4.21) we obtain the *vector constraint equation*:

$$\boxed{D_j K^j{}_k - D_k K^j{}_j = 8\pi T_{\mu\nu} N^\mu h^\nu{}_k\,.} \tag{1.4.24}$$

INCIDENTALLY: It seems that the first author to recognize the special character of the equations $G_{\mu\nu}N^\nu = 0$ was Darmois [185], see [93] for the history of the problem.

1.5 Cauchy data

Let us return to the discussion of the end of Section 1.1. It is appropriate to adapt a more general point of view than that presented there, where we assumed that the initial data were given on an open subset $\mathscr{O}$ of the zero-level set of the function y^0. A correct geometric picture is to start with an n-dimensional hypersurface $\mathscr{S}$, and prescribe initial data there; the case where $\mathscr{S}$ is $\mathscr{O}$ is thus a special case of this construction. At this stage there are two attitudes one may wish to adopt: the first is that $\mathscr{S}$ is a subset of the spacetime $\mathscr{M}$ — this is essentially what we assumed in Section 1.4. Another way of looking at this is to consider $\mathscr{S}$ as a hypersurface of its own, equipped with an embedding

$$i : \mathscr{S} \to \mathscr{M} \,.$$

The most convenient approach is to go back and forth between those points of view, and we will do this without further notice whenever useful for the discussion at hand.

A *vacuum initial data set* $(\mathscr{S}, h, K)$ is a triple where $\mathscr{S}$ is an n–dimensional manifold, h is a Riemannian metric on $\mathscr{S}$, and K is a symmetric two-covariant tensor field on $\mathscr{S}$. Further (h, K) are supposed to satisfy the vacuum constraint equations (1.4.20) and (1.4.24), perhaps (but not necessarily so) with a non-zero cosmological constant Λ.

It follows directly from its definition that the metric h is uniquely defined by the spacetime metric g once that $\mathscr{S} \subset \mathscr{M}$ (or $i(\mathscr{S}) \subset \mathscr{M}$) has been prescribed; the same applies to the extrinsic curvature tensor K. So a hypersurface in a vacuum spacetime defines a unique vacuum initial data set.

Let us show that specifying K is equivalent to prescribing the "time-derivatives" of the space-part g_{ij} of the resulting spacetime metric g. Suppose, indeed, that a spacetime (M, g) has been constructed (not necessarily vacuum) such that K is the extrinsic curvature tensor of $\mathscr{S}$ in $(\mathscr{M}, g)$. Consider any domain $\mathscr{O} \subset \mathscr{S}$ of coordinates y^i, let $y^0 = 0$ be a function vanishing on $\mathscr{S}$ with non-vanishing gradient there, and construct coordinates $(y^\mu) = (y^0, y^i)$ in some $\mathscr{M}$–neighborhood of $\mathscr{U}$ such that $\mathscr{S} \cap \mathscr{U} = \mathscr{O}$; those coordinates could be wave-coordinates, as described at the end of Section 1.1, but this is not necessary at this stage. Since y^0 is constant on $\mathscr{S}$ the one-form dy^0 annihilates $T\mathscr{S}$, so does the one-form $g(N, \cdot)$, and since $\mathscr{S}$ has codimension one it follows that dy^0 must be proportional to $g(N, \cdot)$:

$$N_\alpha dy^\alpha = N_0 dy^0$$

on $\mathscr{O}$. The normalization $-1 = g(N, N) = g^{\mu\nu} N_\mu N_\nu = g^{00}(N_0)^2$ gives

$$N_\alpha dy^\alpha = \frac{1}{\sqrt{|g^{00}|}} dy^0 \,.$$

Next,

$$\begin{aligned}
K_{ij} &:= g(\nabla_i N, \partial_j) = \nabla_i N_j \\
&= \partial_i N_j - \Gamma^\mu{}_{ji} N_\mu \\
&= -\Gamma^0_{ji} N_0 \\
&= -\frac{1}{2} g^{0\sigma} \Big(\partial_j g_{\sigma i} + \partial_i g_{\sigma j} - \partial_\sigma g_{ij} \Big) N_0 \,.
\end{aligned} \qquad (1.5.1)$$

This shows that to calculate K_{ij} we need to know $g_{\mu\nu}$ *and* $\partial_0 g_{ij}$ at $y^0 = 0$. Reciprocally, (1.5.1) can be rewritten as

$$\begin{aligned}
\partial_0 g_{ij} &= \frac{2}{g^{00} N_0} K_{ij} \\
&\quad + \text{terms determined by the } g_{\mu\nu}\text{'s and their } \textit{space}\text{–derivatives}\,,
\end{aligned}$$

so that the knowledge of the $g_{\mu\nu}$'s and of the K_{ij}'s at $y^0 = 0$ allows one to calculate $\partial_0 g_{ij}$.

We conclude that K_{ij} is the geometric counterpart of the $\partial_0 g_{ij}$'s.

INCIDENTALLY: It is sometimes said that the metric components $g_{0\alpha}$ have a *gauge character*. By this it is usually meant that the fields under consideration do not have any intrinsic meaning, and their values can be changed using the action of some family of transformations, relevant to the problem at hand, without changing the geometric, or physical, information carried by those objects. In our case the relevant transformations are the coordinate ones, and things are made precise by the following proposition:

PROPOSITION 1.5.2 *Let g_{AB}, $\tilde g_{AB}$ be two metrics such that*

$$g_{ij}|_{\{y^0=0\}} = \tilde g_{ij}|_{\{y^0=0\}}\,, \qquad K_{ij}|_{\{y^0=0\}} = \tilde K_{ij}|_{\{y^0=0\}}\,. \tag{1.5.2}$$

Then there exists a coordinate transformation ϕ defined in a neighborhood of $\{y^0 = 0\}$ which preserves (1.5.2) such that

$$g_{0\mu}|_{\{y^0=0\}} = (\phi^*\tilde g)_{0\mu}|_{\{y^0=0\}}\,. \tag{1.5.3}$$

Furthermore, for any metric g there exist local coordinate systems $\{\bar y^\mu\}$ such that $\{y^0 = 0\} = \{\bar y^0 = 0\}$ and, if we write $g = \bar g_{\mu\nu} d\bar y^\mu d\bar y^\nu$ etc. *in the barred coordinate system, then*

$$\begin{gathered} g_{ij}|_{\{y^0=0\}} = \bar g_{ij}|_{\{\bar y^0=0\}}\,, \quad K_{ij}|_{\{y^0=0\}} = \bar K_{ij}|_{\{\bar y^0=0\}}\,, \\ \bar g_{00}|_{\{y^0=0\}} = -1\,, \quad \bar g_{0i}|_{\{y^0=0\}} = 0\,. \end{gathered} \tag{1.5.4}$$

REMARK 1.5.3 We can actually always achieve $\bar g_{00} = -1$, $\bar g_{0i} = 0$ in a whole neighborhood of $\mathscr{S}$: this is done by shooting geodesics normally to $\mathscr{S}$, choosing y^0 to be the affine parameter along those geodesics, and by transporting the coordinates y^i from $\mathscr{S}$ by requiring them to be constant along the normal geodesics. The coordinate system will break down wherever the normal geodesics start intersecting, but the implicit function theorem guarantees that there will exist a neighborhood of $\mathscr{S}$ on which this does not happen. The resulting coordinates are called *Gauss coordinates.* While Gauss coordinates are geometrically natural, in these coordinates the Einstein equations do not appear to have good properties from the PDE point of view.

PROOF: It suffices to prove the second claim: for if $\bar\phi$ is the transformation that brings g to the form (1.5.4), and $\tilde\phi$ is the corresponding transformation for $\tilde g$, then $\phi := \tilde\phi \circ \bar\phi^{-1}$ will satisfy (1.5.3).

Let us start by calculating the change of the metric coefficients under a transformation of the form

$$y^0 = \varphi \bar y^0\,, \qquad y^i = \bar y^i + \psi^i \bar y^0\,. \tag{1.5.5}$$

If $\varphi > 0$ then clearly

$$\{y^0 = 0\} = \{\bar y^0 = 0\}\,.$$

Further, one has

$$\begin{aligned} g\Big|_{\{y^0=0\}} &= \Big(g_{00}(dy^0)^2 + 2g_{0i}dy^0dy^i + g_{ij}dy^idy^j\Big)\Big|_{\{y^0=0\}} \\ &= \Big(g_{00}(\bar y^0 d\varphi + \varphi d\bar y^0)^2 + 2g_{0i}(\bar y^0 d\varphi + \varphi d\bar y^0)(d\bar y^i + \bar y^0 d\psi^i + \psi^i d\bar y^0) \end{aligned}$$

$$
\begin{aligned}
& +g_{ij}(d\bar y^i+\bar y^0 d\psi^i+\psi^i d\bar y^0)(d\bar y^j+\bar y^0 d\psi^j+\psi^j d\bar y^0)\Big)\Big|_{\{y^0=0\}} \\
= \;& \Big(g_{00}(\varphi d\bar y^0)^2+2g_{0i}\varphi d\bar y^0(d\bar y^i+\psi^i d\bar y^0) \\
& +g_{ij}(d\bar y^i+\psi^i d\bar y^0)(d\bar y^j+\psi^j d\bar y^0)\Big)\Big|_{\{y^0=0\}} \\
= \;& \Big((g_{00}\varphi^2+2g_{0i}\psi^i+g_{ij}\psi^i\psi^j)(d\bar y^0)^2 \\
& +2(g_{0i}\varphi+g_{ij}\psi^j)d\bar y^0 d\bar y^i+g_{ij}d\bar y^i d\bar y^j\Big)\Big|_{\{y^0=0\}} \\
=: \;& \bar g_{\mu\nu}d\bar y^\mu d\bar y^\nu\,.
\end{aligned}
$$

We shall apply the above transformation twice: first we choose $\varphi = 1$ and

$$\psi^i = h^{ij} g_{0j}\,,$$

where h^{ij} is the matrix inverse to g_{ij}; this leads to a metric with $\bar g_{0i} = 0$. We then apply, to the new metric, a second transformation of the form (1.5.5) with the new $\psi^i = 0$, and with a φ chosen so that the final g_{00} equals minus one. □

1.6 Solutions global in space

In order to globalize the existence Theorem 1.3.1 *in space*, we will show that

1. coordinate transformations of initial data extend to coordinate transformations of the spacetime metric, and that
2. two solutions differing only by the values $g_{0\alpha}|_{\{y^0=0\}}$ are (locally) isometric.

The question, what is the right *spacetime*-existence-and-uniqueness statement, deserves a chapter of its own, and will be addressed in Chapter 2.

For this, suppose that g and $\tilde g$ both solve the vacuum Einstein equations on overlapping subsets of a globally hyperbolic region $\mathscr U$, with the *same* Cauchy data (h, K) on $\mathscr U \cap \mathscr S$. Here $\mathscr S$ is a spacelike hypersurface and we assume that $\mathscr U \cap \mathscr S$ is a Cauchy surface for $\mathscr U$. We also assume that each metric is expressed in a single coordinate system in its domain of definition, say coordinates $\{x^\mu\}$ for g and $\{\tilde x^\mu\}$ for $\tilde g$, such that the initial data surface $\mathscr S$ is given by the equation $\{x^0 = 0\}$ for g, and $\{\tilde x^0 = 0\}$ for $\tilde g$. The coordinates x^μ and $\tilde x^\mu$ are not assumed to be harmonic in what follows.

Let the space-coordinates x^i, associated with the metric g on $\mathscr S$, be defined on a subset $\mathscr O$ of $\mathscr S$, and let the space-coordinates $\tilde x^i$, likewise associated with $\tilde g$, be defined on a subset $\widetilde{\mathscr O}$ of $\mathscr S$. Let $\widetilde{\mathscr V}$ be the intersection

$$\widetilde{\mathscr V} = \mathscr O \cap \widetilde{\mathscr O}$$

(compare Figure 1.6.1).

By definition, "same Cauchy data" means that there exists a *space-coordinate* transformation

$$(x^0 = 0, x^i) \mapsto (\tilde x^0 = 0, \tilde x^i)\,, \tag{1.6.1}$$

such that on the overlap region $\widetilde{\mathscr V}$ it holds

$$h_{ij} = \frac{\partial \tilde x^k}{\partial x^i}\frac{\partial \tilde x^k}{\partial x^j}\tilde h_{k\ell}\,, \quad K_{ij} = \frac{\partial \tilde x^k}{\partial x^i}\frac{\partial \tilde x^k}{\partial x^j}\tilde K_{k\ell}\,, \tag{1.6.2}$$

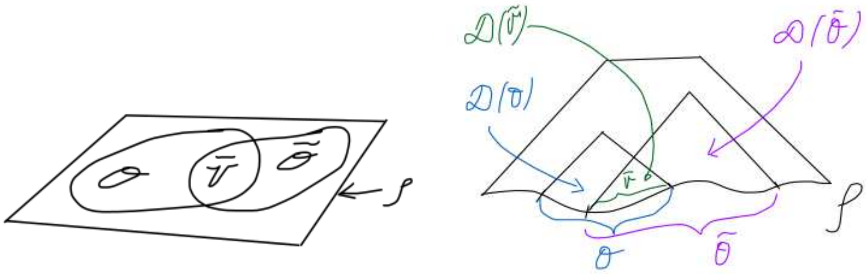

Figure 1.6.1: Two overlapping coordinate patches on $\mathscr{S}$. There exists a coordinate transformation transforming the $g_{\alpha\beta}$'s into the $\tilde g_{\alpha\beta}$'s on the domain of dependence $\mathscr{D}(\widetilde{\mathscr{V}})$ of $\widetilde{\mathscr{V}}$. Only the future of $\mathscr{S}$ is shown, the situation to the past of $\mathscr{S}$ is completely analogous.

where $h_{ij} = g_{ij}(x^0 = 0, \cdot)$, $\tilde h_{ij} = \tilde g_{ij}(\tilde x^0 = 0, \cdot)$, similarly for the extrinsic curvature tensor.

One can then introduce wave coordinates in a, perhaps small, neighborhood of $\widetilde{\mathscr{V}}$, globally hyperbolic both for g and $\tilde g$, by solving

$$\Box_g y^A = 0\,, \qquad \Box_{\tilde g} \tilde y^A = 0\,, \tag{1.6.3}$$

using the *same* initial data for y^A and $\tilde y^A$. For definiteness, the owner of the metric g can solve the first equation in (1.6.3) with initial data

$$y^0|_{x^0=0} = 0\,, \quad y^i|_{x^0=0} = x^i\,. \tag{1.6.4}$$

The initial derivatives $\partial y^A/\partial x^0|_{x^0=0}$ can then be algebraically determined by furthermore requiring (compare (1.3.16))

$$g^{\alpha\beta}\frac{\partial y^0}{\partial x^\alpha}\frac{\partial y^0}{\partial x^\beta}\Big|_{x^0=0} = -1\,, \quad g^{\alpha\beta}\frac{\partial y^0}{\partial x^\alpha}\frac{\partial y^i}{\partial x^\beta}\Big|_{x^0=0} = 0\,. \tag{1.6.5}$$

Equivalently,

$$g^{00}\frac{\partial y^0}{\partial x^0}\frac{\partial y^0}{\partial x^0}\Big|_{x^0=0} = -1\,, \quad g^{00}\frac{\partial y^i}{\partial x^0}\Big|_{x^0=0} = -g^{0i}\Big|_{x^0=0}\,. \tag{1.6.6}$$

These algebraic equations have smooth solutions when the initial data hypersurface $\{x^0 = 0\}$ is assumed to be spacelike everywhere, as then g^{00} has no zeros there.

After inverting the map $x^\alpha \mapsto y^A$ in a spacetime neighborhood of the coordinate patch where the coordinates x^i are defined, she can calculate the metric coefficients in harmonic coordinates y^A:

$$g^{AB}(y^C) = \left(g^{\alpha\beta}\frac{\partial y^A}{\partial x^\alpha}\frac{\partial y^A}{\partial x^\alpha}\right)\left(x^\mu(y^C)\right)\,. \tag{1.6.7}$$

By passing to a further subset of $\mathscr{U}$ if necessary, while leaving $\mathscr{U}\cap\mathscr{S}$ unaffected, she can ensure that $\mathscr{U}$ is globally hyperbolic with Cauchy surface $\mathscr{U}\cap\mathscr{S}$.

The owner of the metric $\tilde{g}$ will solve the second equation in (1.6.3) with initial data on $\widetilde{\mathscr{V}}$ given as

$$\tilde{y}^0|_{\tilde{x}^0=0} = 0\,, \quad \tilde{y}^i|_{\tilde{x}^0=0} = x^i\,, \tag{1.6.8}$$

where the coordinates x^i have to be expressed in terms of the coordinates $\tilde{x}^j$ by inverting the map (1.6.1). The initial derivatives $\partial\tilde{y}^A/\partial\tilde{x}^0|_{\tilde{x}^0=0}$ are then algebraically determined from the tilde-version of (1.6.5),

$$-1 = \tilde{g}^{\alpha\beta}\frac{\partial\tilde{y}^0}{\partial\tilde{x}^\alpha}\frac{\partial\tilde{y}^0}{\partial\tilde{x}^\beta}\bigg|_{\tilde{x}^0=0}\,, \qquad 0 = \tilde{g}^{\alpha\beta}\frac{\partial\tilde{y}^0}{\partial\tilde{x}^\alpha}\frac{\partial\tilde{y}^i}{\partial\tilde{x}^\beta}\bigg|_{\tilde{x}^0=0}\,; \tag{1.6.9}$$

equivalently,

$$\tilde{g}^{00}\frac{\partial\tilde{y}^0}{\partial\tilde{x}^0}\frac{\partial\tilde{y}^0}{\partial\tilde{x}^0}\bigg|_{\tilde{x}^0=0} = -1\,, \quad \tilde{g}^{00}\frac{\partial\tilde{y}^i}{\partial\tilde{x}^0}\bigg|_{x^0=0} = -\tilde{g}^{0j}\frac{\partial x^i}{\partial\tilde{x}^j}\bigg|_{\tilde{x}^0=0}\,. \tag{1.6.10}$$

Proceeding as before, he can calculate

$$\tilde{g}^{AB}(\tilde{y}^C) = \left(\tilde{g}^{\alpha\beta}\frac{\partial\tilde{y}^A}{\partial\tilde{x}^\alpha}\frac{\partial\tilde{y}^A}{\partial\tilde{x}^\alpha}\right)(\tilde{x}^\mu(\tilde{y}^C))\,. \tag{1.6.11}$$

In this way one obtains two solutions of the harmonically-reduced Einstein equations (1.1.20) with *the same initial data.*

Suppose first, for simplicity, that all the fields involved are smooth. The uniqueness part of Theorem 1.3.1 shows that

$$g^{AB}(y^C) \equiv \tilde{g}^{AB}(y^C)$$

on any globally hyperbolic subset of $\mathscr{U}$ on which the spacetime coordinates $x^\alpha(y^A)$ and $\tilde{x}^\alpha(\tilde{y}^A)$ are simultaneously defined, with $y^A = \tilde{y}^A$. Composing the maps

$$x^\alpha \mapsto y^\alpha = \tilde{y}^\alpha \mapsto \tilde{x}^\alpha$$

(where the middle map is the identity map $y^\alpha \mapsto \tilde{y}^\alpha = y^\alpha$), it follows from (1.6.7) and (1.6.11) that

$$\tilde{g}^{\mu\nu}(x^\gamma) = \left(\tilde{g}^{\alpha\beta}\frac{\partial\tilde{y}^A}{\partial\tilde{x}^\alpha}\frac{\partial\tilde{y}^A}{\partial\tilde{x}^\alpha}\right)(\tilde{x}^\mu(x^\gamma))\,, \tag{1.6.12}$$

on the globally hyperbolic subset of $\mathscr{U}$ as above. That is to say, the metric g can be obtained from $\tilde{g}$ by a coordinate transformation there.

Patching together solutions defined in local coordinates using the procedure above allows us to construct a solution of our equations on *some* globally hyperbolic neighborhood of any initial data hypersurface $\mathscr{S}$. We emphasise that no completeness, compactness, or asymptotic hypotheses are needed for this.

Summarising, we have proved:

THEOREM 1.6.1 *Any smooth vacuum initial data set $(\mathscr{S}, h, K)$ admits a globally hyperbolic vacuum development.*

REMARK 1.6.2 The solutions are locally unique, in a sense made clear by the proof: given two solutions constructed as above, there exists a globally hyperbolic neighborhood of the initial data surface on which the solutions coincide. Since there is a lot of arbitrariness in the construction above, related with the choices of various coordinate patches, coordinate systems, their overlaps, it is far from clear whether any kind of global uniqueness holds. As already mentioned, we will return to this in Section 2.1.

When the initial data are not smooth, the question arises whether the metrics constructed in local harmonic coordinates as above will be sufficiently differentiable to apply the uniqueness part of Theorem 1.3.1. Now, the metrics obtained so far are in a space $C^1([0,T],H^s)$, where the Sobolev space H^s is defined using the space-derivatives of the metric. The initial data for the harmonic coordinates y^μ or $\tilde y^\mu$ of (1.6.3) may be chosen to be in $H^{s+1}\times H^s$. However, a rough inspection of (1.6.3) shows that the resulting solutions y^μ and $\tilde y^\mu$ will be only in $C^1([0,T],H^s)$, because of the low regularity of the metric. But then (1.1.8) implies that the components of the metrics in the y^μ or $\tilde y^\mu$ coordinates will be in $C^1([0,T],H^{s-1})$. Uniqueness can only be guaranteed *if* $s-1>n/2+1$, which is one degree of differentiability more than needed for existence.

This was the state of affairs for some fifty-five years until the following simple argument of Planchon and Rodnianski [350]: To make it clear that the functions y^μ are considered to be scalars in (1.6.3), let us write y for each of the y^μ's. Commuting derivatives with $\Box_g$ one finds, for metrics satisfying the vacuum Einstein equations,

$$\Box_g\nabla_\alpha y=\nabla_\mu\nabla^\mu\nabla_\alpha y=[\nabla_\mu\nabla^\mu,\nabla_\alpha]y=\underbrace{R^{\sigma\mu}{}_{\alpha\mu}}_{=R^\sigma{}_\alpha=0}\nabla_\sigma y=0\,.$$

Commuting once more one obtains an evolution equation for the field $\psi_{\alpha\beta}:=\nabla_\alpha\nabla_\beta y$:

$$\Box_g\psi_{\alpha\beta}+\underbrace{\nabla_\sigma R_\beta{}^\lambda{}_\alpha{}^\sigma}_{=0}\nabla_\lambda y+2R_\beta{}^\lambda{}_\alpha{}^\sigma\psi_{\sigma\lambda}=0\,,$$

where the underbraced term vanishes, for vacuum metrics, by a contracted Bianchi identity. So the most offending term in this equation for $\psi_{\alpha\beta}$, involving three derivatives of the metric, disappears when the metric is vacuum. Standard theory of hyperbolic PDEs shows now that the functions $\nabla_\alpha\nabla_\beta y$ are in $C^1([0,T],H^{s-1})$, hence $y\in C^1([0,T],H^{s+1})$, and the transformed metrics are regular enough to invoke uniqueness without having to increase s.

Suppose, now, that an initial data set $(\mathscr S,h,K)$ as in Theorem 1.3.1 is given. Covering $\mathscr S$ by coordinate neighborhoods $\mathscr O_p$, $p\in\mathscr S$, one can use Theorem 1.3.1 to construct globally hyperbolic developments $(\mathscr U_p,g_p)$ of $(\mathscr O_p,h,K)$. By the arguments just given the metrics so obtained will coincide, after performing a suitable coordinate transformation, on globally hyperbolic subsets of their joint domains of definition. Thus, similarly to the smooth case, we can patch the $(\mathscr U_p,g_p)$'s together to a globally hyperbolic Lorentzian manifold with Cauchy surface $\mathscr S$, obtaining:

THEOREM 1.6.3 *Any vacuum initial data set* $(\mathscr{S}, h, K)$ *of differentiability class* $H^{s+1} \times H^s$, $s > n/2$, *admits a globally hyperbolic development.*

Remark 1.6.2 applies as is to the $H^{s+1} \times H^s$ setting.

Chapter 2

The global evolution problem

2.1 Maximal globally hyperbolic developments

In Theorem 1.6.3 we have established *local* existence of solutions of the Cauchy problem for vacuum Einstein equations. There arises the question of *global existence and uniqueness* of solutions.

Now, it is easy to see that there can be no uniqueness for the Cauchy problem, unless some restrictions on the solution are imposed. Consider for example the three manifolds $(-\infty, 1) \times \mathbb{R}^n$, $\mathbb{R} \times \mathbb{R}^n$ and $(\mathbb{R} \times \mathbb{R}^n) \setminus \{(1, \vec{0})\}$ equipped with the obvious flat metric. All three spacetimes are solutions of the vacuum Einstein equations with initial data $(\{0\} \times \mathbb{R}^n, \delta, 0)$, where δ is the Euclidean metric on $\mathbb{R}^n$. The first two are globally hyperbolic developments of the given initial data, but the third is not. And obviously no pair is isometric: e.g,. the second is geodesically complete, while the other two are not. So, to obtain uniqueness, some further conditions are needed.

A possible set of such conditions has been spelled-out in the celebrated Choquet-Bruhat – Geroch theorem [97] (compare [384]), which asserts that to every smooth vacuum general relativistic initial data set (M, h, K) one can associate a smooth solution of the vacuum Einstein equations which is unique, up to isometries, in the class of maximal globally hyperbolic spacetimes. The corresponding result with Sobolev initial data $(h, K) \in H^s \oplus H^{s-1}$, $\mathbb{N} \ni s > n/2 + 1$ has been proved in [126] and reads:

THEOREM 2.1.1 *Consider a vacuum Cauchy data set (M, h, K), where M is an n-dimensional manifold, $h \in H^s_{\mathrm{loc}}(M)$ is a Riemannian metric on M, and $K \in H^{s-1}_{\mathrm{loc}}(M)$ is a symmetric two–tensor on M, satisfying the general relativistic vacuum constraint equations, where $\mathbb{N} \ni s > n/2 + 1$, $n \geq 3$. Then there exists a Lorentzian manifold $(\mathscr{M}, g)$ with a $H^s_{\mathrm{space,loc}}$-metric,* unique up to isometries *within the $H^s_{\mathrm{space,loc}}$ class,* inextendible *in the class of globally hyperbolic spacetimes with a $H^s_{\mathrm{space,loc}}$ vacuum metric and with an embedding $i : M \to \mathscr{M}$ such that $i^*g = h$, and such that K corresponds to the extrinsic curvature tensor of $i(M)$ in $\mathscr{M}$.*

To avoid ambiguities, global hyperbolicity in Theorem 2.1.1 is meant as the requirement that every inextendible causal curve in $\mathscr{M}$ meets $i(M)$ precisely once; compare Appendix A.22, p. 233. A spacetime is said to be maximal globally hyperbolic if it cannot be extended within the category of globally hyperbolic spacetimes.

The manifold $(\mathscr{M}, g)$ of the theorem is called the *maximal globally hyperbolic vacuum development of* (M, h, K); it is yet another classical result of Yvonne Choquet-Bruhat that $(\mathscr{M}, g)$ is independent of s for $s > n/2 + 1$.

While Theorem 2.1.1 is highly satisfactory, it does not quite prove what one wants, because *uniqueness is obtained in the globally hyperbolic class only.* A striking example illustrating that there is a problem here is provided by the following observation (see [124, 140, 324]), which summarises Theorem B.1.6, p. 243 and Remark B.1.7 from Appendix B.1 :

THEOREM 2.1.2 *Consider the Taub spacetime* $(\mathscr{M}_{(t_-,t_+)}, \mathbf{g})$ *of Appendix B.1. Then*

1. $(\mathscr{M}_{(t_-,t_+)}, \mathbf{g})$ *is maximal globally hyperbolic.*
2. *There exists an uncountable number of analytic, vacuum, simply connected, non-isometric maximal extensions of* $(\mathscr{M}_{(t_-,t_+)}, \mathbf{g})$.

From the point of view of the Cauchy problem, the situation is thus the following: Our world "today" is described by an initial data set for the Einstein equations and possibly some further fields. The field equations can be used to evolve the data, and hence predict the future. So while Theorem 2.1.1 shows that the initial data predict only the maximal globally hyperbolic part of the spacetime, Theorem 2.1.2 makes it clear that there is no uniqueness beyond. Hence, Einstein equations fail to predict the future, at least if we start with initial data for the Taub spacetime.

The question arises, whether or not some further conditions can be imposed on the initial data to avoid this problem. For this it is instructive to discuss a few examples.

2.2 Some examples

EXAMPLE 2.2.1 Let (M, h, K) be Cauchy data for a cylindrically symmetric polarized metric:

$$ds^2 = e^{2(U-A)}(-dt^2 + dr^2) + e^{-2U} r^2 d\phi^2 + e^{2U} dz^2 \qquad (2.2.1)$$

$$U = U(t, r)\,, \qquad A = A(t, r)\,,$$
$$M = \{t = 0\} \approx \mathbb{R}^3\,.$$

For metrics of the form (2.2.1) vacuum Einstein equations reduce to a single linear wave equation (in the flat Minkowski metric) for U,

$$\Box U = \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right] U = 0\,, \qquad (2.2.2)$$

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,.$$

More precisely, given any solution U of the wave equation (2.2.2), the function A appearing in the metric can be found by elementary integration of the remaining equations

$$\partial_t A = 2r\partial_t U \partial_r U\,, \quad \partial_r A = r((\partial_t U)^2 + (\partial_r U)^2)\,. \tag{2.2.3}$$

Given Cauchy data (U_0, U_1) for U and a distribution $\rho = \rho(t,r) \in \mathcal{D}'(\mathbb{R}^3)$ on $\mathbb{R}^3$ such that supp $\rho \cap M = \phi$, where $M = \{(t,x,y) \in \mathbb{R}^3 : t = 0\}$, define U_ρ as the unique solution of the problem

$$\Box\, U_\rho = \rho\,, \tag{2.2.4}$$

$$U_\rho(0,r) = U_0(r)\,, \quad \frac{\partial U_\rho}{\partial t}(0,r) = U_1(r)\,.$$

The distribution U_ρ will exist if e.g. $\rho \in H_m(\mathbb{R}^3)$, $(U_0, U_1) \in H_k(M) \oplus H_{k-1}(M)$, for any $m, k \in \mathbb{R}$. In particular if $f \in C^0(\mathbb{R})$, then the distribution ρ given by

$$\rho = f(t)\,\delta_0 \in H_m(\mathbb{R}^3)\,, \qquad m < -3/2 \tag{2.2.5}$$

is an allowed distribution.

Let $\mathscr{M}_\rho$ be the interior of $\mathbb{R}^4 \setminus \{(t,x,y,z) : \rho(t,x,y) \neq 0\}$, let g_ρ be the metric (2.2.1) with $U = U_\rho$, and with a function A obtained by integrating (2.2.3); here one might need to restrict oneself to a simply connected subset of $\mathbb{R}^4 \setminus \{(t,x,y,z) : \rho(t,x,y) \neq 0\}$ if needed. For any, say smooth, (U_0, U_1), the resulting family $(\mathscr{M}_\rho, g_\rho)$ of vacuum spacetimes is thus parametrized by the set of distributions $\rho \in \mathcal{D}'(\mathbb{R}^3)$ subject to the restrictions above, each member $(\mathscr{M}_\rho, g_\rho)$ being a vacuum development of the same initial data. It is clear that for different ρ's one will in general obtain non–isometric spacetimes. $\Box$

The rather trivial non-uniqueness just described arises from the fact that we have pretended that the spacetime is vacuum by removing from our manifold the regions where matter was present. The example shows that in order to achieve any kind of uniqueness it is natural to consider only these developments $(\mathscr{M}, g)$ which contain the maximal globally hyperbolic development $(\mathscr{M}_0, g_0)$ of the data. In other words, there should exist an isometric embedding of $(\mathscr{M}_0, \gamma_0)$ into $(\mathscr{M}, \gamma)$. This restriction, when imposed in the example above, would exclude all the $\mathscr{M}_\rho$'s except for the spacetime obtained by solving (2.2.4) with $\rho = 0$.

The spacetimes $(\mathscr{M}_\rho, g_\rho)$ obtained from ρ of the form (2.2.5) provide a family of examples which suggest that non-uniqueness of solutions might arise in spacetimes with naked singularities. In [120, Appendix E] it is shown that for any smooth function $f(t)$ such that $0 \notin$ supp f, and for any smooth (U_0, U_1), there exists a unique solution U_ρ of (2.2.4) with ρ given by (2.2.5) which is smooth on $\mathbb{R}^3 \setminus$supp ρ. If we set supp f to be, say, the interval $[1, \infty)$, and we vary f, we obtain an infinite dimensional family of non-isometric spacetimes with a "naked singularity sitting on the set" $\{t \in [1,\infty), x = y = 0\}$. The arbitrariness of f represents an arbitrariness introduced by "the singularity", thus f can be thought of as a "boundary condition at the singularity". As already mentioned, these spacetimes are excluded by the criterion that we consider only these developments which are "at least as large as the maximal globally

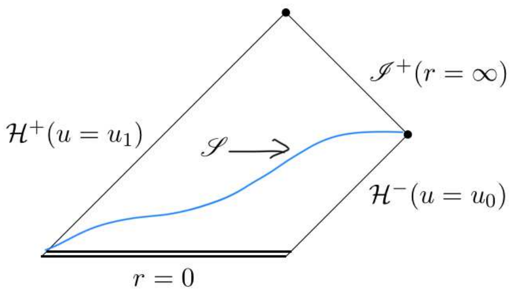

Figure 2.2.1: A projection diagram for a Robinson-Trautman spacetime with ${}^2\mathscr{M} \approx S^2$, $m > 0$.

hyperbolic development". The examples seem, however, to indicate that the occurrence of real singularities might lead to behaviour which is difficult to control.

Example 2.2.2 Let $(B(1), h, K)$ be the standard initial data for Minkowski spacetime restricted to an open unit ball: $B(1) \subset \mathbb{R}^3$, $h_{ij} = \delta_{ij}$, $K_{ij} = 0$. As shown by Bartnik [35], $(B(1), h, K)$ may be extended in an infinite number of ways to distinct vacuum asymptotically flat Cauchy data sets $(\mathbb{R}^3, \tilde{h}, \tilde{K})$. The maximal globally hyperbolic development $(\mathscr{M}, g_0)$ of (M, h, K) is the set $\{-1 < t < 1, 0 \leq r < 1 - |t|\} \subset \mathbb{R}^4$ equipped with the Minkowski metric, and any maximal globally hyperbolic development of $(\mathbb{R}^3, \tilde{h}, \tilde{K})$ will provide an extension of $(\mathscr{M}, g_0)$. □

The last example shows that in any uniqueness theorem it is natural to restrict attention to *inextendible* Cauchy data sets (M, h, K). This condition would exclude the behaviour described there.

There are at least two ways for a Cauchy data set (M, h, K) to be inextendible: one is to assume that (M, h) is complete, another possibility is the occurrence of a singularity at "what would have been ∂M". Let us first consider an example of the latter behaviour:

Example 2.2.3 Consider a Robinson–Trautman spacetime obtained by prescribing some smooth function $\lambda_0 \in C^\infty(S^2)$ at $u = u_0$ as in Appendix B.2, p. 251). Fix $u_0 < u_1 \leq \infty$, consider a smooth spacelike hypersurface $M \subset [u_0, u_1) \times \mathbb{R} \times S^2$, as shown in Figure 2.2.1, let (h, K) be the data induced on M by the RT metric $\mathbf{g}$ (B.2.1). The hypersurface M is inextendible through its "left corner" $= 0$ as in Figure 2.2.1 because of the singularity at $r = 0$ of Robinson–Trautman metrics. In Robinson–Trautman coordinates (u, r, θ, φ), the set $(u_0, u_1) \times \mathbb{R} \times S^2$ is the maximal globally hyperbolic development of (M, h, K). If $u_1 < \infty$ then there exists a neighbourhood of the hypersurface $\{u = u_1\}$ in which the metric is uniquely defined in the vacuum Robinson–Trautman class by (M, h, K). This fails, however, at the horizon $\mathcal{H}^+ = \{``u = \infty"\}$: Theorem B.2.2 shows that there exist infinitely many C^{117} extensions across $\mathcal{H}^+$ of the maximal globally hyperbolic development of (M, h, K). □

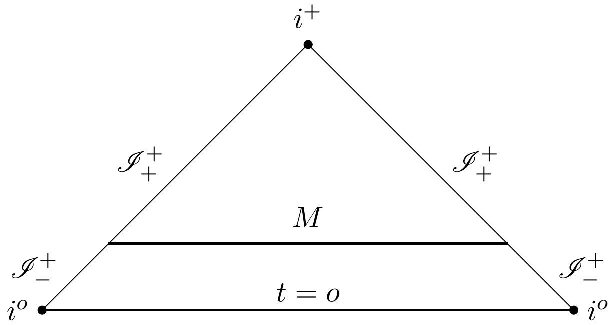

Figure 2.2.2: A "hyperboloidal" initial data surface.

The last example suggests that Cauchy data (M, h, K) which are inextendible through "∂M" because of "singularities sitting on ∂M" lead to non-uniqueness when attempting to go beyond the maximal globally hyperbolic development. It should be, however, pointed out that although this behaviour is generic in the class of Robinson–Trautman spacetimes, the Robinson–Trautman spacetimes themselves are not generic, and it cannot be excluded that this kind of non-uniqueness might be removed in generic situations by imposing some conditions.

EXAMPLE 2.2.4 Let (M, h, K) be "hyperboloidal initial data", as described in Section 3.2.4, p. 114, in particular (M, h) is a complete Riemannian manifold; suppose moreover that (M, h, K) is "smoothly conformally compactifiable" and that the hypotheses of Theorem 2.6.1, p. 52 below hold (*cf.* *e.g.* [12] or Sections 2.6.1 and 3.2.4 for more details). We can choose the time orientation in such a way that the maximal vacuum globally hyperbolic development $(\mathscr{M}, \gamma)$ of (M, h, K) contains at least "a piece" $\mathscr{I}^+_+$ of $\mathscr{I}^+$, where $\mathscr{I}^+_+$ is the part of $\mathscr{I}^+$ to the future of M, *cf.* Figure 2.2.2 (*cf.* Theorem 2.6.1 and [209], compare [378–380]). Using *e.g.* the techniques of [366] one can show [261] that supplementing (M, h, K) by appropriate smooth data on $\mathscr{I}^+_-$ — the part of $\mathscr{I}^+$ in the past of M — one can find a vacuum metric on a neighbourhood $\mathcal{O}$ of $M \cup \mathscr{I}^+_-$. There is arbitrariness in the choice of the "missing data on $\mathscr{I}^+_-$", and different data[1] will lead to non-isometric extension of $(\mathscr{M}, \gamma)$ to the past of M. This example shows that even the requirement of completeness of (M, h) is not sufficient for inextendibility of maximal globally hyperbolic developments. □

Summarising, it seems natural to require that:

- The space of Lorentzian manifolds of interest to physics should contain only those developments $(\mathscr{M}, \gamma)$ of (M, h, K) which contain the unique maximal globally hyperbolic development of (M, h, K);
- if M is compact, then all the initial data metrics of physical interest should be complete;

[1] By choosing (M, h, K) to be the data induced on the standard hyperboloid in Minkowski spacetime one can by this method construct a curious spacetime which is the Minkowski spacetime to the future of a hyperboloid, and not-Minkowski to its past.

- for non-compact M, the allowed initial data metrics should be complete and the initial data sets should be *e.g.* asymptotically flat in a well-controlled sense.

2.3 Strong cosmic censorship

The examples just given show that there exist vacuum spacetimes with non-unique extensions of a maximal globally hyperbolic region. In such examples the spacetime $(\mathscr{M}, \mathbf{g})$ of Theorem 2.1.1 is unique in the class of globally hyperbolic spacetimes, but it can be extended in more than one way to strictly larger vacuum solutions. In such cases the extension always takes places across a *Cauchy horizon* (see Appendix A.22 for the definition).

So one cannot expect uniqueness in general. However, it has been suggested by Penrose [345] that non-uniqueness happens only in very special circumstances. The following result of Isenberg and Moncrief [253, 331, 332] (compare [236, 333]) indicates that this might indeed be the case:

THEOREM 2.3.1 *Let $(\mathscr{M}, \mathbf{g})$ be an analytic spacetime containing an analytic compact Cauchy horizon $\mathscr{H}$. If the null geodesics threading $\mathscr{H}$ are closed, then the Cauchy horizon is a* Killing horizon*; in particular the isometry group of $(\mathscr{M}, \mathbf{g})$ is at least one-dimensional.*

The hypotheses of analyticity, compactness, and closed generators are of course highly restrictive; partial results towards removing the closed-generators hypothesis can be found in [333]. In any case it is conceivable that some kind of local isometries need to occur in spacetimes with Cauchy horizons when those conditions are not imposed; indeed, all known examples have this property. But of course existence of local isometries is a highly non-generic property, even when vacuum equations are imposed [46], so a version of Theorem 2.3.1 without those undesirable hypotheses would indeed establish SCC.

Whether or not Cauchy horizons require Killing vector fields, a loose mathematical formulation of *strong cosmic censorship*, as formulated in [123] following Moncrief and Eardley [330] and Penrose [345], is the following:

CONJECTURE 2.3.2 (Strong cosmic censorship conjecture) *Consider the collection of initial data for, say, electro–vacuum or vacuum spacetimes, with the initial data surface $\mathscr{S}$ being compact, or with asymptotically flat initial data. For generic such data the maximal globally hyperbolic development is inextendible.*

Because of the difficulty of the strong cosmic censorship problem, a full understanding of the issues which arise in this context seems to be completely out of reach at this stage. There is therefore some interest in trying to understand that question under various restrictive hypotheses, *e.g.*, symmetry. The simplest case, of spatially homogeneous spacetimes, has turned out to be surprisingly difficult, because of the intricacies of the dynamics of some of the Bianchi models which we are about to discuss, and has been settled in the affirmative in [155] (compare Theorem 2.3.4 below).

2.3.1 Bianchi A metrics

An important example of the intricate dynamical behavior of solutions of the Einstein equations is provided by the *"Bianchi"* vacuum metrics. The key insight provided by these spacetimes is the supposedly chaotic behavior of large families of metrics in this class when a singularity is approached. This dynamics has been conjectured to be generic; we will return to this issue in Section 2.6.4. In this section we concentrate on Bianchi A models, the reader is referred to [234, 360, 361] for related results on Bianchi B models.

As will be seen shortly, in Bianchi A spacetimes the Einstein evolution equations reduce to a polynomial dynamical system on an algebraic four-dimensional submanifold of $\mathbb{R}^5$. The spatial parts of the Bianchi geometries provide a realization of six, out of eight, homogeneous geometries in three dimensions which form the basis of Thurston's geometrization program.

For our purposes here we define the Bianchi spacetimes as *maximal globally hyperbolic vacuum developments of initial data which are invariant under a simply transitive group of isometries.* Here the transitivity of the isometry group is meant at the level of initial data, and *not* for the spacetime. The name is a tribute to Bianchi, who gave the classification of three dimensional Lie algebras which underline the geometry here. These metrics split into two classes, Bianchi A and Bianchi B, as follows: Let G be a 3-dimensional Lie group, and let Z_i, $i = 1, 2, 3$ denote a basis of left-invariant vector fields on G. Define the *structure constants* γ^k_{ij} by the formula

$$[Z_i, Z_j] = \gamma_{ij}{}^k Z_k \ .$$

The Lie algebra and Lie group are said to be of class A if $\gamma_{ik}{}^k = 0$; class B are the remaining ones. The classes A and B correspond in mathematical terminology to the *unimodular* and *non-unimodular* Lie algebras. A convenient parameterization of the structure constants is provided by the symmetric matrix n^{ij} defined as

$$n^{ij} = \frac{1}{2}\gamma_{kl}{}^{(i}\epsilon^{j)kl}. \qquad (2.3.1)$$

This implies $\gamma_{ij}{}^k = \epsilon_{ijm}n^{km}$. The Bianchi A metrics are then divided into six classes, according to the eigenvalues of the matrix n^{ij}, as described in Table 2.1. For the Bianchi IX metrics, of particular interest to us here, the group G is

Table 2.1: Lie groups of Bianchi class A.

Bianchi type	n_1	n_2	n_3	Simply connected group
I	0	0	0	Abelian $\mathbb{R}^3$
II	+	0	0	Heisenberg
VI_0	0	+	−	Sol (isometries of the Minkowski plane $\mathbb{R}^{1,1}$)
VII_0	0	+	+	universal cover of Euclid (isometries of $\mathbb{R}^2$)
VIII	−	+	+	universal cover of $SL(2,\mathbb{R})$
IX	+	+	+	$SU(2)$

$SU(2)$. Thus, the Taub metrics discussed in Appendix B.1, p. 235 are members of the Bianchi IX family, distinguished by the existence of a further $U(1)$ factor in the isometry group.

Let G be any three-dimensional Lie group, the Lie algebra of which belongs to the Bianchi A class. (As already mentioned, the G's are closely related to the Thurston geometries, see Table 2.1; compare [7, Table 2]). Denote by $\{\sigma^i\}$ the basis dual to $\{Z_i\}$. It is not too difficult to show that both A and B Bianchi metrics can be globally written as

$$\mathbf{g} = -dt^2 + g_{ij}(t)\sigma^i\sigma^j \ , \qquad t \in I \ , \tag{2.3.2}$$

with a maximal time interval I.

There are various ways to write the Einstein equations for a metric of the form (2.3.2). We use the formalism introduced by Wainwright and Hsu [407], which has proven to be most useful for analytical purposes [368, 373, 374]. We follow the presentation in [373].

Let

$$\sigma_{ij} = K_{ij} - \frac{1}{3}\mathrm{tr}_g K g_{ij} \ , \qquad \theta := \mathrm{tr}_g K \ ,$$

be the trace-free part of the extrinsic curvature tensor of the level sets of t. Away from the (isolated) points at which θ vanishes, one can introduce

$$\begin{aligned} \Sigma_{ij} &= \sigma_{ij}/\theta \ , \\ N_{ij} &= n_{ij}/\theta \ , \\ B_{ij} &= 2N_i{}^k N_{kj} - N^k{}_k N_{ij} \ , \\ S_{ij} &= B_{ij} - \frac{1}{3}B^k{}_k\delta_{ij} \ . \end{aligned}$$

Set $S_p = \frac{3}{2}(\Sigma_{22}+\Sigma_{33})$ and $\Sigma_- = \sqrt{3}(\Sigma_{22}-\Sigma_{33})/2$. If we let N_i be the eigenvalues of N_{ij}, the vacuum Einstein equations (a detailed derivation of which can be found in [373]) lead to the following *autonomous, polynomial* dynamical system

$$\begin{aligned} N_1' &= (q - 4S_p)N_1 \ , \\ N_2' &= (q + 2S_p + 2\sqrt{3}\Sigma_-)N_2 \ , \\ N_3' &= (q + 2S_p - 2\sqrt{3}\Sigma_-)N_3 \ , \\ S_p' &= -(2-q)S_p - 3S_+ \ , \\ \Sigma_-' &= -(2-q)\Sigma_- - 3S_- \ , \end{aligned} \tag{2.3.3}$$

where a prime denotes derivation with respect to a new time coordinate τ defined by

$$\frac{dt}{d\tau} = \frac{3}{\theta} \ . \tag{2.3.4}$$

Further,

$$\begin{aligned} q &= 2(S_p^2 + \Sigma_-^2) \ , \\ S_+ &= \frac{1}{2}[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)] \ , \\ S_- &= \frac{\sqrt{3}}{2}(N_3 - N_2)(N_1 - N_2 - N_3) \ . \end{aligned} \tag{2.3.5}$$

The vacuum constraint equations reduce to one equation,

$$S_p^2 + \Sigma_-^2 + \frac{3}{4}[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1)] = 1 \ . \qquad (2.3.6)$$

The points $(N_1, N_2, N_3, S_p, \Sigma_-)$ can be classified according to the values of N_1, N_2, N_3 in the same way as the n^i's in Table 2.1. The sets $N_i > 0$, $N_i < 0$ and $N_i = 0$ are invariant under the flow determined by (2.3.3), and one can therefore classify solutions to (2.3.3)-(2.3.6) accordingly. Bianchi IX solutions correspond, up to symmetries of the system, to points with all N_i's positive, while for Bianchi $VIII$ solutions one can assume that two N_i's are positive and the third is negative.

Points with $N_1 = N_2 = N_3 = 0$ correspond to Bianchi I models. The associated vacuum metrics were first derived by Kasner, and take the form

$$ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2p_i} dx^i \otimes dx^i \ , \quad p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \ . \qquad (2.3.7)$$

An important role in the analysis of (2.3.3) is played by the *Kasner circle*, defined as the set $\{q = 2\}$. These points belong to the configuration space, as determined by (2.3.6), for Bianchi I models, but the equation $q = 2$ is incompatible with (2.3.6) for Bianchi IX metrics. Nevertheless, we shall see shortly that the Kasner circle plays an essential role in the analysis of the Bianchi IX dynamics.

The set $\Sigma_- = 0$, $N_2 = N_3$, together with its permutations, is invariant under the flow of (2.3.3)-(2.3.6). In the Bianchi IX case these are the Taub solutions. In the Bianchi $VIII$ case the corresponding explicit solutions, known as the NUT metrics, have been found by Newman, Tamburino and Unti [336], and they exhibit properties similar to the Bianchi IX Taub solutions.

The *ω-limit* of an orbit γ of a dynamical system is defined as the set of accumulation points of that orbit. In [373, 374], Ringström proves the following:

THEOREM 2.3.3 *The ω-limit set of each non-NUT Bianchi $VIII$ orbit contains at least two distinct points on the Kasner circle. Similarly, non-Taub–NUT Bianchi IX orbits have at least three distinct ω-limit points on the Kasner circle.*

The picture which emerges from a numerical analysis of (2.3.3) (see [54, 159] and references therein) is the following: Every non-Taub–NUT Bianchi IX orbit approaches some point on the Kasner circle; there it performs a “bounce”, after which it eventually approaches another point on the Kasner circle, and so on. Theorem 2.3.3 establishes the validity of this picture. The numerical analysis further suggests that generic orbits will have a dense ω-limit set on the Kasner circle; this is compatible with, but does not follow from, Ringström's analysis. It has been argued that the map which associates to each bounce the nearest point on the Kasner circle possesses chaotic features; this is at the origin of the “mixmaster behavior” terminology, sometimes used in this context. Major progress concerning this issue has been achieved in [41, 292, 364], where existence of orbits exhibiting the above behaviour has been established, but the question of what happens for *all*, or for *generic* orbits remains open.

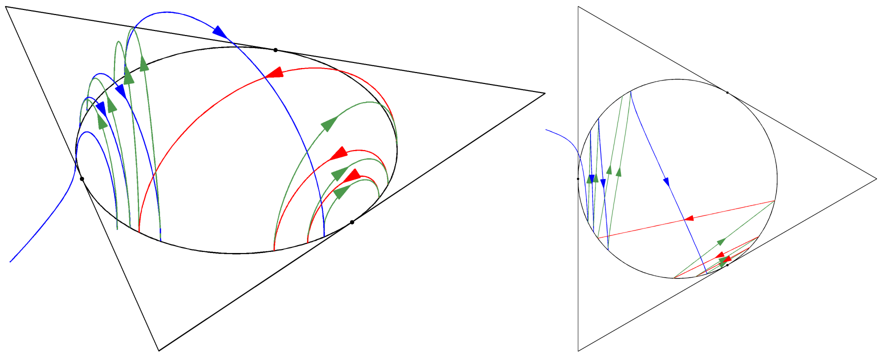

Figure 2.3.1: A few "bounces" in a typical Bianchi IX orbit; figures and numerics by Woei-Chet Lim. The vertical axis represents N_1 (red), N_2 (green), N_3 (blue), with only the biggest of the N_i's plotted. The Kasner circle and the triangle for the Kasner billiard in the (Σ_+, Σ_-)–plane are shown. The projected trajectories can be seen to approach the billiard ones.

The following result of Ringström [373] provides further insight into the geometry of Bianchi IX spacetimes:

THEOREM 2.3.4 *In all maximal globally hyperbolic developments* $(\mathscr{M}, \mathbf{g})$ *of non-Taub–NUT Bianchi IX vacuum initial data or of non-NUT Bianchi VIII vacuum initial data the Kretschmann scalar*

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

is unbounded along inextendible causal geodesics.

Note that the observation of curvature blow-up provides a proof, alternative to that of [155], of the non-existence of Cauchy horizons in generic Bianchi IX models.

2.3.2 Gowdy toroidal metrics

The next simplest case, after the Bianchi models, is that of Gowdy metrics on $\mathbb{T}^3 := S^1 \times S^1 \times S^1$:

$$g = e^{(\tau-\lambda)/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}[e^P d\sigma^2 + 2e^P Q d\sigma d\delta + (e^P Q^2 + e^{-P})d\delta^2], \tag{2.3.8}$$

where $\tau \in \mathbb{R}$ and (θ, σ, δ) are coordinates on $\mathbb{T}^3$, with the functions P, Q and λ depending only on τ and θ. This form of the metric can always be attained [119] when considering maximal globally hyperbolic $U(1) \times U(1)$–symmetric vacuum spacetimes with $\mathbb{T}^3$–Cauchy surfaces and with *vanishing twist constants*:

$$c_a = \epsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_a^\delta \,, \qquad a = 1, 2 \,, \tag{2.3.9}$$

where the X_a's are the Killing vectors generating the $U(1) \times U(1)$ action by isometries. The condition $c_1 = c_2 = 0$ is equivalent to the requirement that the family of planes $\text{span}\{X_1, X_2\}^\perp$ is integrable.

For metrics of the form (2.3.8), the Einstein vacuum equations become a set of *wave-map* equations

$$P_{\tau\tau} - e^{-2\tau} P_{\theta\theta} - e^{2P}(Q_\tau^2 - e^{-2\tau} Q_\theta^2) = 0, \tag{2.3.10}$$
$$Q_{\tau\tau} - e^{-2\tau} Q_{\theta\theta} + 2(P_\tau Q_\tau - e^{-2\tau} P_\theta Q_\theta) = 0, \tag{2.3.11}$$

which are supplemented by ODE's for the function λ:

$$\lambda_\tau = P_\tau^2 + e^{-2\tau} P_\theta^2 + e^{2P}(Q_\tau^2 + e^{-2\tau} Q_\theta^2), \tag{2.3.12}$$
$$\lambda_\theta = 2(P_\theta P_\tau + e^{2P} Q_\theta Q_\tau). \tag{2.3.13}$$

Here we write P_τ for $\partial_\tau P$, etc.

In order to solve these equations, one starts with initial data for P and Q such that

$$\int_{S^1} (P_\theta P_\tau + e^{2P} Q_\theta Q_\tau) d\theta = 0, \tag{2.3.14}$$

this last condition being an integral consequence of (2.3.13) in view of the θ-periodicity imposed. One then solves (2.3.10)-(2.3.11) in order to obtain P and Q. Finally, λ is obtained by integrating (2.3.12)-(2.3.13). Global existence of solutions to (2.3.10)-(2.3.11) was proved in [328] when the initial data are given on a hypersurface $\{\tau = \text{const}\}$, and in [119] for general $U(1) \times U(1)$–symmetric Cauchy surfaces.

The question of SCC in this class of metrics has been settled by Ringström, who proved that the set of smooth initial data for Gowdy models on $\mathbb{T}^3$ that do *not* lead to the formation of Cauchy horizons contains a set which is open and dense within the set of all smooth initial data. More precisely, Ringström's main result (see [375, 377] and references therein) is the following:

THEOREM 2.3.5 *Let $\tau_0 \in \mathbb{R}$ and let $\mathcal{S} = \{(Q(\tau_0), P(\tau_0), Q_\tau(\tau_0), P_\tau(\tau_0))\}$ be the set of smooth initial data for (2.3.10)-(2.3.11) satisfying (2.3.14). There is a subset $\mathcal{G}$ of $\mathcal{S}$ which is open with respect to the $C^2 \times C^1$ topology, and dense with respect to the C^∞ topology, such that the spacetimes of the form (2.3.8) corresponding to initial data in $\mathcal{G}$ are causally geodesically complete in one time direction, incomplete in the other time direction, and the Kretschmann scalar, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, becomes unbounded in the incomplete direction of causal geodesics.*

This result does indeed establish SCC in this class of metrics: to see that the resulting spacetimes are inextendible in the category of C^3 manifolds with C^2 Lorentzian metrics, note that the existence of any such extension would imply existence of geodesics which are incomplete in the original spacetime, and along which every curvature scalar is bounded.

Theorem 2.3.5 is complemented by the results in [86, 141, 329], where infinite dimensional families of (nongeneric) solutions which *are extendible* across a Cauchy horizons are constructed.

The key to the understanding of the global structure of the Gowdy spacetimes is the analysis of the behavior of the functions P and Q as $\tau \to \pm\infty$. The asymptotic behavior of those functions, established by Ringström, can then be translated into statements about the behavior of the spacetime geometry as those limits are approached. A central element of the proof is the existence of a *velocity function*

$$v(\theta) := \lim_{\tau\to\infty} \sqrt{P_\tau^2 + e^{2P}Q_\tau^2}\,.$$

Essential steps in Ringström's analysis are provided by the work on Fuchsian PDEs of Kichenassamy and Rendall [266, 370], as well as the study of the action of Geroch transformations by Rendall and Weaver [371] (compare [86]). See also [146] for the related problem of an exhaustive description of Cauchy horizons in those models.

2.3.3 Other $U(1) \times U(1)$ symmetric models

The existence of two Killing vectors is also compatible with S^3, $L(p,q)$ ("lens" spaces), and $S^1 \times S^2$ topologies. Thus, to achieve a complete understanding of the set of spatially compact initial data with precisely two Killing vectors one needs to extend Ringström's analysis to those cases. There is an additional difficulty that arises because of the occurrence of axes of symmetry, where the $(1+1)$–reduced equations have the usual singularity associated with polar coordinates. Nevertheless, in view of the analysis by Christodoulou and Tahvildar-Zadeh [114, 115] (see also [119]), the global geometry of *generic* maximal globally hyperbolic solutions with those topologies is reasonably well understood. This leads one to expect that one should be able to achieve a proof of SCC in those models using simple abstract arguments, but this remains to be seen.

Recall, finally, that general models with two Killing vectors X_1 and X_2 on $\mathbb{T}^3$ have non-vanishing *twist constants* (2.3.9). The Gowdy metrics are actually "zero measure" in the set of all $U(1) \times U(1)$ symmetric metrics on $\mathbb{T}^3$ because $c_a \equiv 0$ for the Gowdy models. The equations for the resulting metrics are considerably more complicated when the c_a's do not vanish, and only scant rigorous information is available on the global properties of the associated solutions [52, 256, 367]. It seems urgent to study the dynamics of those models, as they are expected to display [53] "oscillatory behavior" as the singularity is approached, in the sense of Section 2.6.4. Thus, they should provide the simplest model in which to study this behavior.

2.3.4 Spherical symmetry

One could think that the simplest possible asymptotically flat model for studying the dynamics of the gravitational field will be obtained by requiring spherical symmetry, since then the equations should reduce to wave equations in only two variables, t and r. Unfortunately, for vacuum spacetimes this turns out to be useless for this purpose because of the Jebsen-Birkhoff theorem [65, 257], which asserts that spherically symmetric vacuum metrics are static. So, if one wishes to maintain spherical symmetry, supplementary fields are needed. The case of a scalar field was studied in a series of intricate papers over thirteen years by Christodoulou, beginning with [107] and culminating in [109] with the verification of the strong cosmic censorship conjecture within the model.

Christodoulou further established "weak cosmic censorship" in this class, an issue to which we return in the next section, and exhibited non-generic examples for which the conclusions of these conjectures fail [108].

The situation changes when electromagnetic fields are introduced. The analysis by Dafermos [168, 169] of the spherically symmetric Einstein-Maxwell-scalar field equations yields a detailed picture of the interior of the black hole for this model, in terms of initial data specified on the event horizon and on an ingoing null hypersurface. When combined with the work by Dafermos and Rodnianski [174] on Price's law, one obtains the following global picture: initial data with a compactly supported scalar field, and containing a trapped surface (see Definition **??**, p. **??**), lead to spacetimes which *either* contain a degenerate (extremal) black hole, *or* develop a Cauchy horizon, with a spacetime metric that can be continued past this horizon *in a* C^0*, but not* C^1 *manner*. It seems that not much is known about the properties of the degenerate solutions, which are presumably non-generic; it would be of interest to clarify that. In any case, the work shows that strong cosmic censorship holds within the class of nondegenerate solutions with trapped surfaces, at the C^1 level, leaving behind the perplexing possibility of continuous extendability of the metric.

The reader is referred to [7, 123, 369] and references therein for further reading on SCC.

2.4 Weak cosmic censorship

The strong cosmic censorship conjecture is an attempt to salvage predictability of Einstein's theory of gravitation. There exists a variant thereof which addresses the fact that we do not seem to observe any of the singularities that are believed to accompany gravitational collapse. The hope is then that, generically, in asymptotically flat spacetimes, any singular behavior that might form as a result of gravitational collapse, such as causality violations, lack of predictability, or curvature singularities, will be *clothed by an event horizon*. For this, one introduces the notion of *future null infinity*, which is an idealized boundary attached to spacetime that represents, loosely speaking, the end points of null geodesics escaping to infinity. The *black hole event horizon* is then the boundary of the past of null infinity. One then wishes the part of the spacetime that lies outside the black hole region to be well-behaved and "sufficiently large". This is the content of the *weak cosmic censorship* conjecture, originally due to Penrose [345], as made precise by Christodoulou [110]: *for generic asymptotically flat initial data, the maximal globally hyperbolic development has a complete future null infinity*. Heuristically this means that, disregarding exceptional sets of initial data, no singularities are observed at large distances, even when the observations are continued indefinitely. One should remark that, despite the names, the strong and weak cosmic censorship conjectures are logically independent; neither follows from the other. Note also that some predictability of Einstein's theory would be salvaged if strong cosmic censorship failed with weak cosmic censorship being verified, since then the failure of predictability would be invisible to outside observers.

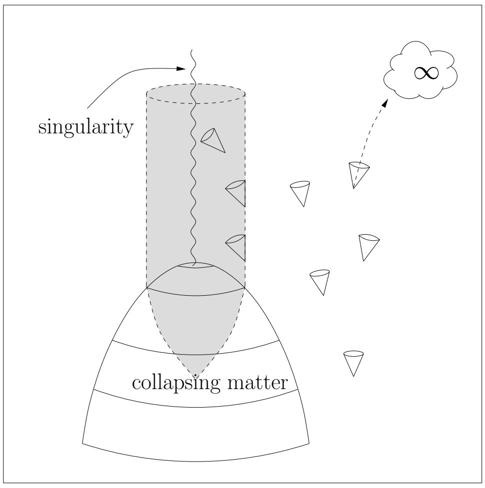

Figure 2.4.1: Light-cones in the Oppenheimer-Snyder collapse. I am grateful to C. Williams for providing the figure.

Both cosmic censorship conjectures are intimately related to the issue of *gravitational collapse*, the dynamical formation of black holes and singularities, first observed for a homogeneous dust model by Oppenheimer and Snyder in 1939 [340], visualized in Figure 2.4.1.

So far the only complete analysis of weak cosmic censorship in a field theoretical model is that of the spherically symmetric scalar field model studied by Christodoulou [108, 109], already mentioned in Section 2.3.4.

INCIDENTALLY: The situation in higher dimensions appears to be more complex. On the positive side, Dafermos and Holzegel [170] have proved non-linear stability for a restricted class of perturbations of the five dimensional Schwarzschild metric. On the other hand, numerical work in [200] suggests failure of weak cosmic censorship near six-dimensional Myers-Perry black holes.

2.5 Stability of vacuum cosmological models

Not being able to understand the dynamics of all solutions, one can ask whether some features of certain particularly important solutions persist under small perturbations of initial data. For example, will geodesic completeness still hold for spacetimes arising from small perturbations of Minkowskian initial data? Or, will a global, all encompassing, singularity persist under perturbations of Bianchi IX initial data? Such questions are the object of stability studies.

2.5.1 $U(1)$ symmetry

Our understanding of models with *exactly one* Killing vector is dramatically poorer than that of $U(1) \times U(1)$ symmetric spacetimes. Here one only has *stability* results, for small perturbations within the $U(1)$ isometry class, in the expanding direction ("away from the singularity"): In [91] Choquet-Bruhat considers $U(1)$ symmetric initial data (h,K) for the vacuum Einstein equations on a manifold of the form $\mathscr{S}\times S^1$, where $\mathscr{S}$ is a compact surface of genus $g>1$. It is assumed that $\mathrm{tr}_h K$ is constant, and that (h,K) are sufficiently close to (h_0,K_0), where h_0 is a product metric

$$h_0=\gamma+dx^2\,,$$

with γ being a metric of constant Gauss curvature on $\mathscr{S}$, and with K_0 proportional to h_0. The sign of the trace of K_0 determines an expanding time direction and a contracting one. Under those conditions, Choquet-Bruhat proves that the solution exists for an infinite proper time in the expanding direction. The analysis builds upon previous work by Choquet-Bruhat and Moncrief [103], where a supplementary polarization condition has been imposed. Not much is known in the contracting direction in those models (see [252]), where "mixmaster behavior"[2] is expected [50, 55]; compare [56].

2.5.2 Future stability of hyperbolic models

The proof of the above result bears some similarity to the *future stability* theorem of Andersson and Moncrief [16], as generalized in [14], for spatially compact hyperbolic models *without any symmetries*. Those authors consider initial data near a negatively curved compact space form, with the extrinsic curvature being close to a multiple of the metric, obtaining future geodesic completeness in the expanding direction. The control of the solution is obtained by studying the Bel-Robinson tensor, and its higher-derivatives analogues. A striking ingredient of the proof is an elliptic-hyperbolic system of equations, used to obtain local existence in time [15].

2.6 Stability of Minkowski spacetime

The idea of stability results is to fix some spacetime (M,γ_0), maximal globally hyperbolic development of some data $(\mathscr{S},h_0,K_0)$, and try to prove that for (h,K) sufficiently close to (h_0,K_0) in some norm the global properties of maximal developments (M,γ) of the $(\mathscr{S},h,K)$'s will mimic those of (M,γ_0).

The first natural question is to enquire about stability of Minkowski spacetime.

2.6.1 Friedrich's stability theorem

The first results of this type have been proved by Friedrich [210], with or without a cosmological constant, and also for the Einstein–Yang–Mills system [211]; here we shall review the vacuum case with zero cosmological constant only. Friedrich's approach takes advantage of the fact that conformal transformations can map infinite domains into finite ones, reducing in this way the global-in-time stability problem to a much simpler short=time stability problem for conformally rescaled fields. Suppose thus that a (spatially non–compact) spacetime

[2] See the discussion after Theorem 2.3.3, and Section 2.6.4.

(M,γ) can be conformally mapped into a spatially compact spacetime $(\tilde M,\tilde\gamma)$ (this can be done *e.g.* for the Minkowski spacetime, with $(\tilde M,\tilde\gamma)$ — the "Einstein cylinder", $\tilde M \approx \mathbb{R}\times S^3$, see Appendix **??**. After such an infinite compression the conformal factor which relates the physical metric γ to the "unphysical" metric $\tilde\gamma$, $\gamma_{\mu\nu} = \Omega^{-2}\tilde\gamma_{\mu\nu}$, will vanish on the boundary $\partial M \equiv \mathscr{I}$ of M in $\tilde M$, which introduces singular terms when one naively rewrites Einstein equations for γ in terms of $\tilde\gamma$ and Ω. Friedrich has observed that one can derive a well posed system of equations for the conformally rescaled fields which is regular even at points at which Ω vanishes, and which is equivalent to vacuum Einstein equations on the set $\Omega > 0$ (*cf.* also [96, 104] for a different "conformally regular" system of equations), which leads to the following [210, 211]:

THEOREM 2.6.1 (Future stability of the "hyperboloidal initial value problem") *Let (g_0,K_0) be the data induced on the unit hyperboloid*

$$\mathscr{S} = \{(t,x,y,z)\in\mathbb{R}^4 : t = \sqrt{1+x^2+y^2+z^2}\} \approx \mathbb{R}^3$$

from the flat metric of Minkowski spacetime. Consider the space X of Cauchy data (h,K) such that

1. *(h,K) are smoothly conformally compactifiable*[3], i.e. *there exist a smooth compact Riemannian manifold $(\tilde{\mathscr{S}},\tilde g)$ with boundary, with Int$(\tilde{\mathscr{S}})\approx\mathscr{S}$, where Int$(\cdot)$ is the interior of $\cdot$, and a smooth (up to boundary) non–negative function Ω on $\tilde{\mathscr{S}}$, vanishing only on $\partial\tilde{\mathscr{S}}$, with $d\Omega(p)\neq 0$ for $p\in\partial\tilde{\mathscr{S}}$, such that we have*

   $$h_{ij} = \Omega^{-2}\tilde h_{ij}\,,$$

   and the fields

   $$\tilde L^{ij} \equiv \Omega^{-3}(h^{ik}h^{j\ell}K_{k\ell} - \frac{1}{3}h^{\ell m}K_{\ell m}h^{ij})\,, \qquad \tilde K \equiv \Omega h^{ij}K_{ij}$$

   are smooth (up to boundary) on $\tilde{\mathscr{S}}$;

2. *the Weyl tensor $C^\alpha{}_{\beta\gamma\delta}$ of the four–dimensional metric, formally calculated from (h,K) using vacuum Einstein equations, vanishes at the conformal boundary $\partial\tilde{\mathscr{S}}$;*

3. *there exist fields Ω_n and Ω_{nn}, smooth (up to boundary) on $\tilde{\mathscr{S}}$, which we identify with tetrad components in the directions normal to $\tilde{\mathscr{S}}$ of the gradient, respectively the Hessian, of Ω, such that*

   $$(\Omega_n^2 - \tilde h^{ij}\Omega_i\Omega_j)|_{\partial\tilde{\mathscr{S}}} = 0\,,$$

   and the tensor field

   $$e_{\alpha\beta} = \nabla_\alpha\nabla_\beta\Omega - \frac{1}{4}\tilde\gamma^{\mu\nu}\nabla_\mu\nabla_\nu\Omega\,\tilde\gamma_{\alpha\beta}$$

   vanishes at $\partial\tilde{\mathscr{S}}$.

[3] In [210] one assumes, roughly speaking, that $(\Omega,\tilde g,\tilde L,\tilde K)\in H_k(\tilde{\mathscr{S}})\oplus H_k(\tilde{\mathscr{S}})\oplus H_{k-1}(\tilde{\mathscr{S}})\oplus H_{k-1}(\tilde{\mathscr{S}})$, $(\Omega_n, d^\alpha_{\beta\gamma\delta}, f_{\alpha\beta})\in H_{k-1}(\tilde{\mathscr{S}})\oplus H_{k-2}(\tilde{\mathscr{S}})\oplus H_{k-2}(\tilde{\mathscr{S}})$, $k\geq 6$; it is rather clear that by not too difficult technical improvements of the existence theorems used in [210] this threshold can be relaxed to $k\geq 5$ and probably even to $k\geq 4$.

Set $d^\alpha{}_{\beta\gamma\delta} \equiv \Omega^{-1}C^\alpha{}_{\beta\gamma\delta}$*,* $f_{\alpha\beta} = \Omega^{-1}e_{\alpha\beta}$*. There exists* $\epsilon > 0$ *such that for all* $(g, K) \in X$ *satisfying*

$$\| \Omega - \Omega_0 \|_{H_6(\tilde{\mathscr{S}})} + \| \tilde{h}_{ij} - \tilde{h}_{0\,ij} \|_{H_6(\tilde{\mathscr{S}})} + \| \tilde{L}_{ij} - \tilde{L}_{0\,ij} \|_{H_5(\tilde{\mathscr{S}})} + \| \tilde{K} - \tilde{K}_0 \|_{H_5(\tilde{\mathscr{S}})}$$

$$+ \| d^\alpha{}_{\beta\gamma\delta} \|_{H_4(\tilde{\mathscr{S}})} + \| \Omega_n - \Omega_{0\,n} \|_{H_5(\tilde{\mathscr{S}})} + \| f_{\alpha\beta} - f_{0\,\alpha\beta} \|_{H_4(\tilde{\mathscr{S}})} < \epsilon$$

(where $H_\ell(\tilde{\mathscr{S}})$ *is the Sobolev space of tensors on* $\tilde{\mathscr{S}}$ *which are square integrable together with all the derivatives up to order* ℓ *on* $\tilde{\mathscr{S}}$ *with respect to the Riemannian measure* $d\mu_{\tilde{g}}$ *of the metric* $\tilde{g}$*, and* Ω_0*,* $\tilde{h}_{0\,ij}$*, etc., denote the corresponding quantities for Minkowski spacetime), the maximal globally hyperbolic development* (M, γ) *is future null and timelike geodesically complete, hence* (M, γ) *is* strongly maximal *to the future.*

By the very nature of Friedrich's construction (*cf.* the discussion of Example 4 in Section 2.2, p. 38) the above theorem guarantees global uniqueness to the future of $\mathscr{S}$ only, and no rigorous results are available about the possibility of supplementing the Cauchy data on $\mathscr{S}$ by Cauchy data on the part of $\mathscr{I}$ which lies to the past of $\mathscr{S}$ to obtain global uniqueness to the past. Moreover, the following features of this theorem deserve further investigation:

1. the rather high differentiability conditions needed for stability,
2. the hypothesis of the vanishing of the Weyl curvature on the conformal boundary $\partial\tilde{\mathscr{S}}$ — the so called Weyl tensor condition,
3. the hypothesis of the vanishing of $e_{\alpha\beta}$ at the conformal boundary,
4. the independence of the various hypotheses above.

Let us recall that the Weyl tensor condition has been shown by Penrose [344] to be necessary for C^k, $k \geq 3$, differentiability of the conformally rescaled fields at the conformal boundary of a spacetime, but the stability results of Christodoulou and Klainerman [112] suggest that the Weyl tensor condition needs not to hold in generic spacetimes obtained by evolution from asymptotically flat (at spacelike infinity) initial data (thus generic $\mathscr{I}$'s obtained in this way will probably *not* be C^3). As discussed in detail in Section 3.2.4, p. 114, the Cauchy data sets satisfying the Weyl tensor condition also turn out to be *non generic* in the space of solutions of the constraint equations which can be constructed by the conformal method. These drawbacks of Friedrich's theorem are more than compensated by the (relative) simplicity of the method. It has been shown in [12] that the vanishing of the space components e_{ij} of $e_{\alpha\beta}$ at $\partial\tilde{\mathscr{S}}$ and smoothness of $\tilde{h}_{ij}$ imply the vanishing of the Weyl tensor at $\partial\tilde{\mathscr{S}}$, under the supplementary hypotheses that the extrinsic curvature of $\tilde{\mathscr{S}}$ is pure trace on $\partial\tilde{\mathscr{S}}$ ($\tilde{L}_{ij}|_{\partial\tilde{\mathscr{S}}} = 0$), and the Cauchy surface $\mathscr{S}$ has constant extrinsic curvature ($h^{ij}K_{ij}$ =const).

2.6.2 The Christodoulou-Klainerman proof

One of the flagship results in mathematical general relativity is nonlinear stability of Minkowski spacetime, first proved by Christodoulou and Klainerman [112]. One starts with an asymptotically flat vacuum initial data set (h, K) on $\mathbb{R}^3$. Under standard asymptotic flatness conditions, for (h, K) sufficiently close to Minkowskian data, the maximal globally hyperbolic development $(\mathscr{M}, \mathbf{g})$ of the data contains a maximal hypersurface, i.e., a hypersurface satisfying $\mathrm{tr}_h K = 0$; this follows from the results in [30, 37, 113]. So without loss of generality one can, in the small data context, assume that the initial data set is maximal.

The precise notion of smallness needed for the Christodoulou-Klainerman theorem is defined as follows: For $p \in \Sigma \approx \mathbb{R}^3$, $a > 0$, consider the quantity

$$Q(a,p) = a^{-1} \int_\Sigma \{\sum_{\ell=0}^{1} (d_p^2 + a^2)^{\ell+1} |\nabla^\ell \mathrm{Ric}|^2 + \sum_{\ell=1}^{2} (d_p^2 + a^2)^{\ell} |\nabla^\ell K|^2\} d\mu_g \,, \quad (2.6.1)$$

where d_p is the geodesic distance function from p, Ric is the Ricci tensor of the metric g, $d\mu_g$ is the Riemannian measure of the metric g and ∇ is the Riemannian connection of g. Let

$$Q_* = \inf_{a>0,\, p\in\Sigma} Q(a,p) \,.$$

Christodoulou and Klainerman prove causal geodesic completeness of $(\mathscr{M}, \mathbf{g})$ provided that Q_* is sufficiently small. The proof proceeds via an extremely involved bootstrap argument involving a foliation by maximal hypersurfaces Σ_t, together with an analysis of the properties of an *optical function* u. In the context here this is a solution of the eikonal equation

$$\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0 \,,$$

the level sets C_u of which intersect Σ_t in spheres that expand as t increases. We have:

THEOREM 2.6.2 (Global Stability of Minkowski spacetime) *Assume that* $(\mathscr{S}, h, K)$ *is maximal, with*[4]

$$h_{ij} = \delta_{ij} + o_3(r^{-1/2}), \quad K_{ij} = o_2(r^{-3/2}) \,. \quad (2.6.2)$$

There is an $\epsilon > 0$ *such that if* $Q_* < \epsilon$*, then the maximal globally hyperbolic development* $(\mathscr{M}, \mathbf{g})$ *of* $(\mathscr{S}, h, K)$ *is geodesically complete. Furthermore,* $\mathscr{M}$ *contains a* maximal *foliation* $\mathscr{S}_t$ *given by the level hypersurfaces of a time function* t*, and a* null *foliation* C_u *given by the level hypersurfaces of an outgoing optical function* u*, such that relative to an adapted null frame* $e_4 = L, e_3 = \underline{L}$ *and* $(e_a)_{a=1,2}$ *we have, along the null hypersurfaces* C_u *the weak peeling decay,*

$$\begin{aligned} \alpha_{ab} &:= R(L, e_a, L, e_b) = O(r^{-7/2}), & 2\beta_a &:= R(L, \underline{L}, L, e_a) = O(r^{-7/2}) \\ 4\rho &:= R(L, \underline{L}, L, \underline{L}) = O(r^{-3}), & 4\sigma &:=^* R(L, \underline{L}, L, \underline{L}) = O(r^{-7/2}) \\ 2\underline{\beta}_a &:= R(L, \underline{L}, L, e_a) = O(r^{-2}), & \underline{\alpha}_{ab} &:= R(\underline{L}, e_a\,, \underline{L}, e_b) = O(r^{-1}) \end{aligned}$$

[4]A function f on $\mathscr{S}$ is $o_k(r^{-\lambda})$ if $|r^{\lambda+i}\nabla^i f|_g \to 0$ as $r \to \infty$ for all $i = 0, \ldots k$.

as $r \to \infty$ with $4\pi r^2 =$Area$(\mathscr{S}_t \cap C_u)$. Finally, $\rho - \bar\rho = O(r^{-7/2})$, with $\bar\rho$ being the average of ρ over the compact 2-surfaces $\mathscr{S}_t \cap C_u$.

The above version of Theorem 2.6.2 is due to Bieri [61, 62]. The original formulation in [112] assumes moreover that

$$h = (1 + 2M/r)\delta + o_4(r^{-3/2}), \quad K = o_3(r^{-5/2}), \tag{2.6.3}$$

and in the definition (2.6.1) a term involving K with $\ell = 0$ is added.

By definition, asymptotically flat initial data sets approach the Minkowskian ones as one recedes to infinity. One therefore expects that at sufficiently large distances one should obtain “global existence”, in the sense that the maximal globally hyperbolic development contains complete *outgoing* null geodesics. This question has been addressed by Klainerman and Nicolò [269–271]; the reader is referred to those references for precise statements of the hypotheses made:

THEOREM 2.6.3 *Consider an asymptotically flat initial data set $(\mathscr{S}, h, K)$, with maximal globally hyperbolic development $(\mathscr{M}, \mathbf{g})$. Let Ω_r denote a conditionally compact domain bounded by a coordinate sphere $S_r \subset \mathscr{S}_{\rm ext}$. There exists $R > 0$ such that for all $r \geq R$ the generators of the boundary $\partial J^+(\Omega_r)$ of the domain of influence $J^+(\Omega_r)$ of Ω_r are future-complete.*

Both in [112] and in [270] one can find detailed information concerning the behavior of null hypersurfaces as well as the rate at which various components of the Riemann curvature tensor approach zero along timelike and null geodesics.

2.6.3 The Lindblad-Rodnianski proof

A completely new proof of stability of Minkowski spacetime has been given by Lindblad and Rodnianski [296, 297]. The method provides less detailed asymptotic information than [112] and [270] on various quantities of interest, but is much simpler. The argument is flexible enough to allow the inclusion of a scalar field, or of a Maxwell field [302, 303] (compare [418] for an analysis along the lines of the Christodoulou-Klainerman approach), and generalizes to higher dimensions [95]. Further it allows the following, rather weak, asymptotic behavior of the initial data:

$$h = (1 + 2m/r)\delta + O(r^{-1-\alpha}), \qquad K = O(r^{-2-\alpha}). \tag{2.6.4}$$

The decay conditions (2.6.4) are weaker than those in [112, 270], but stronger than those in [63].

The analysis of Lindblad and Rodnianski, in the Einstein-Maxwell-scalar field case, proceeds as follows: Consider the Einstein-Maxwell equations with a neutral scalar field:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = T_{\mu\nu} + \hat{T}_{\mu\nu}, \tag{2.6.5}$$

with

$$\hat{T}_{\mu\nu} = \partial_\mu \psi\, \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha \psi\, \partial_\beta \psi), \quad T_{\mu\nu} = 2(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F^{\lambda\rho} F_{\lambda\rho}).$$

We assume that we are on $\mathbb{R}^{n+1}$, so that the Maxwell field F has a global potential A, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The matter field equations read

$$D_\mu F^{\mu\nu} = 0\,, \quad \Box_g \psi = 0\,. \tag{2.6.6}$$

The initial data, denoted by $(\mathring{h}, \mathring{K}, \mathring{A}, \mathring{E}, \psi_0, \psi_1)$, are assumed to satisfy the following asymptotic conditions, for $r = |x| \to \infty$, with some $\alpha > 0$:

$$\begin{array}{l} \mathring{A} = O(r^{\frac{1-n}{2}-\alpha})\,, \quad \mathring{K}_{ij} = O(r^{-\frac{n+1}{2}-\alpha})\,, \quad \mathring{E} = O(r^{-\frac{n+1}{2}-\alpha})\,, \\ \psi_0 := \psi|_{t=0} = O(r^{\frac{1-n}{2}-\alpha})\,, \quad \psi_1 := \partial_t\psi|_{t=0} = O(r^{-\frac{n+1}{2}-\alpha}) \end{array} \tag{2.6.7}$$

One also supposes that the relevant constraint equations hold initially:

$$\begin{cases} \mathring{R} = \mathring{K}^{ij}\mathring{K}_{ij} - \mathring{K}_i{}^i\mathring{K}_j{}^j + 2\mathring{E}_i\mathring{E}^i + \mathring{F}_{ij}\mathring{F}^{ij} + |\nabla\psi_0|^2 + |\psi_1|^2\,, \\ \nabla^j\mathring{K}_{ij} - \nabla_i\mathring{K}_j{}^j = \mathring{F}_{0j}\mathring{F}_i{}^j + \nabla_i\psi_0\,\psi_1 \quad, \\ \nabla_i\mathring{F}^{0i} = 0\,, \end{cases} \tag{2.6.8}$$

where $\mathring{R}$ is the scalar curvature of the metric $\mathring{h}$.

The strategy is to impose *globally* the wave coordinates condition

$$\partial_\mu\left(g^{\mu\nu}\sqrt{|\det g|}\right) = 0 \quad \forall\nu = 0, ..., n, \tag{2.6.9}$$

as well as the *Lorenz gauge*,

$$\partial_\mu\left(\sqrt{|\det g|}A^\mu\right) = 0\,. \tag{2.6.10}$$

Writing $\Box_g = g^{\alpha\beta}\partial_\alpha\partial_\beta$, the dynamical equations are rewritten as

$$\tilde{\Box}_g \begin{pmatrix} h^1_{\mu\nu} \\ A_\sigma \\ \psi \end{pmatrix} = \begin{pmatrix} S_{\mu\nu} - 2\partial_\mu\psi\partial_\nu\psi \\ S_\sigma \\ 0 \end{pmatrix} - \begin{pmatrix} \tilde{\Box}_g h^0_{\mu\nu} \\ 0 \\ 0 \end{pmatrix}, \tag{2.6.11}$$

where the source terms $S_{\mu\nu}$ and S_σ are bilinear in the derivatives of the fields, with coefficients depending upon the metric. Furthermore,

$$h^1_{\mu\nu} = h_{\mu\nu} - h^0_{\mu\nu}\,, \text{ with } h^0_{\mu\nu}(t) = \begin{cases} \chi(r/t)\chi(r)\frac{2m}{r}\delta_{\mu\nu}, & n = 3; \\ 0, & n \geq 4, \end{cases} \tag{2.6.12}$$

where $\chi \in C^\infty$ is any function such that $\chi(s)$ equals 1 for $s \geq 3/4$ and 0 for $s \leq 1/2$. The proof relies heavily on the structure of the nonlinear terms in wave coordinates.

Recall that there exists an extensive literature on wave equations in $3+1$ dimensions with nonlinearities satisfying the *null condition* [267, 268], but the above nonlinearities *do not* satisfy that condition. The argument works only because one can treat on a different footing different components of h. Indeed, for solutions of the wave equation on Minkowski spacetime, the derivatives in directions tangent to the light cones decay faster than the transverse ones. But the wave coordinate condition can be used to express the transverse derivatives of some components of $g_{\mu\nu}$ in terms of tangential derivatives of the remaining ones. This gives one supplementary control of the nonlinearities.

Recall also that global existence on $\mathbb{R}^{n+1}$ with $n \geq 4$, for small initial data, of solutions of quasi-linear wave equations, under structure conditions compatible with the above, has been proved in [241, 290][5], see also [106] for odd $n \geq 5$, but the analysis there assumes fall-off of initial data near spatial infinity incompatible with the Einstein constraints[6].

We have:

THEOREM 2.6.4 *Consider smooth initial data $(\mathring{h}, \mathring{K}, \mathring{A}, \mathring{E}, \psi_0, \psi_1)$ on $\mathbb{R}^n$, $n \geq 3$, satisfying* (2.6.7) *and* (2.6.8). *Let $N \in \mathbb{N}$, suppose that $N_n := N + [\frac{n+2}{2}] - 2 \geq 6 + 2[\frac{n+2}{2}]$, and set*

$$\begin{aligned} E_{N_n,\gamma}(0) = \ & \textstyle\sum_{0\leq \mathrm{i}\leq N_n} \Big(\big|\big|(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I h^1_0\big|\big|^2_{L^2} + \Big|\Big|(1+r)^{1/2+\gamma+|I|}\nabla^I \mathring{K}\Big|\Big|^2_{L^2} \\ & + \Big|\Big|(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I \mathring{A}\Big|\Big|^2_{L^2} + \Big|\Big|(1+r)^{1/2+\gamma+|I|}\nabla^I \mathring{E}\Big|\Big|^2_{L^2} \\ & + \|(1+r)^{1/2+\gamma+|I|}\nabla\, \nabla^I\psi_0\|_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\nabla^I\psi_1\|_{L^2}\Big) \,. \end{aligned} \tag{2.6.13}$$

There exist constants $\varepsilon_0 > 0$ and $\gamma_0(\varepsilon_0)$, with $\gamma_0(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$, such that if

$$\sqrt{E_{N_n,\gamma}(0)} + m \leq \varepsilon_0, \tag{2.6.14}$$

for some $\gamma > \gamma_0$, then the maximal globally hyperbolic development of the initial data is geodesically complete.

The proof by Lindblad and Rodnianski is an ingenious and intricate analysis of the coupling between the wave-coordinates gauge and the evolution equations. One makes a clever guess of how the fields decay in space and time, encoded in the following weighted energy functional,

$$\begin{aligned} \mathscr{E}^{\mathrm{Matter}}_{N_n}(t) = \sup_{0\leq\tau\leq t} & \sum_{Z\in\mathscr{Z},|I|\leq N_n} \\ & \int_{\Sigma_\tau} \Big(\big|\partial Z^I h^1\big|^2 + \big|\partial Z^I A\big|^2 + \big|\partial Z^I \psi\big|^2\Big)\, w(q)\, d^n x\,, \end{aligned} \tag{2.6.15}$$

where

$$w(q) = \left\{ \begin{array}{ll} 1 + (1+|q|)^{1+2\gamma}, & q > 0; \\ 1 + (1+|q|)^{-2\mu}, & q < 0, \end{array} \right. \tag{2.6.16}$$

with $q = r - t$, $\mu > 0$ and $0 < \gamma < 1$. Here the Z's are the following generators of the conformal Lorentz group, first used to study the decay of solutions of the Minkowski wave equation by Klainerman [267]

$$\partial_\alpha, \quad x_\alpha \frac{\partial}{\partial x^\beta} - x_\beta \frac{\partial}{\partial x^\alpha}\,, \quad x^\alpha \frac{\partial}{\partial x^\alpha}\,, \tag{2.6.17}$$

[5]Those works build upon [267, 268]; however the structure conditions in [267, 268] are not compatible with the Einstein equations.

[6]In [241, 290] compactly supported data are considered. In the theorem for general quasi-linear systems given in [106] the initial data are in a Sobolev space which requires fall-off at infinity faster than $r^{-n-3/2}$. This should be compared with a fall-off of $g_{\mu\nu} - \eta_{\mu\nu}$ *not faster* than r^{-n+2} required by the positive energy theorem.

One argues by continuity: one chooses $0 < \delta < \frac{1}{4}$, and one considers the maximal time T so that the inequality

$$\mathcal{E}^{\text{Matter}}_{N_n}(t) \leq 2C_{N_n}\varepsilon^2(1+t)^{2\delta} \tag{2.6.18}$$

holds for $0 \leq t \leq T$. A sophisticated method, using the Klainerman-Sobolev inequalities [267], together with a new weighted energy inequality, allows one to show that (2.6.18) then holds for $0 \leq t \leq T$ with a smaller constant on the right-hand-side, contradicting maximality of T, and thus proving global existence.

A long standing question in the study of asymptotically flat spacetimes is that of the existence of an asymptotic expansion of the metric as one recedes to infinity along outgoing null cones, see [209, 212, 344]. Neither the analysis of [296, 297], nor that in [112, 270], provides sufficient information. A breakthrough result is due to Hintz and Vasy [235], who prove that initial data which are polyhomogeneous at spatial infinity lead to spacetimes which are polyhomogeneous at null infinity. Their proof provides in fact yet another proof of stability of Minkowski spacetime. A precise relation between the logarithmic terms at null infinity with the asymptotic behaviour of data at spatial infinity remains lacking, it would be of interest to settle this.

2.6.4 The mixmaster conjecture

The *most important question* in the study of the Cauchy problem is that of the *global properties of the resulting spacetimes*. So far we have seen examples of geodesically complete solutions (e.g., small perturbations of Minkowski spacetime), or all-encompassing singularities (e.g., generic Bianchi models), or of Cauchy horizons (e.g., Taub–NUT metrics). The geodesically complete solutions are satisfying but dynamically uninteresting, while the strong cosmic censorship conjecture expresses the hope that Cauchy horizons will almost never occur. So it appears essential to have a good understanding of the remaining cases, presumably corresponding to singularities. Belinski, Khalatnikov and Lifschitz [50] suggested that, near singularities, at each space point the dynamics of the gravitational field resembles that of generic Bianchi metrics, as described in Section 2.3.1. Whether or not this is true, and in which sense, remains to be seen; in any case the idea, known as *the BKL* conjecture, provided guidance — and still does — to a significant body of research on general relativistic singularities; see [29, 182, 183, 196] and references therein. This then leads to the mathematical challenge of making sense of the widely abused soundbite:

the singularity in generic gravitational collapse is spacelike, local, and oscillatory.

Here

1. *spacelike* is supposed to mean that strong cosmic censorship holds.

   The term

2. *local* refers to the idea that, near generic singularities, there should exist coordinate systems in which the metric asymptotes to a solution of equations in which *spatial* derivatives of appropriately chosen fields have been neglected.[7]

   Finally,

3. *oscillatory* is supposed to convey the idea that the approximate solutions will actually be provided by the Bianchi IX metrics.

While BKL put emphasis on Bianchi IX models, some other authors seem to favor Bianchi $\mathrm{VI}_{-1/9}$, or not-necessarily Bianchi, oscillations [50, 55, 183, 232, 405]. This line of thinking can be summarised in the following, somewhat loose, conjecture:

Conjecture 2.6.5 (Mixmaster conjecture) *Let $n+1 \leq 10$. There exist open sets of maximal globally hyperbolic vacuum metrics for which some natural geometric variables undergo oscillations of increasing complexity along incomplete inextendible geodesics.*

The *BKL conjecture* would be a more precise version of the above, claiming moreover genericity of the behavior, and pointing out to the Bianchi IX dynamics as the right model. Those properties are even more speculative, and have therefore been ignored in Conjecture 2.6.5.

The only examples so far of oscillatory singularities *which are not spatially homogeneous* are due to Berger and Moncrief [56]. In that work a solution-generating transformation has been applied to Bianchi IX metrics, resulting in non-homogeneous solutions governed by the "oscillatory" functions arising from a non-Taub Bianchi IX metric. The resulting metrics are parameterised by a finite number of parameters and have at least one but no more than two Killing vectors. The analysis complements the numerical evidence for oscillatory behavior in $U(1)$ symmetric models presented in [57].

How this scenario is affected by the occurrence of "spikes" observed in some models [158, 293–295, 371, 383], or by the "weak-null" singularities [173] predicted in the "mass inflation" scenario of Israel and Poisson [352], remains to be seen. It has been suggested that the oscillatory behavior disappears in spacetime-dimensions higher than ten [183, 191], and large families of non-oscillatory solutions with singularities have indeed been constructed in [184].

The main rigorous evidence for a relatively large class of vacuum[8] spacetimes with singularities which are *spacelike and local* in the sense above is Ringström's

[7] The resulting truncated equations should then presumably resemble the equations satisfied by spatially homogeneous metrics. However, different choices of quantities which are expected to be time-independent will lead to different choices of the associated notion of homogeneity; for instance, in [50] the types Bianchi VIII and IX are singled out; the notion of genericity of those types within the Bianchi A class is read from Table 2.1 as follows: "something that can be non-zero is more generic than something that is". On the other hand, the analysis in [232] seems to single out Bianchi $\mathrm{VI}_{-1/9}$ metrics.

[8] See, however, [17, 184] for a class of spacetimes with sources; [184] also covers vacuum in space dimensions $n \geq 10$.

Theorem 2.3.5, p. 47, describing generic Gowdy metrics, but the resulting singularities are *not* oscillatory. A large class of metrics with a similar non-oscillatory behaviour and without any isometries has been constructed in [145, 277]. This is not in contradiction with the conjecture, since neither the Gowdy metrics nor the metrics in [145, 277] arise from generic initial data. As such, the numerical studies of [53] suggest that the switching-on of the twist constants in Gowdy metrics will indeed generically lead to some kind of oscillatory behavior.

Chapter 3

The constraint equations

A set (M, g, K), where (M, g) is a Riemannian manifold, and K is a symmetric tensor field on M, is called a *vacuum initial data set* if the vacuum constraint equations (1.4.19) and (1.4.24) hold:

$$D_j K^j{}_k = D_k K^j{}_j \,, \tag{3.0.1a}$$

$$R(g) = 2\Lambda + |K|_g^2 - (\mathrm{tr}_g K)^2 \,. \tag{3.0.1b}$$

Here, as before, Λ is the cosmological constant.

Note that we have been writing h for the initial data metric in Chapter 1, but we will use the symbol g for this metric throughout the current chapter.

3.1 The conformal method

The object of this section is to present the *conformal method* for constructing solutions of (3.0.1). This method works best when $\mathrm{tr}_g K$ is constant throughout M, which is assumed to be connected:

$$\partial_i (\mathrm{tr}_g K) = 0 \,. \tag{3.1.1}$$

(We shall see shortly that (3.1.1) leads to a decoupling of the equations (3.0.1), in a sense which will be made precise.) Hypersurfaces M in a spacetimes $\mathscr{M}$ satisfying (3.1.1) are known as *constant mean curvature (CMC) surfaces*. Equation (3.1.1) is sometimes viewed as a "gauge condition", in the following sense: if we require (3.1.1) to hold on all hypersurfaces M_τ within a family of hypersurfaces in the spacetime, then this condition restricts the freedom of choice of the associated time function t which labels those hypersurfaces. Unfortunately there exist spacetimes in which no CMC hypersurfaces exist [33, 249]. Now, the conformal method is the only method known which produces *all* solutions satisfying a reasonably mild "gauge condition", it is therefore regrettable that condition (3.1.1) is a restrictive one.

INCIDENTALLY: The conformal method seems to go back to Lichnerowicz [291] (see [93] for the history of the problem), except that Lichnerowicz proposes a different treatment of the vector constraint equation there. The associated analytical aspects have been implemented in various contexts: asymptotically flat [113], asymptotically hyperbolic [9, 11, 12], or spatially compact [243]; see also [39, 105, 244, 416].

There exist a few other methods for constructing solutions of the constraint equations which do not require constant mean curvature: the "thin sandwich approach" of Baierlain, Sharp and Wheeler [27], further studied in [38, 49]; the gluing methods of Corvino and Schoen [84, 130, 132, 144, 161, 386] presented in Section 3.5; the conformal gluing technique of Joyce [260], as extended by Isenberg, Mazzeo and Pollack [247, 249]; the quasi-spherical construction of Bartnik [35, 393] and its extension due to Smith and Weinstein [396, 397]. One can also use the implicit function theorem, or variations thereof [102, 250, 255], to construct solutions of the constraint equations for which (3.1.1) does not necessarily hold. In [39] the reader will find a presentation of alternative approaches to constructing solutions of the constraints, covering work done up to 2003. □

3.1.1 The Yamabe problem

At the heart of the conformal method lies the *Yamabe problem*. From the general relativistic point of view, this correspond to special initial data where K vanishes; such initial data are called *time symmetric*. For such data (3.0.1b) becomes

$$R(g) = 2\Lambda\,. \tag{3.1.2}$$

In other words, g is a metric of constant scalar curvature.

There is a classical method, usually attributed to Yamabe [415], which allows one to construct metrics satisfying (3.1.2) by conformal deformation: given a metric $\tilde g$ one sets

$$g_{ij} = \phi^{\frac{4}{n-2}}\tilde g_{ij}\,, \tag{3.1.3}$$

then (3.1.2) becomes

$$\Delta_{\widetilde g}\phi - \frac{n-2}{4(n-1)}\widetilde R\phi = -\frac{n-2}{2(n-1)}\Lambda\phi^{\frac{n+2}{n-2}}\,. \tag{3.1.4}$$

One thus obtains a metric of constant scalar curvature 2Λ when a strictly positive solution ϕ can be found.

Equation 3.1.4 is known as the *Yamabe equation*, and the problem of finding positive solutions of this equation on compact manifolds is known as the *Yamabe problem*. The final solution, that such deformations always exist when Λ is suitably restricted (we will return to this issue shortly), has been given by Schoen [387]. Previous key contributions include [23, 404], and a comprehensive review of the problem can be found in [287]. A completely different solution has been devised by Bahri [26]. A surprising development is the proof of existence of a dimensional-threshold for compactness of the set of solutions of the Yamabe equation on compact manifolds carrying positive scalar curvature [75, 76, 265]: compactness holds if and only if $n \le 24$.

The idea is then to do something similar in general relativity, exploiting the fact that the Yamabe problem has already been solved. For this we need, first, to understand the behaviour of the vector constraint equation under conformal transformations.

INCIDENTALLY: Regardless of whether the manifold is compact or not, the *Yamabe number* of a metric is defined as

$$Y(M,g) = \inf_{u\in C^\infty_c,\ u\not\equiv 0}\frac{\int_M\big(\frac{4(n-1)}{n-2}|du|^2_g + Ru^2\big)d\mu_g}{\big(\int_M u^{\frac{2n}{n-2}}d\mu_g\big)^{(n-2)/n}}\,. \tag{3.1.5}$$

where C_c^∞ denotes the space of compactly supported smooth functions. The number $Y(M,g)$ depends only upon the conformal class of g: Indeed, if $\tilde g_{ij} = u^{4/(n-2)} g_{ij}$, then on the set where $u > 0$ we have $d\mu_{\tilde g} = u^{2n/(n-2)} d\mu_g$, and (C.2.16), p. 268, gives

$$\begin{aligned}\int_{u>0} \tilde R \, d\mu_{\tilde g} &= \int_{u>0} \left(Ru^2 - \frac{4(n-1)}{n-2} u \Delta_g u \right) d\mu_g \\ &= \int_{u>0} \left(Ru^2 + \frac{4(n-1)}{n-2} |du|_g^2 \right) d\mu_g \,. \end{aligned} \tag{3.1.6}$$

The conformal invariance of $Y(M,g)$ readily follows.

If $Y(M,g) > 0$ we say that g is in the *positive Yamabe class*, etc.

When M is compact, one can show that there exists a conformal rescaling so that $\widetilde R$ is strictly positive [287] if and only if g is in the positive Yamabe class, similarly for the zero and negative Yamabe class cases. We note that non-existence of positive conformal factors relating two metrics whose scalar curvatures are constant but have different signs follows immediately by integrating equation (3.1.4) over M.)

Note that $Y(M,g)$ is positive if and only if we have a Sobolev-type inequality: for all functions with compact support,

$$\|u\|^2_{L^{\frac{2n}{n-2}}} \le C \int_M \left(\frac{4(n-1)}{n-2} |Du|^2 + Ru^2 \right) , \tag{3.1.7}$$

and the optimal constant is then $C = (Y(M,g))^{-1}$. See also [3]. □

3.1.2 The vector constraint equation

As is made clear by the name, the *conformal method* exploits the properties of (3.0.1) under conformal transformations: consider a metric $\widetilde g$ related to g by a conformal rescaling:

$$\widetilde g_{ij} = \phi^\ell g_{ij} \quad \Longleftrightarrow \quad \widetilde g^{ij} = \phi^{-\ell} g^{ij} \,. \tag{3.1.8}$$

This implies

$$\begin{aligned}\widetilde\Gamma^i{}_{jk} &= \frac12 \widetilde g^{im} (\partial_j \widetilde g_{km} + \partial_k \widetilde g_{jm} - \partial_m \widetilde g_{jk}) \\ &= \frac12 \phi^{-\ell} g^{im} (\partial_j(\phi^\ell g_{km}) + \partial_k(\phi^\ell g_{jm}) - \partial_m(\phi^\ell g_{jk})) \\ &= \Gamma^i{}_{jk} + \frac{\ell}{2\phi} (\delta^i_k \partial_j \phi + \delta^i_j \partial_k \phi - g_{jk} D^i \phi) \,, \end{aligned} \tag{3.1.9}$$

where, as before, D denotes the covariant derivative of g.

We start by analysing what happens with (3.0.1a). The idea is to gain insight into this equation by decomposing K in its trace-free part and a trace part. For this, let $\widetilde D$ denote the covariant derivative operator of the metric $\widetilde g$, and consider any trace-free symmetric tensor field $\widetilde L^{ij}$, we have

$$\begin{aligned}\widetilde D_i \widetilde L^{ij} &= \partial_i \widetilde L^{ij} + \widetilde\Gamma^i{}_{ik} \widetilde L^{kj} + \widetilde\Gamma^j{}_{ik} \widetilde L^{ik} \\ &= D_i \widetilde L^{ij} + (\widetilde\Gamma^i{}_{ik} - \Gamma^i{}_{ik}) \widetilde L^{kj} + (\widetilde\Gamma^j{}_{ik} - \Gamma^j{}_{ik}) \widetilde L^{ik} \,.\end{aligned}$$

Now, from (3.1.9) we obtain

$$\begin{aligned}\widetilde{\Gamma}^i{}_{ik} &= \Gamma^i{}_{ik} + \frac{\ell}{2\phi}(\delta^i_k\partial_i\phi + \delta^i_i\partial_k\phi - g_{ik}D^i\phi) \\ &= \Gamma^i{}_{ik} + \frac{n\ell}{2\phi}\partial_k\phi\,, \qquad (3.1.10)\end{aligned}$$

and we are assuming that we are in dimension n. As $\widetilde{L}$ is traceless we obtain

$$\begin{aligned}\widetilde{D}_i\widetilde{L}^{ij} &= D_i\widetilde{L}^{ij} + \frac{n\ell}{2\phi}\partial_k\phi\widetilde{L}^{kj} + \frac{\ell}{2\phi}(\delta^j_k\partial_i\phi + \delta^j_i\partial_k\phi - \underbrace{g_{ik}D^j\phi)\widetilde{L}^{ik}}_{\sim g_{ik}\widetilde{L}^{ik}=0} \\ &= D_i\widetilde{L}^{ij} + \frac{(n+2)\ell}{2\phi}\partial_k\phi\widetilde{L}^{kj} \\ &= \phi^{-\frac{(n+2)\ell}{2}}D_i(\phi^{\frac{(n+2)\ell}{2}}\widetilde{L}^{ij})\,. \qquad (3.1.11)\end{aligned}$$

It follows that

$$\widetilde{D}_i\widetilde{L}^{ij} = 0 \quad \Longleftrightarrow \quad D_i(\phi^{\frac{(n+2)\ell}{2}}\widetilde{L}^{ij}) = 0\,. \qquad (3.1.12)$$

This observation leads to the following:

Suppose that the CMC condition (3.1.1) holds, set

$$L^{ij} := K^{ij} - \frac{\mathrm{tr}_g K}{n}g^{ij}\,. \qquad (3.1.13)$$

Then L^{ij} is symmetric and trace-free whenever K^{ij} satisfies the vector constraint equation (3.0.1a).

On the other hand, let τ be a constant, and let $\widetilde{L}^{ij}$ be symmetric, trace-free, and $\widetilde{g}$*–divergence free*: by definition, this means that

$$\widetilde{D}_i\widetilde{L}^{ij} = 0\,.$$

Setting

$$\begin{aligned}L^{ij} &:= \phi^{\frac{(n+2)\ell}{2}}\widetilde{L}^{ij} \qquad (3.1.14a)\\ K^{ij} &:= L^{ij} + \frac{\tau}{n}g^{ij}\,, \qquad (3.1.14b)\end{aligned}$$

the tensor field K^{ij} satisfies (3.0.1a). (A convenient choice of ℓ is given in (3.1.16) below).

More generally, assuming neither vacuum nor $d(\mathrm{tr}_g K) = 0$, with the rescaling $\widetilde{g}_{ij} = \phi^\ell g_{ij}$ and with the definitions (3.1.14) we will have

$$\begin{aligned}8\pi J^j &:= D_i(K^{ij} - \mathrm{tr}_g K g^{ij}) \\ &= D_i(\phi^{\frac{(n+2)\ell}{2}}\widetilde{L}^{ij}) - \frac{n-1}{n}D^j\tau \\ &= \phi^{\frac{(n+2)\ell}{2}}\tilde{D}_i\widetilde{L}^{ij} - \frac{n-1}{n}\phi^{-\ell}\tilde{D}^j\tau\,. \qquad (3.1.15)\end{aligned}$$

With the choice

$$\ell = -\frac{4}{n-2}\,, \qquad (3.1.16)$$

motivated by (3.1.3), this can also be written as the following equation for $\tilde{L}$ when τ and J^i have been prescribed:

$$\tilde{D}_i\widetilde{L}^{ij} = 8\pi\phi^{\frac{2(n+2)}{n-2}}J^i + \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\tilde{D}^j\tau\,. \qquad (3.1.17)$$

3.1.3 The scalar constraint equation

To analyse the scalar constraint equation (3.0.1b) we shall use the formula (C.2.14) of Appendix C [1]

$$R(g)\phi^{-\ell} = \widetilde{R} + \frac{(n-1)\ell}{\phi}\Delta_{\widetilde{g}}\phi - \frac{(n-1)\ell\{(n-2)\ell+4\}}{4\phi^2}|d\phi|^2_{\widetilde{g}}\,, \qquad (3.1.18)$$

where $\widetilde{R}$ is the scalar curvature of $\widetilde{g}$. Clearly it is convenient to choose

$$\ell = -\frac{4}{n-2}\,, \qquad (3.1.19)$$

as then the last term in (3.1.18) drops out. In order to continue we use (3.1.14) to calculate

$$\begin{aligned}
|K|^2_g - (\mathrm{tr}_g K)^2 &= g_{ik}g_{jl}K^{ij}K^{kl} - \tau^2 \\
&= g_{ik}g_{jl}(L^{ij} + \frac{\tau}{n}g^{ij})(L^{kl} + \frac{\tau}{n}g^{kl}) - \tau^2 \\
&= \underbrace{g_{ik}}_{=\phi^{-\ell}\widetilde{g}_{ik}} \ g_{jl} \ \underbrace{L^{ij}}_{=\phi^{(n/2+1)\ell}\widetilde{L}^{ij}} \ L^{kl} - \tau^2(1-\frac{1}{n}) \\
&= \phi^{n\ell}\widetilde{g}_{ik}\widetilde{g}_{jl}\widetilde{L}^{ij}\widetilde{L}^{kl} - \tau^2(1-\frac{1}{n})\,,
\end{aligned}$$

giving thus

$$|K|^2_g - (\mathrm{tr}_g K)^2 = \phi^{n\ell}|\widetilde{L}|^2_{\widetilde{g}} - \frac{n-1}{n}\tau^2\,. \qquad (3.1.20)$$

Equations (3.0.1b), (3.1.9) and (3.1.20) with ℓ given by (3.1.19) finally yield

$$\Delta_{\widetilde{g}}\phi - \frac{n-2}{4(n-1)}\widetilde{R}\phi = -\tilde{\sigma}^2\phi^{\frac{2-3n}{n-2}} + \beta\phi^{\frac{n+2}{n-2}}\,, \qquad (3.1.21)$$

where

$$\tilde{\sigma}^2 := \frac{n-2}{4(n-1)}|\widetilde{L}|^2_{\widetilde{g}}\,, \quad \beta := \left[\frac{n-2}{4n}\tau^2 - \frac{n-2}{2(n-1)}\Lambda\right]. \qquad (3.1.22)$$

In dimension $n=3$ this equation is known as the *Lichnerowicz equation*:

$$\boxed{\Delta_{\widetilde{g}}\phi - \frac{\widetilde{R}}{8}\phi = -\tilde{\sigma}^2\phi^{-7} + \beta\phi^5\,.} \qquad (3.1.23)$$

We note that $\tilde{\sigma}^2$ is positive, as the notation suggests, while β is a constant, non-negative if $\Lambda = 0$, or in fact if $\Lambda \le 0$.

The strategy is now the following: let $\widetilde{g}$ be a given Riemannian metric on M, and let $\widetilde{L}^{ij}$ be any symmetric transverse $\widetilde{g}$-divergence free tensor field. We

[1] Note that the relationship between g and $\tilde{g}$ in Appendix C is formally identical to the one here, namely (3.1.8), but in (C.2.14) the Ricci scalar $\tilde{R}$ is calculated in terms of $\Delta_g\phi$. Here we need R in terms of $\Delta_{\widetilde{g}}\phi$. For this we write $g_{ij} = \phi^{\tilde{\ell}}\tilde{g}_{ij}$, and use (C.2.14) with g there interchanged with $\widetilde{g}$ and ℓ there replaced by $\tilde{\ell}$. This gives (3.1.18), keeping in mind that $\tilde{\ell}$ is the negative of ℓ in the current section.

then solve (if possible) (3.1.21) for ϕ, and obtain a vacuum initial data set by calculating g using (3.1.8), and by calculating K using (3.1.14).

More generally, the energy density of matter fields is related to the geometry through the formula

$$16\pi\mu \quad := \quad R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 - 2\Lambda\,. \tag{3.1.24}$$

If μ has been prescribed, this becomes an equation for ϕ

$$\Delta_{\widetilde{g}}\phi - \frac{n-2}{4(n-1)}\widetilde{R}\phi \quad = \quad -\tilde{\sigma}^2\phi^{\frac{2-3n}{n-2}} + \beta\phi^{\frac{n+2}{n-2}} - \frac{4\pi(n-2)}{(n-1)}\phi^{\frac{n+2}{n-2}}\mu\,. \tag{3.1.25}$$

See Section 3.1.9, p. 101 for comments on the powers of ϕ arising in (3.1.25).

3.1.4 The vector constraint equation on compact manifolds

In order to solve the Lichnerowicz equations we need the transverse-traceless tensor (TT tensor) field $\widetilde{L}$, and so to obtain an exhaustive construction of CMC initial data sets we have to give a prescription for constructing such tensors. It is a non-trivial fact [72] (compare [188]) that the space of TT tensors is always infinite dimensional in dimension larger than two.

EXAMPLE 3.1.3 An *ad hoc* example of TT tensor field on *three-dimensional non-conformally flat* manifolds is provided by the Bach tensor, see Appendix C.5. Another one is provided by the Ricci tensor on manifolds with constant scalar curvature.

Further examples are provided by metrics with symmetries, as observed by Bobby Beig (private communication): Suppose that X and Y are orthogonal Killing vectors, then the tensor field

$$K_{ij} = X_{(i}Y_{j)} \tag{3.1.26}$$

is readily seen to be transverse ($D_iK^i{}_j = 0$) and traceless.

A systematic prescription how to construct TT tensors has been given by York: here one starts with an arbitrary symmetric traceless tensor field $\tilde{B}^{ij}$, which will be referred to as the *seed field*. One then writes

$$\widetilde{L}^{ij} = \tilde{B}^{ij} + \tilde{C}(Y)^{ij}\,, \tag{3.1.27}$$

where $\tilde{C}(Y)$ is the *conformal Killing operator*:

$$\tilde{C}(Y)^{ij} := \widetilde{D}^iY^j + \widetilde{D}^jY^i - \frac{2}{n}\widetilde{D}_kY^k\widetilde{g}^{ij}\,. \tag{3.1.28}$$

The requirement that $\widetilde{L}^{ij}$ be divergence free becomes then an equation for the vector field Y:

$$\widetilde{D}_i\widetilde{L}^{ij} = 0 \quad \Longleftrightarrow \quad \widetilde{L}(Y)^j := \widetilde{D}_i(\widetilde{D}^iY^j + \widetilde{D}^jY^i - \frac{2}{n}\widetilde{D}_kY^k\widetilde{g}^{ij}) = -\widetilde{D}_i\tilde{B}^{ij}\,. \tag{3.1.29}$$

INCIDENTALLY: While (3.1.29) looks complicated at first sight, it is rather natural: we want to produce transverse traceless tensors by solving an elliptic differential equation. Since the condition of being divergence-free is already a first order equation, and it is not elliptic, then the lowest possible order of such an equation will be two. Now, the divergence operation turns two-contravariant tensor fields to vector fields, so the most straightforward way of ensuring ellipticity is to seek an equation for a vector field. The simplest object that we obtain by differentiating a vector field is the tensor field $\widetilde{D}^i Y^j$; in order to achieve the desired symmetries we need to symmetrise and remove the trace, which leads to the conformal Killing operator (3.1.28). □

The operator L defined in (3.1.29) is known as the *conformal vector Laplacian.* Equation (3.1.29) is a second order linear partial differential equation for Y, the solvability of which can be easily analysed. In this section we shall consider spatially compact manifolds M. We will give an existence proof for (3.1.29):

THEOREM 3.1.5 *For any smooth symmetric traceless tensor field $\tilde{B}^{ij}$ there exists a smooth vector field Y such that (3.1.29) holds.*

PROOF: We provide a sketch of the proof here; a more detailed exposition can be found in the next section. Recall that a *conformal Killing vector* for the metric $\widetilde{g}$ is a nontrivial solution of the equation $\widetilde{C}(Y) = 0$. When $(M, \widetilde{g})$ does not admit any conformal Killing vectors, Theorem 3.1.5 follows immediately from Theorem 3.1.11 below, together with (3.1.39) and (3.1.41). Indeed, surjectivity means precisely that the equation

$$\widetilde{L}(Y) = Z$$

has a solution for any Z.

Let H_k be as defined at the beginning of Section 3.1.5, see (3.1.31) below, When conformal Killing vectors exist, then for any $k \geq 0$ the image of

$$\widetilde{L} : H_{k+2} \to H_k$$

is the L^2-orthogonal of the kernel of $\widetilde{L}^*$. Indeed, let Z be orthogonal to that image. We have

$$\begin{aligned}
\int_M Z_i \widetilde{D}_j \tilde{B}^{ij} &= -\int_M \widetilde{D}_j Z_i \tilde{B}^{ij} \\
&= -\frac{1}{2}\int_M (\widetilde{D}_j Z_i + \widetilde{D}_i Z_j)\tilde{B}^{ij} \qquad (\tilde{B} \text{ is symmetric}) \\
&= -\frac{1}{2}\int_M \underbrace{(\widetilde{D}_j Z_i + \widetilde{D}_i Z_j - \frac{2}{n}\widetilde{D}_k Z^k \widetilde{g}_{ij})}\tilde{B}^{ij} \qquad (\tilde{B} \text{ is trace-free}) .
\end{aligned} \tag{3.1.30}$$

This will vanish for all symmetric trace-free tensor fields $\tilde{B}$ if and only if the underbraced term vanishes; this is to say, if Z is a conformal Killing vector field.

Since $\widetilde{L}$ is formally self-adjoint (see (3.1.38) below), we conclude that the kernel of $\widetilde{L}^* = \widetilde{L}$ is the space of conformal Killing vectors, and thus the image of $\widetilde{L}$ is exactly the subspace of those vector fields in H_k which are L^2-orthogonal to the space of conformal Killing vectors.

It also follows from (3.1.30) that right-hand side of (3.1.29) is orthogonal to the space of Killing vectors. We conclude that $-\widetilde{D}_j\tilde{B}^{ij}$ lies in the image of $\widetilde{L}$, and so there exist solutions of (3.1.29). These solutions are not unique since conformal Killing vectors are in the kernel of the operator $\tilde{L}$, but the non-uniqueness does not change the tensor field $\tilde{L}^{ij}$ defined in (3.1.27). □

A property essentially equivalent to Theorem 3.1.5 is the existence of the *York splitting*, also known in the mathematical literature as the *Berger-Ebin splitting*:

THEOREM 3.1.6 *On any compact Riemannian manifold* (M,g) *the space of symmetric tensors, say* $\Gamma S^2 M$*, splits* L^2*-orthogonally as*

$$\Gamma S^2 M = C^\infty g \oplus TT \oplus \mathrm{Im} C\,,$$

where $C^\infty g$ *are tensors proportional to the metric,* TT *denotes the space of transverse traceless tensors, and* $\mathrm{Im}C$ *is the image of the conformal Killing operator defined in (3.1.28).*

PROOF: : Given any symmetric two-covariant tensor field A let ψ denote the trace of A divided by n, set

$$B_{ij} = A_{ij} - \psi g_{ij}\,.$$

Then B_{ij} is symmetric and traceless. Similarly to (3.1.29), we let Y be any solution of the equation

$$L(Y)^i = D_j\tilde{B}^{ij}\,.$$

Here, of course, $C(Y)^{ij} := D^iY^j + D^jY^i - \frac{2}{n}D_kY^kg^{ij}$ and $L(Y)^i = D_iC(Y)^{ij}$. Then $B_{ij} - C(Y)_{ij}$ is transverse and traceless, and we have indeed

$$A_{ij} = \underbrace{\psi g_{ij}}_{\in C^\infty \times g} + \underbrace{B_{ij} - C(Y)_{ij}}_{\in TT} + \underbrace{C(Y)_{ij}}_{\in \mathrm{Im}C}\,.$$

The L^2-orthogonality of the factors is easily verified; compare (3.1.30). □

3.1.5 Some linear elliptic theory

The main ingredients of the existence proof which we will present shortly are the following:

1. *Function spaces*: one uses the spaces H_k, $k \in \mathbb{N}$, defined as the completion of the space of smooth tensor fields on M with respect to the norm

$$\|u\|_k := \sqrt{\sum_{0\le\ell\le k}\int_M |D^\ell u|^2 d\mu}\,, \tag{3.1.31}$$

where $D^\ell u$ is the tensor of ℓ-th covariant derivatives of u with respect to some covariant derivative operator D. For compact manifolds[2] this space is identical with that of fields in L^2 such that their distributional derivatives of order less than or equal to k are also in L^2. Again for compact manifolds, different choices of measure $d\mu$ (as long as it remains absolutely continuous with respect to the coordinate one), of the tensor norm $|\cdot|$, or of the connection D, lead to the same space, with equivalent norm.

INCIDENTALLY: Recall that if $u \in L^2$ then $\partial_i u = \rho_i$ in a distributional sense if for every smooth compactly supported vector field we have

$$\int_M X^i \rho_i = -\int_M D_i X^i u\,.$$

More generally, let L be a linear differential operator of order m and let L^* be its *formal* L^2 adjoint, which is the operator obtained after differentiating by parts:

$$\int_M \langle u, L^* v\rangle := \int_M \langle Lu, v\rangle\,, \quad \forall u, v \in C^m_c\,;$$

here C^m_c is the space of C^m compactly supported fields. (Incidentally, the reader will note by comparing the last two equations that the formal adjoint of the derivative operator is *the negative* of the divergence operator.) Then, for $u \in L^1_{\rm loc}$ (this is the space of measurable fields u which are Lebesgue-integrable on any compact subset of the manifold), the distributional equation $Lu = \rho$ is said to hold if for all smooth compactly supported v's we have

$$\int_M \langle u, L^* v\rangle = \int_M \langle \rho, v\rangle\,.$$

One sometimes talks about *weak solutions* rather than distributional ones. □

The spaces H_k are Hilbert spaces with the obvious scalar product:

$$\langle u, v\rangle_k = \sum_{0\le \ell\le k} \int_M \langle D^\ell u, D^\ell v\rangle d\mu\,.$$

The Sobolev embedding theorem [25] asserts that H_k functions are, locally, of $C^{k'}$ differentiability class, where k' is the largest integer satisfying

$$k' < k - n/2\,. \tag{3.1.32}$$

On a compact manifold the result is true globally,

$$H_k \subset C^{k'}\,, \tag{3.1.33}$$

with the inclusion map being continuous:

$$\|u\|_{C^{k'}} \le C\|u\|_{H_k}\,. \tag{3.1.34}$$

[2]For non-compact manifolds this is not always the case, compare [24].

2. *Orthogonal complements in Hilbert spaces:* Let H be a Hilbert space, and let E be a *closed linear subspace* of H. Then (see, *e.g.*, [410]) we have the direct sum
$$H = E \oplus E^{\perp} \,. \tag{3.1.35}$$
This result is sometimes called *the projection theorem.*

3. *Rellich-Kondrashov compactness:* we have the obvious inclusion
$$H_k \subseteq H_{k'} \quad \text{if} \quad k > k' \,.$$
The *Rellich-Kondrashov theorem* (see, *e.g.*, [1, 25, 220, 278]) asserts that, on compact manifolds, this inclusion is *compact.* Equivalently,[3] if u_n is any sequence satisfying $\|u_n\|_k \le C$, and if $k' < k$, then there exists a subsequence u_{n_i} and $u_\infty \in H_k$ such that u_{n_i} converges to u_∞ in $H_{k'}$ topology as i tends to infinity.

4. *Elliptic regularity:* If $Y \in L^2$ satisfies $LY \in H_k$ in a distributional sense, with L — an elliptic operator of order m with smooth coefficients, then $Y \in H_{k+m}$, and Y satisfies the equation in the classical sense. Further, on compact manifolds for every k there exists a constant C_k such that
$$\|Y\|_{k+m} \le C_k(\|LY\|_k + \|Y\|_0) \,. \tag{3.1.36}$$

Our aim is to show that solvability of (3.1.29) can be easily studied using the above basic facts. We start by verifying ellipticity of L. Recall that the *symbol* σ of a linear partial differential operator L of the form
$$L = \sum_{0 \le \ell \le m} a^{i_1 \ldots i_\ell} D_{i_1} \ldots D_{i_\ell} \,,$$
where the $a^{i_1 \ldots i_\ell}$'s are linear maps from fibers of a bundle E to fibers of a bundle F, is defined as the map
$$T^*M \ni p \mapsto \sigma(p) := a^{i_1 \ldots i_m} p_{i_1} \ldots p_{i_m} \,.$$
Thus, every derivative D_i is replaced by p_i, and all terms other than the top order ones are ignored. An operator is said to be *elliptic*[4] if the symbol is an isomorphism of fibers for all $p \neq 0$. In our case (3.1.29) the operator L acts on vector fields and produces vector fields, with
$$TM \ni Y \to \sigma(p)(Y) = p_i(p^i Y^j + p^j Y^i - \frac{2}{n} p_k Y^k \widetilde{g}^{ij})\partial_j \in TM \,. \tag{3.1.37}$$
(The indices on p^i have been raised with the metric $\widetilde{g}$.) To prove bijectivity of $\sigma(p)$, $p \neq 0$, it suffices to verify that $\sigma(p)$ has trivial kernel. Assuming $\sigma(p)(Y) = 0$, a contraction with p_j gives
$$p_j p_i(p^i Y^j + p^j Y^i - \frac{2}{n} p_k Y^k \widetilde{g}^{ij}) = |p|^2 p_j Y^j (2 - \frac{2}{n}) = 0 \,,$$

[3] In this statement we have also made use of the *Tichonov-Alaoglu* theorem, which asserts that bounded sets in Hilbert spaces are weakly compact; *cf.*, *e.g.* [410].

[4] See [2, 334] for more general notions of ellipticity.

hence $p_jY^j = 0$ for $n > 1$ since $p \neq 0$. Contracting instead with Y_j and using the last equality we obtain

$$Y_j p_i(p^iY^j + p^jY^i - \frac{2}{n}p_kY^k\widetilde{g}^{ij}) = |p|^2|Y|^2 = 0\,,$$

and $\sigma(p)$ has no kernel, as desired.

Recall that the *formal adjoint* L^* of L is defined by integration by parts:

$$\int \langle u, Lv\rangle = \int \langle L^*u, v\rangle$$

for all smooth compactly supported fields u, v. (Note that the definition of a self-adjoint operator further requires an equality of domains, an issue which is, fortunately, completely ignored in the formal definition.)

Our next step is to show that the conformal vector Laplacian L is formally self-adjoint. For this, we continue by the calculation of L^*: So, let X and Y be smooth, or C^2, we have

$$\begin{aligned}
\int_M X_i L(Y)^i d\mu_g &= \int_M X_i D_j(D^iY^j + D^jY^i - \frac{2}{n}g^{ij}D_kY^k)\,d\mu_g \\
= \ & -\int_M D_jX_i(D^iY^j + D^jY^i - \frac{2}{n}g^{ij}D_kY^k)\,d\mu_g \\
= \ & -\frac{1}{2}\int_M (D_jX_i + D_jX_i)\underbrace{(D^iY^j + D^jY^i - \frac{2}{n}g^{ij}D_kY^k)}_{\text{symmetric in } i \text{ and } j}\,d\mu_g \\
= \ & -\frac{1}{2}\int_M (D_jX_i + D_jX_i - \frac{2}{n}D_kX^kg_{ij})\underbrace{(D^iY^j + D^jY^i - \frac{2}{n}g^{ij}D_kY^k)}_{\text{trace free}}\,d\mu_g \\
= \ & -\int_M (D_jX_i + D_jX_i - \frac{2}{n}D_kX^kg_{ij})D^iY^j\,d\mu_g \\
= \ & \int_M D^i(D_jX_i + D_jX_i - \frac{2}{n}D_kX^kg_{ij})Y^j\,d\mu_g \\
= \ & \int_M L(X)^jY_j\,d\mu_g\,.
\end{aligned} \tag{3.1.38}$$

We have thus shown that the conformal vector Laplacian is *formally self adjoint*, as claimed:

$$L^* = L\,. \tag{3.1.39}$$

We further note that the fourth line in (3.1.38) implies

$$\int_M Y_iL(Y)^i = -\frac{1}{2}\int_M |C(Y)|^2\,, \tag{3.1.40}$$

in particular if Y is C^2 then

$$L(Y) = 0 \quad \Longleftrightarrow \quad C(Y) = 0\,. \tag{3.1.41}$$

But most Riemannian manifolds have no conformal Killing vector fields [44]. We thus see that Riemannian manifolds for which L has a non-trivial kernel are very special.

REMARK 3.1.8 The existence of non-trivial conformal Killing vectors implies the existence of conformal isometries of (M,g). A famous theorem of Lelong-Ferrand – Obata [288, 339] (compare [282]) shows that, on compact manifolds in dimensions greater than or equal to three, there exists a conformal rescaling such that Y is a Killing vector, except if (M,g) is conformally isometric to S^n with a round metric. In the former case (the conformally rescaled) (M,g) has a non-trivial isometry group, which imposes restrictions on the topology of M, and forces g to be very special. For instance, the existence of non-trivial Lie group of isometries of a compact manifold implies that M admits an S^1 action, which is a serious topological restriction, and in fact is not possible for "most" topologies (see, *e.g.*, [201, 202], and also [203] and references therein for an analysis in dimension four). It is also true that even if M admits S^1 actions, then there exists an open and dense set of metrics, in a $C^{k(n)}$ topology, or in a $H^{k'(n)}$ topology, with appropriate $k(n)$, $k'(n)$ [44], for which no nontrivial solutions of the over-determined system of equations $C(Y)=0$ exist. □

In order to continue we shall need a somewhat stronger version of (3.1.36):

PROPOSITION 3.1.9 *Let L be an elliptic operator of order m on a compact manifold. If there are no non-trivial smooth solutions of the equation $L(Y)=0$, then (3.1.36) can be strengthened to*

$$\|Y\|_{k+m} \le C'_k \|L(Y)\|_k \,. \tag{3.1.42}$$

REMARK 3.1.10 Equation (3.1.42) implies that L has trivial kernel, which shows that the condition on the kernel is necessary.

PROOF: Suppose that the result does not hold, then for every $n \in \mathbb{N}$ there exists $Y_n \in H_{k+m}$ such that

$$\|Y_n\|_{k+m} \ge n \|L(Y_n)\|_k \,. \tag{3.1.43}$$

Multiplying Y_n by an appropriate constant if necessary we can suppose that

$$\|Y_n\|_{L^2} = 1\,. \tag{3.1.44}$$

The basic elliptic inequality (3.1.36) gives

$$\|Y_n\|_{k+m} \le C_2(\|L(Y_n)\|_k + \|Y_n\|_0) \le \frac{C_2}{n}\|Y_n\|_{k+m} + C_2\,,$$

so that for n such that $C_2/n \le 1/2$ we obtain

$$\|Y_n\|_{k+m} \le 2C_2\,.$$

It follows that Y_n is bounded in H_{k+m}; further (3.1.43) gives

$$\|L(Y_n)\|_k \le \frac{2C_2}{n}\,. \tag{3.1.45}$$

By the Rellich-Kondrashov compactness we can extract a subsequence, still denoted by Y_n, such that Y_n converges in L^2 to some vector field $Y_* \in H_{k+m}$. Continuity of the norm together with L^2 convergence implies that

$$\|Y_*\|_{L^2} = 1\,, \tag{3.1.46}$$

so that $Y_* \neq 0$. One would like to conclude from (3.1.45) that $L(Y_*) = 0$, but that is not completely clear because we do not know whether or not

$$L(Y_*) = \lim_{n\to\infty} L(Y_n)\,.$$

Instead we write the distributional equation: for every smooth X we have

$$\int_M \langle L(Y_n), X\rangle = \int_M \langle Y_n, L^*(X)\rangle\,.$$

Now, $L(Y_n)$ tends to zero in L^2 by (3.1.45), and Y_n tends to Y_* in L^2, so that passing to the limit we obtain

$$0 = \int_M \langle Y_*, L^*(X)\rangle\,.$$

It follows that Y_* satisfies $L(Y_*) = 0$ in a distributional sense. Elliptic regularity implies that Y_* is a smooth solution of $L(Y_*) = 0$, it is non-trivial by (3.1.46), a contradiction. □

We are ready to prove now:

THEOREM 3.1.11 *Let L be an elliptic partial differential operator of order m on a compact manifold and suppose that the equations $L(u) = 0$, $L^*(v) = 0$ have no non-trivial smooth solutions, where L^* is the formal adjoint of L. Then for any $k \geq 0$ the map*

$$L : H_{k+m} \to H_k$$

is an isomorphism.

PROOF: An element of the kernel is necessarily smooth by elliptic regularity, it remains thus to show surjectivity. We start by showing that the image of L is closed: let Z_n be a Cauchy sequence in $\operatorname{Im} L$, then there exists $Z_\infty \in L^2$ and $Y_n \in H_{k+m}$ such that

$$L(Y_n) = Z_n \stackrel{L^2}{\to} Z_\infty\,.$$

Applying (3.1.42) to $Y_n - Y_\ell$ we find that Y_n is Cauchy in H_{k+m}, therefore converges in H_{k+m} to some element $Y_\infty \in H_{k+m}$. By continuity of L the sequence $L(Y_n)$ converges to $LC(Y_\infty)$ in L^2, hence $Z_\infty = LY_\infty$, as desired.

Consider, first, the case $k = 0$. By the orthogonal decomposition theorem we have now

$$L^2 = \operatorname{Im} L \oplus (\operatorname{Im} L)^\perp\,,$$

and if we show that $(\operatorname{Im} L)^\perp = \{0\}$ we are done. Let, thus, $Z \in (\operatorname{Im} L)^\perp$, this means that

$$\int_M \langle Z, L(Y)\rangle = 0 \tag{3.1.47}$$

for all $Y \in H_{m+2}$. In particular (3.1.47) holds for all smooth Y, which implies that $L^*(Z) = 0$ in a distributional sense. Now, the symbol of L^* is the transpose of the symbol of L, which shows that L^* is also elliptic. We can thus use elliptic regularity to conclude that Z is smooth, and $Z = 0$ follows.

The result in L^2 together with elliptic regularity immediately imply the result in H_k. □

3.1.6 The scalar constraint equation on compact manifolds, $\tau^2 \geq \frac{2n}{(n-1)}\Lambda$

Theorem 3.1.11, together with Equation (3.1.41) and Remark 3.1.8, gives a reasonably complete description of the solvability of (3.1.29). We simply note that if $\tilde{B}^{ij}$ there is smooth, then the associated solution will be smooth by elliptic regularity. To finish the presentation of the conformal method we need to address the question of existence of solutions of the Lichnerowicz equation (3.1.21).

A complete description can be obtained when the constant β defined in (3.1.22) satisfies

$$\beta \equiv \left[\frac{n-2}{4n}\tau^2 - \frac{n-2}{2(n-1)}\Lambda\right] \geq 0\,, \tag{3.1.48}$$

this will certainly be the case if $\Lambda \leq 0$. In this case, to emphasise positivity we will write

$$\frac{n-2}{4n}\tau_\Lambda^2$$

for β, thus rewriting (3.1.21) as

$$\Delta_{\tilde{g}}\phi - \frac{n-2}{4(n-1)}\widetilde{R}\phi = -\tilde{\sigma}^2\phi^{\frac{2-3n}{n-2}} + \frac{n-2}{4n}\tau_\Lambda^2\phi^{\frac{n+2}{n-2}}\,. \tag{3.1.49}$$

As already pointed out in Section 3.1.1, the case $\sigma = 0$ corresponds to the so-called *Yamabe* equation; in this case solutions of (3.1.49) produce metrics with constant scalar curvature $-(n-1)\tau_\Lambda^2$. We will take it for granted that one can first deform the metric conformally so that

$$\boxed{\tilde{R} \text{ is constant,}}$$

and we will assume that this has been done. It should be recognised that making use of the solution of the Yamabe problem sweeps the real difficulties under the carpet. Nevertheless, there remains some work to do even after the Yamabe part of the problem has been solved.

In what follows we will assume smoothness of all objects involved. More recently, these equations have been studied with metrics of low differentiability [90, 308]; this was motivated in part by work on the evolution problem for "rough initial data" [272–274, 398]. Boundary value problems for the constraint equations, with nonlinear boundary conditions motivated by black holes, were considered in [178, 309].

In order to provide a complete answer to the question of solvability of (3.1.49), as first done by Isenberg [243], we start by showing that (3.1.49) has no solutions in several cases: For this, suppose that there exists a solution, and integrate (3.1.49) over M:

$$\int_M \left(\frac{n-2}{4(n-1)}\widetilde{R}\phi - \tilde{\sigma}^2\phi^{\frac{2-3n}{n-2}} + \frac{n-2}{4n}\tau_\Lambda^2\phi^{\frac{n+2}{n-2}}\right) = 0\,.$$

Since we want ϕ to be positive, there are obvious obstructions for this equation to hold, and hence for existence of positive solutions: for example, if $\tilde{\sigma}^2 \equiv 0$ and

$\tau_\Lambda^2 = 0$ then there can be a positive solution only if $\widetilde{R}$ vanishes (and then ϕ is necessarily constant, e.g. by an appropriate version of the maximum principle). Analysing similarly other possibilities one finds:

PROPOSITION 3.1.12 *Suppose that either*

1. $\tilde{\sigma}^2 \equiv 0$, $\tau_\Lambda^2 = 0$, *but* $\tilde{R} \neq 0$; *or*
2. $\tilde{\sigma}^2 \equiv 0$, $\tau_\Lambda^2 \neq 0$ *but* $\tilde{R} \geq 0$; *or*
3. $\tau_\Lambda^2 = 0$, $\tilde{\sigma}^2 \not\equiv 0$, *but* $\tilde{R} \leq 0$.

Then (3.1.49) has no positive solutions.

We emphasize that the non-existence result is *not* a failure of the conformal method to produce solutions, but a *no-go* result; we will return to this issue in Proposition 3.1.25 below.

It turns out that there exist positive solutions for all other cases. This will be proved using the *monotone iteration scheme*, which we are going to describe now. For completeness we start by proving a simple version of the *maximum principle*:

PROPOSITION 3.1.13 *Let* (M, g) *be compact, let* $C^0(M) \ni c < 0$ *and let* $u \in C^2(M)$. *If*

$$\Delta u + cu \geq 0\,, \tag{3.1.50}$$

then $u \leq 0$. *If equality in (3.1.50) holds at some point, then* $u \equiv 0$.

PROOF: Suppose that u has a strictly positive maximum at p. In local coordinates around p we then have

$$g^{ij}\partial_i\partial_j u - g^{ij}\Gamma^k{}_{ij}\partial_k u \geq -cu\,.$$

The second term on the left-hand side vanishes at p because ∂u vanishes at p, the first term is non-positive because at a maximum the matrix of second partial derivatives is non-positive definite. On the other hand the right-hand side is strictly positive, which gives a contradiction. If equality holds in (3.1.50) then both u and minus u are non-positive, hence the result. □

We note without proof the following [220, Theorem 8.19]:

PROPOSITION 3.1.14 *Under the remaining hypotheses of Proposition 3.1.13, its conclusions remain true if* $c \leq 0$ *and* $c \not\equiv 0$. □

Consider, now, the operator

$$L = \Delta_{\widetilde{g}} + c$$

for some function $c < 0$. The symbol of L reads

$$\sigma_L(p) = g^{ij}p_ip_j \neq 0 \quad \text{if} \quad p \neq 0\,,$$

which shows that L is elliptic. It is well-known that $\Delta_{\widetilde{g}}$ is formally self-adjoint (with respect to the measure $d\mu_{\widetilde{g}}$), and Proposition 3.1.13 allows us to apply Theorem 3.1.11 to conclude existence of H_{k+2} solutions of the equation

$$Lu = \rho \tag{3.1.51}$$

for any $\rho \in H_k$; u is smooth if ρ and the metric are.

Returning to the Lichnerowicz equation (3.1.49), let us rewrite this equation in the form

$$\Delta_{\widetilde{g}}\phi = F(\phi, x) . \tag{3.1.52}$$

A C^2 function ϕ_+ is called a *super-solution* of (3.1.52) if

$$\Delta_{\widetilde{g}}\phi_+ \le F(\phi_+, x) . \tag{3.1.53}$$

Similarly a C^2 function ϕ_- is called a *sub-solution* of (3.1.52) if

$$\Delta_{\widetilde{g}}\phi_- \ge F(\phi_-, x) . \tag{3.1.54}$$

A solution is both a sub-solution and a super-solution. This shows that a necessary condition for existence of solutions is the existence of sub- and super-solutions. It turns out that this condition is also sufficient, modulo an obvious inequality between ϕ_- and ϕ_+:

THEOREM 3.1.15 *Suppose that (3.1.52) admits a sub-solution ϕ_- and a super-solution ϕ_+ satisfying*

$$\phi_- \le \phi_+ .$$

If $\partial_\phi F$ is continuous, then there exists a C^2 solution ϕ of (3.1.52) such that

$$\phi_- \le \phi \le \phi_+ .$$

(ϕ is smooth if F is.)

REMARK 3.1.16 The proof here uses compactness of M in an essential way. Perhaps somewhat surprisingly, the monotone iteration scheme works without any compactness, completeness, or other global conditions on (M, g), and requires only existence of continuous sub- and supersolutions satisfying the required inequalities in a weak sense, see Section 3.2.2, p. 106. □

PROOF: The argument is known as the *monotone iteration scheme*, or the *method of sub- and super-solutions.* We set

$$\phi_0 = \phi_+ ,$$

and our aim is to construct a sequence of functions such that

$$\phi_- \le \phi_n \le \phi_+ , \tag{3.1.55a}$$
$$\phi_{n+1} \le \phi_n . \tag{3.1.55b}$$

We start by chosing c to be a positive constant large enough so that the function

$$\phi \to F_c(\phi, x) := F(\phi, x) - c\phi$$

is monotone decreasing for $\phi_- \le \phi \le \phi_+$. This can clearly be done on a compact manifold. By what has been said we can solve the equation

$$(\Delta_{\widetilde{g}} - c)\phi_{n+1} = F_c(\phi_n, x)\,.$$

Clearly (3.1.55a) holds with $n = 0$. Suppose that (3.1.55a) holds for some n, then

$$\begin{aligned}
(\Delta_{\widetilde{g}} - c)(\phi_{n+1} - \phi_+) &= F_c(\phi_n, x) - \underbrace{\Delta_{\widetilde{g}}\phi_+}_{\le F(\phi_+,x)} + c\phi_+ \\
&\ge F_c(\phi_n, x) - F_c(\phi_+, x) \ge 0\,, \qquad (3.1.56)
\end{aligned}$$

by monotonicity of F_c. The maximum principle (the reader is warned that c in (3.1.56) is opposite in sign to the c in Proposition 3.1.13) gives

$$\phi_{n+1} \le \phi_+\,,$$

and induction establishes the second inequality in (3.1.55a). Similarly we have

$$\begin{aligned}
(\Delta_{\widetilde{g}} - c)(\phi_- - \phi_{n+1}) &= \underbrace{\Delta_{\widetilde{g}}\phi_-}_{\ge F(\phi_-,x)} - c\phi_- - F_c(\phi_n, x) \\
&\ge F_c(\phi_-, x) - F_c(\phi_n, x) \ge 0\,,
\end{aligned}$$

and (3.1.55a) is established. Next, we note that (3.1.55a) implies (3.1.55b) with $n = 0$. To continue the induction, suppose that (3.1.55b) holds for some $n \ge 0$, then

$$(\Delta_{\widetilde{g}} - c)(\phi_{n+2} - \phi_{n+1}) = F_c(\phi_{n+1}, x) - F_c(\phi_n, x) \ge 0\,,$$

again by monotonicity of F_c, and (3.1.55b) is proved.

Since ϕ_n is monotone decreasing and bounded there exists ϕ such that ϕ_n tends pointwise to ϕ as n tends to infinity. Continuity of F gives

$$F_n := F(\phi_n, x) \to F_\infty = F(\phi, x)\,,$$

again pointwise. By the Lebesgue dominated theorem F_n converges to F_∞ in L^2. The elliptic inequality (3.1.36) gives

$$\begin{aligned}
\|\phi_n - \phi_m\|_{H^2} &\le C_2(\|(\Delta_{\widetilde{g}} - c)(\phi_n - \phi_m)\|_{L^2} + \|\phi_n - \phi_m\|_{L^2}) \\
&= C_2(\|F_c(\phi_{n-1}, \cdot) - F_c(\phi_{m-1}, \cdot)\|_{L^2} + \|\phi_n - \phi_m\|_{L^2})\,.
\end{aligned}$$

which implies that the sequence $(\phi_n)_{n\in\mathbb{N}}$ is Cauchy in H^2. Completeness of H^2 implies that there exists $\phi_\infty \in H^2$ such that $\phi_n \to \phi_\infty$ in H^2. Recall that from any sequence converging in L^2 we can extract a subsequence, say $(\phi_{n_i})_{i\in\mathbb{N}}$, converging pointwise almost everywhere. Thus ϕ_{n_i} converges pointwise to ϕ, and pointwise almost everywhere to ϕ_∞, which implies that $\phi = \phi_\infty$ almost everywhere. Redefining ϕ on a zero-measure set if necessary, we conclude that $\phi \in H^2$.

Continuity of $\Delta_{\widetilde{g}} - c$ on H^2 shows that

$$(\Delta_{\widetilde{g}} - c)\phi = \lim_{n\to\infty}(\Delta_{\widetilde{g}} - c)\phi_n = F_c(\phi, x) = F(\phi, x) - c\phi\,,$$

so that ϕ satisfies the equation, as desired. The remaining claims follow from elliptic regularity theory. □

In order to apply Theorem 3.1.15 to the Lichnerowicz equation (3.1.49) we need appropriate sub- and super-solutions. The simplest guess is to use constants, and we start by exploring this possibility. Setting $\phi_- = \epsilon$ for some small constant $\epsilon > 0$, we need

$$0 = \Delta_{\tilde{g}}\epsilon \geq F(\epsilon, x) \equiv \frac{n-2}{4(n-1)}\widetilde{R}\epsilon - \tilde{\sigma}^2\epsilon^{\frac{2-3n}{n-2}} + \frac{n-2}{4n}\tau_\Lambda^2\epsilon^{\frac{n+2}{n-2}} \tag{3.1.57}$$

for ϵ small enough. Since $2-3n$ is negative and $\frac{n+2}{n-2}$ is larger than one, we find:

Lemma 3.1.17 *A sufficiently small positive constant is a subsolution of (3.1.49) if*

1. $\widetilde{R} < 0$, *or if*
2. $\tilde{\sigma}^2 > 0$.

Next, we set $\phi_+ = M$, with M a large constant, and we need to check that

$$0 \leq \frac{n-2}{4(n-1)}\widetilde{R}M - \tilde{\sigma}^2 M^{\frac{2-3n}{n-2}} + \frac{n-2}{4n}\tau_\Lambda^2 M^{\frac{n+2}{n-2}} . \tag{3.1.58}$$

We see that:

Lemma 3.1.18 *A sufficiently large positive constant is a supersolution of (3.1.49) if*

1. $\widetilde{R} > 0$, *or if*
2. $\tau_\Lambda^2 > 0$.

As an immediate Corollary of the two Lemmata and of Theorem 3.1.15 one has:

Corollary 3.1.19 *The Lichnerowicz equation can always be solved if $\widetilde{R}$ is strictly negative and $\tau_\Lambda \neq 0$.*

Before proceeding further it is convenient to classify the metrics on M as follows: we shall say that $g \in \mathscr{Y}^+$ if g can be conformally deformed to achieve positive scalar curvature. We shall say that $g \in \mathscr{Y}^0$ if g can be conformally rescaled to achieve zero scalar curvature but $g \notin \mathscr{Y}^+$. Finally, we let $\mathscr{Y}^-$ be the collection of the remaining metrics. It is known that all classes are non-empty, and that every metric belongs to precisely one of the classes.

One then has the following result:

Theorem 3.1.20 (Isenberg [243]) *The following table summarizes whether or not the Lichnerowicz equation (3.1.49) admits a positive solution:*

	$\tilde\sigma^2 \equiv 0,\ \tau_\Lambda = 0$	$\tilde\sigma^2 \equiv 0,\ \tau_\Lambda \neq 0$	$\tilde\sigma^2 \not\equiv 0,\ \tau_\Lambda = 0$	$\tilde\sigma^2 \not\equiv 0,\ \tau_\Lambda \neq 0$
$\widetilde{g} \in \mathscr{Y}^+$	*no*	*no*	*yes*	*yes*
$\widetilde{g} \in \mathscr{Y}^0$	*yes*	*no*	*no*	*yes*
$\widetilde{g} \in \mathscr{Y}^-$	*no*	*yes*	*no*	*yes*

For initial data in the class $(\mathscr{Y}^0, \sigma \equiv 0, \tau_\Lambda = 0)$ *all solutions are constants, and any positive constant is a solution. In all other cases the solutions are unique.*

PROOF: All the "no" entries are covered by Proposition 3.1.12. The "yes" in the first column follows from the fact that constants are (the only) solutions in this case.

To cover the remaining "yes" entries, let us number the rows and columns of the table as in a matrix T_{ij}. Then T_{32} and T_{34} are the contents of Corollary 3.1.19.

In the positive Yamabe class, Lemma 3.1.18 shows that a sufficiently large constant provides a supersolution. A small constant provides a subsolution if $\tilde\sigma^2$ has no zeros; this establishes T_{13} and T_{14} for strictly positive $\tilde\sigma^2$. However, it could happen that $\tilde\sigma^2$ has zeros. To cover this case, as well as the zero-Yamabe-class case T_{24}, we use a mixture of an unpublished argument of E. Hebey [230] and of that in [310]. Similarly to several claims above, this applies to the following general setting: Let h, a, and f be smooth functions on a compact Riemannian manifold M, with $h \geq 0$, $a \geq 0$ and $f \geq 0$. Consider the equation

$$\Delta_{\widetilde{g}} u - hu = fu^\alpha - au^{-\beta}\,, \tag{3.1.59}$$

with $\alpha > 1$ and $\beta > 0$. We further require $f + h \not\equiv 0$ and $a \not\equiv 0$. (All those hypotheses are satisfied in T_{13}, T_{14}, and T_{24}.) Then there exists a function u_1 such that

$$\Delta_{\widetilde{g}} u_1 - (h+f)u_1 = -a\,.$$

The function u_1 is strictly positive by the maximum principle. For $t > 0$ sufficiently small the function $u_t = tu_1$ is a subsolution of (3.1.59): indeed, from $ta \leq at^{-\beta}u_1^{-\beta}$ and $ftu_1 \geq ft^\alpha u_1^\alpha$ for t small enough we conclude that

$$\Delta_{\widetilde{g}} u_t - hu_t = -ta + tu_1 f \geq -at^{-\beta}u_1^{-\beta} + ft^\alpha u_1^\alpha\,.$$

The existence of a solution follows again from Theorem 3.1.15.

Uniqueness in all $\tilde{R} \geq 0$ cases, except T_{21}, follows from the fact that the function $\phi \mapsto F(\phi, x)$, defined in (3.1.57), is monotonously increasing for non-negative $\tilde{R}$: indeed, let ϕ_1 and ϕ_2 be two solutions of (3.1.52), then

$$\Delta_{\tilde{g}}(\phi_2 - \phi_1) + \underbrace{(- \int_{\phi_1}^{\phi_2} \partial_\phi F(\phi, x) d\phi)}_{=:c}(\phi_2 - \phi_1) = 0\,.$$

It follows from the monotonicity properties of F that $c \leq 0$. The maximum principle, Proposition 3.1.14, gives $\phi_1 = \phi_2$ whenever the function c is *not* identically zero.

To prove uniqueness when $\tilde{R} < 0$, suppose that there exist two distinct solutions ϕ_a, $a = 1, 2$; exchanging the ϕ_a's if necessary we can without loss of generality assume that the set where $\phi_2 > \phi_1$ is non-empty. By construction, the scalar curvature R of the metric $g := \phi_2^{\frac{4}{n-2}} \tilde{g}$ satisfies

$$\frac{n-2}{4(n-1)} R = \tilde{\sigma}^2 - \frac{n-2}{4n} \tau_\Lambda^2\,. \tag{3.1.60}$$

Because the whole construction is conformally covariant, the function

$$\phi := \frac{\phi_1}{\phi_2}$$

satisfies again (3.1.49) with respect to the metric g:

$$\Delta_g \phi - \frac{n-2}{4(n-1)} R\phi = -\tilde{\sigma}^2 \phi^{\frac{2-3n}{n-2}} + \frac{n-2}{4n} \tau_\Lambda^2 \phi^{\frac{n+2}{n-2}}\,. \tag{3.1.61}$$

In view of (3.1.60), this can be rewritten as

$$\Delta_g \phi = -\tilde{\sigma}^2 (\phi^{\frac{2-3n}{n-2}} - \phi) + \frac{n-2}{4n} \tau_\Lambda^2 (\phi^{\frac{n+2}{n-2}} - \phi)\,. \tag{3.1.62}$$

By choice, the minimum value of ϕ, say a, is strictly smaller than one. At the point where the minimum is attained we obtain

$$0 \leq \Delta_g \phi = \underbrace{-\tilde{\sigma}^2 (a^{-\frac{3n-2}{n-2}} - a)}_{I} + \underbrace{\frac{n-2}{4n} \tau_\Lambda^2 (a^{\frac{n+2}{n-2}} - a)}_{II}\,. \tag{3.1.63}$$

But both I and II are strictly negative for $a < 1$, which gives a contradiction, and establishes uniqueness. □

Remark 3.1.21 The conformal covariance properties of the conformal method, already used in the proof above, are significant enough to warrant emphasising. Consider, thus, a set $(\widetilde{g}, \widetilde{L}, \phi, \beta)$, where $\widetilde{L}$ is a $\widetilde{g}$-TT tensor, and ϕ solves the Lichnerowicz equation in the metric $\widetilde{g}$:

$$\Delta_{\widetilde{g}} \phi - \frac{n-2}{4(n-1)} \widetilde{R} \phi = -\frac{n-2}{4(n-1)} |\widetilde{L}|_{\widetilde{g}}^2 \phi^{\frac{2-3n}{n-2}} + \beta \phi^{\frac{n+2}{n-2}}\,. \tag{3.1.64}$$

Here β is a function which might or might not be given by (3.1.22). Let ψ be a strictly positive function, then $\hat{L}_{ab} := \psi^{-2}\widetilde{L}_{ab}$ is a TT tensor for the metric $\hat{g} := \psi^{\frac{4}{n-2}}\widetilde{g}$, and the function $\hat{\phi} := \psi^{-1}\phi$ is a solution of the Lichnerowicz equation in the metric $\hat{g}$:

$$\Delta_{\hat{g}}\hat{\phi} - \frac{n-2}{4(n-1)}\hat{R}\hat{\phi} = -\frac{n-2}{4(n-1)}|\hat{L}|^2_{\hat{g}}\hat{\phi}^{\frac{2-3n}{n-2}} + \beta\hat{\phi}^{\frac{n+2}{n-2}}\,, \tag{3.1.65}$$

where $\hat{R}$ is the curvature scalar of the metric $\hat{g}$.

To prove (3.1.65), the following formula is useful:

$$\Delta_{\hat{g}}\hat{\phi} - \frac{n-2}{4(n-1)}\hat{R}\hat{\phi} = \psi^{-\frac{n+2}{n-2}}\left(\Delta_{\widetilde{g}}(\psi\hat{\phi}) - \frac{n-2}{4(n-1)}\widetilde{R}\psi\hat{\phi}\right)\,. \tag{3.1.66}$$

INCIDENTALLY: The calculation leading to (3.1.66) proceeds as follows:

$$\begin{aligned}
\widetilde{g}_{ab} &= \psi^{-\frac{4}{n-2}}\hat{g}_{ab}\,,\\
\sqrt{\det\widetilde{g}} &= \psi^{-\frac{2n}{n-2}}\sqrt{\det\hat{g}}\,,\\
\widetilde{g}^{ab} &= \psi^{\frac{4}{n-2}}\hat{g}^{ab}\,,\\
\Delta_{\widetilde{g}}(\psi\hat{\phi}) &= \hat{\phi}\Delta_{\widetilde{g}}\psi + \psi\Delta_{\widetilde{g}}\hat{\phi} + 2\widetilde{g}^{ab}\partial_a\psi\partial_b\hat{\phi}\\
&= \hat{\phi}\Delta_{\widetilde{g}}\psi + \psi\Delta_{\widetilde{g}}\hat{\phi} + 2\psi^{\frac{4}{n-2}}\hat{g}^{ab}\partial_a\psi\partial_b\hat{\phi}\,,\\
\Delta_{\widetilde{g}}\hat{\phi} &= \frac{1}{\sqrt{\det\widetilde{g}}}\partial_a(\sqrt{\det\widetilde{g}}\widetilde{g}^{ab}\partial_b\hat{\phi})\\
&= \frac{\psi^{\frac{2n}{n-2}}}{\sqrt{\det\hat{g}}}\partial_a(\psi^{-\frac{2n}{n-2}}\sqrt{\det\hat{g}}\psi^{\frac{4}{n-2}}\hat{g}^{ab}\partial_b\hat{\phi})\\
&= \frac{\psi^{\frac{2n}{n-2}}}{\sqrt{\det\hat{g}}}\partial_a(\psi^{-2}\sqrt{\det\hat{g}}\hat{g}^{ab}\partial_b\hat{\phi})\\
&= \psi^{\frac{4}{n-2}}\big(\Delta_{\hat{g}}\hat{\phi} - \frac{2}{\psi}\hat{g}^{ab}\partial_a\psi\partial_b\hat{\phi}\big)\,,\\
\Delta_{\widetilde{g}}(\psi\hat{\phi}) &= \hat{\phi}\Delta_{\widetilde{g}}\psi + \psi^{\frac{n+2}{n-2}}\Delta_{\hat{g}}\hat{\phi}\,.
\end{aligned}$$

To finish the calculation one uses the transformation formula (C.2.16):

$$\hat{R} = \psi^{-\frac{n+2}{n-2}}\left(-\frac{4(n-1)}{n-2}\Delta_{\widetilde{g}}\psi + \tilde{R}\psi\right)\,. \tag{3.1.67}$$

□

REMARK 3.1.23 The question of stability of solutions of the Lichnerowicz equation with $\Lambda > (n-1)\tau^2/2n$ (so that $\beta < 0$) has been addressed in [194, 195, 355]. Surprisingly enough, in [194] stability is established in dimensions $n \leq 5$, but examples are constructed where stability fails when $n \geq 6$; see also [?]. □

As a Corollary of Theorem 3.1.20 one obtains:

THEOREM 3.1.24 *Any compact Riemannian manifold (M, g) carries some vacuum initial data set.*

PROOF: We can construct non-trivial solutions of the vector constraint equation using the method of Section 3.1.4, which takes us to the last two columns of the table of Theorem 3.1.20. Choosing some $\tau_\Lambda^2 > 0$ we can then solve the Lichnerowicz equation whatever the Yamabe type of g by the last column in that table. □

As already pointed out, we have the following trivial observation which highlights the importance of Isenberg's Theorem 3.1.20:

PROPOSITION 3.1.25 *All solutions of the vacuum constraint equation can be constructed by the conformal method.*

PROOF: If (M, g, K) is a CMC vacuum initial data set, the result is established by setting $Y = 0$, $\phi = 1$, $\widetilde{L}^{ij} = K^{ij} - \frac{\mathrm{tr}K}{n} g^{ij}$. □

INCIDENTALLY: A natural question is whether the set of solutions to the constraint equations forms a manifold. This was first considered by Fischer and Marsden [204], who identified an obstruction arising from symmetries (compare [18, 19, 327]) and provided a Fréchet manifold structure away from a singular set. Banach manifold structures have been constructed in [133], and a Hilbert manifold structure for asymptotically flat or asymptotically hyperbolic initial data sets in [36, 189, 316, 317]. Note that the constructions in the last references clearly generalise to more general classes of data. □

3.1.7 The scalar constraint equation on compact manifolds, $\tau^2 < \frac{2n}{(n-1)}\Lambda$

Theorem 3.1.20 gives an exhaustive and rather simple picture of CMC initial data on compact manifolds when $\frac{2n}{(n-1)}\Lambda \leq \tau^2$. The situation for Λ's exceeding this bound is rather different, with several questions remaining open.

It follows immediately from the scalar constraint equation that when $\tau = 0 = \sigma^2$ but $\Lambda > 0$, then we are in the positive case of the Yamabe problem. More generally, a necessary condition for existence of solutions of the Lichnerowicz equation in the current case is that g be of positive Yamabe type. Indeed, supposing that a solution ϕ exists, we can always make a conformal transformation of the metric so that $\phi \equiv 1$. But then the Lichnerowicz equation (3.1.21) gives

$$\frac{n-2}{4(n-1)}\widetilde{R} = \tilde{\sigma}^2 - \beta > 0\,,$$

since $\beta < 0$, whence the result.

It follows in particular that it is always possible to conformally rescale the seed data so that, say,

$$\tilde{R} = n(n-1)\,, \tag{3.1.68}$$

and we will often use this normalisation in what follows.

Obvious examples of three dimensional compact manifolds carrying a metric with positive scalar curvature are given by

$$S^3/\Gamma\,, \quad S^2 \times S^1\,, \tag{3.1.69}$$

where Γ is a discrete subgroup of $O(3)$ without fixed points. The quotient manifolds S^3/Γ are called *spherical manifolds*, see [21] for a complete list.

It turns out that a complete description of the possible topologies of three dimensional compact manifolds carrying metrics with positive scalar curvature can be given using the *connected sum* construction, which proceeds as follows: consider any two manifolds M_a, $a = 1, 2$. Consider two sets $B_a \subset M_a$, each diffeomorphic to a ball in $\mathbb{R}^n$. One then defines the manifold $M_1 \# M_2$, called the *connected sum of* M_1 *and* M_2, as the set

$$(M_1 \setminus B_1) \cup ([0,1] \times S^{n-1}) \cup (M_2 \setminus B_2)$$

in which the sphere ∂B_1 is identified in the obvious way with $\{0\} \times S^{n-1}$, and the sphere ∂B_2 is identified with $\{1\} \times S^{n-1}$. In other words, one removes balls from the M_a's and connects the resulting spherical boundaries with a "neck" $[0,1] \times S^{n-1}$.

Consider, then, two manifolds (M_a, g_a) with positive scalar curvature. Gromov and Lawson [224] have shown how to construct a metric of positive scalar curvature on $M_1 \# M_2$. This implies that any compact, orientable three-manifold which is a connected sum of spherical manifolds and of copies of $S^2 \times S^1$ carries a metric of positive scalar curvature. The resolution of the Poincaré conjecture by Perelman [346–348] completes previous work of Schoen-Yau [390] and Gromov-Lawson [225] on this topic, and proves the converse: these are the only compact three-manifolds with positive scalar curvature.

Under the current conditions, a pointwise obstruction to existence of solutions of the Lichnerowicz equation can be derived as follows [231]: Let p_0 be a point where the minimum of ϕ is attained, set $\epsilon := \phi(p_0)$. To emphasize the current sign, we define

$$\Lambda_\tau := \left[-\frac{n-2}{4n}\tau^2 + \frac{n-2}{2(n-1)}\Lambda \right] \geq 0\,. \tag{3.1.70}$$

This coincides with $-\beta$ as defined in (3.1.22), p. 65.

At the minimum the Laplacian of ϕ is non-negative, and there the Lichnerowicz equation gives

$$0 \leq \Delta_{\widetilde{g}}\phi\Big|_{p_0} = \frac{n-2}{4(n-1)}\widetilde{R}\epsilon - \sigma^2\epsilon^{(2-3n)/(n-2)} - \Lambda_\tau\epsilon^{(n+2)/(n-2)}\,. \tag{3.1.71}$$

But the right-hand side is negative since $\Lambda_\tau \geq 0$ if σ^2 is sufficiently large, which gives an obstruction to existence. Setting $a := \epsilon^{4/(n-2)}$, (3.1.71) becomes

$$\begin{aligned} 0 &\leq \frac{n-2}{4(n-1)}\widetilde{R} - \sigma^2\epsilon^{-4(n-1)/(n-2)} - \Lambda_\tau\epsilon^{4/(n-2)} \\ &= \frac{n-2}{4(n-1)}\widetilde{R} - \sigma^2 a^{-(n-1)} - \Lambda_\tau a =: G(a)\,. \end{aligned}$$

Hence, the function G so defined needs to be positive at a. In particular the maximum of G must be positive. To determine this maximum, we note that the condition of the vanishing of G' gives

$$(n-1)\sigma^2 a^{-n} = \Lambda_\tau\,.$$

Inserting into G, we find that the maximum of G will be non-positive if

$$\Lambda_\tau \sigma^{2/(n-1)} \geq (n-1)^{1/(n-1)} \left(\frac{n-2}{4n(n-1)} \widetilde{R}_+ \right)^{n/(n-1)} , \quad (3.1.72)$$

where $\widetilde{R}_+$ is the positive part of $\widetilde{R}$, holds everywhere. We conclude that:

PROPOSITION 3.1.27 *There* cannot *be a positive minimum of ϕ, and hence a positive solution, if (3.1.72) holds everywhere.*

In other words, violation of (3.1.72) *somewhere* is a *necessary* condition for existence of strictly positive solutions.

Note that the above calculation can be used to obtain a lower bound on ϕ when σ^2 has no zeros or when $\widetilde{R}$ is strictly positive.

One can also obtain integral, instead of pointwise, conditions for non-existence of solutions, see [231, Theorems 2.1 and 2.2] for details.

After a conformal rescaling of the metric so that $\tilde{R}$ is (a positive) constant, the operator appearing at the left-hand side of (3.1.21), p. 65 is obviously coercive. This allows us to apply to the problem at hand some key results from [195, 355] and obtain:

THEOREM 3.1.28 (Premoselli [355]) *Let $\Lambda_\tau > 0$, and let M be a compact manifold of dimension $n \in \{3, 4, 5\}$ and constant positive scalar curvature.*

1. *For every seed data set $(\tilde{g}, \tilde{L})$ with $\tilde{L} \not\equiv 0$ there exists $\theta_* \in \mathbb{R}^+$ such that the Lichnerowicz equation with seed data $(\tilde{g}, \theta\tilde{L})$, $\theta \in \mathbb{R}^+$, has*
   - *(a) no solution for $\theta \geq \theta_*$;*
   - *(b) exactly one solution when $\theta = \theta_*$;*
   - *(c) at least two solutions for $\theta < \theta_*$.*
2. *For every set S of, say smooth, seed data satisfying (3.1.68) such that $\inf_S \sup_M \tilde{\sigma}^2 > 0$, the associated collection of solutions of the constraint equations is compact.*

PROOF: Point 1. is a special case of [355, Theorem 2.1]. Concerning point 2.,[5] the argument for sequences $\tilde{\sigma}^2{}_k$ in [195] generalises immediately to the current setting, including perturbations g_k of the metric. Indeed, that last generalisation requires only uniform bounds on the geometry of the manifold and on the Green functions, which hold in the current setting. □

REMARK 3.1.29 Premoselli's theorem above builds upon, and extends, previous work of Hebey, Pacard and Pollack [231]. In [231, Corollaries 3.1 and 3.2] the reader will find several criteria for existence, essentially amounting to the requirement that σ^2 be small and without zeros. For example, on compact

[5]We are grateful to Bruno Premoselli for useful comments concerning this point.

manifolds such that $\widetilde{R} \geq 0$ but not identically zero, there exists a constant C depending upon $\tilde{g}$ and n such that, if $\Lambda_\tau > 0$ and $\sigma^2 > 0$ and

$$\Lambda_\tau^{n-1} \int_M \sigma^2 \leq C \,,$$

then a solution exists. This is proved using the Mountain Pass Lemma [359]. □

A non-trivial example of the behaviour described by Theorem 3.1.28 is presented in Section 3.1.8 below.

Delaunay metrics

An interesting class of metrics on $S^1 \times S^{n-1}$ with positive scalar curvature is provided by the *Delaunay metrics* which, for $n \geq 3$, take the form

$$g = u^{4/(n-2)}(dy^2 + \mathring{g}_{n-1}) \,, \tag{3.1.73}$$

with $u = u(y)$ and where, as before, $\mathring{g}_{n-1}$ is the unit round metric on S_p^{n-1}. The metrics are spherically symmetric, hence conformally flat. It is shown in [389] that the solutions of the Yamabe equation are spherically symmetric as well, so that the constant scalar curvature condition $R(g) = n(n-1)$ reduces to an ODE for u:

$$u'' - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0. \tag{3.1.74}$$

Solutions are determined by two parameters which correspond respectively to a minimum value ε for u, with

$$0 \leq \varepsilon \leq \bar{\varepsilon} = (\frac{n-2}{n})^{\frac{n-2}{4}} \,, \tag{3.1.75}$$

called the *Delaunay parameter* or *neck size*, and a translation parameter along the cylinder. An ODE analysis [315] shows that all positive solutions are periodic. The degenerate solution with $\varepsilon = 0$ corresponds to the round metric on a sphere from which two antipodal points have been removed. The solution with $\varepsilon = \bar{\varepsilon}$ corresponds to the rescaling of the cylindrical metric so that the scalar curvature has the desired value.

The Delaunay metrics provide an example of countable non-uniqueness of solutions of the Yamabe equation on $S^1 \times S^{n-1}$: for any $T > 0$ and $\ell \in \mathbb{N}^*$ there exists a solution u_ℓ of (3.1.74) with period T/ℓ. Each such function u_ℓ provides a metric with constant scalar curvature $n(n-1)$ on the manifold on which the coordinate y of (3.1.73) is T-periodic.

INCIDENTALLY: The ODE (3.1.74) was first studied by Fowler [206, 207], however the name arises from an analogy with the Delaunay surfaces, which are complete, periodic CMC surfaces of revolution in $\mathbb{R}^3$ [186].

One can view the Delaunay metrics as singular solutions of the Yamabe equation on (S_p^n, g_0). There exists a number of uniqueness results in this context: it is known that no solution with a single singular point exists, and that any solution with exactly two isolated singular points must be conformally equivalent to a Delaunay metric.

It is also known that conformally flat metrics, with constant positive scalar curvature, and with an *isolated singularity of the conformal factor* are necessarily asymptotic to a Delaunay metric; in fact, in dimensions $n = 3, 4, 5$ the conformal flatness condition is not needed [307]. Specifically, in spherical coordinates about an isolated singularity of the conformal factor, there is a half-Delaunay metric which g converges to, exponentially fast in r, along with all of its derivatives. This fact is used in [312, 314, 315, 353, 362] where complete, constant scalar curvature metrics, conformal to the round metric on $S_p \setminus \{p_1, \dots, p_k\}$ were studied and constructed. This is one instance of the more general "singular Yamabe problem". □

INCIDENTALLY: The time-evolution of time-symmetric Delaunay data leads to the *Kottler–Schwarzschild–de Sitter metrics* of Section B.3, p. 262 in $n+1$ dimensions, with cosmological constant $\Lambda > 0$ and mass $m \in \mathbb{R}$:

$$ds^2 = -V dt^2 + V^{-1} dr^2 + r^2 \mathring{g}_{n-1}, \quad \text{where} \quad V = V(r) = 1 - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2}\,, \tag{3.1.76}$$

where $\ell > 0$ is related to the cosmological constant Λ by the formula $2\Lambda = n(n-1)/\ell^2$. Comparing (3.1.76) and (3.1.73) we find

$$r = u^{\frac{2}{n-2}}\,, \quad r\frac{dy}{dr} = V^{-1/2}\,, \tag{3.1.77}$$

which allows us to determine y as a function of r on any interval of r's on which V has no zeros.

To avoid a singularity lying at finite distance on the level sets of t one needs $m > 0$. Equation (3.1.76) provides then a spacetime metric satisfying the Einstein equations with cosmological constant $\Lambda > 0$ and with well behaved spacelike hypersurfaces when one restricts the coordinate r to an interval (r_b, r_c) on which $V(r)$ is positive; such an interval exists if and only if

$$\left(\frac{2}{(n-1)(n-2)}\right)^{n-2} \Lambda^{n-2} m^2 n^2 < 1\,. \tag{3.1.78}$$

When $n = 3$ this translates to the condition that $9m^2\Lambda < 1$. □

3.1.8 Bifurcating solutions of the constraint equations

The question arises, what more can be said about the structure of the set of solutions as in Premoselli's Theorem 3.1.28. In this section we present a toy model where a complete description can be carried out. The setting is that of all $U(1) \times SO(3)$ general relativistic initial data on $S^1 \times S^2$, which is a direct generalisation of the Delaunay metrics by including a non-trivial extrinsic curvature tensor, first discussed in [149]. We follow the presentation in [138]. See [69] for a numerical analysis of a related model.

Thus, we choose the initial data metrics

$$g = \phi^4 \mathring{g} \tag{3.1.79}$$

to be conformal to

$$\mathring{g} \equiv g_{\mathring{T},\mathring{R}} := \left(\frac{\mathring{T}}{2\pi}\right)^2 d\psi^2 + \frac{2}{\mathring{R}} d\Omega^2\,, \tag{3.1.80}$$

where ψ is a 2π-periodic coordinate on S^1, $\mathring{T}$ and $\mathring{R}$ are positive constants, while $d\Omega^2$ is the unit round metric on S^2. The metric $g_{\mathring{T},\mathring{R}}$ has scalar curvature $\mathring{R}$. A constant rescaling of $g_{\mathring{T},\mathring{R}}$ can be absorbed in a redefinition of $\mathring{T}$ and $\mathring{R}$ which leaves invariant the product $\mathring{T}^2\mathring{R}$. We will write $\mathring{g}$ instead of $g_{\mathring{T},\mathring{R}}$ when the explicit values of $\mathring{T}$ and $\mathring{R}$ are not essential.

Following [149], the extrinsic curvature tensor K will be taken of the form

$$K = \frac{2\alpha\phi^{-2}}{\sqrt{6}}\left(\left(\frac{\mathring{T}}{2\pi}\right)^2 d\psi^2 - \frac{1}{\mathring{R}}d\Omega^2\right) + \frac{\tau}{3}g =: \phi^{-2}\mathring{L} + \frac{\tau}{3}g\,, \qquad (3.1.81)$$

where $\alpha > 0$ and τ are non-negative constants. This is the general form for a $U(1)\times SO(3)$-invariant two-covariant symmetric tensor, except for the condition

$$\alpha > 0$$

which has been made to exclude the Delaunay metrics. Note that the "seed tensor field" $\mathring{L}$ is $\mathring{g}$-transverse and traceless. The multiplicative normalisation factor in $\mathring{L}$ has been chosen so that $|\mathring{L}|_{\mathring{g}} = |\alpha|$.

The general relativistic constraint equations will be satisfied by (g, K), with $g = \phi^4\mathring{g}$, if and only if

$$\Delta_{\mathring{g}}\phi - \frac{\mathring{R}}{8}\phi = -\frac{\lambda_\tau}{8}\phi^5 - \frac{\alpha^2}{8}\phi^{-7}\,, \qquad (3.1.82)$$

keeping in mind our hypothesis that

$$\boxed{\frac{\lambda_\tau}{8} := \frac{\Lambda}{4} - \frac{\tau^2}{12} > 0\,.} \qquad (3.1.83)$$

Recall that when λ_τ is negative and $\alpha^2 \neq 0$ the solutions are unique by Isenberg's Theorem 3.1.20. This implies that ϕ inherits then the symmetries of the metric $\mathring{g}$, and hence is constant. We will see that this is not the case anymore when λ_τ is positive.

We note the following, where we allow $\mathring{R}$, α and λ_τ *not to be* constant:

PROPOSITION 3.1.32 *Consider (3.1.82) on a compact Riemannian manifold with continuous functions $\mathring{R}$, α and λ_τ satisfying $\lambda_\tau > 0$. If*

$$\min\alpha^2 \min\lambda_\tau^2 \geq \frac{4}{3^3}\max\mathring{R}^3\,, \qquad (3.1.84)$$

then (3.1.82) has no positive solutions unless $\mathring{R}$, α^2 and λ_τ are all positive constants and the inequality in (3.1.84) is an equality, in which case there is a unique positive solution, which is constant:

$$\phi = \left(\frac{2\mathring{R}}{3\lambda_\tau}\right)^{1/4}.$$

PROOF: Write (3.1.82) as $\Delta\phi = F$. A simple analysis of the polynomial $\phi \mapsto \alpha^2 - \mathring{R}\phi^8 + \lambda_\tau\phi^{12}$ gives $F \geq 0$ when (3.1.84) holds. Multiplying the equation by ϕ and integrating by parts gives $\phi = \phi_0 = \text{const}$, $F(\phi_0) \equiv 0$, and the result readily follows. □

The key observation for the further analysis of the problem at hand is that the conformal factor ϕ depends only upon the coordinate ψ running around the S^1 factor of $M = S^1 \times S^2$. This follows from the difficult results of [258] on uniqueness and non-uniqueness of solutions of classes of semilinear equations on S^3 with isolated singularities on the north and south pole:

THEOREM 3.1.33 *Positive solutions ϕ of (3.1.82) depend at most upon ψ.*

Because of the conformal covariance of the problem, we can always rescale $\mathring{g}$ so that a constant solution of the Lichnerowicz equation, whenever one exists, equals one. After such a rescaling we will obtain

$$\mathring{R} = \lambda_\tau + \alpha^2 \,. \tag{3.1.85}$$

This normalisation will be often used in what follows.

Assuming (3.1.85), Theorem 3.1.33 reduces the problem to finding all 2π-periodic solutions $\phi = \phi(\psi)$ of the Lichnerowicz equation:

$$\begin{aligned} \frac{(2\pi)^2}{\mathring{T}^2}\frac{d^2\phi}{d\psi^2} &= -\frac{1}{8}\big(-(\alpha^2+\lambda_\tau)\phi + \lambda_\tau\phi^5 + \alpha^2\phi^{-7}\big) \\ &= -\frac{1}{8\phi^7}(\phi^4-1)(\lambda_\tau\phi^8 - \alpha^2(1+\phi^4)) =: -\frac{dV}{d\phi}(\phi)\,. \end{aligned} \tag{3.1.86}$$

In order to account for the invariance of the problem under rigid rotations of S^1 we will require

$$\partial_\psi\phi(0) = 0\,, \tag{3.1.87}$$

which can always be fulfilled by an adequate choice of the origin on the circle. With this normalisation each family of solutions differing from each other by a rotation of S^1 appears either as one solution, when ϕ is constant, or as two solutions, one where 0 is a local maximum of ϕ and one where 0 is a local minimum of ϕ.

The conserved energy for (3.1.86) reads

$$H(\phi,\dot\phi) = \frac{1}{2}\dot\phi^2 - \frac{\alpha^2+\lambda_\tau}{16}\phi^2 + \frac{\lambda_\tau}{48}\phi^6 - \frac{\alpha^2}{48}\phi^{-6} =: \frac{1}{2}\dot\phi^2 + V(\phi)\,, \tag{3.1.88}$$

where a dot denotes a derivatives with respect to

$$t := \frac{\mathring{T}}{2\pi}\psi\,. \tag{3.1.89}$$

Keeping in mind our assumptions $\phi > 0$, $\alpha^2 \neq 0$ and $\lambda_\tau > 0$, the equation $dV/d\phi = 0$ can be written as

$$(y-1)(x^2 + x^2y - 2y^2) = 0\,, \text{ where } y = \phi^4\,, \tag{3.1.90}$$

and where we set

$$x := \frac{\sqrt{2}|\alpha|}{\sqrt{\lambda_\tau}} > 0\,. \tag{3.1.91}$$

The positive solutions are $y = 1$ and $y = \frac{x}{4}(x + \sqrt{x^2+8})$, distinct unless $x = 1$. Representative plots of V can be found in Figure 3.1.1.

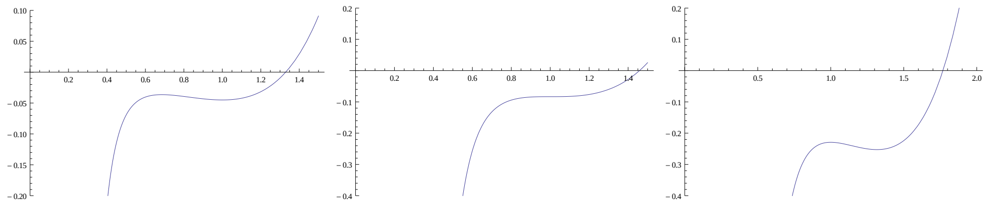

Figure 3.1.1: Typical form of the potential $V(\phi)$ with $|\alpha| < \sqrt{\lambda_\tau/2}$ (left), $|\alpha| = \sqrt{\lambda_\tau/2}$ (middle) and $|\alpha| > \sqrt{\lambda_\tau/2}$ (right).

When $|\alpha| = \sqrt{\lambda_\tau/2}$ the only solution which remains bounded away from zero for all times is $\phi \equiv 1$. This case corresponds precisely to that already covered in Proposition 3.1.32, and thus from now on we assume that

$$|\alpha| \neq \sqrt{\lambda_\tau/2} \text{ or, equivalently, } x \neq 1\,.$$

It is convenient, first, to drop the requirement of periodicity of ϕ. For this consider (3.1.86), where the ψ-derivatives are replaced by derivatives with respect to the parameter $t \in \mathbb{R}$ of (3.1.89). The nature of the solutions

$$\mathbb{R} \ni t \mapsto \phi(t)$$

can easily be understood by inspection of the potentials in Figure 3.1.1, or of the phase portrait in the $(\phi, \dot\phi)$ plane of Figure ??. •3.1.1 It should be clear from Figure 3.1.1 that the critical point $(\phi = 1, \dot\phi = 0)$ is stable if and only if

•*3.1.1: two very large figures commented out because too long to display and compile*

$$|\alpha| < \sqrt{\lambda_\tau/2}\,.$$

Subsequently, the analysis has to be carried-out separately according to whether or not this inequality is satisfied.

The case $|\alpha| < \sqrt{\lambda_\tau/2}$:

The solution $\phi(\psi) \equiv 1$ has energy

$$H_1 = V(1) = -\frac{1}{24}\left(2\alpha^2 + \lambda_\tau\right)\,. \tag{3.1.92}$$

The second critical point $(\phi = \phi_2 < 1, \dot\phi = 0)$ has energy which we will denote by $H_2 = H_2(\alpha, \lambda_\tau)$. An analytic expression for H_2 can be obtained but is not very enlightening:

$$H_2 = -\frac{\lambda_\tau\sqrt{x}}{48} \times \frac{x^4 + 6x^2 + 16 + (x^3+2x)\sqrt{x^2+8}}{\left(x + \sqrt{x^2+8}\right)^{3/2}}\,. \tag{3.1.93}$$

There is an orbit corresponding to a non-trivial homoclinic solution with energy $H(\phi, \dot\phi) = H_2$ which asymptotes to ϕ_2 as t tends both to plus and minus infinity. On both pictures of Figure **??**, the relevant orbit lies on the piece of the red curve that closes up.

All orbits lying in the conditionally compact set, say $\Pi \subset \{(\phi, \dot\phi) \in \mathbb{R}^2\}$, enclosed by this homoclinic orbit are periodic. These are the only orbits with ϕ bounded and bounded away from zero, and hence the only ones of interest to us as solutions of the Lichnerowicz equation on $S^1 \times S^2$ leading to a spatially compact vacuum data set with the same topology.

The periodic orbits oscillate between $\phi_{\min}(\alpha, \lambda_\tau, E)$ and $\phi_{\max}(\alpha, \lambda_\tau, E)$, where

$$E := H(\phi, \dot\phi)$$

denotes the energy of the solution. Figure **??** shows that the function

$$E \mapsto \phi_{\min}(\alpha, \lambda_\tau, E)$$

is monotonously decreasing to $\phi_{\min}(\alpha, \lambda_\tau) = \phi_2$, while $E \mapsto \phi_{\max}(\alpha, \lambda_\tau, E)$ is monotonously increasing to a value $\phi_{\max}(\alpha, \lambda_\tau)$. There is a bound

$$\phi_{\max}(\alpha, \lambda_\tau) \le \sqrt{3}$$

which is approached as $\alpha \to 0$. It is attained on the solution with $\alpha = 0$ with energy $E = H_2$, for which $\phi_2 = 0$; this solution closes-off $\mathbb{R} \times S^2$ to a smooth round S^3. A plot of $\phi_{\min}(\alpha, \lambda_\tau)$ can be found in Figure 3.1.2. We note $\lim_{(|\alpha|/\lambda_\tau)\to 0} \phi_{\min} = 0$.

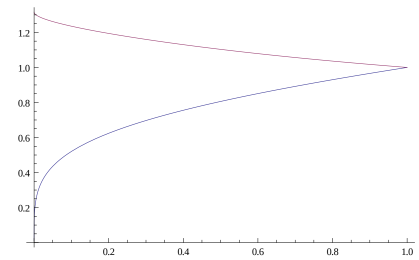

Figure 3.1.2: $\phi_{\min}(\alpha, \lambda_\tau)$ and $\phi_{\max}(\alpha, \lambda_\tau)$ as functions of the scaled variable $x = \sqrt{2}|\alpha|/\sqrt{\lambda_\tau}$.

The discussion so far applies to solutions on $\mathbb{R} \times S^2$. Each such solution with minimal period T leads to a solution of the Lichnerowicz equation on $S^1 \times S^2$ with metric $g_{nT,\mathring{R}}$, for any integer $n \ge 1$, by replacing the S^1-factor by its n-fold cover. The question that arises is then, which values of T are realised by the solutions above. To answer this one needs to understand the *period function*, defined as the function which to a periodic orbit with energy E associates its minimal period $T(\alpha, \lambda_\tau, E)$. For any such orbit with ϕ varying between $\phi_{\min}(\alpha, \lambda_\tau, E)$ and $\phi_{\max}(\alpha, \lambda_\tau, E)$ the period equals

$$T = \sqrt{2} \int_{\phi_{\min}(\alpha,\lambda_\tau,E)}^{\phi_{\max}(\alpha,\lambda_\tau,E)} \frac{d\phi}{\sqrt{E - V(\phi)}}, \tag{3.1.94}$$

where the turning points $\phi_{\min}(\alpha, \lambda_\tau, E)$ and $\phi_{\max}(\alpha, \lambda_\tau, E)$ are found by solving the equations

$$V(\phi_{\min}(\alpha, \lambda_\tau, E)) = E = V(\phi_{\max}(\alpha, \lambda_\tau, E)),$$

with $\phi_{\min}(\alpha, \lambda_\tau, E)) \in [\phi_2(\alpha, \lambda_\tau), 1]$ and $\phi_{\max}(\alpha, \lambda_\tau, E)) \in [1, \infty)$. Since in our case V is a real analytic function of ϕ, the real analytic version of the implicit function theorem shows that away from the critical level sets of H the functions $E \mapsto \phi_{\min}$ and $E \mapsto \phi_{\max}$ are real analytic.

When E approaches the energy of the stable critical point, ϕ_s, where $V''(\phi_s) > 0$, the period approaches that of linearized oscillations around ϕ_s:

$$T \to \frac{2\pi}{\sqrt{V''(\phi_s)}}. \tag{3.1.95}$$

In particular, when $|\alpha| < \sqrt{\lambda_\tau/2}$ the stable critical point is $\phi_1 = 1$ and one has

$$T \to T_1(\alpha, \lambda_\tau) = \frac{2\sqrt{2}\pi}{\sqrt{\lambda_\tau - 2\alpha^2}}. \tag{3.1.96}$$

Near to and away from the critical point $\phi = 1$ the function T is differentiable, with the sign of the derivative of T with respect to E determined by the sign of the Chicone test function [89]

$$N = (G')^4 \left(\frac{G}{(G')^2}\right)'', \tag{3.1.97}$$

where $G(\phi) = V(\phi) - V(1)$ is the potential normalised so that $G(1) = 0$, on the interval $[\phi_{\min}(\alpha, \lambda_\tau)), \phi_{\min}(\alpha, \lambda_\tau))]$. N can be computed and takes the following form:

$$N = \frac{\lambda_\tau^3(\phi^2 - 1)^4}{768\phi^{22}} P(\phi),$$

where P is a polynomial of degree 28 in ϕ which is conveniently computed with e.g. MATHEMATICA, and which can be checked to be positive in the range of variables of interest. Hence N is non-negative on the interval

$$[\phi_{\min}(\alpha, \lambda_\tau)), \phi_{\max}(\alpha, \lambda_\tau))],$$

thus proving that the period function is increasing with E.

When moving continuously amongst the solutions so that their energy E tends to H_2, the period of the solutions grows to infinity since the (bounded) solution with $E = H_2$ is a homoclinic orbit.

A plot of the period function $E \mapsto T(\alpha, \lambda_\tau, E)$ can be found in Figure 3.1.3 for $\alpha = 0.2$ and $\lambda_\tau = 1$.

The case $|\alpha| = \sqrt{\lambda_\tau/2}$

leads to only one periodic positive solution, namely the constant one.

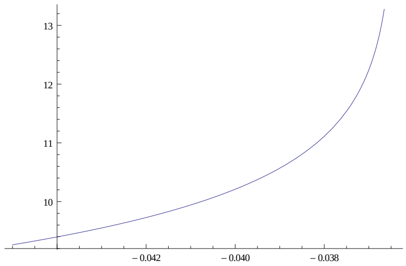

Figure 3.1.3: Values of the period of oscillation with $\alpha = 0.2$ and $\lambda_\tau = 1$. In this case one has $H_1 \simeq -0.045$ and $H_2 \simeq -0.0364$.

The case $|\alpha| > \sqrt{\lambda_\tau/2}$:

The analysis is very similar to that of the case $|\alpha| < \sqrt{\lambda_\tau/2}$. We now have $x \in (1, \infty)$. The stable point becomes ϕ_2 and a calculation shows that

$$V''(\phi_2) = \frac{\lambda_\tau}{8}\sqrt{x^2+8}\left(3x - \sqrt{x^2+8}\right)$$

which is clearly positive if and only if $x \in (1, \infty)$. When the energy of a periodic solution approaches H_2, its period approaches that of the solutions of the linearized problem around ϕ_2:

$$T \to T_2(\alpha, \lambda_\tau) = \frac{2\pi}{\sqrt{V''(\phi_2)}} = \frac{4\sqrt{2}\pi}{\sqrt{\lambda_\tau}(x^2+8)^{1/4}\left(3x - \sqrt{x^2+8}\right)^{1/2}}. \quad (3.1.98)$$

As expected, the period of small oscillations goes to infinity as x tends to 1 since the critical points of V (namely 1 and ϕ_2) merge to a single degenerate critical point.

The proof of the monotonicity of the period function $T(\alpha, \lambda_\tau, E)$ with respect to the energy E of the solution translates without modifications: one proves that the Chicone test function N of (3.1.97) is positive on the interval $[1, \infty)$.

Similarly, the period $T(\alpha, \lambda_\tau, E)$ goes to infinity as $E \to H_1$. An example plot of T is given in Figure 3.1.4.

The bifurcation analysis

We are ready now to carry out a bifurcation analysis of solutions of the equation (3.1.82). We fix $\mathring{g}$ (i.e. $\mathring{T}$ and $\mathring{R}$), λ_τ and let α be our varying bifurcation parameter. In particular, we no longer assume the conformal gauge $\mathring{R} = \alpha^2 + \lambda_\tau$. From Theorem 3.1.33, ϕ depends at most on ψ, so (3.1.82) reduces to

$$\frac{(2\pi)^2}{\mathring{T}^2}\frac{d^2\phi}{d\psi^2} = -\frac{1}{8}\left(\lambda_\tau\phi^5 + \alpha^2\phi^{-7} - \mathring{R}\phi\right). \quad (3.1.99)$$

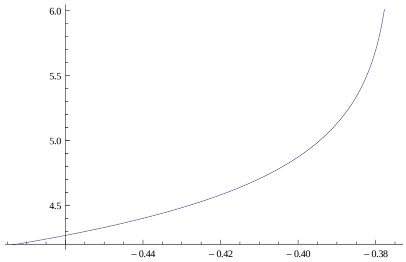

Figure 3.1.4: The minimal period as a function of the energy when $\alpha = 2$ and $\lambda_\tau = 1$. Here $H_2 \simeq -0.4735$ and $H_1 = -0.375$.

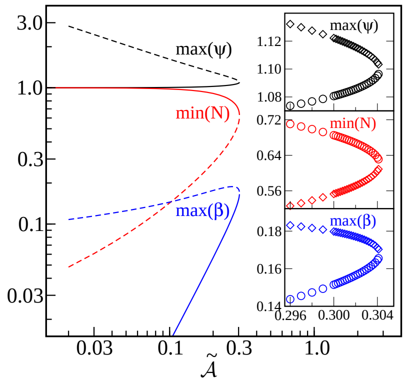

Figure 3.1.5: Example of a fold bifurcation in numerical solutions of constraint equations in the conformal-thin-sandwich formulation, from [349]; ψ is the conformal factor.

As seen in Proposition 3.1.32, there is no solution to (3.1.99) if $\alpha^2 > \frac{4}{27\lambda_\tau^2}\mathring{R}^3$, and a unique solution when $\alpha^2 = \alpha^2_{\max} := \frac{4}{27\lambda_\tau^2}\mathring{R}^3$ which is constant:

$$\phi \equiv \phi_0 = \left(\frac{2\mathring{R}}{3\lambda_\tau}\right)^{1/4}.$$

In accordance with the terminology of [354, Remark 2.3.2], the point $\phi \equiv \phi_0$ is a subcritical fold bifurcation. For lower values of α, we get two branches of constant solutions going down to $\alpha = 0$:

$$\begin{cases} \phi & \equiv \phi_+(\alpha) = \left(\frac{\mathring{R}}{3\lambda_\tau} + \frac{1}{3\lambda_\tau}\left(\frac{N(\alpha)}{2^{1/3}} + \frac{2^{1/3}\mathring{R}^2}{N(\alpha)}\right)\right)^{1/4}, \\ \phi & \equiv \phi_-(\alpha) = \left(\frac{\mathring{R}}{3\lambda_\tau} - \frac{1}{3\lambda_\tau}\left(\bar{j}\frac{2^{1/3}\mathring{R}^2}{N(\alpha)} + j\frac{N(\alpha)}{2^{1/3}}\right)\right)^{1/4}, \end{cases}$$

where $j = (-1 + i\sqrt{3})/2$ and N is given by

$$N(\alpha) := \left(2\mathring{R}^3 - 27\alpha^2\lambda_\tau^2 + 3\sqrt{3}\sqrt{-4\mathring{R}^3\alpha^2\lambda_\tau^2 + 27\alpha^4\lambda_\tau^4}\right)^{1/3}.$$

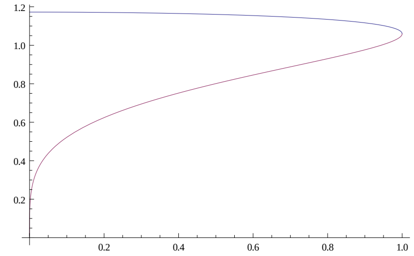

Figure 3.1.6: A plot of $\phi_+(\alpha)$ (blue) and $\phi_-(\alpha)$ (red) with $\lambda_\tau = 1$ and $\mathring{R} = \left(\frac{27}{4}\right)^{1/3}$ as a function of α.

Note that N has nonzero imaginary part. Nevertheless $\phi_\pm$ are real with $0 < \phi_- < \phi_+$. A plot of these solutions is given in Figure 3.1.6. From the shape of the potential, we see that ϕ_- is unstable while ϕ_+ is stable within the class of solutions defined on intervals of $\mathbb{R}$. To find potential bifurcation points, we look for ϕ's solving (3.1.99) such that the linearization of (3.1.99), namely

$$\frac{(2\pi)^2}{\mathring{T}^2}\frac{d^2\xi}{d\psi^2} = -\frac{1}{8}\left(5\lambda_\tau\phi^4 - 7\alpha^2\phi^{-8} - \mathring{R}\right)\xi\,, \qquad (3.1.100)$$

admits a non-trivial solution ξ. We introduce the following function spaces:

$$C^k_{\text{even}}(S^1,\mathbb{R}) := \{\xi \in C^k(S^1,\mathbb{R})\,, \xi \text{ is an even function of } \psi\}\,. \qquad (3.1.101)$$

These function spaces will be important to suppress the S^1-translation-invariance of the set of solutions. We assume first that ϕ is constant.

Proposition 3.1.34 *Bifurcations on the curve $\phi \equiv \phi_+(\alpha)$ occur for the following values of α:*

$$\alpha_{k,\pm} = \pm\frac{2}{3\sqrt{3}\lambda_\tau}\left(\mathring{R} + \left(\frac{2\pi}{\mathring{T}}\right)^2 k^2\right)\sqrt{\mathring{R} - 2\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2}\,, \qquad (3.1.102)$$

where $k \in \mathbb{N}$ is such that $\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2 < \frac{\mathring{R}}{2}$. The values $\alpha_{0,\pm} = \pm\alpha_{\max}$ correspond to fold bifurcations described earlier, while α_k with $k > 1$ correspond to pitchfork bifurcations à la Crandall-Rabinowitz [167]. There are no bifurcations on the curve $\phi \equiv \phi_-(\alpha)$.

We note that the values $\alpha_{k,\pm}$ can be rewritten as

$$\alpha_{k,\pm} = \pm\frac{2}{3\sqrt{3}\lambda_\tau}\sqrt{\mathring{R}^3 - 2\left(\frac{2\pi}{\mathring{T}}\right)^6 k^6 - 3\mathring{R}\left(\frac{2\pi}{\mathring{T}}\right)^4 k^4}\,,$$

from which it follows that all values $\alpha_{k,\pm}$ lie in the range $[-\alpha_{\max}, \alpha_{\max}]$.

Proof: Since ϕ is constant, the right hand side of (3.1.100) is constant. Since ξ is 2π-periodic, this imposes the condition

$$\frac{1}{8}\left(5\lambda_\tau\phi^4 - 7\alpha^2\phi^{-8} - \mathring{R}\right) = k^2\frac{(2\pi)^2}{\mathring{T}^2}\,, \qquad (3.1.103)$$

for some $k \in \mathbb{N}$. The corresponding solution ξ is then, up to multiplication by a constant,

$$\xi = \cos\left(k(\psi - \psi_0)\right) .$$

Values of α and ϕ for which (3.1.99) and (3.1.103) hold can be found as follows: We introduce the polynomials

$$\begin{aligned} P(X) &= -\tfrac{1}{8}\left(\lambda_\tau X^3 + \alpha^2 - \mathring{R}X^2\right) , \\ Q(X) &= \left(\tfrac{2\pi}{\mathring{T}}\right)^2 k^2 X^2 + \tfrac{1}{8}\left(\mathring{R}X^2 + 7\alpha^2 - 5\lambda_\tau X^3\right) , \end{aligned}$$

which are obtained, for P, by multiplying the right hand side of (3.1.99) by ϕ^7 and setting $X = \phi^4$, and similarly for Q by multiplying (3.1.103) by ϕ^8 and setting $X = \phi^4$. The resultant of P and Q is given by

$$-\frac{\alpha^4\lambda_\tau}{4096}\left(8\left(\frac{2\pi}{\mathring{T}}\right)^6 k^6 + 12\left(\frac{2\pi}{\mathring{T}}\right)^4 k^4\mathring{R} - 4\mathring{R}^3 + 27\alpha^2\lambda_\tau^2\right) .$$

It is zero when $\alpha = 0$ or when $\alpha = \alpha_{k,\pm}$ (see (3.1.102)). This means that when $\alpha = \alpha_{k,\pm}$, P and Q have a common root given by

$$X_k = \frac{2}{3\lambda_\tau}\left(\mathring{R} + \left(\frac{2\pi}{\mathring{T}}\right)^2 k^2\right) . \tag{3.1.104}$$

This value of X corresponds to $\phi_k := X_k^{1/4} = \phi_+(\alpha_{k,\pm})$.

It can be checked that

$$V''(\phi_k) = \left(\frac{2\pi}{\mathring{T}}\right)^2 k^2 ,$$

so, for all values of $k > 0$, ϕ_k is a stable local minimum for V. This proves that the bifurcation points along both branches $\phi_\pm(\alpha)$ of constant solutions are located only on the curve $\phi \equiv \phi_+(\alpha)$.

We now check that Proposition C.8.3, p. 274, applies in this case. To get rid of the S^1-invariance, we restrict the space of solutions to the Banach space $C^2_{\text{even}}(S^1, \mathbb{R})$ of (3.1.101) and restrict ourselves to the study of solutions ϕ to (3.1.99) belonging to this space. This restriction is actually not important since any solution ϕ to (3.1.99) admits a point ψ_0 where $\phi'(\psi_0) = 0$. It follows from the Cauchy-Lipschitz theorem that $\phi(\psi_0 + \delta\psi) = \phi(\psi_0 - \delta\psi)$, $\forall \delta\psi \in \mathbb{R}$. Translating the solution, we can assume that $\phi \in C^2_{\text{even}}(S^1, \mathbb{R})$.

We let

$$F : C^2_{\text{even}}(S^1, \mathbb{R}) \cap \{\phi > 0\} \times \mathbb{R} \to C^0_{\text{even}}(S^1, \mathbb{R})$$

be the following operator:

$$F(\phi, \alpha) := \frac{(2\pi)^2}{\mathring{T}^2}\frac{d^2\phi}{d\psi^2} + \frac{1}{8}\left(\lambda_\tau\phi^5 + \alpha^2\phi^{-7} - \mathring{R}\phi\right) .$$

At points $(\phi_k, \alpha_{k,\pm})$, the linearization of F has a 2-dimensional kernel generated by the following two vectors

$$\begin{aligned} v_1 := (\delta\phi_1, \delta\alpha_1) &= (-2\alpha_{k,\pm}\phi_k, 5\lambda_\tau\phi_k^{12} - \mathring{R}\phi_k^8 - 7\alpha_{k,\pm}^2)\,, \\ v_2 := (\delta\phi_2, \delta\alpha_2) &= (\cos(k\psi)\,, 0)\,. \end{aligned}$$

The derivative

$$D_\phi F(\phi_k, \alpha_{k,\pm}) = \frac{(2\pi)^2}{\mathring{T}^2}\left(\frac{d^2}{d\psi^2} + k^2\right)$$

has one-dimensional kernel generated by $\delta\phi_2 = \cos(k\psi)$, and its image is the kernel of the map

$$f \mapsto \int_{S^1} f(\psi)\cos(k\psi)d\psi.$$

This is the reason why we restrict to the space of even functions, otherwise the kernel of $D_\phi F(\phi_k, \alpha_{k,\pm})$ would be two-dimensional, similarly for the cokernel, thus failing to satisfy the assumptions of [167, Theorem 1]. The only condition that remains to be verified is that

$$F''(\phi_k, \alpha_{k,\pm})(v_1, v_2) \notin R(F'(\phi_k, \alpha_{k,\pm})).$$

This actually follows from a straightforward calculation:

$$\begin{aligned} F''(\phi_k, \alpha_{k,\pm})(v_1, v_2) &= \tfrac{\alpha_{k,\pm}}{4}\left(7\mathring{R} - \tfrac{7\alpha^2}{\phi_k^8} - 55\phi_k^4\lambda_\tau\right)\cos(k\psi) \\ &= -\tfrac{\alpha_{k,\pm}}{16}\left(142\left(\tfrac{2\pi}{\mathring{T}}\right)^2 k^2 + 121\mathring{R}\right)\cos(k\psi)\,. \end{aligned}$$

This finishes the proof of Proposition 3.1.34. □

Our next step is to obtain a better understanding of the curves of non-constant solutions. To label the branches solutions, we define the *index* of a solution. Given a non-constant solution ϕ, we have, for all $\psi \in S^1$, $(\phi(\psi)\,, \dot{\phi}(\psi)) \neq (\phi_+(\alpha)\,, 0)$. So a non-constant solution ϕ is a curve in $\mathbb{R}^2 \setminus \{(\phi_+(\alpha)\,, 0)\}$. We define its index as the class of ϕ in

$$\pi_1(\mathbb{R}^2 \setminus \{(\phi_+(\alpha)\,, 0)\}) \simeq \mathbb{Z}\,.$$

This index is constant along a curve of solutions, except at the bifurcation points on the curve $\alpha \mapsto \phi_+(\alpha)$ where the index is not defined. Each solution on the curve $\alpha \mapsto \phi_-(\alpha)$ has index zero while each bifurcation point $(\alpha_{k,\pm}, \phi_+(\alpha_{k,\pm}))$ is the limit point of two curves of non-constant solutions with index k:

PROPOSITION 3.1.35 *For all $k \geq 1$ such that $\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2 < \frac{\mathring{R}}{2}$ there exist two curves*

$$(\alpha_{k,-}, \alpha_{k,+}) \mapsto \phi_{k,\pm}(\alpha) \in C^2_{\text{even}}(S^1, \bar{R})$$

of solutions to (3.1.99) of index k which are $2\pi/k$-periodic. The curves are obtained one from the other as follows:

$$\phi_{k,-}(\psi) \equiv \phi_{k,+}\left(\psi + \frac{\pi}{k}\right)\,.$$

No bifurcations occur on these curves except at the points $\alpha_{k,\pm}$. These solutions, together with the solutions lying on the curves $\alpha \mapsto \phi_\pm(\alpha)$, exhaust the set of solutions to (3.1.99).

Before proving this proposition, we need the following lemma:

LEMMA 3.1.36 *The period $T(\alpha, E)$ of the solutions to (3.1.99) with energy E depends analytically on (α, E) for $\alpha \in (-\alpha_{\max}, \alpha_{\max})$ and $E \in (V(\phi_+(\alpha)), V(\phi_-(\alpha)))$.*

PROOF: The proof is based on a rewriting of (3.1.94):

$$T(\alpha, E) = \sqrt{2}\int_{\phi_{\min}(\alpha,E)}^{\phi_{\max}(\alpha,E)} \frac{d\phi}{\sqrt{E - V(\phi,\alpha)}},$$

where $\phi_{\min}(\alpha, E) < \phi_{\max}(\alpha, E)$ are the two solutions to $V(\phi, \alpha) = E$ in the range $(\phi_-(\alpha), \infty)$. Note that since

$$\frac{\partial V}{\partial \phi}(\phi_{\min}(\alpha, E), E), \ \frac{\partial V}{\partial \phi}(\phi_{\max}(\alpha, E), E) \neq 0,$$

the analytic implicit function theorem shows that $\phi_{\min}$ and $\phi_{\max}$ are analytic functions in α and E. Given α_0 and E_0 satisfying the assumptions of the lemma, we choose an arbitrary value $\phi_0 \in (\phi_{\min}(\alpha_0, E_0), \phi_{\max}(\alpha_0, E_0))$. Given (α, E) close to (α_0, E_0), we split (3.1.94) as follows:

$$T(\alpha, E) = \sqrt{2}\left[\int_{\phi_{\min}(\alpha,E)}^{\phi_0} \frac{d\phi}{\sqrt{E - V(\phi,\alpha)}} + \int_{\phi_0}^{\phi_{\max}(\alpha,E)} \frac{d\phi}{\sqrt{E - V(\phi,\alpha)}}\right].$$

We show how to rewrite the first integral so that its analyticity in the vicinity of (α_0, E_0) becomes apparent. Note that

$$\begin{aligned} E - V(\phi,\alpha) &= V(\phi_{\min}(\alpha, E), \alpha) - V(\phi, \alpha) \\ &= -(\phi - \phi_{\min}(\alpha, E)) \textstyle\int_0^1 \frac{\partial V}{\partial \phi}(\lambda\phi + (1-\lambda)\phi_{\min}(\alpha, E))d\lambda \\ &= -x_-^2 \textstyle\int_0^1 \frac{\partial V}{\partial \phi}(\lambda x_-^2 + \phi_{\min}(\alpha, E), \alpha)d\lambda, \end{aligned}$$

where we set $\phi = x_-^2 + \phi_{\min}(\alpha, E)$. So,

$$\begin{aligned} &\textstyle\int_{\phi_{\min}(\alpha,E)}^{\phi_0} \frac{d\phi}{\sqrt{E-V(\phi,\alpha)}} \\ &\quad = 2\textstyle\int_0^{\sqrt{\phi_0 - \phi_{\min}(\alpha,E)}} \frac{dx_-}{\sqrt{-\int_0^1 \frac{\partial V}{\partial\phi}(\lambda x_-^2 + \phi_{\min}(\alpha,E), \alpha)d\lambda}}. \end{aligned}$$

A similar rewriting of the second integral yields

$$\begin{aligned} &\textstyle\int_{\phi_0}^{\phi_{\max}(\alpha,E)} \frac{d\phi}{\sqrt{E-V(\phi,\alpha)}} \\ &\quad = 2\textstyle\int_0^{\sqrt{\phi_{\max}(\alpha,E) - \phi_0}} \frac{dx_+}{\sqrt{\int_0^1 \frac{\partial V}{\partial\phi}((1-\lambda)x_+^2 + \phi_{\max}(\alpha,E), \alpha)d\lambda}}, \end{aligned}$$

where $\phi = \phi_{\max}(\alpha, E) - x_+^2$. The function

$$(E, \alpha, x_-) \mapsto -\int_0^1 \frac{\partial V}{\partial \phi}(\lambda x_-^2 + \phi_{\min}(\alpha, E)\,, \alpha) d\lambda$$

is clearly analytic and positive for all $x_+ \in \left[0, \sqrt{\phi_{\max}(\alpha, E) - \phi_0}\right]$ since

$$\frac{\partial V}{\partial \phi}(\phi_{\min}(\alpha, E)\,, \alpha) < 0\,.$$

This is enough to conclude that

$$\begin{aligned}
&\int_{\phi_{\min}(\alpha,E)}^{\phi_0} \frac{d\phi}{\sqrt{E - V(\phi, \alpha)}} \\
&\quad = 2\int_0^{\sqrt{\phi_0 - \phi_{\min}(\alpha,E)}} \frac{dx_-}{\sqrt{-\int_0^1 \frac{\partial V}{\partial \phi}(\lambda x_-^2 + \phi_{\min}(\alpha, E)\,, \alpha) d\lambda}}
\end{aligned}$$

is analytic in (α, E) in a neighborhood of (α_0, E_0). Similar arguments apply for the second integral. □

Proof of Proposition 3.1.35. We first remark that since the energy H defined in (3.1.88) is conserved, solutions $\phi(\psi)$ to (3.1.99) with index k are actually periodic with minimal period $2\pi/k$. We select a non-constant solution (α_0, ϕ_0) with index k and energy E_{α_0}. Since the index is locally constant, all solutions nearby, potentially with a different α, are $2\pi/k$-periodic.

From our previous analysis, for α between $\pm\alpha_{\max}$ the period $T_{\alpha,\lambda_\tau,\mathring{R}}(E)$ of a solution ϕ with energy $E = H(\phi, \dot\phi)$ is strictly increasing with respect to E. Since we are restricting ourselves to solutions belonging to $C^2_{\mathrm{even}}(S^1, \mathbb{R})$, we have $\dot\phi(0) = 0$ so $E = V(\phi(0))$. Note that since the derivative of T with respect to E is strictly positive, for all α near α_0 there exists a unique value $E_\alpha \in (V(\phi_+(\alpha))\,, V(\phi_-(\alpha)))$ of the energy so that the solution with energy E_α has period $2\pi/k$. E_α depends smoothly on α by Lemma 3.1.36. We let $\phi_-^*(\alpha)$ denote the unique solution $\phi > \phi_+(\alpha)$ to $V(\phi) = V(\phi_-(\alpha))$. From the shape of the potential V, there exist exactly two values $\phi_{\min}(\alpha, k)$, $\phi_{\max}(\alpha, k)$ so that

$$\phi_-(\alpha) < \phi_{\min}(\alpha, k) < \phi_+(\alpha) < \phi_{\max}(\alpha, k) < \phi_-^*(\alpha)\,,$$

and

$$V(\phi_{\min}(\alpha, k)) = V(\phi_{\max}(\alpha, k)) = E_\alpha\,.$$

These two values map smoothly to two solutions of (3.1.99). Thus we have proven that near a value α_0 for which there exists a solution ϕ_0 with index k, there exist two and only two distinct curves of solutions with index k.

Note that a $2\pi/k$-periodic solution increases from $\phi_{\min}(\alpha, k)$ to $\phi_{\max}(\alpha, k)$ in an interval of length π/k and then decreases from $\phi_{\max}(\alpha, k)$ to $\phi_{\min}(\alpha, k)$ in the same amount of time. Hence, translating the solution ϕ with $\phi(0) = \phi_{\min}(\alpha, k)$ by π/k we get the solution $\phi(0) = \phi_{\max}(\alpha, k)$ and vice versa.

Let

$$I_k \subset \mathbb{R}$$

denote the set of values for which there exists a pair of solutions of index k. The previous analysis shows that I_k is an open subset. Assume that I_k contains a boundary point α_∞ which is not $\alpha_{k,\pm}$, Let $\alpha_i \in I_k$ be such that $\alpha_i \to \alpha_\infty$. The corresponding functions ϕ_i with period $2\pi/k$ all have $\phi_i(0) \in (\phi_-(\alpha_i)\,,\phi^*_-(\alpha_i))$. Without loss of generality, we can assume that $\phi_i(0)$ converges to some limit $\phi_\infty(0) \in [\phi_-(\alpha_\infty)\,,\phi^*_-(\alpha_\infty)]$. $\phi_\infty(0)$ cannot be $\phi_-(\alpha_\infty)$ nor $\phi^*_-(\alpha_\infty)$ since the period of the functions ϕ_i's would grow unbounded.

If $\phi_\infty(0) \neq \phi_+(\alpha_\infty)$, by continuity of the period with respect to initial data, the solution ϕ_∞ to (3.1.99) is periodic with period $2\pi/k$. The previous argument shows that α_∞ is an interior point of I_k, a contradiction. Thus the only endpoints of I_k are on the curve $\phi \equiv \phi_+(\alpha)$, this is to say bifurcation points we found in Proposition 3.1.34.

The question now arises, whether new solutions occur with values of α larger than α_k, or smaller, or both. We will see that, for all k such that $\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2 < \frac{\mathring{R}}{2}$, I_k contains an interval of the form $(\alpha_{k,+}-\epsilon, \alpha_{k,+})$. Since (3.1.99) only depends on α^2, I_k also contains the interval $(\alpha_{k,-}, \alpha_{k,-}+\epsilon)$. We let

$$(-\delta,\delta) \ni t \mapsto (\alpha(t)\,,\phi_k(t))$$

denote a differentiable curve of non-constant solutions passing through $(\alpha_{k,+},\tilde\phi_k)$ at the value $t=0$ of parameter t, where

$$\tilde\phi_k := \left[\frac{2}{3\lambda_\tau}\left(\mathring{R}+\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2\right)\right]^{1/4},$$

compare (3.1.104). The existence of this curve has been established in Proposition 3.1.34. We expand α and ϕ_k in terms of the parameter t as follows:

$$\left\{\begin{array}{ll} \alpha(t) & = \alpha_{k,+} + t\tilde\alpha^1_{k,+} + t^2\tilde\alpha^2_{k,+} + t^3\tilde\alpha^3_{k,+} + O(t^4)\,, \\ \phi_k(t) & = \tilde\phi_k + t\tilde\phi^1_k + t^2\tilde\phi^2_k + t^3\tilde\phi^3_k + O(t^4)\,, \end{array}\right. \tag{3.1.105}$$

where $\tilde\phi^1_k = \cos(k\psi)$ and insert this development in (3.1.99). From the terms linear in t in (3.1.99), it follows that $\tilde\alpha^1_{k,+}=0$. Looking at terms of order t^2, we find that $\tilde\phi^2_k = \lambda\cos(2k\psi)+\epsilon\cos(k\psi)+\mu$, where

$$\lambda = \frac{1}{12k^2}\left(\frac{3\lambda_\tau}{2}\right)^{1/4} \frac{4\left(\frac{\mathring{T}}{2\pi}\right)^2\mathring{R}-3k^2}{\left(\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2+\mathring{R}\right)^{1/4}},$$

and

$$\mu = -\frac{(6\lambda_\tau)^{1/4}}{8}\times$$
$$\frac{\left(4\mathring{R}-3\left(\frac{2\pi}{\mathring{T}}\right)^2k^2\right)\sqrt{2}\sqrt{\left(\frac{2\pi}{\mathring{T}}\right)^2k^2+\mathring{R}}+\sqrt{\lambda_\tau}\sqrt{\mathring{R}-2\left(\frac{2\pi}{\mathring{T}}\right)^2k^2}\,\tilde\alpha^2_{k,+}}{\left(\left(\frac{2\pi}{\mathring{T}}\right)^2k^2+\mathring{R}\right)^{3/4}\left(\frac{2\pi}{\mathring{T}}\right)^2k^2}.$$

There is no loss of generality in assuming that $\epsilon = 0$ since this can be reabsorbed in the definition of t. More importantly, μ depends on $\tilde{\alpha}^2_{k,+}$, this is why we need to consider terms cubic in t in (3.1.99). The expression for $\tilde{\alpha}^2_{k,+}$ is obtained by setting to zero the coefficient of $t^3 \cos(k\psi)$ in (3.1.99):

$$\tilde{\alpha}^2_{k,+} = -\frac{\sqrt{2}}{3\sqrt{\lambda_\tau}} \frac{10\mathring{R}^2 - 9\mathring{R}\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2 - 12\left(\frac{2\pi}{\mathring{T}}\right)^4 k^4}{\sqrt{\left(\mathring{R} - 2\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2\right)\left(\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2 + \mathring{R}\right)}}.$$

The numerator of this expression is decreasing on $[0, \infty)$ when seen as a function of k so it is bounded from below from the value it takes when $\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2 = \frac{\mathring{R}}{2}$, which is $5\mathring{R}/2$. Since we have $\alpha = \alpha_{k,+} + \tilde{\alpha}^2_{k,+} t^2 + O(t^3)$ with $\tilde{\alpha}^2_{k,+} < 0$, we deduce that $\alpha(t) < \alpha_{k,+}$ for small values of the parameter t. This concludes the proof of the claim.

I_k being connected with endpoints $\alpha_{k,\pm}$, we conclude that $I_k = (\alpha_{k,-}, \alpha_{k,+})$. □

An illustration of the last two propositions is given in Figure 3.1.7.

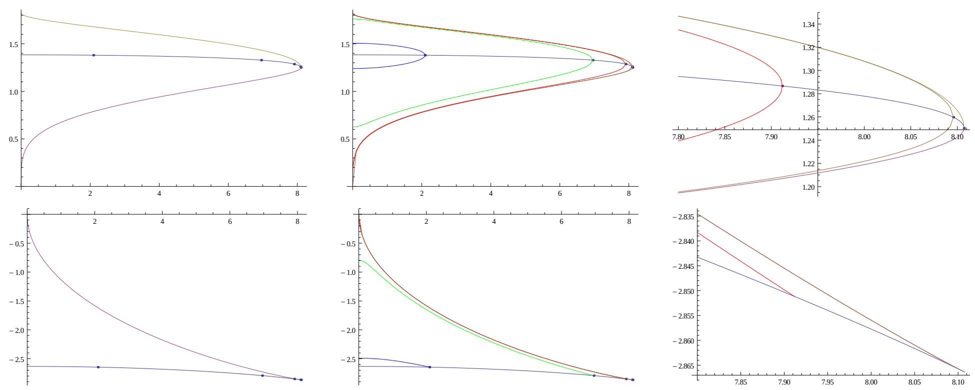

Figure 3.1.7: An illustration of Propositions 3.1.34 and 3.1.35 with $\mathring{T} = 2\pi$, $\mathring{R} = 33$ and $\lambda_\tau = 9$. On all plots α varies along the horizontal axis. On the first line, the plot on the left shows the three curves of $\phi_-(\alpha)$ (magenta), $\phi_+(\alpha)$ (marine blue) and $\phi^*_-(\alpha)$ (yellow). Dots indicate the position of the bifurcation points. Where curves merge, there are actually two points which almost coincide. One corresponds to the fold bifurcation while the second one is a pitchfork bifurcation. All other points are pitchfork bifurcations. The second plot shows the value at the origin of the solutions of index 1 (brown), 2 (red), 3 (green) and 4 (blue). And the third plot is a zoom of the second one near $\alpha_{\max}$. The graphs on the second line show the energy $H(\phi, \dot{\phi})$ of the solutions.

Summarising, we have proved:

Theorem 3.1.37 *Let $\mathring{T}, \mathring{R} \in \mathbb{R}^+$, and let $\lambda_\tau > 0$ be defined in (3.1.83). Con-*

sider a metric $\mathring{g} = g_{\mathring{T},\mathring{R}}$*, and define* $k_{\max}$ *to be the largest integer* k *such that*

$$2\left(\frac{2\pi}{\mathring{T}}\right)^2 k^2 < \mathring{R}\,.$$

Depending on the value of $\alpha \in \mathbb{R}$*, Equation (3.1.99) has, up to translation in the* S^1*-direction,*

- *no solutions if* $\alpha \notin [-\alpha_{\max}, \alpha_{\max}]$*, where* $\alpha_{\max} = \frac{2}{3\sqrt{3}\lambda_\tau}\mathring{R}^{3/2}$*,*
- *only one solution if* $\alpha = \pm\alpha_{\max}$*,*
- *two constant solutions and* k *non constant* $SO(3)$*-symmetric solutions of index* 1*, ...,* k *when* $\alpha \in (\alpha_{k,-}, \alpha_{k+1,-}] \cup [\alpha_{k+1,+}, \alpha_{k,+})$ *if* $k < k_{\max}$ *or when* $\alpha \in (\alpha_{k,-}, \alpha_{k,+})$ *if* $k = k_{\max}$*.*

It follows from the (generalised) Birkhoff Theorem that the spacetimes obtained by evolution of the current initial data are quotients of either the Nariai spacetime or the Schwarzschild-de Sitter spacetime.

The set of solutions of the constraint equations just constructed shows explicitly that:

PROPOSITION 3.1.38 *There exist spatially compact CMC initial data sets at which the set of solutions of the vacuum constraint equations is* not *a manifold.*

It turns out that the bifurcation behaviour above will not occur in generic situations, and is closely related to the fact that the initial data just constructed have non-trivial KIDs (cf. Definition 3.5.8, p. 138 below). Indeed, the set of solutions of vacuum constraint equations forms a manifold at initial data without KIDs, as follows from the implicit function theorem, since the linearisation of the constraint map is an invertible operator away from such data.

More information about the structure of the initial data set at data with isometries can be found in [18, 19].

3.1.9 Matter fields

The conformal method easily extends to CMC constraint equations for some non-vacuum initial data, e.g. the Einstein-Maxwell system [243] where one obtains results very similar to those of Theorem 3.1.20. However, other important examples, such as the Einstein-scalar field system [99–101, 231], require more effort and are not as fully understood.

Recall that the energy density μ and the energy-momentum density J of matter fields is related to the geometry through the formulae

$$16\pi\mu \;:=\; R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 - 2\Lambda \qquad (3.1.106)$$

$$8\pi J^i \;:=\; D_i(K^{ij} - \mathrm{tr}_g K g^{ij})\,. \qquad (3.1.107)$$

If μ and J have been prescribed, this becomes the system of equations (3.1.17) and (3.1.25) for ϕ and $\widetilde{L}$

$$\begin{aligned}\Delta_{\widetilde{g}}\phi - \frac{n-2}{4(n-1)}\widetilde{R}\phi &= -\sigma^2\phi^{\frac{2-3n}{n-2}} + \beta\phi^{\frac{n+2}{n-2}} - \frac{4(n-2)}{(n-1)}\phi^{\frac{n+2}{n-2}}\pi\mu\,, \quad (3.1.108)\\ \tilde{D}_i\widetilde{L}^{ij} &= 8\pi\phi^{\frac{2(n+2)}{n-2}}J^i + \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\tilde{D}^j\tau\,. \quad (3.1.109)\end{aligned}$$

It is important to realize that the conformal method has no physical contents, and is an ansatz for constructing solutions of the constraint equations. The question of scaling properties of μ and J under conformal transformations is thus largely a matter of convenience. For instance, *when* $d\tau = 0$, a convenient prescription for J is to set

$$\tilde{J}^i = \phi^{\frac{2(n+2)}{n-2}}J^i\,, \qquad (3.1.110)$$

and to view $\tilde{J}$ as free data, for then (3.1.109) decouples from (3.1.108). There is then a natural rescaling of μ which arises from the *dominant energy condition* $\mu^2 \geq g_{ij}J^iJ^j$: since $g_{ij} = \phi^{\frac{4}{n-2}}\widetilde{g}_{ij}$, under (3.1.110) we have

$$g_{ij}J^iJ^j = \phi^{\frac{4}{n-2}-\frac{4(n+2)}{n-2}}\widetilde{g}_{ij}\tilde{J}^i\tilde{J}^j = \phi^{-\frac{4(n+1)}{n-2}}\widetilde{g}_{ij}\tilde{J}^i\tilde{J}^j\,,$$

and so the dominant energy condition will be covariant under these rescalings if we set

$$\tilde{\mu} = \phi^{\frac{2(n+1)}{n-2}}\mu\,, \qquad (3.1.111)$$

viewing $\tilde{\mu}$ as the free data, and μ as the derived ones. The scaling (3.1.110)-(3.1.111) is known to us from [92], where it has been termed *York scaling.* With those definitions (3.1.108)-(3.1.109) become

$$\begin{aligned}\Delta_{\widetilde{g}}\phi - \frac{n-2}{4(n-1)}\widetilde{R}\phi &= -\sigma^2\phi^{\frac{2-3n}{n-2}} + \beta\phi^{\frac{n+2}{n-2}}\\ &\quad -\frac{4(n-2)}{(n-1)}\phi^{-\frac{n}{n-2}}\pi\tilde{\mu}\,. \quad (3.1.112)\\ \tilde{D}_i\widetilde{L}^{ij} &= 8\pi\tilde{J}^i + \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\tilde{D}^j\tau\,. \quad (3.1.113)\end{aligned}$$

Maxwell fields

Let the (spacelike) initial data hypersurface $\mathscr{S}$ be given by the equation $x^0 = 0$, define

$$\alpha := \frac{1}{\sqrt{-g^{00}}}\,, \qquad (3.1.114)$$

so that the future directed unit normal N to $\mathscr{S}$ has covariant components

$$N_\mu dx^\mu = -\alpha dx^0\,.$$

When constructing initial data involving Maxwell equations, one needs to keep in mind that the Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 4\pi J^\nu_M\,, \qquad \nabla_\mu \star F^{\mu\nu} = 0\,, \qquad (3.1.115)$$

where $\star F^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, and where J^μ_M denotes the charge-current four-vector density, imply constraints equations on the electric and magnetic fields E^i and B^i :

$$E^i\partial_i := N_\mu F^{i\mu}\partial_i = \alpha F^{0i}\partial_i\,, \qquad B^i\partial_i := \alpha \star F^{i0}\partial_i\,. \tag{3.1.116}$$

Indeed, we then have

$$D_iE^i = 4\pi\alpha J^0_M\,, \qquad D_iB^i = 0\,. \tag{3.1.117}$$

Since $D_iB^i = \partial_i(\sqrt{\det g_{k\ell}}B^i)/\sqrt{\det g_{k\ell}}$, the second equation above is immediately covariant under conformal rescalings of the metric if B^i is taken of the form

$$B^i := \phi^{-\frac{2n}{n-2}}\tilde B^i\,, \tag{3.1.118}$$

where $\tilde B^i$ is divergence-free in the metric $\tilde g$. Likewise the first equation in (3.1.117) will be conformally covariant if

$$E^i := \phi^{-\frac{2n}{n-2}}\tilde E^i\,, \quad \alpha J^i_M := \phi^{-\frac{2n}{n-2}}\tilde\alpha\tilde J^i_M\,, \quad \tilde D_i\tilde E^i = 4\pi\tilde\alpha\tilde J^0_M\,. \tag{3.1.119}$$

The energy-density of the Maxwell fields is (keeping in mind that $n = 3$ here)

$$\begin{aligned}\mu &:= \frac{1}{8\pi}\left(g_{ij}E^iE^j + g_{ij}B^iB^j\right)\\ &= \phi^{-8}\underbrace{\frac{1}{8\pi}\left(\tilde g_{ij}\tilde E^i\tilde E^j + \tilde g_{ij}\tilde B^i\tilde B^j\right)}_{=:\tilde\mu}\,.\end{aligned} \tag{3.1.120}$$

This motivates the York scaling (3.1.111). One similarly checks the York-scaling property for the energy-momentum three-vector J^i.

3.2 Non-compact initial data sets: an overview

So far we have been considering initial data sets on compact manifolds. However, there exist *noncompact* classes of data which are of interest. Indeed, apart from compact manifolds, several classes of general relativistic initial data sets have been studied with various degrees of completeness, including:

1. manifolds with *asymptotically flat* ends,
2. manifolds with *asymptotically hyperbolic* ends,
3. manifolds with ends of *cylindrical type*,
4. manifolds with *asymptotically periodic* ends,
5. manifolds with *cylindrically bounded* ends,
6. manifolds with *asymptotically conical* ends.

The aim of this section is to review those models.

Some reminders might be in order. A vacuum initial data set (M, g, K) is a triple consisting of an n-dimensional manifold M, a Riemannian metric on g, and a symmetric two-covariant tensor K. One moreover requires that the vacuum constraint equations hold:

$$R(g) = 2\Lambda + |K|_g^2 - (\mathrm{tr}_g K)^2 \,, \tag{3.2.1}$$

$$D_j K^j{}_k - D_k K^j{}_j = 0 \,. \tag{3.2.2}$$

The reader will note that we have allowed for a non-zero cosmological constant $\Lambda \in \mathbb{R}$. The hypothesis that $\Lambda = 0$ is adequate when describing isolated gravitating systems such as the solar system; $\Lambda > 0$ seems to be needed in cosmology in view of the observations of the rate of change of the Hubble constant [372, 413]; finally, a negative cosmological constant appears naturally in many models of theoretical physics, such as string theory or supergravity. For those reasons it is of interest to consider all possible values of Λ.

A *CMC initial data set* is one where $\mathrm{tr}_g K$ is constant; data are called *maximal* if $\mathrm{tr}_g K$ is identically zero. A *time-symmetric* data set is one where K vanishes identically. In this case (3.2.1)-(3.2.2) reduces to the requirement that the scalar curvature of g be constant:

$$R(g) = 2\Lambda \,.$$

3.2.1 Non-compact manifolds with constant positive scalar curvature

The topological classification of compact three-manifolds with positive scalar curvature generalises to the following non-compact setting: One says that a Riemannian metric g has bounded geometry if g has bounded sectional curvatures and injectivity radius bounded away from zero. Using Ricci flow (the short-time existence of which is guaranteed in the setting by the work of Shi [394]), one has [60]:

THEOREM 3.2.1 *Let $\mathscr{S}$ be a connected, orientable three-manifold which carries a complete Riemannian metric of bounded geometry and uniformly positive scalar curvature. Then there is a finite collection $\mathcal{F}$ of spherical manifolds such that $\mathscr{S}$ is an (infinite) connected sum of copies of $S^1 \times S^2$ and members of $\mathcal{F}$.*

The cylinders $(\mathbb{R} \times S^{n-1}, dx^2 + \mathring{g}_{n-1})$ provide examples of non-compact manifolds with positive scalar curvature. The underlying manifold can be viewed as S^n from which the north and south poles have been removed. In Theorem 3.2.1 they are viewed as an infinite connected sum of $S^1 \times S^2$.

INCIDENTALLY: Completing the initial data $(\mathbb{R} \times S^2, dx^2 + \mathring{g}_2)$ with a suitable extrinsic curvature tensor, when $\Lambda = 0$ the corresponding evolution leads to the interior Schwarzschild metric: for $t < 2m$

$$g = -\frac{1}{\frac{2m}{t} - 1} dt^2 + \left(\frac{2m}{t} - 1\right) dx^2 + t^2 \mathring{g}_2 \,. \tag{3.2.3}$$

When $\Lambda > 0$, vanishing extrinsic curvature leads to the *Nariai metrics* [335]:

$$g = \frac{1}{\Lambda}\left(-dt^2 + \cosh^2 t\, d\rho^2 + \mathring{g}_2\right) \tag{3.2.4}$$

(compare [73]). Further related examples are discussed in Section 3.1.8. □

Another class of positive scalar constant curvature metrics on $\mathbb{R} \times S^{n-1}$ is provided by the *Delaunay metrics* of Section 3.1.7, when the coordinate y of (3.1.73) is not periodically identified, but runs over $\mathbb{R}$.

Example 3.2.3 The Delaunay metrics provide an example of complete spherically symmetric metrics with positive scalar curvature. Large classes of metrics with the last set of properties can be constructed as follows: Recall that (see (C.2.14), p. 268, with g and $\tilde{g}$ there interchanged)

$$g_{ij} = \phi^{\frac{4}{n-2}}\tilde{g}_{ij} \quad \Longrightarrow \quad R = \phi^{-\frac{4}{n-2}}\left(\widetilde{R} - \frac{4(n-1)}{(n-2)\phi}\Delta_{\tilde{g}}\phi\right). \tag{3.2.5}$$

So if $\tilde{g}$ is the flat Euclidean metric δ, then g will have non-negative scalar curvature if and only if

$$\Delta_\delta \phi \le 0\,.$$

Smooth spherically symmetric solutions of this inequality which are regular at the origin and which asymptote to one at infinity can be obtained by setting

$$\phi = 1 + \frac{1}{r^{n-2}}\int_0^r f(s)s^{n-1}ds + \int_r^\infty f(t)tdt\,, \tag{3.2.6}$$

where f is any smooth positive function such that $\int_0^\infty f(r)r^{n-1}dr$ is finite. (The solution will be be asymptotically flat in the usual sense if, e.g., f is compactly supported.) Indeed, we have

$$\phi' = -\frac{(n-2)}{r^{n-1}}\int_0^r f(s)s^{n-1}ds\,, \tag{3.2.7}$$

hence

$$\Delta_\delta\phi = r^{-(n-1)}\partial_r(r^{n-1}\partial_r\phi) = -(n-2)f\,,$$

and so the sign of the scalar curvature of $\phi^{4/(n-2)}\delta$ is determined by that of $-f$.

In the region where f vanishes we have $R = 0$, which provides vacuum regions. Connected regions of non-zero f can be thought of as a central star, or shells of matter.

It is interesting to enquire about existence of spherically symmetric *minimal surfaces* for such metrics. Now a strict definition of a minimal surface is the requirement of minimum area amongst nearby competing surfaces, but any critical point of the area functional is also often called "minimal", and we will follow this practice.

In the case under consideration, the area of spheres of constant radius is proportional to $(r^2\phi^{4/(n-2)})^{(n-1)/2}$, and so the area will have vanishing derivative with respect to r if and only if

$$\begin{aligned}(r\phi^{2/(n-2)})' = 0 \quad &\Longleftrightarrow \quad -\frac{2r\phi'}{(n-2)} = \phi \\ &\Longleftrightarrow \quad \underbrace{\frac{1}{r^{n-2}}\int_0^r f(s)s^{n-1}ds}_{=:h(r)} = \underbrace{1 + \int_r^\infty f(t)tdt}_{=:g(r)}\,. \end{aligned} \tag{3.2.8}$$

This formula can be used to construct solutions containing minimal spheres as follows: Suppose that the function f of (3.2.6) is constant, say $f = f_0 > 0$, on an interval $[0, r_1]$. Then $h(r) = f_0 r^2/n$ increases while $g(r) = g(0) - f_0 r^2/2$ decreases, so equality will be achieved precisely once somewhere before r_1 if

$$\frac{f_0 r_1^2}{n} > 1 + \int_{r_1}^{\infty} f(r) r dr\,. \tag{3.2.9}$$

The value of $r \in [0, r_1]$ at which the equality in (3.2.8) holds provides our first "minimal" surface (which actually locally maximizes area). We then let f drop smoothly to zero on $[r_1, r_2]$, for some $r_2 > r_1$, and keep f equal to zero on $[r_2, r_3)$, with r_3 possibly equal to ∞. On $[r_1, r_2]$ the function h continues to increase while the function g continues to decrease, so there cannot be any further minimal surfaces in this interval. On $[r_2, r_3)$ the function

$$h(r) = r^{-(n-2)} \int_0^{r_2} f(s) s^{n-1} ds$$

decreases as $r^{-(n-2)}$ while $g(r)$ remains constant, so equality in (3.2.8) will be attained before r_3 if

$$r_3^{-(n-2)} \int_0^{r_2} f(s) s^{n-1} ds < 1 + \int_{r_3}^{\infty} f(r) r\, dr\,. \tag{3.2.10}$$

Note that the choice of r_3 does not affect (3.2.9) insofar as f vanishes on $[r_2, r_3)$.

In particular if we choose $r_3 = \infty$, we can first choose any central value f_0, and then choose r_1 so that $f_0 r_1^2/n > 2$. Choosing the intermediate region $[r_1, r_2]$ small enough so that the integral at the right-hand side is smaller than one, it follows from (3.2.9) that the resulting metric will have precise one locally maximal sphere somewhere before r_1. For $r > r_2$ the metric is the (asymptotically flat) space-Schwarzschild metric with a second minimal sphere somewhere in $[r_2, \infty)$.

□

3.2.2 The barrier method on general manifolds

We show here, following [148], how to establish existence of solutions to certain linear and semilinear elliptic equations which arise in various geometric settings on non-compact manifolds.

Barrier functions

In order to construct solutions of the Lichnerowicz equation we will use the monotone iteration scheme, as adapted to general manifolds in Theorem 3.2.7 below. This requires sub- and supersolutions of the equation, also referred to as *barriers*, and the aim of this subsection is to show how to construct such barriers in situations of interest.

It is convenient to start with some terminology. Let L be a partial differential operator with formal L^2-adjoint L^*, and let ϕ be a continuous function. We will say that

$$L\phi \geq \psi \textit{ in a weak sense,}$$

or *weakly*, if for every compactly supported smooth function we have

$$\int_M \phi L^* \varphi \geq \int_M \psi \varphi\,. \tag{3.2.11}$$

The simplest example of usefulness of continuous weak barriers is provided by linear equations:

PROPOSITION 3.2.4 *Let $(M, \widetilde{g})$ be a smooth Riemannian manifold, and h any nonnegative smooth function on M. Suppose that f is smooth and there exist two C^0 functions $\underline{\tilde{\phi}} \le \overline{\tilde{\phi}}$ which satisfy*

$$(\Delta_{\widetilde{g}} - h)\underline{\tilde{\phi}} \ge f, \qquad (\Delta_{\widetilde{g}} - h)\overline{\tilde{\phi}} \le f$$

in the weak sense. Then there exists a smooth function u such that

$$(\Delta_{\widetilde{g}} - h)u = f \quad \text{and} \quad \underline{\tilde{\phi}} \le u \le \overline{\tilde{\phi}}. \tag{3.2.12}$$

PROOF: Choose an exhaustion of M by a sequence of compact manifolds with smooth boundary M_j. Because of the sign of h, the inhomogeneous Dirichlet problem

$$(\Delta_{\widetilde{g}} - h)u_j = f, \quad u|_{\partial M_j} = \underline{\tilde{\phi}}\Big|_{\partial M_j}$$

is uniquely solvable for every j. By the standard (weak) comparison principle, $\underline{\tilde{\phi}} \le u_j \le \overline{\tilde{\phi}}$ on M_j.

Letting $j \to \infty$, we see that the sequence $\{u_j\}$ is uniformly bounded on every compact set $K \subset M$. Using local elliptic estimates and the Arzela-Ascoli theorem on each M_i, a diagonalisation argument shows that some subsequence $u_{j'}$ converges in C^∞ on every compact set. The limit function u satisfies the correct equation and is sandwiched between the two barriers $\underline{\tilde{\phi}}$ and $\overline{\tilde{\phi}}$. □

Note that there is no a priori reason for this solution to be unique, although this may be true in certain circumstances.

A useful aspect of Proposition 3.2.4 is that the barriers need only be continuous rather than C^2. The simplest situation in which such a more relaxed hypothesis may arise is the following:

LEMMA 3.2.5 *Suppose that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are two subsolutions for the equation $(\Delta_{\widetilde{g}} - h)u = f$. Then $\tilde{\phi} = \max\{\tilde{\phi}_1, \tilde{\phi}_2\}$ is also a subsolution in the sense that if u is a solution to this equation on a domain D and if $u \ge \tilde{\phi}$ on ∂D then $u \ge \tilde{\phi}$ on D. Similarly if $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are supersolutions, then $\tilde{\phi} = \min\{\tilde{\phi}_1, \tilde{\phi}_2\}$ is also a supersolution.*

PROOF: Observe that $u \ge \tilde{\phi} \ge \tilde{\phi}_j$ on ∂D, hence $u \ge \tilde{\phi}_j$ on D. Since this is true for $j = 1, 2$, we have $u \ge \tilde{\phi}$ on D too. □

In many applications one of the subsolutions, say $\tilde{\phi}_2$, is typically only defined on some open subset of M rather than on the whole space, so the argument above does not quite work. We therefore need to formulate this result in a slightly more general way.

As in the proof of Proposition 3.2.4, it suffices to consider barriers on a compact manifold with boundary, since when M is noncompact we construct solutions on a exhaustion of M by compact manifolds with boundary M_j, and then extract a convergent sequence using Arzela-Ascoli.

Thus let M be a compact manifold with boundary, and suppose that $\partial M = \partial_1 M \cup \partial_2 M$ is a union of two components (which may themselves decompose further). Suppose that $\mathscr{U}$ is a relatively open set M containing $\partial_2 M$, but which has closure disjoint from $\partial_1 M$. Let ϕ_1 be a subsolution for the operator $L = \Delta - h$ which is defined on all of M, and ϕ_2 a subsolution for L which is only defined on $\mathcal{U}$. We assume that ϕ_1 and ϕ_2 are continuous and subsolutions in the weak sense.

Consider the open set $\mathcal{V} = \{\phi_2 > \phi_1\}$, and let us suppose that

$$\partial_2 M \subset \mathscr{V} \subset \overline{\mathscr{V}} \subset \mathscr{U}.$$

Define the function

$$\psi = \begin{cases} \max\{\phi_1, \phi_2\} & \text{in } \mathscr{U}, \\ \phi_1 & \text{in } M \setminus \mathscr{U}. \end{cases}$$

LEMMA 3.2.6 *This function ψ is continuous and a weak global subsolution for L on M.*

PROOF: The continuity of ψ is clear from the fact that $\phi_1 > \phi_2$ in a neighbourhood of the 'inner' boundary of $\mathscr{U}$, i.e. $\partial\mathscr{U} \setminus \partial_2 M$.

Next, suppose that u is a solution defined on all of M and that $u \geq \psi$ on ∂M. Thus $u \geq \phi_1$ on $\partial_1 M$ and $u \geq \phi_2$ on $\partial_2 M$. By asumption, $\phi_2 \geq \phi_1$ on $\partial_2 M$ so $u \geq \phi_1$ on all of ∂M, hence since ϕ_1 is a subsolution, $u \geq \phi_1$ on all of M. In particular, $u \geq \phi_1$ on the set $Y = \{\phi_1 = \phi_2\}$.

The set Y is compactly contained in $\mathscr{U}$, and furthermore, $\partial\overline{\mathscr{V}} = Y \cup \partial_2 M$. This means that $u \geq \phi_2$ on $\partial\overline{\mathscr{V}}$, hence $u \geq \phi_2$ on all of $\mathscr{V}$. Putting these facts together yields that $u \geq \psi$ on all of M . □

Examples of barrier functions

Now, suppose that $(M, \widetilde{g})$ is a complete manifold with a finite number of ends, each of one of the six types listed on p. 103. We illustrate how the situation above arises by describing standard types of sub- and supersolutions for the problem

$$(\Delta_{\widetilde{g}} - h)u = f \tag{3.2.13}$$

on each of these types of ends. Here f is a $C^{0,\mu}$ function which satisfies certain weighted decay conditions which are implicit in each case. In each of these geometries, the end E is a product $\mathbb{R}^+ \times N$, where N is a compact manifold, but there is a different asymptotic structure each time.

1. *Asymptotically conic ends* (this includes asymptotically Euclidean ends): Here $\tilde{g}$ approaches the conic metric $g_c := dr^2 + r^2 h$ for some metric h on N in the following sense. Using a fixed coordinate system $(y_1, \ldots, y_{n-1})$ on N, augmented by $r = y_0 \geq 1$, we assume that

$$\widetilde{g}_{ij} - (g_c)_{ij} = o(1)\ , \quad \partial_k(\widetilde{g}_{ij} - (g_c)_{ij}) = o(r^{-1}). \tag{3.2.14}$$

Then

$$u_{\pm} = \pm C\|r^{\alpha+2} f\|_{L^\infty} r^{-\alpha}$$

are sub- and supersolutions of (3.2.13) when $r \geq r_0$ and $C \gg 0$, provided $\alpha \in (0, n-2)$.

2. *Conformally compact (asymptotically hyperbolic) ends*: We now assume that that $x \in (0, x_0]$ and that $\overline{g} = x^2 \widetilde{g}$ has components approaching those of $dx^2 + \mathring{g}$ as $x \to 0$, where $\mathring{g}$ is a Riemannian metric on N, and that the derivatives of the coordinate components of $\overline{g}$ are $o(x^{-1})$. Now, if $\nu \in (0, n-1)$, the functions

$$u_\pm = \pm \|x^{-\nu} f\|_{L^\infty} x^\nu$$

are sub- and supersolutions of (3.2.13) when $x_0 \ll 1$ and C is sufficiently large.

3. *Asymptotically cylindrical ends*: Assume that on $[x_0, \infty) \times N$ the metric components of $\widetilde{g}$ and their first coordinate derivatives approach those of $dx^2 + \mathring{g}$, where $\mathring{g}$ is a Riemannian metric on N. We emphasize that no decay rate is required. Assume moreover that

$$h \geq \eta^2 > 0,$$

for some constant η. Then, if $\nu \in (-\eta, \eta)$, the functions

$$u_\pm = \pm C \|e^{\nu x} f\|_{L^\infty} e^{-\nu x}$$

are sub- and supersolutions of (3.2.13) for $x_0 \gg 1$.

4. *Cylindrically bounded ends* (this includes asymptotically periodic ends): Consider a metric $\widetilde{g}$ on $[x_0, \infty) \times N$. To obtain exponentially decaying sub- and supersolutions we need to ensure that

$$(\Delta_{\widetilde{g}} - h) e^{-\nu x} \leq -C e^{-\nu x} \tag{3.2.15}$$

for some constant C. Now

$$(\Delta_{\widetilde{g}} - h) e^{-\nu x} = (-\nu \Delta_{\widetilde{g}} x + \nu^2 |dx|_{\widetilde{g}} - h) e^{-\nu x} \ , \tag{3.2.16}$$

and so (3.2.15) holds if

$$h \geq C - \nu \Delta_{\widetilde{g}} x + \nu^2 |dx|_{\widetilde{g}} \ . \tag{3.2.17}$$

For example, this holds when $|\nu|$ is small enough provided there exists a constant $\epsilon > 0$ such that

$$h \geq \epsilon \ , \quad \Delta_{\widetilde{g}} x \leq \epsilon^{-1} \ , \quad |dx|_{\widetilde{g}} \leq \epsilon^{-1} \ ,$$

In particular, this holds for any cylindrically bounded metric (including conformally asymptotically cylindrical and conformally asymptotically periodic metrics) provided $h \geq \eta^2 > 0$.

An alternative construction of barriers

All the above sub- and supersolutions take constant values at the boundary $\{x_0\} \times N$ of E, and have the extra property that the gradient of $\mp u_\pm$ points into E at the boundary (in the cylindrical cases, one must assume that $\nu > 0$ for this to hold). If this sort of normal derivative condition holds, then there is an alternate proof that these can be used to construct weak barriers.

Suppose that $(M, \widetilde{g})$ is the union of a smooth compact manifold with boundary M_0 with a finite number of ends E_ℓ. Suppose too that on each end E_ℓ there are sub- and supersolutions $u_{\ell,-} < 0 < u_{\ell,+}$ of (3.2.13) which take constant values on ∂M_0, and such that $\mp\nabla u_{\ell,\pm}$ is nonvanishing along ∂E_ℓ and points into E_ℓ. Possibly multiplying the $u_{\ell,\pm}$ by large constants, we assume that all $u_{\ell,\pm}$ take the same constant value $\alpha_\pm$ on ∂M_0.

Let u_0 be the solution of (3.2.13) on M_0 with $u_0 = 0$ on ∂M_0. Choose a large constant C so that $|\nabla u_{\ell,\pm}| \geq C$ on ∂M_0 gradient of each of the $u_\pm$ on ∂M_0 is everywhere larger than $\sup_{\partial M_0} |\nabla u_0|$. Then the functions

$$\phi_\pm = \begin{cases} u_0 + C\alpha_\pm & \text{on } M_0 \\ Cu_{\ell,\pm} & \text{on } E_\ell, \end{cases} \tag{3.2.18}$$

are weak sub- and supersolutions of (3.2.13) on the entire manifold M. Indeed, the choice of C guarantees that the distributional second derivatives of $\phi_\pm$ have the appropriate signs at ∂M_0.

The monotone iteration scheme on non-compact manifolds

Let us show how barrier functions can be used to solve semilinear elliptic equations *without any compactness or asymptotic conditions* on the manifold:

THEOREM 3.2.7 (Monotone iteration scheme, version 2) *Let $(M, \widetilde{g})$ be a smooth Riemannian manifold and $F : M \times \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function. Suppose that $\underline{\tilde{\phi}} \leq \overline{\tilde{\phi}}$ are continuous functions which satisfy*

$$\Delta_{\widetilde{g}} \underline{\tilde{\phi}} \geq F(\cdot, \underline{\tilde{\phi}}) \;, \qquad \Delta_{\widetilde{g}} \overline{\tilde{\phi}}(z) \leq F(\cdot, \overline{\tilde{\phi}})$$

weakly. Then there exists a smooth function $\tilde{\phi}$ on M such that

$$\Delta_{\widetilde{g}} \tilde{\phi} = F(\cdot, \tilde{\phi}) \;, \qquad \underline{\tilde{\phi}} \leq \tilde{\phi} \leq \overline{\tilde{\phi}} \;.$$

As in the linear case, we do not assert that the solution is unique, and there are examples which show that uniqueness may fail.

PROOF: When M is compact, we proceed as follows. Let $\underline{\alpha} = \inf_M \underline{\tilde{\phi}}$ and $\overline{\alpha} = \sup_M \overline{\tilde{\phi}}$. We continue as in the proof of Theorem 3.1.15, p. 76: Rewrite the equation as

$$(\Delta_{\widetilde{g}} - A^2)\tilde{\phi} = F_A(z, \tilde{\phi}),$$

where

$$F_A(z, \tilde{\phi}) := F(z, \tilde{\phi}) - A^2\tilde{\phi}.$$

Choose A so large that the function F_A satisfies $\partial F_A(z,\mu)/\partial\tilde{\phi} < 0$ for almost every $\mu \in [\underline{\alpha}, \overline{\alpha}]$.

Now set $\tilde{\phi}_0 = \underline{\tilde{\phi}}$, and define the sequence of functions $\tilde{\phi}_j$ by

$$(\Delta_{\widetilde{g}} - A^2)\tilde{\phi}_{j+1} = F_A(z, \tilde{\phi}_j).$$

To see that this is well-defined for every j, note simply that $\Delta_{\widetilde{g}} - A^2$ is invertible and furthermore, as in the proof of Theorem 3.1.15, by the maximum principle and induction,

$$\underline{\tilde{\phi}} = \tilde{\phi}_0 \leq \tilde{\phi}_1 \leq \tilde{\phi}_2 \leq \ldots < \overline{\tilde{\phi}},$$

for all j, which implies that F_A is monotone for the same constant A (which depends only on $\underline{\tilde{\phi}}$ and $\overline{\tilde{\phi}}$). Even though $\tilde{\phi}_0$ is only continuous, standard elliptic regularity shows that $\tilde{\phi}_1 \in C^{0,\alpha}$ and that $\tilde{\phi}_j \in C^{2,\alpha}$ for $j \geq 2$.

We have produced a sequence which is monotone and uniformly bounded away from 0 and ∞, so it is straightforward to extract a subsequence which converges in $C^{2,\alpha}$ for some $0 < \alpha < 1$. If $F \in C^\infty$, then the subsequence converges in C^∞ too.

All of this works equally well if M is a compact manifold with boundary. To be concrete, we require at each stage that $\tilde{\phi}_j = \underline{\tilde{\phi}}$ on ∂M and we obtain a solution in the limit which satisfies the same boundary conditions.

Now consider a general manifold $(M, \widetilde{g})$. As in Lemma 3.2.5, choose an exhaustion M_j of M by compact submanifolds with smooth boundary. For each j, choose A_j so large that $\partial F(z,\mu)/\partial\tilde{\phi} - A_j^2 < 0$ on M_j for almost every

$$\mu \in [\inf_{M_j} \underline{\tilde{\phi}}, \sup_{M_j} \overline{\tilde{\phi}}]\ .$$

We may as well assume that A_j is a nondecreasing sequence.

Using the first part of the proof, for each j we can solve the equation

$$(\Delta_{\widetilde{g}} - A_j^2)\tilde{\phi}_j = F_{A_j}(z, \tilde{\phi}_j), \quad \tilde{\phi}_j\Big|_{\partial M_j} = \underline{\tilde{\phi}}$$

Notice that by adding $A_j^2\tilde{\phi}_j$ to both sides, the functions $\tilde{\phi}_j$ all satisfy the same equation and are all trapped between the two fixed barrier functions $\underline{\tilde{\phi}}$ and $\overline{\tilde{\phi}}$, albeit on an expanding sequence of domains. Elliptic estimates for the fixed equation $\Delta\tilde{\phi} = F(z, \tilde{\phi})$ may now be used to obtain uniform a priori estimates for derivatives of $\tilde{\phi}_j$ on any fixed compact set. From this we can use Arzela-Ascoli and a diagonalization argument to find a subsequence which converges in $C^{2,\alpha}$ (or C^∞) on any compact set to a limit function which satisfies the equation and which lies between the same two barrier functions. □

3.2.3 Asymptotically flat manifolds

One of the most widely studied class of Lorentzian manifolds are the *asymptotically flat spacetimes* which model isolated gravitational systems. Now, there exist several ways of defining asymptotic flatness, all of them roughly equivalent in vacuum. In this section we describe the Cauchy data point of view, which appears to be the least restrictive in any case.

So, a spacetime $(\mathscr{M}, \mathbf{g})$ will be said to possess an *asymptotically flat end* if $\mathscr{M}$ contains a spacelike hypersurface $\mathscr{S}_{\rm ext}$ diffeomorphic to $\mathbb{R}^n \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R. An end comes thus equipped with a set of Euclidean coordinates $\{x^i, i = 1, \ldots, n\}$, and one sets $r = |x| := \big(\sum_{i=1}^n (x^i)^2\big)^{1/2}$. One then assumes that there exists a constant $\alpha > 0$ such that, in local coordinates on $\mathscr{S}_{\rm ext}$ obtained from $\mathbb{R}^n \setminus B(R)$, the metric g induced by $\mathbf{g}$ on $\mathscr{S}_{\rm ext}$, and the second fundamental form K of $\mathscr{S}_{\rm ext}$, satisfy the fall-off conditions, for some $k > 1$,

$$g_{ij} - \delta_{ij} = O_k(r^{-\alpha})\,, \qquad K_{ij} = O_{k-1}(r^{-1-\alpha})\,, \tag{3.2.19}$$

where we write $f = O_k(r^\beta)$ if f satisfies

$$\partial_{k_1} \ldots \partial_{k_\ell} f = O(r^{\beta-\ell})\,, \quad 0 \le \ell \le k\,. \tag{3.2.20}$$

The PDE aspects of the problem require furthermore (g, K) to lie in certain weighted Hölder or Sobolev spaces defined on $\mathscr{S}$. More precisely, the above decay conditions should be implemented by conditions on the Hölder continuity of the fields; alternatively, the above equations should be understood in an integral sense. The constraint equations can be conveniently treated in both Hölder and Sobolev spaces, but one should keep in mind that L^2-type Sobolev spaces are better suited for solving the evolution equations.

Incidentally: The analysis of elliptic operators such as the Laplacian on weighted functional spaces was initiated by Nirenberg and Walker [338]; see also [31, 94, 298–301, 318–320, 341] as well as [92]. □

The conformal method works again very well for asymptotically flat initial data sets. The approach is very similar to the one for compact manifolds, with two important distinctions: on non-compact manifolds the embeddings $H_k \subset H_m$ and $C^{k,\alpha} \subset C^{m,\alpha}$, for $k \ge m$, are not compact anymore. Furthermore, to obtain good mapping properties for elliptic operators one needs to introduce weighted Sobolev or Hölder spaces. The reader is referred to the original references for details [81, 92, 94, 102, 105, 113, 309, 310].

CMC initial data can only be asymptotically flat if $\Lambda = \tau = 0$. The Lichnerowicz equation simplifies then to

$$\Delta_{\tilde g}\phi - \frac{n-2}{4(n-1)}\widetilde{R}\phi = -\sigma^2\phi^{(2-3n)/(n-2)}\,, \tag{3.2.21}$$

where

$$\sigma^2 := \frac{n-2}{4(n-1)}|\widetilde{L}|^2_{\tilde g}\,. \tag{3.2.22}$$

The treatment of TT tensors is essentially identical to that on compact manifolds. In fact, the analysis is somewhat simpler because there are no conformal Killing vectors which decay to zero as one recedes to infinity [113, 310], so the conformal vector Laplacian has no kernel on weighted Sobolev spaces with decay.

Concerning the Lichnerowicz equation (3.2.21), suppose that there exists a positive solution of this equation, then the conformally rescaled metric $g = \phi^{4/(n-2)}\tilde{g}$ has non-negative scalar curvature $R = |L|^2_g$, with L being an appropriate rescaling of $\tilde{L}$. Thus, a necessary condition for existence of positive solutions of (3.2.21) is that there exist metrics of non-negative scalar curvature in the conformal class $[\tilde{g}]$ of $\tilde{g}$. A conformally invariant criterion for this has been proposed in [81] but, as emphasised e.g. in [214, 309], the statement in [81] is inaccurate. In [214, 309] a corrected version has been provided, as follows:

Recall that the *Yamabe number* of a metric is defined by the equation

$$Y(M,g) = \inf_{u\in C^\infty_b,\ u\not\equiv 0} \frac{\int_M (|Du|^2 + \frac{n-2}{4(n-1)}Ru^2)}{\left(\int_M u^{2n/(n-2)}\right)^{(n-2)/n}}\,. \tag{3.2.23}$$

where C^∞_b denotes the space of compactly supported smooth functions. As shown in Section 3.1.1, p. 62, the number $Y(M,g)$ depends only upon the conformal class of g. We have the following key fact:

THEOREM 3.2.9 (Maxwell, Friedrich [214, 309]) *For asymptotically flat manifolds, there exists a conformal rescaling so that $\widetilde{R}$ is non-negative if and only if $Y(M,g) > 0$.*

In view of the above, any asymptotically flat manifold which carries a maximal initial data set has $Y(M,g) > 0$. In particular $\mathbb{R}^3$ with the flat metric is in the positive Yamabe class. Examples with $Y(M,g) \le 0$ can be found in [214].

Suppose, then, that we can perform the conformal rescaling that makes $\widetilde{R} \ge 0$. Setting

$$\phi = 1 + u\,,$$

the requirement that g has vanishing scalar curvature translates into an equation for u:

$$\Delta_{\tilde{g}} u - \frac{n-2}{4(n-1)}\widetilde{R}u = -\frac{n-2}{4(n-1)}\widetilde{R}\,. \tag{3.2.24}$$

Because $\widetilde{R}$ is non-negative now, there is no difficulty in finding a solution u decaying to zero at infinity, with suitable weighted regularity. Note that u is strictly positive by the maximum principle, in particular $1+u$ has no zeros and asymptotes to one. Replacing $\tilde{g}$ by $(1+u)^{4/(n-2)}\tilde{g}$, the new $\tilde{g}$ is again asymptotically flat, and (3.2.21) simplifies to

$$\Delta_{\tilde{g}}\phi = -\sigma^2\phi^{(2-3n)/(n-2)}\,. \tag{3.2.25}$$

Since

$$0 = \Delta_{\tilde{g}} 1 \ge -\sigma^2 = -\sigma^2 1^{(2-3n)/(n-2)}\,,$$

the constant function

$$\phi_- := 1$$

provides a subsolution which asymptotes to one. To obtain a supersolution we use a variation of Hebey's trick, as used in our treatment of compact manifolds: Let v be a solution approaching zero in the asymptotically flat regions of

$$\Delta_{\tilde{g}} v = -\sigma^2\,. \tag{3.2.26}$$

Then v is strictly positive by the maximum principle. Let $\phi_+ = 1 + v$, then

$$\Delta_{\tilde{g}}\phi_+ = -\sigma^2 \le -\sigma^2(1+v)^{(2-3n)/(n-2)} = -\sigma^2\phi_+^{(2-3n)/(n-2)}\,, \qquad (3.2.27)$$

so ϕ_+ is indeed a supersolution. Since $\phi_+ := 1 + v \ge 1 =: \phi_-$, we can use the monotone iteration scheme of Theorem 3.2.7 to obtain a solution. Note that ϕ_- and ϕ_+ both asymptote to one, so the solution also will.

This provides a complete description of vacuum, asymptotically flat, CMC initial data.

3.2.4 Asymptotically hyperboloidal initial data

Asymptotically hyperboloidal initial data arise naturally when studying spacetimes with a negative cosmological constant, in which case the asymptotic behaviour is modelled on that of a $t = 0$-slice of anti-de Sitter spacetime:

$$g \to_{r\to\infty} b := \frac{dr^2}{\frac{r^2}{\ell^2}+1} + r^2 d\Omega^2\,, \quad K \to_{r\to\infty} 0\,, \qquad (3.2.28)$$

where $\ell > 0$ is a constant related to the cosmological constant as

$$\frac{1}{\ell^2} = -\frac{2\Lambda}{n(n-1)}\,. \qquad (3.2.29)$$

Suitable rates have of course to be imposed, using weighted Hölder or weighted Sobolev spaces.

Such data also arise on hypersurfaces extending to the radiation zone in spacetimes with vanishing cosmological constant, in a way reminiscent of hyperboloids in Minkowski spacetime, cf. Example 1.4.1, p. 23. This justifies considering initial data sets with

$$g \to_{r\to\infty} \frac{dr^2}{r^2+1} + r^2 d\Omega^2\,, \quad K \to_{r\to\infty} \text{const} \neq 0\,. \qquad (3.2.30)$$

INCIDENTALLY: Existence and asymptotic properties of hyperboloidal initial data sets have been exhaustively analysed in [9–12], we provide an overview below; see also [5, 6]. An alternative elegant construction can be found in [219].

A Hilbert manifold structure on the set of solutions of asymptotically hyperboloidal initial data has been constructed in [189]; compare [133].

Gluing constructions involving asymptotically hyperboloidal initial data sets can be found in [132, 134, 136, 245]

A dense set of hyperboloidal initial data with simplified asymptotic behaviour has been exhibited in [177]. □

Asymptotically hyperboloidal initial data sets can be conveniently described within the conformal framework, as introduced by Penrose [344] to describe the behaviour of physical fields at null or timelike infinity.

Given a, say vacuum, smooth "physical" spacetime $(\widetilde{\mathscr{M}}, \tilde{\gamma})$, which we assume here to be without boundary, one associates to it a smooth "unphysical spacetime" $(\mathscr{M},\gamma)$ with boundary $\partial\mathscr{M}$ and a smooth function Ω on $\mathscr{M}$, such that $\widetilde{\mathscr{M}}$ is a subset of $\mathscr{M}$ and

$$\Omega|_{\widetilde{\mathscr{M}}} > 0\,, \quad \gamma_{\mu\nu}|_{\widetilde{\mathscr{M}}} = \Omega^2 \tilde\gamma_{\mu\nu}\,, \tag{3.2.31}$$

$$\Omega|_{\partial\mathscr{M}} = 0\,, \tag{3.2.32}$$

$$d\Omega(p) \neq 0 \quad \text{for} \quad p \in \partial\mathscr{M}\,. \tag{3.2.33}$$

This is the same as Definition **??**, p. **??** of Appendix **??**; we have repeated it here for the convenience of the reader because of a different notation for the metric here and because, as opposed to there, in this section tilded quantities denote the physical ones, while non-tilded quantities denote the unphysical, conformally rescaled ones.

It is common usage in general relativity to use the symbol $\mathscr{I}$ for $\partial\mathscr{M}$, and we shall often do so. If $\mathscr{S}$ is a hypersurface in $\mathscr{M}$, by $\mathscr{I}^+$ we shall denote the connected component of $\mathscr{I}$ which intersects the causal future of $\mathscr{S}$.

The hypothesis of smoothness of $(\mathscr{M},\gamma,\Omega)$ and the fact that $(\widetilde{\mathscr{M}},\tilde\gamma)$ is vacuum imposes several restrictions on various fields. Indeed, let us define

$$P_{\mu\nu} = \tfrac{1}{2}\,(R_{\mu\nu} - \tfrac{1}{6}\,R\gamma_{\mu\nu})\,, \tag{3.2.34}$$

with an analogous definition for the tilded quantities. In spacetime dimension four, for metrics $\tilde\gamma$ satisfying the vacuum Einstein equations with cosmological constant Λ one has

$$\frac{\Lambda}{6}\tilde\gamma_{\mu\nu} = \tilde P_{\mu\nu} = P_{\mu\nu} - \tfrac{1}{\Omega}\nabla_\mu\,\nabla_\nu\,\Omega + \tfrac{1}{2\Omega^2}\,\nabla^\alpha\,\Omega\,\nabla_\alpha\,\Omega\,\gamma_{\mu\nu}\,, \tag{3.2.35}$$

where ∇_μ is the covariant derivative of the metric $\gamma_{\mu\nu}$. Equations (3.2.32) and (3.2.35) imply

$$\nabla^\alpha\,\Omega\,\nabla_\alpha\,\Omega|_{\partial\mathscr{M}} = \frac{1}{3}\Lambda\,, \tag{3.2.36}$$

$$(\nabla_\mu\,\nabla_\nu\,\Omega - \tfrac{1}{4}\,\nabla^\alpha\,\nabla_\alpha\,\Omega\,\gamma_{\mu\nu})|_{\partial\mathscr{M}} = 0\,. \tag{3.2.37}$$

When $\Lambda = 0$, the tangential components of the tensor appearing in (3.2.37) are known as *the shear* of the hypersurface $\{\Omega = 0\}$. We have thus established an observation due to Penrose:

PROPOSITION 3.2.11 *In vacuum, $\mathscr{I} := \partial\mathscr{M}$ is*

$$\begin{cases} \textit{timelike}, & \Lambda < 0; \\ \textit{null, with vanishing shear}, & \Lambda = 0; \\ \textit{spacelike}, & \Lambda > 0. \end{cases} \tag{3.2.38}$$

Suppose that $\mathscr{S} \subset \mathscr{M}$ is a spacelike hypersurface in $(\mathscr{M},\gamma)$, let

$$\widetilde{\mathscr{S}} = \mathscr{S} \cap \widetilde{\mathscr{M}}\,, \qquad \partial\mathscr{S} = \partial\widetilde{\mathscr{S}} = \mathscr{S} \cap \partial\mathscr{M}\,,$$

and let g_{ij}, K_{ij}, respectively $\tilde g_{ij}, \tilde K_{ij}$, be the induced metric and extrinsic curvature of $\mathscr{S}$ in $(\mathscr{M},\gamma)$, respectively $\widetilde{\mathscr{S}}$ in $(\widetilde{\mathscr{M}},\tilde\gamma)$. If we denote by L^{ij} and $\tilde L^{ij}$ the traceless part of $K^{ij} = g^{ik}g^{j\ell}K_{k\ell}$, $\tilde K^{ij} = \tilde g^{ik}\tilde g^{j\ell}\tilde K_{k\ell}$,

$$L^{ij} = K^{ij} - \tfrac{1}{3}\,K\,g^{ij}\,, \quad K = g^{ij}\,K_{ij}\,,$$

$$\widetilde{L}^{ij} = \tilde{K}^{ij} - \tfrac{1}{3}\,\tilde{K}\,\widetilde{g}^{ij}\,, \quad \tilde{K} = \widetilde{g}^{ij}\,\tilde{K}_{ij}\,, \tag{3.2.39}$$

one finds

$$\widetilde{L}^{ij} = \Omega^3\,L^{ij}\,, \quad |\widetilde{L}|_{\widetilde{g}} = \Omega\,|L|_g\,,$$

$$\tilde{K} = \Omega K - 3\,n^\alpha\,\Omega_{,\alpha}\,, \tag{3.2.40}$$

where n^α is the unit normal to $\mathscr{S}$ for the metric γ, and $|\cdot|_h$ denotes the tensor norm in a Riemannian metric h. Since n^α is timelike and $\nabla\Omega(p)$ is null for $p \in \partial\mathscr{S}$, the trace

$$\tilde{K}|_{\partial\mathscr{S}} = -3n^\alpha\,\Omega_{,\alpha}|_{\partial\mathscr{S}} \tag{3.2.41}$$

has constant sign, because the scalar product of two non–vanishing non–spacelike vectors cannot change sign. From (3.2.32) we also have

$$g_{ij}|_{\widetilde{\mathscr{S}}} = \Omega^2\,\widetilde{g}_{ij}\,.$$

In what follows we will assume that

$$\Lambda = 0\,.$$

Then $\nabla\Omega$ is null non–vanishing at $\partial\mathscr{S}$ by (3.2.36), and (3.2.40)–(3.2.41) imply

$$D^i\,\Omega\,D_i\,\Omega|_{\partial\mathscr{S}} = \left(\frac{\tilde{K}}{3}\right)^2\Bigg|_{\partial\mathscr{S}} > 0\,, \tag{3.2.42}$$

where D_i is the Riemannian connection of the metric g_{ij}. To summarize, necessary conditions for an initial data set $(\widetilde{\mathscr{S}}, \widetilde{g}, \tilde{K})$ to arise from an "extended initial data set $(\mathscr{S}, g, K)$ intersecting a smooth $\mathscr{I}$" are

C1. There exists a Riemannian manifold $(\mathscr{S}, g)$ with boundary, with $g \in C^k(\mathscr{S})$, such that

$$\mathscr{S} = \widetilde{\mathscr{S}} \cup \partial\mathscr{S}\,.$$

Moreover there exists a function $\Omega \in C^k(\mathscr{S})$ such that

$$g_{ij}|_{\widetilde{\mathscr{S}}} = \Omega^2\,\widetilde{g}_{ij}\,, \tag{3.2.43}$$

$$\Omega|_{\partial\mathscr{S}} = 0\,, \qquad |D\Omega|_{g|\partial\mathscr{S}} > 0\,. \tag{3.2.44}$$

C2. The symmetric tensor field $\tilde{K}^{ij}$ satisfies, for some $\tilde{K} \in C^{k-1}(\mathscr{S})$, $\widetilde{L}^{ij} \in C^{k-1}(\mathscr{S})$,

$$\tilde{K}^{ij} = \widetilde{L}^{ij} + \tfrac{1}{3}\,\tilde{K}\,\widetilde{g}^{ij}\,, \quad \tilde{K} = \widetilde{g}_{ij}\,\tilde{K}^{ij}\,, \tag{3.2.45}$$

$$\tilde{K}|_{\partial\mathscr{S}} \text{ is nowhere vanishing}\,, \tag{3.2.46}$$

$$|\widetilde{L}|_{\widetilde{g}|\partial\mathscr{S}} = 0\,. \tag{3.2.47}$$

The question arises, how to construct such data sets? This involves constructing solutions of the vacuum scalar constraint equation,

$$\bar{R}(\widetilde{g}) + \tilde{K}^2 - \tilde{K}_{ij}\,\tilde{K}^{ij} = 0\,, \tag{3.2.48}$$

where $\widetilde{R}(\widetilde{g})$ is the Ricci scalar of the metric $\widetilde{g}$, and the vacuum vector constraint equation,

$$\widetilde{D}_i(\tilde{K}^{ij} - \tilde{K}\,\widetilde{g}^{ij}) = 0\,, \tag{3.2.49}$$

where $\widetilde{D}$ is the Riemannian connection of the metric $\widetilde{g}$, under appropriate asymptotic conditions. We will apply the conformal method, after adding the convenient assumption

C3.

$$\widetilde{D}_i\tilde{K} \equiv 0\,. \tag{3.2.50}$$

Recall that under (3.2.50) the scalar and the vector constraint equations decouple, and the Choquet-Bruhat-Lichnerowicz-York conformal method allows one to construct solutions of (3.2.48)–(3.2.49). An initial data set satisfying C1–C3 will be called a C^k *hyperboloidal initial data set* (smooth if $k=\infty$), while conditions C1– C2 will be called Penrose's C^k conditions. Without loss of generality we may normalize $\tilde{K}$ so that

$$\tilde{K} = 3\,, \tag{3.2.51}$$

and (3.2.48)–(3.2.49) can be rewritten as

$$\widetilde{R}(\widetilde{g}) + 6 = \widetilde{L}_{ij}\,\widetilde{L}^{ij} \tag{3.2.52}$$

$$\widetilde{D}_i\,\widetilde{L}^{ij} = 0\,. \tag{3.2.53}$$

To construct such data we start with a set of "seed fields" (g_{ij}, A^{ij}), where g_{ij} is any smooth Riemannian metric on $\widetilde{\mathscr{S}}$ extending smoothly to $\partial\widetilde{\mathscr{S}}$, and A^{ij} is any symmetric, traceless tensor field on $\widetilde{\mathscr{S}}$ extending smoothly to $\partial\widetilde{\mathscr{S}}$, via the following procedure: Let x be any defining function for $\partial\widetilde{\mathscr{S}}$, *i.e.* a function satisfying $x \in C^\infty(\widetilde{\mathscr{S}})$, $x \geq 0$, $x(p) = 0 \Longleftrightarrow p \in \partial\widetilde{\mathscr{S}}$, and $dx \neq 0$ at $\partial\widetilde{\mathscr{S}}$. Let $A^{ij} \in C^\infty(\widetilde{\mathscr{S}})$ be symmetric traceless and let X be any solution of the equation

$$D_i\left[x^{-3}\left(D^iX^j + D^jX^i - \frac{2}{3}D_kX^kg^{ij}\right)\right] = -D_i\left(x^{-2}A^{ij}\right)\,,$$

define

$$L^{ij} \equiv \frac{\Omega^2}{x^3}\left(D^iX^j + D^jX^i - \frac{2}{3}D_kX^kg^{ij}\right) + \frac{\Omega^2}{x^2}A^{ij}\,, \tag{3.2.54}$$

where Ω is a solution of the equation

$$\Omega\Delta_g\Omega - \frac{3}{2}|D\Omega|^2_g + \frac{1}{4}\Omega^2(R(g) - |L|^2_g) + \frac{3}{2} = 0\,, \tag{3.2.55}$$

satisfying $\Omega \geq 0$, with Ωx^{-1} approaching one as $x \to 0$, and where $|L|^2_g = g_{ij}g_{kl}L^{ik}L^{jl}$. Setting

$$\begin{aligned} \tilde{g}_{ij} &= \Omega^{-2}g_{ij}\,, \\ \tilde{K}^{ij} &= \Omega^3 L^{ij} + \tilde{g}^{ij}\,, \end{aligned}$$

one obtains a CMC solution of the vacuum constraint equations satisfying (3.2.51).

For simplicity we shall assume from now on that $\partial\widetilde{\mathscr{S}} \approx S^2$ – the two dimensional sphere. In [11] the following has been shown:

1. For any (g, A) as above one can find a solution X to (3.2.54) such that

$$\begin{aligned} \left(\frac{x}{\Omega}\right)^2 L^{ij} = \quad & U^{ij} + x^2 \log x\, U^{ij}_{\log}, \\ & U^{ij}, U^{ij}_{\log} \in C^\infty(\widetilde{\mathscr{S}})\,. \end{aligned}$$

   Given any g there exists an open dense set (in the $C^\infty(\widetilde{\mathscr{S}})$ topology) of A's for which $U^{ij}_{\log}|_{\partial\widetilde{\mathscr{S}}} \not\equiv 0$ (however, there exists an infinite dimensional closed subspace of A's for which $U^{ij}_{\log}|_{\partial\widetilde{\mathscr{S}}} \equiv 0$). If $U^{ij}_{\log}|_{\partial\widetilde{\mathscr{S}}} \equiv 0$, then $U^{ij}_{\log} \equiv 0$ and thus $x^2\Omega^{-2}L^{ij} \in C^\infty(\widetilde{\mathscr{S}})$. Let us also note that in an orthonormal frame e_i such that $e_A \parallel \partial\widetilde{\mathscr{S}}$, $A = 2, 3$, if we write, in a neighbourhood of $\partial\widetilde{\mathscr{S}}$,

$$L^{ij} = L^{ij}_0(v) + xL^{ij}_1(v) + \cdots\,,$$

   where v denote coordinates on $\partial\widetilde{\mathscr{S}}$, then we have $L^{1i}_0 \equiv 0$, while both $L^{AB}_0(v) - \frac{1}{2}L^{CD}_0 h_{CD}h^{AB}$ and $L^{AB}_1(v) - \frac{1}{2}L^{CD}_1 h_{CD}h^{AB}$ are freely specifiable tensor fields on $\partial\widetilde{\mathscr{S}}$. X is unique in an appropriate class of functions, *cf.* [11] for details.

2. For any (g, A) as above one can find a solution $\Omega \in \mathcal{A}_{phg}$ of equation (3.2.55), where $\mathcal{A}_{phg}$ denotes the space of polyhomogeneous functions on $\widetilde{\mathscr{S}}$. This means that there exists a sequence $\{N_j\}_{j=0}^\infty$ with $N_0 = N_1 = N_2 = N_3 = 0$, $N_4 = 1$ and functions $\Omega_{i,j} \in C^\infty(\bar{M})$ such that

$$\Omega \sim \sum_{i\geq 0}\sum_{j=0}^{N_i} \Omega_{i,j}x^i \log^j x\,, \tag{3.2.56}$$

   where "$\sim$" means "asymptotic to", in the sense that, for any desired n, Ω minus a truncated sum of the form given by the right hand side of (3.2.56) vanishes faster than x^n, and that this property is preserved under differentiation in the obvious way. According to standard terminology, functions with these properties are called *polyhomogeneous*, *cf. e.g.* [240]. For an open dense set of (g, A)'s we have $\Omega_{4,1}|_{\partial\widetilde{\mathscr{S}}} \not\equiv 0$. If $\Omega_{4,1}|_{\partial\widetilde{\mathscr{S}}} \equiv 0$, then $\Omega \in C^\infty(\widetilde{\mathscr{S}})$.

Suppose now that one has initial data such that the log terms described above do not vanish. In such a case the metric will immediately pick up log terms when time–evolved with Einstein equations, so that at later times there will be no decomposition of the three dimensional metric into a smooth up to the boundary background and a conformal factor. This shows that it is natural to consider the above construction under the condition that the seed fields are polyhomogeneous rather than smooth. One can show [11] that for any polyhomogeneous Riemannian metric g on $\widetilde{\mathscr{S}}$ and for any uniformly bounded polyhomogeneous symmetric tensor field A^{ij} on $\widetilde{\mathscr{S}}$ there exist solutions (X, Ω) of (3.2.54) and (3.2.55) such that L^{ij} given by (3.2.54) is polyhomogeneous and uniformly bounded on $\widetilde{\mathscr{S}}$, and Ω/x is polyhomogeneous, uniformly bounded, and uniformly bounded away from zero on $\widetilde{\mathscr{S}}$.

When $L^{ij} \equiv 0$ and the seed metric is smooth up to the boundary, the obstructions to smoothness of Ω have been analysed in detail in [12]. In that reference it has been shown, in particular, that $\Omega_{4,1}|_{\partial\widetilde{\mathscr{S}}}$ vanishes if the Weyl tensor of the unphysical (conformally rescaled) spacetime metric is bounded near $\partial\widetilde{\mathscr{S}}$. In [10] we have extended that analysis to the case $L^{ij} \not\equiv 0$. In order to present our results it is useful to define two tensor fields $\sigma^{\pm}$ defined on the conformal boundary $\partial\widetilde{\mathscr{S}}$ of the initial data surface:

$$\sigma^{\pm}_{AB} \equiv \left(\lambda_{AB} - \frac{h^{CD}\lambda_{CD}}{2}h_{AB}\right) \pm \left(K_{AB} - \frac{h^{CD}K_{CD}}{2}h_{AB}\right)\Big|_{\partial\widetilde{\mathscr{S}}}, \qquad (3.2.57)$$

which we shall call the *shear tensors of* $\partial\widetilde{\mathscr{S}}$. Here h_{AB} is the induced metric on $\partial\widetilde{\mathscr{S}}$, λ_{AB} is the extrinsic curvature of $\partial\widetilde{\mathscr{S}}$ in $(\widetilde{\mathscr{S}}, g)$, while K_{ij} can be thought of as the extrinsic curvature of $\widetilde{\mathscr{S}}$ in the conformally rescaled, unphysical spacetime metric.

Let us say that a spacetime admits a polyhomogeneous $\mathscr{I}$ if the conformally rescaled metric is polyhomogeneous at the conformal boundary; *i.e.*, in local coordinates the components of the conformally rescaled metric are bounded and polyhomogeneous. In the case of Cauchy data constructed as described above starting from smooth seed fields, the results of [10] linking the geometry of the boundary of the initial data surface with the geometry of the resulting spacetime can be summarized as follows:

1. Suppose that neither σ^+ nor σ^- vanishes. Then *there exists no development of the initial data with a smooth or polyhomogeneous* $\mathscr{I}$.

2. Suppose that $\sigma^+ \equiv 0$ or $\sigma^- \equiv 0$; changing the time orientation if necessary we may without loss of generality assume that $\sigma^+ \equiv 0$. Let $K^{\log}_{ij}$ denote "the logarithmic part" of K_{ij}:

$$K_{ij} = \hat{K}_{ij} + x^2 \log x \, K^{\log}_{ij},$$
$$\hat{K}_{ij} \in C^\infty(\widetilde{\mathscr{S}}), \; K^{\log}_{ij} \in C^0(\bar{M}) \cap \mathcal{A}_{phg}.$$

Then the following holds:

(a) If $K^{\log}_{1A}|_{\partial\widetilde{\mathscr{S}}} \not\equiv 0$, then if there exists a development with a polyhomogeneous $\mathscr{I}$, it is essentially polyhomogeneous, *i.e.*, no development with a smooth $\mathscr{I}$ exists.
The vanishing of $K^{\log}_{1A}|_{\partial\widetilde{\mathscr{S}}}$ is actually equivalent (under the present assumptions) to the vanishing at the conformal boundary of the Weyl tensor of the conformally rescaled metric.

(b) Suppose instead that $K^{\log}_{1A}|_{\partial\widetilde{\mathscr{S}}} \equiv 0$ and $K^{\log}_{11}|_{\partial\widetilde{\mathscr{S}}} \equiv 0$. Then *there exists a development which admits a smooth conformal boundary.*

It should be stressed that the results linking the log terms with the non–vanishing of the Weyl tensor proved in [10] show that the occurrence of shear and of at least some of the log terms in asymptotic expansions of physical fields at $\mathscr{I}$ is not an artefact of a bad choice of a conformal factor, or of a pathological choice of the initial data hypersurface (within the class of uniformly bounded from above and uniformly bounded away from zero, locally C^2, conformal factors and C^1 deformations of the initial data hypersurface which fix $\partial\widetilde{\mathscr{S}}$): if $\mathscr{I}$ is *not* shear–free (by which we mean that none of the shear tensors $\sigma^{\pm}$ vanish), then no conformal transformation will make it shear free. Similarly if the Weyl tensor does not vanish at ∂M, then no “gauge transformation” in the above sense will make it vanish (*cf.* [10] for a more detailed discussion).

The conditions for smoothness–up–to–boundary of an initial data set can be expressed as *local* conditions on the boundary on the seed fields (g_{ij}, A^{ij}). Let (x, v^A) be a Gauss coordinate system near $\partial\widetilde{\mathscr{S}}$; the interesting case is the one in which one of the shear tensors of the conformal boundary vanishes, which corresponds to the condition that, changing A_{ij} to $-A_{ij}$ if necessary,

$$\left(\lambda_{AB} - \frac{\lambda}{2}h_{AB}\right)\Big|_{\partial\widetilde{\mathscr{S}}} = \left(A_{AB} - \frac{1}{2}h^{CD}A_{CD}h_{AB}\right)\Big|_{\partial\widetilde{\mathscr{S}}} . \qquad (3.2.58)$$

It can be shown that without loss of generality in the construction of the initial data one can assume that

$$A_{1j}\Big|_{\partial\widetilde{\mathscr{S}}} = 0 ,$$

and in what follows we shall assume that this condition holds – the equations below would have been somewhat more complicated without this condition. Similarly, it is useful to choose a “conformal gauge” such that

$$\lambda \equiv h^{AB}\lambda_{AB}\Big|_{\partial\widetilde{\mathscr{S}}} = 0 .$$

Then the conditions for smoothness up to the boundary of Ω and L^{ij} reduce to

$$\left[\mathcal{D}^A\mathcal{D}^B\lambda_{AB} + R_{AB}\lambda^{AB}\right]\Big|_{\partial\widetilde{\mathscr{S}}} = 0 , \qquad (3.2.59)$$

where $\mathcal{D}$ is the covariant derivative operator of the metric h induced from g on $\partial\widetilde{\mathscr{S}}$, R_{ij} is the Ricci tensor of g, and

$$\left[\partial_x A_{AB} - \frac{1}{2}h^{CD}\partial_x A_{CD}h_{AB}\right]\Big|_{\partial\widetilde{\mathscr{S}}} = 0 . \qquad (3.2.60)$$

Failure of (3.2.59) or (3.2.60) will lead to occurrence of some log terms in the initial data set.

The overall picture that emerges from the results of [10, 112, 153, 235, 342, 411] and from the results described here is that the usual hypotheses of smoothness of $\mathscr{I}$ are overly restrictive. These results make it clear that a possible self–consistent setup for an analysis of the gravitational radiation is that of *polyhomogeneous* rather than smooth functions on the conformally completed manifold, *i.e.*, functions that have asymptotic expansions in terms of powers of x and $\log x$ rather than of x only. It should, however, be stressed that even though the fact that the physical fields $(\tilde g, \tilde K)$ satisfy the constraint equations guarantees the existence of some vacuum development $(\mathscr{M}, \tilde\gamma)$, it is by no means obvious that in the case when *e.g.* $\sigma^+ \equiv 0$, the existence of some kind of compactification of $(\widetilde{\mathscr{S}}, \tilde g, \tilde K)$ implies the existence of some useful conformal completion of $(\mathscr{M}, \tilde\gamma)$. Nevertheless we expect that the methods of [235] can be used to show that polyhomogeneous initial data of the kind constructed in [11], as described above, for which the shear of $\mathscr{I}$ vanishes will lead to space-times with metrics which along lightlike directions admit expansions in terms of $r^{-j}\log^i r$, rather than in terms of r^{-j} as postulated in [70, 382] (*cf.* [147] for a more detailed discussion of that question).

The results presented here immediately lead to the following question: how much *physical* generality does one lose by restricting oneself to Cauchy data which satisfy the conditions (3.2.59)–(3.2.60)? These conditions are similar in spirit to those of Bondi *et al.* [70], who impose conditions on the r^{-2} terms in the "free part of the metric" at $u = 0$ to avoid the occurrence of $r^{-j}\log^i r$ terms in the metric at later times. By doing so, or by imposing (3.2.59)–(3.2.60), one gains the luxury of working with smooth conformal completions, avoiding all the complications which arise due to the occurrence of log terms — but, then, does one overlook some *physically significant* features of radiating gravitating systems?

To obtain a real understanding of gravitational radiation, it is therefore necessary to establish what asymptotic conditions are appropriate from a physical point of view. The following are some criteria which might be considered as physically desirable:

1. existence of a well defined notion of total energy;

2. existence of a well defined notion of angular momentum;

3. existence of a development $(\mathscr{M}, \gamma)$ of the initial data set which admits a $\mathscr{I}$ with a reasonable regularity;

4. existence of a development of the data up to timelike infinity i^+.

In some situations it might be appropriate to impose only part of the above conditions. On the other hand it might perhaps be appropriate to add to the above the requirement that the function spaces considered include those data sets which arise by evolution from generic initial data which are asymptotically flat at spatial infinity.

We would like to emphasize that *it is not known* what regularity conditions on the conformally compactified metric are *necessary* for any of the above criteria to hold.

3.2.5 Asymptotically cylindrical initial data

One of the simplest solutions of the constraint equations is the cylinder $\mathbb{R} \times S^{n-1}$ with the standard product metric and with extrinsic curvature tensor K proportional to the metric. The vacuum development of such data, when the cosmological constant Λ is zero, is the interior Schwarzschild metric; when $\Lambda > 0$, one is led to the Nariai metric. When $\Lambda = 0$, examples of vacuum spacetimes containing asymptotically cylindrical ends are provided by the static slices of extreme Kerr solutions or of the Majumdar-Papapetrou solutions (both to be presented shortly), as well as those CMC slices in the Schwarzchild-Kruskal-Szekeres spacetimes which are asymptotic to slices of constant area radius $r < 2m$.

Initial data involving asymptotically cylindrical ends are used in numerical studies of Einstein equations [40, 228, 229], where they are known as "trumpet initial data". Various constructions thereof have been given in in [157, 179, 180, 227, 408, 409], with a systematic PDE analysis in [149, 150]. Non-CMC cylindrical ends have been constructed in [193, 286].

Incidentally: Studies of the Yamabe problem on manifolds with ends of cylindrical type shows that it is natural, and indeed necessary, to consider not only initial data sets where the metric is asymptotically cylindrical, but also metrics which are asymptotically periodic in space. In fact, constant-scalar-curvature metrics which are asymptotically cylindrical are more difficult to construct than the asymptotically periodic ones.

For the constant scalar curvature problem, the asymptotically (space-)periodic metrics are asymptotic to the *Delaunay* metrics, cf. [78, 152, 154, 279, 315]), which in the relativistic setting are the metrics induced on the static slices of the maximally extended Schwarzschild-de Sitter solutions. We refer also to [78, 79, 88, 152, 154, 279, 307, 312–315, 353, 362, 388] for the construction, and properties, of complete constant positive scalar curvature metrics with *asymptotically Delaunay* ends.

Exactly periodic, not necessarily time-symmetric, initial data can be obtained by lifting solutions of the constraint equations from $S^1 \times N$ to the cyclic cover $\mathbb{R} \times N$, where N is any compact manifold. In particular, the lifts to $\mathbb{R} \times S^2$ of initial data sets for the Gowdy metrics [119, 221] on $S^1 \times S^2$ provide a large family of non-CMC space-periodic solutions.

Further existence results for asymptotically space-periodic solutions of the constraint equations can be found in [149, 150, 160].

A simple example of a metric with two cylindrical ends with *toroidal transverse topology* is provided by Bianchi I metrics in which two directions only have been compactified, leading to a spatial topology $\mathbb{R} \times \mathbb{T}^2$.

Asymptotically cylindrical ends and degenerate Killing horizons

Asymptotically cylindrical ends, or cousins thereof, arise naturally in the presence of *degenerate Killing horizons.* We describe here some key examples.

The flagship one is provided by the *Majumdar-Papapetrou* black holes, in which the spacetime metric $\mathbf{g}$ and the electromagnetic potential A take the form

$$\mathbf{g} = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)\,, \tag{3.2.61}$$
$$A = u^{-1}dt\,. \tag{3.2.62}$$

The *standard MP black holes* are obtained if the coordinates x^μ of (3.2.61)–(3.2.62) cover the range $\mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, I$, with the function u taking the form

$$u = 1 + \sum_{i=1}^{I} \frac{m_i}{|\vec{x} - \vec{a}_i|}\,, \tag{3.2.63}$$

for some strictly positive constants m_i. Introducing radial coordinates centered at a puncture $\vec{a}_i$, the metric g induced on the slices $t = \text{const}$ by (3.2.61) is

$$g = \frac{m_i^2}{r^2}(1 + O(r))(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2))\;. \tag{3.2.64}$$

The new coordinate $x = -\ln r$ leads to a manifestly asymptotically cylindrical metric:

$$g = m_i^2(1 + O(e^{-x}))(dx^2 + \underbrace{d\theta^2 + \sin^2\theta\, d\varphi^2}_{=:\breve{h}})\;. \tag{3.2.65}$$

In this example the slices $t = \text{const}$ are totally geodesic, and it follows from the scalar constraint equation (with Maxwell sources) that the scalar curvature of g is positive everywhere. Furthermore the scalar curvature of the metric $\breve{h}$ defined in (3.2.65) equals two, while R approaches $2/m_i^2$ as one moves out along the i^{th} cylindrical end.

A very similar analysis applies on the slices of constant time in the Kastor-Traschen metrics [263], solutions of the vacuum Einstein equations with positive cosmological constant.

A slightly more general setting is that of metrics with ends which are conformal to asymptotically cylindrical ends. Such ends arise in extreme Kerr metrics. Indeed, the *extreme Kerr metrics* in Boyer-Lindquist coordinates take the form,

$$\begin{aligned}\mathbf{g} \;=\; & -dt^2 + \frac{2mr}{r^2 + m^2\cos^2\theta}(dt - m\sin^2\theta d\varphi)^2 + (r^2 + m^2)\sin^2\theta d\varphi^2 \\ & + \frac{r^2 + m^2\cos^2\theta}{(r-m)^2}dr^2 + (r^2 + m^2\cos^2\theta)d\theta^2\;. \end{aligned}\tag{3.2.66}$$

The metric induced on the slices $t = \text{const}$ reads, keeping in mind that $r > m$,

$$\begin{aligned} g \;=\; & \frac{r^2 + m^2\cos^2\theta}{(r-m)^2}dr^2 + (r^2 + m^2\cos^2\theta)d\theta^2 \\ & + \frac{(r^2 + m^2)^2 - (r-m)^2 m^2 \sin^2\theta}{r^2 + m^2\cos^2\theta}\sin^2\theta d\varphi^2\;. \end{aligned}\tag{3.2.67}$$

Introducing a new variable $x \in (-\infty, \infty)$ defined as

$$dx = -\frac{dr}{r-m} \quad \Longrightarrow \quad x = -\ln(r-m) \, ,$$

so that x tends to infinity as r approaches m from above, the metric g in (3.2.67) exponentially approaches

$$m^2(1+\cos^2\theta)\Big(dx^2 + \underbrace{d\theta^2 + \frac{4\sin^2\theta}{(1+\cos^2\theta)^2}d\varphi^2}_{=:\breve h}\Big) \tag{3.2.68}$$

as $x \to \infty$. We thus see that the degenerate Kerr spacetimes contain CMC slices with *asymptotically conformally cylindrical* ends.

INCIDENTALLY: Recall that the scalar curvature K of a metric of the form $d\theta^2 + e^{2f}d\varphi^2$ equals

$$K = -2(f'' + (f')^2) \, .$$

Hence the transverse part $\breve h$ of the limiting conformal metric appearing in (3.2.68) has scalar curvature

$$K = -\frac{4\cos(2\theta)}{(\cos^2\theta+1)^2} \, ,$$

which is negative on the northern hemisphere and positive on the southern one. □

INCIDENTALLY: The metric induced on sections of the event horizon of the Kerr metric reads

$$ds^2 = (R^2 + a^2\cos^2\theta)\, d\theta^2 + \frac{(R^2+a^2)^2\sin^2\theta}{R^2+a^2\cos^2\theta}d\varphi^2 \, , \tag{3.2.69}$$

where $R = m \pm \sqrt{m^2-a^2}$.

Comparing with (3.2.68), we see that the limiting transverse metric, as one recedes to infinity along the cylindrical end of the extreme Kerr metric, can be obtained from (3.2.69) by setting $a = m$:

$$\breve h = m^2\Big((1+\cos^2\theta)d\theta^2 + \frac{4\sin^2\theta}{1+\cos^2\theta}d\varphi^2\Big) \, . \tag{3.2.70}$$

The scalar curvature of the metric (3.2.69) is [285]

$$\frac{(R^2+a^2)(3a^2\cos^2\theta - R^2)}{(R^2+a^2\cos^2\theta)^3} \, .$$

It follows that the metric (3.2.70) has scalar curvature

$$\frac{2(3\cos^2\theta - 1)}{m^2(1+\cos^2\theta)^3} \, ,$$

and the reader will note that K changes sign.

We note that the slices $t = \text{const}$ are maximal, and the scalar constraint equation shows that $R \geq 0$. This example clearly exhibits the lack of correlation between the sign of the limit $\lim_{x\to\infty} R$ and that of the scalar curvature of the transverse part of the asymptotic metric, even when the constraint equations hold. □

3.3 TT tensors on locally conformally flat manifolds

3.3.1 Beig's potentials

When the initial data metric is conformal to a flat metric, the problem of constructing solutions of the vector constraint equations reduces to finding TT tensors on Euclidean space. (Localised such tensors can then be carried over to locally conformally flat manifolds.) In three space-dimensions, a complete description of such tensor fields in terms of a third-order potential has been provided by Beig [43]. Our presentation follows [45].

Let g_{ij} denote the Euclidean metric and D^i the associated covariant derivative. Let h_{ij} be a symmetric, transverse and traceless tensor on a simply connected region,

$$\partial_i h^i{}_j = 0 = h^i{}_i \,. \tag{3.3.1}$$

Beig shows that there exists a symmetric traceless "third-order potential" u_{ij} such that

$$h_{m\ell} = P(u)_{m\ell} \,, \tag{3.3.2}$$

where

$$P(u)_{m\ell} := \frac{1}{2}\epsilon_m{}^{ij}\partial_i(\Delta u_{j\ell} - 2\partial_{(\ell}D^n u_{j)n} + \frac{1}{2}g_{j\ell}D^n D^k u_{nk}) \,. \tag{3.3.3}$$

The converse is also true: given any symmetric trace-free tensor u_{ij}, the tensor field $P(u)$ defined by (3.3.3) is symmetric, transverse and traceless. Indeed, the last property and the vanishing of the divergence on the first index are obvious. The symmetry requires some work, which we leave as an exercice to the reader.

One way to see how (3.3.3) arises is to note that $P(u)$ is, apart from a numerical factor, the linearisation at the flat metric of the Cotton-York tensor in the direction of the trace-free tensor u. See [216, 259] for another perspective on this.

In order to prove the existence of a potential as in (3.3.3), let us define

$$\tau_{ijk} := \epsilon_{ij}{}^l h_{lk} \,. \tag{3.3.4}$$

Since $D_{[i}\tau_{jk]l} = \frac{1}{3}\epsilon_{ijk}D_m h^m{}_l = 0$, there exists a tensor field U_{ij} such that

$$\tau_{ijk} := D_{[i}U_{j]k} \,. \tag{3.3.5}$$

Symmetry of h_{ij} implies that all traces of τ_{ijk} vanish, which implies in turn that

$$D_{[l}U_i{}^{[k}\delta_{j]}{}^{l]} = 0 \,. \tag{3.3.6}$$

Hence there exists a tensor field U_{ijk}, which can be chosen to be antisymmetric in jk, so that

$$D_{[i}U_{j]}{}^{kl} + U_{[i}{}^{[k}\delta_{j]}{}^{l]} = 0 \,. \tag{3.3.7}$$

From tracelessness of h_{ij} one finds $\tau_{[ijk]} = 0$, which shows that there exists a vector field V_i such that

$$-\frac{1}{3}U_{[jk]} + D_{[j}V_{k]} = 0 \,. \tag{3.3.8}$$

Equations (3.3.7)-(3.3.8) together with some algebra give

$$D_{[l}(2U_{ij]}{}^k - 3V_i\delta_{j]}{}^k) = 0\,, \tag{3.3.9}$$

which implies existence of a potential V_{ij}:

$$\frac{2}{3}U_{[ij]}{}^k - V_{[i}\delta_{j]}{}^k + D_{[i}V_{j]}{}^k = 0\,. \tag{3.3.10}$$

Setting

$$u_{ij} := -3V_{(ij)} + \delta_{ij}V_k{}^k\,,$$

a lengthy calculation shows that

$$\begin{aligned} U_{ij} &= 3D_iV_j + \frac{1}{2}g_{ij}D^kD^lu_{kl} + \Delta u_{ij} \\ &\quad - 2D^kD_{(i}u_{j)k} - D_iD_jV_d{}^d\,. \end{aligned} \tag{3.3.11}$$

This shows that neither $V_{[ij]}$, nor $V_i{}^i$, nor V_i contribute to $D_{[i}U_{j]k}$ and we finally obtain (3.3.3).

Whenever h_{ij} is defined on star-shaped region containing the origin we can set

$$\sigma_{ijk}(\vec{x}) := \int_0^1 \epsilon_{ij}{}^\ell h_{\ell k}(\lambda\vec{x})\lambda(1-\lambda)^2 d\lambda\,. \tag{3.3.12}$$

We then obtain the following explicit formula for u in terms of σ:

$$u_{j\ell} = 2x^mx^nx_{(j}\sigma_{\ell)mn} + r^2x^m\sigma_{m(j\ell)}\,. \tag{3.3.13}$$

(We note that this is clearly symmetric, and tracelessness is not very difficult to check.)

To prove (3.3.12), we have to successively write down expressions for (i) U_{ij}, (ii) (U_{ijk}, V_i), and (iii) V_{ij}, at each step using formula (3.3.4), and take the symmetric, tracefree part of $-3V_{ij}$ at the end. In going from (i) to (ii) and (ii) to (iii) one uses the identities

$$\int_0^1\int_0^1 F(\lambda\lambda' x)\lambda\lambda'^2 d\lambda d\lambda' = \int_0^1 F(\lambda x)\lambda(1-\lambda)d\lambda\,, \tag{3.3.14}$$

and

$$\int_0^1\int_0^1 F(\lambda\lambda' x)\lambda(1-\lambda)\lambda'^3 d\lambda d\lambda' = \int_0^1 F(\lambda x)\lambda\frac{(1-\lambda)^2}{2}d\lambda\,, \tag{3.3.15}$$

respectively. The rest is index gymnastics.

By a straightforward analysis of (3.3.12) for $\sigma \geq -4$, or by Propositions 3.3.2 and 3.3.3 below regardless of the value of $\sigma \in \mathbb{R}$, we find that if $h_{ij} = O(r^\sigma)$ for large r, then u_{ij} can be chosen to be $O(r^{\sigma+3})$ when $\sigma \notin \{-4,-3,-2\}$, or $O(r^{\sigma+3}\ln r)$ otherwise. Furthermore, if h_{ij} is compactly supported, then u_{ij} can also be chosen to be compactly supported.

EXERCICE 3.3.1 As an example, consider a tensor field h_{ij} describing a plane gravitational wave in TT-gauge propagating in direction $\vec{k}$,

$$h_{ij}(\vec{k}) = \Re\big(H_{ij}e^{i\vec{k}\cdot\vec{x}}\big)\,, \quad \partial_\ell H_{ij} = 0 = H^i{}_i = H_{ij}k^j\,, \tag{3.3.16}$$

with possibly complex constant coefficients H_{ij}, where $\Re$ denotes the real part. Then

$$\begin{aligned}
\sigma_{ijk} &= \Re\big(\epsilon_{ij\ell}H^\ell{}_k\int_0^1 e^{i\lambda\vec{k}\cdot\vec{x}}(\lambda-2\lambda^2+\lambda^3)d\lambda\big) \\
&= \Re\big(W\epsilon_{ij\ell}H^\ell{}_k\big)\,, \quad \text{where} &(3.3.17)\\
W(\vec{x}) &= \frac{2ie^{i\vec{k}\cdot\vec{x}}(\vec{k}\cdot\vec{x}+3i)-\vec{k}\cdot\vec{x}(\vec{k}\cdot\vec{x}-4i)+6}{(\vec{k}\cdot\vec{x})^4} &(3.3.18)\\
&\quad \text{(which tends to } 1/12 \text{ when } \vec{k}\cdot\vec{x} \text{ tends to zero)}\,,\\
u_{j\ell} &= \Re\Big(W\big(2x^m x^i x_{(j}\epsilon_{\ell)mk}H^k{}_i - r^2x^i\epsilon_{ik(j}H^k{}_{\ell)}\big)\Big)\,. &(3.3.19)
\end{aligned}$$

□

Third-order potentials other than (3.3.12) are possible, differing by an element of the kernel of P, which we describe now. We again assume a simply connected region in $\mathbb{R}^3$. We follow through the steps of the argument starting from (3.3.5) with $\tau_{ijk} = 0$, which implies the existence of a potential M_i such that

$$U_{ij} = D_iM_j\,. \tag{3.3.20}$$

Next, from (3.3.7), there exists an antisymmetric tensor field $M^{kl} = M^{[kl]}$ such that

$$U_j{}^{kl} + M^{[k}\delta_j{}^{l]} = D_jM^{kl}\,. \tag{3.3.21}$$

Equation (3.3.8) implies the existence of a function ϕ such that

$$V_i - \frac{1}{3}M_i = D_i\phi\,. \tag{3.3.22}$$

Inserting into (3.3.10) we find that the terms involving M_i cancel so that

$$D_{[i}\left(\frac{2}{3}M_{j]}{}^k + V_{j]}{}^k - \phi\,\delta_{j]}{}^k)\right) = 0\,. \tag{3.3.23}$$

Consequently

$$V_{ij} = -\frac{2}{3}M_{ij} + \phi\,\delta_{ij} + D_iN_j\,, \tag{3.3.24}$$

so that

$$V_{(ij)} - \frac{1}{3}V_k{}^k = D_{(i}N_{j)} - \frac{1}{3}D_kN^k\,. \tag{3.3.25}$$

Setting $\lambda_i = -3N_i/2$, we conclude that any tensor field satisfying $P(u) = 0$ on a simply connected region can be written as

$$u_{ij} = D_i\lambda_j + D_j\lambda_i - \frac{2}{3}D^k\lambda_kg_{ij}\,. \tag{3.3.26}$$

As an application of this formula, consider the family of potentials

$$u_{ij} = \ln(1+r^2)(D_i\lambda_j + D_j\lambda_i - \frac{2}{3}D^k\lambda_k g_{ij}) . \qquad (3.3.27)$$

We have just seen that tensors of the form (3.3.27) *with the* $\ln(1+r^2)$ *term removed* form the kernel of P for any λ_i. This implies that if $\lambda \sim O(r^\sigma)$ for large r then we obtain a (non-trivial) family of $h_{ij} \sim O(r^{\sigma-4})$, for all $\sigma \in \mathbb{R}$.

It is of interest to enquire about the asymptotics of the tensor field u in general. The following result has obvious generalizations to p-forms on $\mathbb{R}^n$ $(n > 3)$ with $1 < p < n$:

PROPOSITION 3.3.2 *Let* $\omega_{ij}(x) = \omega_{[ij]}(x)$ *be a closed 2-form on* $\mathbb{R}^3$ *with* $\omega_{ij} = O(r^\sigma)$, $\alpha \in \mathbb{R}$. *Then there exists a 1-form* $\omega_i(x)$ *with* $\partial_{[i}\omega_{j]} = \omega_{ij}$ *satisfying* $\omega_i(x) = O(r^{1+\sigma})$ *if* $\sigma \neq -2$, $\omega_i(x) = O(r^{-1}\ln r)$ *otherwise.*

PROOF: Consider first the case $\sigma \geq -2$. Then

$$\omega_i(x) = 2\,x^j \int_0^1 \omega_{ji}(\lambda x)\lambda d\lambda = O(r^{1+\alpha}) \quad \text{when } \sigma > -2 , \qquad (3.3.28)$$

and $\omega_i(x) = O(r^{-1}\ln r)$ when $\sigma = -2$. To see this, use spherical coordinates (r,θ,φ) in the argument of ω_{ij} and substitute s/r for λ. When $\sigma < -2$, consider

$$\mu_i(x) = -2\,x^j \int_1^\infty \omega_{ji}(\lambda x)\lambda d\lambda , \qquad (3.3.29)$$

which converges and has the right decay at infinity, but blows up at the origin. The previous expression ω_i is still defined and, in the annulus $\overline{B(2,0)} \setminus B(1,0)$, differs from μ_i by a closed 1-form. Since this set is simply connected, the difference $\Delta_i := \omega_i - \mu_i$ satisfies $\Delta_i = \partial_i f$ for some function f. Now extend f smoothly to a function F on all of $B(2,0)$. Then the 1-form given by $\omega_i + \partial_i F$ in the interior and by μ_i in the exterior satisfies our requirements. □

An essentially identical argument shows the following result, which actually also follows from standard results in algebraic topology [71, Corollary 4.7.1]):

PROPOSITION 3.3.3 *If* ω_{ij} *has compact support, then* ω_i *can also be chosen with compact support.*

A construction of similar potentials for higher-spin constraint equations can be found in [8, 259].

3.3.2 Bowen-York tensors

In numerical relativity, a rich class of initial data sets can be constructed by choosing the initial data metric g_{ij} to be flat, adding a TT tensor K_{ij}, and solving numerically the Lichnerowicz equation. Whence the usefulness of explicit tensor field which are transverse and traceless with respect to a flat metric and which display significant physical properties.

One checks by a direct calculation that the following tensor fields on $\mathbb{R}^3\setminus\{0\}$, essentially due to Bowen and York [74] (compare [47, 48]), are transverse and traceless with respect to the Euclidean flat metric g_{ij}:

$$
\begin{aligned}
{}^1k_{ij}(\vec{P}) &= \frac{3}{2r^2}[P_in_j + P_jn_i - (g_{ij} - n_in_j)(P,n)]\,, && (3.3.30)\\
{}^2k_{ij}(\vec{S}) &= \frac{3}{r^3}[\epsilon_{cda}S^kn^dn_j + \epsilon_{cdb}S^kn^dn_i\,,] && (3.3.31)\\
{}^3k_{ij}(C) &= \frac{C}{r^3}[3n_in_j - g_{ij}]\,, && (3.3.32)\\
{}^4k_{ij}(\vec{Q}) &= \frac{3}{2r^4}[-Q_in_j - Q_jn_i - (g_{ij} - 5n_in_j)(Q,n)]\,, && (3.3.33)
\end{aligned}
$$

where $(\vec{P}, \vec{S}, C, \vec{Q})$ are constant vectors, the x^i's are Cartesian coordinates, $r^2 := g_{ij}x^ix^j \equiv (x,x)$, and $n^i := x^i/r$.

We can solve the Lichnerowicz equation with a linear combination of those tensors, subject to the boundary conditions that the conformal factor ϕ goes to one at infinity and blows up near the origin as $1/r$. The resulting physical initial-data set has then two asymptotically flat ends, one near infinity and another one near $r = 0$. Such ends are necessarily separated by a marginally outer trapped surface [197], which indicates the presence of black hole regions in the resulting spacetime [137]. The origin of $\mathbb{R}^3$ is referred-to as a *puncture* within this scheme.

The vector $\vec{P}$ coincides with the ADM momentum of the resulting initial data, as measured in the large-r asymptotic-region. The vector $\vec{S}$ is the ADM angular momentum in the large-r asymptotic region. Loosely speaking, the tensor field (3.3.30) can be thought-of as describing a "source" at $r = 0$ with linear momentum $\vec{P}$ and vanishing angular momentum, while the tensor field (3.3.31) is associated with a source at $r = 0$ with zero ADM momentum and angular momentum $\vec{S}$.

The tensor ${}^4k_{ij}$ of (3.3.33) is obtained by acting on ${}^1k_{ij}$ by spherical inversion at the sphere $r = 1$. One can check that, after solving the Lichnerowicz equation using ${}^4k_{ij}$ one obtains a linear momentum $\vec{Q}$ measured in the asymptotically flat region near $r = 0$.

The remaining "C-quantity" of (3.3.32) does not seem to have a clear physical interpretation.

One can also take linear combinations of the above tensors with singularities centered at distinct punctures, leading to initial data sets with several asymptotically flat ends. Depending upon the parameters chosen, one may obtain black hole solutions with event horizons which have several connected components [148].

INCIDENTALLY: In [48] the reader will find a construction of large classes of TT tensors on asymptotically flat Riemannian manifolds with leading-order behaviour described by a linear combination of (3.3.30)-(3.3.33). □

3.3.3 Beig-Krammer tensors

Denote by

$$L_{ij} = R_{ij} - \frac{1}{4} g_{ij} R \qquad (3.3.34)$$

the Schouten tensor. Given a vector field ξ and a function ρ set

$$j_j(\xi;\rho) = -D^i(D_{[i}\xi_{j]}\rho)) + \frac{2}{3}(D_j(D\xi))\rho + \frac{2}{3}D_jD_i(\xi^i\rho) + \frac{2}{9}D_j((D\xi)\rho) + 4L_{ji}\xi^i\rho\,. \qquad (3.3.35)$$

Note that j_j vanishes when ρ does.

We have:

THEOREM 3.3.5 (Beig & Krammer [47]) *Consider two vector fields ξ and E on a three-dimensional locally conformally flat manifold, where ξ is a conformal Killing vector field. Set*

$$\rho := \operatorname{div} E\,.$$

Then the symmetric, trace-free tensor field defined as

$$\begin{aligned} k_{ij} \equiv (BY)_{ij}(\xi;E) :=& -\xi^k D_{(i}D_{j)}E_k - 2\xi_{(i}\Delta E_{j)} + g_{ij}\xi^k\Delta E_k + 2\xi_k D_{(i}D^k E_{j)} + \\ &+ 2\xi_{(i}D_{j)}(DE) - \frac{4}{3}g_{ij}\xi^k D_k(DE) + \frac{4}{3}(D\xi)D_{(i}E_{j)} - 8F_{c(i}D^kE_{j)} + 4F^k{}_{(i}D_{j)}E_k - \\ &- \frac{4}{9}g_{ij}(D\xi)(DE) + 4g_{ij}F^{cd}D_kE_d + 4E_{(i}D_{j)}D\xi - \frac{4}{3}g_{ij}E^kD_k(D\xi) + 9\xi^kL_{c(i}E_{j)} + \\ &+ \xi_{(i}L_{j)k}E^k + 4L\,\xi_{(i}E_{j)} - 2g_{ij}\,L\,\xi^kE_k\,, \end{aligned} \qquad (3.3.36)$$

where $L = g^{ij}L_{ij}$ and $F_{ij} = D_{[i}\xi_{j]}$, satisfies

$$D^i k_{ij} = j_j(\xi;\rho)\,. \qquad (3.3.37)$$

From the perspective of vacuum initial data sets, the key point of Theorem 3.3.5 is that every conformal Killing vector field ξ and *divergence-free* vector field E ($\rho = 0$) provide a transverse-traceless tensor field $(BY)_{ij}(\xi;E)$.

It turns out that the Bowen-York tensors (3.3.30)-(3.3.33) fit nicely into the Beig-Krammer scheme. Indeed, consider the following conformal Killing vector fields on $\mathbb{R}^3$:

$$\begin{aligned} {}^1\xi^i(\vec\pi) &= \pi^i\,, & (3.3.38)\\ {}^2\xi^i(\vec\sigma) &= \epsilon^i{}_{jk}\sigma^j x^k\,, & (3.3.39)\\ {}^3\xi^i(\zeta) &= \zeta x^i\,, & (3.3.40)\\ {}^4\xi^i(\vec\gamma) &= (x,x)\gamma^i - 2(x,\gamma)x^i\,, & (3.3.41) \end{aligned}$$

where $\pi^i, \sigma^i, \zeta, \gamma^i$ are constants. Let E be the Coulomb solution of $\operatorname{div} E = 4\pi\delta_0$, i.e.

$$E^i = \frac{n^i}{r^2}. \qquad (3.3.42)$$

Then [47]:

$$
\begin{aligned}
{}^1k(\vec{P}) &= -\frac{1}{2}BY({}^4\xi(\vec{P}),E)\,, && (3.3.43)\\
{}^2k(\vec{S}) &= -BY({}^2\xi(\vec{S}),E)\,, && (3.3.44)\\
{}^3k(C) &= -BY({}^3\xi(C),E)\,, && (3.3.45)\\
{}^4k(\vec{Q}) &= \frac{1}{2}BY({}^1\xi(\vec{Q}),E)\,. && (3.3.46)
\end{aligned}
$$

In particular the image of the map

$$(\xi,E)\mapsto BY(\xi,E)\,,$$

with E satisfying $\operatorname{div} E=0$, is not trivial.

3.4 Non-CMC data

One can consider the conformal method *without* assuming CMC data. As before, the free conformal data consist of a manifold M, a Riemannian metric $\tilde{g}$ on M, a trace-free symmetric tensor $\tilde{\sigma}$, and a *mean curvature function* τ. The fields (g,K) defined as

$$
\begin{aligned}
g &= \phi^q\tilde{g}\,, \qquad \text{where } q=\frac{4}{n-2}\,, && (3.4.1)\\
K &= \phi^{-2}(\tilde{\sigma}+\tilde{C}(Y))+\frac{\tau}{n}\phi^q g\,, && (3.4.2)
\end{aligned}
$$

where ϕ is positive, will then solve the constraint equations with matter energy-momentum density (μ,J) if and only if the function ϕ and the vector field Y solve the equations

$$\operatorname{div}_{\tilde{g}}(\tilde{C}(Y)+\tilde{\sigma})=\frac{n-1}{n}\phi^{q+2}\widetilde{D}\tau+8\pi\phi^{q_J}\tilde{J}\,, \tag{3.4.3}$$

$$\Delta_{\tilde{g}}\phi-\frac{1}{q(n-1)}R(\tilde{g})\phi+\frac{1}{q(n-1)}|\tilde{\sigma}+\tilde{C}(Y)|^2_{\tilde{g}}\phi^{-q-3}-\frac{1}{qn}\tau^2\phi^{q+1}=16\pi\phi^{q_\mu}\tilde{\mu}\,. \tag{3.4.4}$$

Here q_J and q_μ are exponents which can be chosen in a manner which is convenient for the problem at hand. A possible choice is obained by inserting the *York scaling* given in (3.1.110)-(3.1.111) into (3.1.108)-(3.1.109); this is convenient e.g. for the Einstein-Maxwell constraints in dimension $n=3$. Finally, the symbol $\widetilde{D}$ denotes the covariant derivative of $\tilde{g}$, and $\tilde{C}(Y)$ is the *conformal Killing operator* of $\tilde{g}$:

$$\tilde{C}(Y)_{ij}=\widetilde{D}_iY_j+\widetilde{D}_jY_i-\frac{2}{n}\tilde{g}_{ij}\widetilde{D}_kY^k\,. \tag{3.4.5}$$

When $d\tau\not\equiv 0$, the vector constraint equation does not decouple from the scalar one, and one needs to find simultaneously the solution (ϕ,Y) to both equations above.

One can invoke the implicit function theorem to construct solutions of the above when τ is bounded away from zero and $d\tau$ is sufficiently small, near a solution at which the linearized operator is an isomorphism. Other techniques have also been used in this context in [4, 98, 250, 251]. A non-existence theorem for a class of near-CMC conformal data has been established in [254].

The first general result without assuming a small gradient is due to Holst, Nagy, and Tsogtgerel [238, 239] who assumed non-vanishing matter source, $\tilde\mu \not\equiv 0$. Maxwell [311] has extended their argument to include the vacuum case, leading to:

THEOREM 3.4.1 (Holst, Nagy, Tsogtgerel [238, 239], Maxwell [311]) *Let $(M, \widetilde{g}_{ij})$ be a three dimensional, smooth, compact Riemannian manifold of positive Yamabe type without conformal Killing vectors, and let $\tilde\sigma^{ij}$ be a symmetric transverse traceless tensorfield. If the seed tensor $\tilde\sigma^{ij}$ and the matter sources $|\tilde J|_{\widetilde g} \le \tilde\mu$ are sufficiently small, then there exists a scalar field $\phi > 0$ and a vector field Y solving the system*

$$\begin{aligned} \Delta_{\widetilde g}\phi - \tfrac18 \widetilde R\phi &= -\tfrac18|\tilde\sigma|^2_{\widetilde g}\phi^{-7} + \tfrac1{12}\tau^2\phi^5 - 2\pi\tilde\mu\phi^{-3}\,, \\ \widetilde D_i(\widetilde D^i Y^j + \widetilde D^j Y^i - \tfrac23 \widetilde D_k Y^k \widetilde g^{ij}) &= 8\pi \tilde J^j + \tfrac23\phi^6\widetilde D^j\tau\,, \end{aligned} \tag{3.4.6}$$

and hence providing a solution

$$(g_{ij}, K^{ij}) = (\phi^4\widetilde g_{ij}, \phi^{-10}(\tilde\sigma^{ij} + \widetilde D^i Y^j + \widetilde D^j Y^i - \frac23 \widetilde D_k Y^k \widetilde g^{ij}) + \frac{\tau}{3}\phi^{-4}\widetilde g^{ij})$$

of the constraint equations in vacuum ($\mu = 0 = J$) or with sources

$$(\mu, J^i) = (\phi^{-\frac{2(n+1)}{n-2}}\tilde\mu, \phi^{-\frac{2(n+2)}{n-2}}\tilde J^i)$$

(compare (3.1.111) and (3.1.110), p. 102).

The reader is referred to [311] for further general statements concerning the problem at hand.

The Dahl-Gicquaud-Humbert obstruction equation

An important development for the constructions of non-CMC initial data is the discovery of the *obstruction equation* by M. Dahl, R. Gicquaud, and E. Humbert in [175]. The key observation in that work is the following: Let τ be a positive function on M. Then, *non-existence* of solutions of the equation

$$D^i(D_iW_j + D_jW_i - \frac{2}{n}\,D^kW_k\,g_{ij}) = \kappa\,|L(W)|_g\,d\ln\tau \tag{3.4.7}$$

for all

$$\kappa \in (0, 2\sqrt{(n-1)/n}]$$

implies the *existence* of solutions of the constraint equations with

$$\tau = \mathrm{trace}_g(K).$$

Assuming that (M, g) is compact, the authors of [175] further show the following:

1. Suppose that there exists a constant $\alpha > 0$ such that the Ricci tensor of g satisfies $R_{ij} \le -\alpha g_{ij}$ in the sense of quadratic forms. Then there are no solutions of (3.4.7) for functions τ such that

$$|d\ln\tau| \le \sqrt{\frac{n}{2(n-1)}}\alpha\,. \tag{3.4.8}$$

   Equivalently, for every smooth function τ satisfying (3.4.8) there exists a solution of the constraint equations obtained using the conformal method. Moreover, the set of such solutions is compact.

2. There exist manifolds (M,g), functions τ and real numbers $\kappa \in (0, 2\sqrt{(n-1)/n}]$ for which a solution of the obstruction equation (3.4.7) exists.

See also [176] for a related non-existence result.

3.5 Gluing techniques

The *gluing techniques* can be regarded as a singular perturbation method. The techniques are used to produce new solutions by gluing together old ones.

3.5.1 Linearised gravity

We start the presentation with an elementary gluing procedure for linearised gravity on Minkowski spacetime, which can be carried-out using the potentials of Section 3.3.1.

As seen in Section 1.2, for vacuum initial data defined globally on $\mathbb{R}^3$ we can, and will, without loss of generality assume that we are in the transverse traceless gauge,

$$D^i h_{ij} = h^i{}_i = 0\,. \tag{3.5.1}$$

We note that non-vacuum initial data, or vacuum initial data on e.g. $\mathbb{R}^3 \setminus B(R)$ for some $R > 0$, can be brought to a gauge where the trace-free part of h_{ij} is divergence-free, but where the trace does not necessarily vanish. As such, the results below apply to general transverse traceless tensors on a Euclidean background, and thus they apply in particular to the trace-free part of h_{ij} in the transverse gauge; the analysis of the trace part of h_{ij} needs then separate treatment.

As a first step towards gluing, we prove a *shielding result*, which shows that transverse-traceless tensors can be deformed to zero on a set $\tilde\Omega \setminus \overline{\Omega}$, which can be chosen as small as desired, while remaining unchanged in $\overline{\Omega}$:

PROPOSITION 3.5.1 (Beig & Chruściel [45]) *Let Ω, $\tilde\Omega$ and $\widehat\Omega$ be open subsets of $\mathbb{R}^3$ such that*

$$\overline{\Omega} \subset \tilde\Omega \subset \widehat\Omega\,, \tag{3.5.2}$$

with $\widehat\Omega$ simply connected. Every vacuum initial data set for the linearised gravitational field $(\widehat\Omega, h_{ij}, k_{ij})$ in the gauge (3.5.1) can be deformed to a new vacuum initial data set $(\tilde h_{ij}, \tilde k_{ij})$ which coincides with (h_{ij}, k_{ij}) on $\overline{\Omega}$ and vanishes outside of $\tilde\Omega$.

PROOF: Let (u_{ij}, v_{ij}) denote the corresponding Beig potentials of Section 3.3.1, thus

$$(h_{ij}, k_{ij}) = (P(u)_{ij}, P(v)_{ij})\,, \tag{3.5.3}$$

where P is the third-order differential operator of (3.3.3). Let χ_Ω be any smooth function which is identically equal to one on Ω and which vanishes outside of $\widetilde{\Omega}$. Then the initial data set

$$(\tilde{h}_{ij}, \tilde{k}_{ij}) = (P(\chi_\Omega u)_{ij}, P(\chi_\Omega v)_{ij}) \tag{3.5.4}$$

satisfies the vacuum constraint equations everywhere, coincides with (h_{ij}, k_{ij}) in Ω and vanishes outside of $\widetilde{\Omega}$. □

When Ω is bounded, the new fields $(\tilde{h}_{ij}, \tilde{k}_{ij})$ can clearly be chosen to vanish outside of a bounded set. For example, consider a plane wave solution as in (3.3.16), p. 127. Multiplying the potentials (3.3.19) by a cut-off function $\chi_{B(R_1)}$, which equals one on $B(R_1)$ and vanishes outside of $B(R_2)$, provides compactly supported gravitational data which coincide with the plane-wave ones in $B(R_1)$. (Alternatively one can replace $\vec{k}\cdot\vec{x}$ in the first line of (3.3.17), or in (3.3.18)-(3.3.19), by $\vec{k}\cdot\vec{x}\ \chi_{B(R_1)}(\vec{x})$. This would lead to a tensor h_{ij} which is constant outside of $B(R_2)$, and hence an initial data set which is flat outside of $B(R_2)$.) In the limit $\vec{k} = 0$, so that h_{ij} is constant on $B(R_1)$ and, e.g., $k_{ij} = 0$, one obtains data which are Minkowskian both in $B(R_1)$ and outside of $B(R_2)$, and describe a burst of gravitational radiation localised in a spherical annulus. Note that the Minkowskian coordinates for the interior region are distinct from the ones for the outside region. The closest full-theory configuration to this would be Bartnik's time symmetric initial data set [36] which are flat inside a ball of radius R_1, and which can be Corvino-Schoen deformed to be Schwarzschildean outside of the ball of radius R_2; here R_2 will be much larger than R_1 in general, but can be made as close to R_1 as desired by making the free data available in Bartnik's construction sufficiently small.

For Ω's which are not bounded it is interesting to enquire about fall-off properties of the shielded field. This will depend upon the geometry of Ω and the fall-off of the initial field:

For cone-like geometries, as considered in [84, 135], and with $h_{\mu\nu} = O(1/r)$, the gravitational field in the screening region will fall-off again as $O(1/r)$. This is rather surprising, as the gluing approach of [84] leads to a loss of decay even for the linear problem. One should, however, keep in mind that the transition to the TT-gauge for a linearised correction to the metric which falls-off as $1/r$ is likely to introduce $\ln r/r$ terms in the transformed metric, which will then propagate to the gluing region.

As another example, consider the set $\Omega = (a, b) \times \mathbb{R}^2$, which is not covered by the methods of [84]. Our procedure in this case applies but if $h_{\mu\nu} = O(1/r)$, and if the cut-off function is taken to depend only upon the first variable of the product $\Omega = (a, b) \times \mathbb{R}^2$, one obtains a gravitational field $\tilde{h}_{\mu\nu}$ vanishing outside a slab $\widetilde{\Omega} = (c, d) \times \mathbb{R}^2$, with $[a, b] \subset (c, d)$, which might grow as $r^2 \ln r$ when receding to infinity within the slab.

Proposition 3.5.1provides a "shielding" result. This can, however, be used to glue linearised field across a gluing region, as follows: Consider two linearised vacuum initial data sets $(\widehat{\Omega}_1, h_1, k_1)$ and $(\widehat{\Omega}_2, h_2, k_2)$ in the TT-gauge such that

$$\widehat{\Omega}_1 \cap \widehat{\Omega}_2 \neq \emptyset .$$

For simplicity let us assume that both $\widehat{\Omega}_i$'s are simply connected, though of course simple connectedness of suitable neighborhoods of the intersection region would suffice. We can use use Proposition 3.5.1 to screen each of the fields to zero across the gluing region. Adding the resulting new fields provides the desired glued configuration.

3.5.2 Conformal gluings

From the point of view of general relativity, the gluing techniques seem to go back to the work of Joyce [260], who used the space-Schwarzschild metric as a building block for perfoming connected sums of constant scalar curvature together with the conformal method to correct for the error introduced. A version of this procedure suitable for the whole set of general relativistic constraint equations has been developed by Isenberg, Mazzeo and Pollack in [246, 248]. The construction of [248] allowed one to combine initial data sets by taking a connected sum of their underlying manifolds, to add wormholes (by performing codimension-three surgery on the underlying manifold) to a given initial data set, and to replace arbitrary small neighborhoods of points in an initial data set with asymptotically hyperbolic ends.

We have:

THEOREM 3.5.2 (Isenberg, Mazzeo, Pollack [248, 249]) *Let (M, g, K) be a smooth, solution of the Einstein constraint equations, with constant mean curvature τ, where M is not necessarily connected. Furthermore, (M, g, K) may be either compact, or contain a finite number of asymptotically Euclidean or asymptotically hyperbolic ends. Let $p_1, p_2 \in M$, and assume that (M, g, K) is nondegenerate in the sense that the linearisation of the conformal constraint equations has no kernel, and that $K \not\equiv 0$. Let $\hat{M}$ be constructed from M by adding a neck connecting the two points p_1 and p_2. Then there is a one-parameter family of vacuum initial data $(\hat{M}, g_T, K_T)$ with constant mean curvature τ such that, for any $\epsilon > 0$, (g_T, K_T) approach (g, K), as T tends to infinity away from balls of radius ϵ centered at the p_i's.*

REMARK 3.5.3 The hypothesis of non-degeneracy above requires non-existence of conformal Killing vector fields, which is a generic condition by [68]. It further requires triviality of the kernel of the linearisation of the Lichnerowicz equation, which is expected to be satisfied in generic situations. We note that for time symmetric initial data this question reduced to that of the triviality of the kernel of the linearisation of the Yamabe equation, which has been shown to hold for generic metrics on compact manifolds in [265].

In [249] this gluing construction was extended to only require that the mean curvature be constant in a small neighborhood of the point about which one wanted to perform a connected sum. This can be used to show the following:

THEOREM 3.5.4 (Isenberg, Mazzeo, Pollack [249]) *Let M be any closed n-dimensional manifold, and $p \in M$. Then $M \setminus \{p\}$ admits an asymptotically flat initial data set satisfying the vacuum constraint equations.*

A similar result, where the vacuum hypothesis is replaced by the dominant energy condition, has been previously established in [412].

3.5.3 "PP^*-gluings"

The drawback of the gluing constructions just described is that the conformal method introduces a global deformation of the original initial data. A method which avoids this problem has been devised by Justin Corvino in his thesis [161]; compare [165]; we return to this in Section 3.5.5. In these last two works, gluing theorems across annuli are developed for asymptotically flat initial data.

INCIDENTALLY: The Corvino-Schoen gluing methods [161, 165] provide a powerful tool for constructing new general relativistic initial data, with interesting properties, out of old ones [132, 144, 152]. Key applications include the construction of vacuum spacetimes with smooth asymptotic structure in lightlike directions [131, 162], the construction of localised "spacetime bridges" and "wormholes" [143], and the construction of many-body initial data sets [144]; see Section 3.5.6, p. 144.

The main usefulness of this alternative technique in general relativity lies in the fact that, away from the small set about which one fuses the two solutions, the new solution is identical to the original ones. This gives one control on the physical properties of the glued initial data. Furthermore, because of the finite speed of propagation of signals, the resulting solution coincides with the original one in the domain of dependence of the regions where the metric remained unchanged.

The basic setup is that of two initial data sets which are close to each other on a domain $\Omega \subset \mathscr{S}$. One further assumes that Ω has exactly two boundary components, with each component of Ω separating $\mathscr{S}$ into two. The reader can think of Ω as an annulus in $\mathscr{S} = \mathbb{R}^n$, or the region between the cones in $\mathbb{R}^n$ of Figure 3.6.1, p. 148 below. The basic idea is to use the inverse function theorem to construct a new initial data set which will coincide with the first initial data set near a component of the boundary, and with the second initial data set near the other component of $\partial\Omega$.

Let us denote by $\mathcal{C}$ the map which to a pair (K, g) assigns the right-hand sides of the constraint equations:

$$\mathcal{C}(g, K) := \begin{pmatrix} 2(-\nabla^j K_{ij} + \nabla_i \,\mathrm{tr} K) \\ R(g) - |K|^2 + (\mathrm{tr} K)^2 - 2\Lambda \end{pmatrix} . \tag{3.5.5}$$

We denote by P the linearisation of $\mathcal{C}$. Setting $h = \delta g$ and $Q = \delta K$, one has:

$$P(Q, h) = \begin{pmatrix} -K^{pq}\nabla_i h_{pq} + K^q{}_i(2\nabla^j h_{qj} - \nabla_q h^l{}_l) \\ \quad -2\nabla^j Q_{ij} + 2\nabla_i \,\mathrm{tr} Q - 2(\nabla_i K^{pq} - \nabla^q K^p{}_i) h_{pq} \\ \\ -\Delta(\mathrm{tr} h) + \mathrm{div}\,\mathrm{div}\, h - \langle h, \mathrm{Ric}\,(g)\rangle + 2K^{pl}K^q{}_l h_{pq} \\ \quad -2\langle K, Q\rangle + 2\mathrm{tr} K(-\langle h, K\rangle + \mathrm{tr} Q) \end{pmatrix} . \tag{3.5.6}$$

The order of the differential operators that appear in P is

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix},$$

which can be written in the Agmon-Douglis-Nirenberg form (*cf., e.g.* [334, p. 210])

$$\begin{pmatrix} s_1+t_1 & s_1+t_2 \\ s_2+t_1 & s_2+t_2 \end{pmatrix},$$

with $s_1=-1$, $s_2=0$, $t_1=t_2=2$; here it is understood that an operator of order 0 is also an operator of order 2 with vanishing coefficients in front of the first and second derivatives. It follows that the symbol P' of the principal part of P in the sense of Agmon-Douglis-Nirenberg reads

$$P'(x,\xi)(Q,h)=\begin{pmatrix} 2(-\xi^s\delta^t_i+\xi_i g^{st}) & -K^{pq}\xi_i+2K^q{}_i\xi^p-K^l{}_i\xi_l g^{pq} \\ 0 & -|\xi|^2g^{pq}+\xi^p\xi^q \end{pmatrix}\begin{pmatrix} Q_{st} \\ h_{pq}\end{pmatrix}.$$

The formal L^2-adjoint of P takes the form

$$P^*(Y,N)=\begin{pmatrix} 2(\nabla_{(i}Y_{j)}-\nabla^lY_lg_{ij}-K_{ij}N+\mathrm{tr}K\;Ng_{ij}) \\ \\ \nabla^lY_lK_{ij}-2K^l{}_{(i}\nabla_{j)}Y_l+K^q{}_l\nabla_qY^lg_{ij}-\Delta Ng_{ij}+\nabla_i\nabla_jN \\ +(\nabla^pK_{lp}g_{ij}-\nabla_lK_{ij})Y^l-N\mathrm{Ric}\,(g)_{ij}+2NK^l{}_iK_{jl}-2N(\mathrm{tr}\;K)K_{ij} \end{pmatrix}. \tag{3.5.7}$$

From (3.5.7) we obtain the Agmon-Douglis-Nirenberg symbol $P^{*\prime}$ of the principal part of P^*,

$$P^{*\prime}(x,\xi)(Y,N)=\begin{pmatrix} 2(\xi_{(i}\delta^l_{j)}-\xi^lg_{ij}) & 0 \\ K_{ij}\xi^l-2K^l{}_{(i}\xi_{j)}+K^{pl}\xi_qg_{ij} & \xi_i\xi_j-|\xi|^2g_{ij}\end{pmatrix}\begin{pmatrix} Y_l \\ N\end{pmatrix}. \tag{3.5.8}$$

We have the following key observation:

PROPOSITION 3.5.6 *The operator $L:=PP^*$ is elliptic in the sense of Agmon-Douglis-Nirenberg* (cf., e.g. *[334, Definition 6.1.1, p. 210]).*

For the proof we need the following:

LEMMA 3.5.7 *Suppose that* $\dim M\geq 2$*, then $P^{*\prime}(x,\xi)$ is injective for $\xi\neq 0$.*

PROOF: We define a linear map α from the space S_2 of two-covariant symmetric tensors into itself by the formula

$$\alpha(S)=S-(\mathrm{tr}S)g\,. \tag{3.5.9}$$

Let $\xi\neq 0$, if (Y,N) is in the kernel of $P^{*\prime}(x,\xi)$ then

$$\alpha(\xi_{(i}Y_{j)})=0\,,$$

so that $\xi_{(i}Y_{j)}=0$, and $Y=0$. It follows that

$$\alpha(\xi_i\xi_j)N=0\,,$$

which implies $N = 0$. □

PROOF OF PROPOSITION 3.5.6: The differential order of the various entries of L is

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} s_1 + t_1 & s_1 + t_2 \\ s_2 + t_1 & s_2 + t_2 \end{pmatrix},$$

with $s_1 = -1$, $s_2 = 0$, $t_1 = 3$, $t_2 = 4$. Now, $P'(x, \xi)$ is of the form

$$E := \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

while $P^{*\prime}(x, \xi)$ can be written as

$$\begin{pmatrix} -{}^tA & 0 \\ -{}^tB & {}^tD \end{pmatrix},$$

where ${}^tX$ denotes the transpose of X. Let $\xi \neq 0$; by Lemma 3.5.7 ${}^tA$ and ${}^tD$ are injective (hence A and D are surjective) , which implies that ${}^tE$ is injective (hence E is surjective). This shows that $E\,{}^tE$ is bijective: indeed, $E\,{}^tEX = 0$ implies ${}^tXE\,{}^tEX = 0$, which is the same as $|{}^tEX|^2 = 0$, hence $X = 0$. It is straightforward to check that the Agmon-Douglis-Nirenberg symbol of PP^*, defined as the symbol built from those terms which are precisely of order $s_i + t_j$, equals

$$P'(x, \xi)P^{*\prime}(x, \xi) = E\,{}^tE \begin{pmatrix} -I & 0 \\ 0 & 1 \end{pmatrix},$$

and its bijectivity for $\xi \neq 0$ follows. This is precisely the ellipticity condition of Agmon, Douglis, and Nirenberg, whence the result. □

To continue, we consider the linearised equation:

$$P(\delta K, \delta g) = (\delta J, \delta \rho). \tag{3.5.10}$$

Whenever PP^* is an isomorphism, solutions of (3.5.10) can be obtained by solving the equation

$$PP^*(Y, N) = (\delta J, \delta \rho) \tag{3.5.11}$$

for a function N and a vector field Y, and setting

$$(\delta K, \delta g) = P^*(Y, N).$$

Since PP^* is elliptic, the only essential obstruction to solving (3.5.11) is the kernel of PP^*. Now, because of the form of the operator, the kernel of PP^* typically coincides with the kernel of P^*, with the latter contained in $\operatorname{Ker} PP^*$ in any case.

To continue, a definition will be useful:

DEFINITION 3.5.8 *Given an open set Ω, the set of vacuum* Killing Initial Data *(KIDs) on Ω, denoted $\mathscr{K}(\Omega)$, is the set of all solutions of the equation*

$$P^*_{(g,K)|_\Omega}(Y, N) = 0. \tag{3.5.12}$$

Static *vacuum KIDs are defined to be those solutions of (3.5.12) for which $Y \equiv 0 \equiv K$.*

In order to analyse the set of KIDs in an asymptotically hyperbolic setting, it is convenient to rewrite the KID equation (3.5.12) as

$$\begin{aligned} \nabla_{(i}Y_{j)} &= K_{ij}N\,, && (3.5.13)\\ \nabla_i\nabla_j N &= (\mathrm{Ric}\,(g)_{ij} - 2K^l{}_iK_{jl} + \mathrm{tr}_g K K_{ij} - K^{ql}K_{ql}g_{ij})N + \Delta N g_{ij} \\ &\quad -(\nabla^p K_{lp}g_{ij} - \nabla_l K_{ij})Y^l + 2K^l{}_{(i}\nabla_{j)}Y_l\,. && (3.5.14) \end{aligned}$$

Taking traces, we obtain

$$\Delta N = -\frac{1}{n-1}\Big((R+(\mathrm{tr}_g K)^2 - nK^{ql}K_{ql})N - (n\nabla^p K_{lp} - \nabla_l \mathrm{tr}_g K)Y^l\Big)\,, \quad (3.5.15)$$

which allows one to eliminate the second derivatives of N from the right-hand side of (3.5.14), leading to

$$\begin{aligned} \nabla_i\nabla_j N = &\\ &\Big(\mathrm{Ric}\,(g)_{ij} - 2K^l{}_iK_{jl} + \mathrm{tr}_g K K_{ij} + \frac{1}{1-n}(R + (\mathrm{tr}_g K)^2 - K^{ql}K_{ql})g_{ij}\Big)N \\ &+(\nabla_l K_{ij} - (\nabla^p K_{lp} - \nabla_l \mathrm{tr}_g K)g_{ij})Y^l + 2K^l{}_{(i}\nabla_{j)}Y_l\,. \quad (3.5.16) \end{aligned}$$

In the time-symmetric case with $K \equiv 0$, the KID equations decouple. The key part of the equations is then an equation for N, called the *static KID equation*,

$$\nabla_i\nabla_j N = \Big(\mathrm{Ric}(g)_{ij} + \frac{R}{1-n}g_{ij}\Big)N\,. \quad (3.5.17)$$

When $K \equiv 0$, (3.5.13) requires Y to be a (possibly trivial) Killing vector of g.

REMARK 3.5.9 A non-trivial solution (N, Y) of the KID equations generates a spacetime Killing vector field in the domain of dependence of $(\Omega, g|_\Omega, K|_\Omega)$ [327]. Indeed, given a set (M, g, N, Y) with $N \not\equiv 0$, where (M, g) is a Riemannian manifold, N is a function on M and Y is a vector field on M, we define the *Killing development* of (M, g, N, Y) as the spacetime

$$(\mathscr{M}, \mathbf{g}) := (\mathbb{R}\times M, -N^2dt^2 + g_{ij}(dx^i + Y^idt)(dx^j + Y^jdt))\,. \quad (3.5.18)$$

A calculation shows that the Killing development is vacuum if and only if (N, Y) satisfies the vacuum KID equations.

Strictly speaking, to obtain a Lorentzian metric one should remove from M the set of points where N vanishes. However, in many situations of interest one can extend the metric smoothly across the set of zeros of N by passing to a different coordinate system, in which case it is convenient not to remove the zeros.

The vector field

$$X := \partial_t$$

is a Killing vector for the metric $\mathbf{g}$ of (3.5.18). Note that the future-directed unit normal to the level sets of t is

$$T := N^{-1}(\partial_t - Y^i\partial_i)\,, \quad (3.5.19)$$

so that X decomposes as

$$X = NT + Y\,. \tag{3.5.20}$$

Thus N determines the component of X normal to the level sets $\mathscr{S}_t := \{t\} \times M$ of g within $\mathscr{M}$, and Y the tangential part.

The Lie algebra structure on the set of Killing vectors induces a Lie algebra structure on the set of KIDs; this is analysed in detail in [44, 305], where non-vacuum spacetimes are also considered. □

From a geometric point of view one expects that solutions with symmetries should be rare. This was made rigorous in [46], where it is shown that the generic behaviour among solutions of the constraint equations is the absence of KIDs on any open set. On the other hand, one should note that essentially every explicit solution has symmetries. In particular, both the flat initial data for Minkowski space, and the initial data representing the constant time slices of Schwarzschild have KIDs.

3.5.4 A toy model: divergenceless vector fields

To illustrate how this works in a simpler setting, consider the Maxwell constraint equation for a source-free (that is to say, divergence-free) electric field E,

$$P(E) := \mathrm{div} E = 0\,. \tag{3.5.21}$$

The formal adjoint of the divergence operator is the negative of the gradient, so that the "Maxwell KID equation" in this case reads

$$P^*(u) \equiv -\nabla u = 0\,. \tag{3.5.22}$$

The gradient operator has no kernel on a domain with smooth boundary if u is required to vanish on $\partial\Omega$; in fact, the vanishing at a single point of the boundary would suffice. So the equation

$$\mathrm{div} E = \rho \tag{3.5.23}$$

can be solved by solving the Laplace equation for u,

$$PP^*(u) \equiv -\mathrm{div}\nabla u \equiv -\Delta u = \rho\,, \tag{3.5.24}$$

with zero Dirichlet data.

Consider, then, the following toy problem:

Problem 3.5.10 Let E_i, $i = 1, 2$, be two source-free electric fields on $\mathbb{R}^n$. Find a source-free electric field E which coincides with E_1 on a ball $B(R_1)$ of radius R_1 and coincides with E_2 outside a ball of radius $R_2 > R_1$.

Note that if $E_2 \equiv 0$, and if we can solve Problem 3.5.10, we will have screened away the electric field E_1 without introducing any charges in the system: this is the screening of the electric field with an electric field. We will also have constructed an infinite dimensional space of compactly supported divergence free vector fields as E_1 varies, with complete control of E in $B(R_1)$.

As a first attempt to solve the problem, let χ be a radial cut-off function which equals one near the sphere $S(R_1)$ and which equals zero near $S(R_2)$. Set

$$E_\chi = \chi E_1 + (1-\chi)E_2 \,.$$

Since both E_i are divergence-free we have

$$\rho_\chi := \mathrm{div} E_\chi = \nabla\chi \cdot (E_1 - E_2)\,,$$

and there is no reason for ρ_χ to vanish. However, if u solves the equation

$$PP^*(u) \equiv -\Delta u = -\rho_\chi \tag{3.5.25}$$

with vanishing boundary data, then

$$E = E_\chi + P^*(u) \tag{3.5.26}$$

will be divergence free:

$$\mathrm{div} E = \mathrm{div}(E_\chi + P^*(u)) = \rho_\chi + PP^*(u) = 0\,.$$

Now, on $S(R_i)$ we have

$$E|_{S(R_i)} = E_i - \nabla u\,, \tag{3.5.27}$$

and there is no reason why this should coincide with E_i. We conclude that this approach fails to solve the problem.

Replacing Dirichlet data by Neumann data will only help if both $E_i|_{S(R_i)}$ are purely radial, as suitable Neumann data will only guarantee continuity of the normal components of E.

It turns out that there is trick to make this work in whole generality: modify (3.5.24) by introducing weight-functions ψ which vanish very fast at the boundary. An example, which will lead to solutions on the annulus which can be extended to the whole of $\mathbb{R}^n$ in a high-but-finite differentiability class, is provided by the functions

$$\psi = (r-R_1)^\sigma (R_2 - r)^\sigma\,, \quad r \in (R_1, R_2)\,, \tag{3.5.28}$$

with some large positive number σ. Another useful example, which will lead to smoothly-extendable solutions, is

$$\psi = (r-R_1)^\alpha (r-R_2)^\alpha \exp\Big(-\frac{s}{(r-R_1)(R_2-r)}\Big)\,, \quad r \in (R_1, R_2)\,, \tag{3.5.29}$$

with $\alpha \in \mathbb{R}$ and $s > 0$. (The prefactors involving α in (3.5.29) are useful when constructing a consistent functional-analytic set-up, but are essentially irrelevant as far as the blow-up rate of ψ near the $S(R_i)$'s is concerned.)

As such, instead of (3.5.24) consider the equation

$$P(\psi^2 P^*(u)) \equiv -\mathrm{div}(\psi^2 \nabla u) = -\rho_\chi\,. \tag{3.5.30}$$

Solutions of (3.5.30) could provide a solution of Problem 3.5.10 if one replaces (3.5.26) by

$$E = E_\chi + \psi^2 P^*(u) = E_\chi - \psi^2 \nabla u\,. \tag{3.5.31}$$

Now, solutions of (3.5.30) are, at least formally, minima of the functional

$$I = \int_\Omega \frac{1}{2}\psi^2 |\nabla u|^2 + \rho_\chi u \,. \tag{3.5.32}$$

Supposing that minimisation would work, one will then obtain a solution u so that $\psi \nabla u$ is in L^2. Since ψ goes to zero at the boundary very fast, ∇u is likely to blow up. In an ideal world, in which "L^2" is the same as "bounded", ∇u will behave as ψ^{-1} near the boundary. The miracle is that (3.5.30) involves $\psi^2 \nabla u$, with one power of ψ spare, and so the derivatives of u would indeed tend to zero as $\partial\Omega$ is approached.

This naive analysis of the boundary behaviour turns out to be essentially correct: choosing the exponential weights (3.5.29), $\psi^2 \nabla u$ will extend smoothly by zero across the boundaries when the E_i's are smooth. A choice of power-law weights (3.5.28) will lead to extensions of differentiability class determined by the exponent σ, with a loss of a finite number of derivatives due to the fact that L^2 functions are not necessarily bounded, and that there is a loss of differentiability when passing from Sobolev-differentiability to classical derivatives.

It then remains to show that minimisation works in a carefully chosen space. This requires so-called "coercitivity inequalities". For the functional (3.5.32) the relevant inequality is the following *weighted Poincaré inequality:*

$$\int_\Omega \psi^2 |u|^2 \le C \int_\Omega \varphi^2 \psi^2 |\nabla u|^2 \,. \tag{3.5.33}$$

Here $\varphi = (r - R_1)^{-1}(R_2 - r)^{-1}$ when ψ is given by (3.5.28) with $\sigma \neq -1/2$, and $\varphi = (r - R_1)^{-2}(R_2 - r)^{-2}$ for the exponential weights ψ given by (3.5.29) with $s \neq 0$, cf. e.g. [132, 161].

There is an obvious catch here, namely (3.5.33) cannot possibly be true since it is violated by constants. However, (3.5.33) holds on the subspace, say F, of functions which are L^2-orthogonal to constants, when using the weights described above. This turns out to be good enough for solving Problem 3.5.10. For then one can carry out the minimisation on F, finding a minimum $u \in F$. The function u will solve the equation up to an L^2-projection of the equation on constants. In other words, we will have

$$\int_\Omega f(-\mathrm{div}(\psi^2 \nabla u) + \rho_\chi) = 0 \,, \tag{3.5.34}$$

for all differentiable f such that $\int_\Omega f = 0$. Now, integrating the equation (3.5.30) against the constant function $f \equiv 1$ we find

$$\begin{aligned} \int_\Omega (\mathrm{div}(\psi^2 \nabla u) - \rho_\chi) &= \int_\Omega \mathrm{div}(\psi^2 \nabla u - E_\chi) \\ &= \int_{\partial\Omega} (\psi^2 \nabla u - E_\chi) \cdot m = \int_{S(R_1)} E_1 \cdot n - \int_{S(R_2)} E_2 \cdot n \,, \end{aligned} \tag{3.5.35}$$

where m is the outer-directed normal to $\partial\Omega$, and n is the radial vector $\vec{x}/|\vec{x}|$. Since the E_i's are divergence-free it holds that

$$\int_{S(R_i)} E_i \cdot n = \int_{B(R_i)} \mathrm{div}(E_i) = 0 \,.$$

It follows that the right-hand side of (3.5.35) vanishes, hence (3.5.34) holds for all differentiable functions f, and u is in fact a solution of (3.5.30).

(Strictly speaking, when solving (3.5.30) by minimisation using the inequality (3.5.33), Equation (3.5.30) should be replaced by

$$\mathrm{div}(\varphi^2\psi^2\nabla u) = \rho_X \,. \tag{3.5.36}$$

This does not affect the discussion so far, and only leads to a shift of the powers α and σ in (3.5.28)-(3.5.29).)

We conclude that the answer to Problem 3.5.10 is yes, as already observed in [164, 188].

One can likewise solve variants of Problem 3.5.10 with gluing regions which are not annuli, e.g. a difference of two coaxial cones with distinct apertures as in the Carlotto-Schoen Theorem 3.6.1 below. If one of the glued vector fields is taken to be trivial, one obtains configurations where the electric field extends all the way to infinity in open cones and vanishes in, e.g., a half-space.

The gluing construction for the linearised relativistic constraint equations proceeds essentially in the same way. There, in addition to the weighted Poincaré inequality (3.5.33) one also needs a *weighted Korn inequality* for vector fields X:

$$\int_\Omega \psi^2|X|^2 \le C\int_\Omega \varphi^2\psi^2|S(X)|^2 \,, \tag{3.5.37}$$

where $S(X)$ is the symmetric two-covariant vector field defined as

$$S(X)_{ij} = \frac{1}{2}(\nabla_i X_j + \nabla_j X_i)\,. \tag{3.5.38}$$

In (3.5.37) one needs to assume $\sigma \notin \{-n/2, -n2/-1\}$ for the weight (3.5.28), and $s \neq 0$ when ψ is given by (3.5.29). If the metric g has non-trivial Killing vectors, which are solutions of the equation $S(X) = 0$, then (3.5.37) will hold for vector fields X which are in a closed subspace transverse to the space of Killing vectors, with a constant depending upon the subspace.

As already mentioned the space of KIDs is trivial for generic metrics [68], so that the problem of solving modulo kernel does not arise in generic situations.

The full non-linear gluing problem for the scalar curvature, or for vacuum initial data, is solved using the above analysis of the linearised equation together with a tailor-made version of the inverse function theorem, see [132, 161] for details. There it is also shown how to treat problems where existence of KIDs cannot be ignored.

3.5.5 Corvino's theorem

Beyond Euclidean space itself, the constant time slices of the Schwarzschild spacetime form the most basic examples of asymptotically flat, scalar flat manifolds. One long-standing open problem [34, 391] in the field had been whether there exist scalar flat metrics on $\mathbb{R}^n$ which are not globally spherically symmetric but which are spherically symmetric in a neighborhood of infinity and hence, by Birkhoff's theorem, Schwarzschild there.

Corvino resolved this by showing that he could deform any asymptotically flat, scalar flat metric to one which is exactly Schwarzschild outside of a compact set.

THEOREM 3.5.11 ([161]) *Let (M, g) be a smooth Riemannian manifold with zero scalar curvature containing an asymptotically flat end $\mathscr{S}_{\text{ext}} = \{|x| > R_0 > 0\}$. Then there is a $R > R_0$ and a smooth metric $\bar{g}$ on M with zero scalar curvature such that $\bar{g}$ is equal to g in $M \setminus \mathscr{S}_{\text{ext}}$ and $\bar{g}$ coincides on $\{|x| > R\}$ with the metric induced on a standard time-symmetric slice in the Schwarzschild solution. Moreover the mass of $\bar{g}$ can be made arbitrarily close to that of g by choosing R sufficiently large.*

Underlying this result is a gluing construction where the deformation has compact support. The ability to do this is a reflection of the underdetermined nature of the scalar curvature operator.

An elementary illustration of how an underdetermined system can lead to compactly supported solutions is given by the construction of compactly supported transverse-traceless tensors on $\mathbb{R}^3$ in Appendix B of [162] (see also [181] and Section 3.3.1).

An additional challenge in proving Theorem 3.5.11 is the presence of KIDs on the standard slice of the Schwarzschild solution. If the original metric had ADM mass $m(g)$, a naive guess could be that the best fitting Schwarzschild solution would be the one with precisely the same mass. However the mass, and the coordinates of the center of mass, are in one-to-one correspondence with obstructions arising from KIDs. To compensate for this co-kernel in the linearized problem, Corvino uses these ($n+1$ in dimension n) degrees of freedom as effective parameters in the geometric construction. The final solution can be chosen to have its ADM mass arbitrarily close to the initial one.

INCIDENTALLY: Corvino's technique has been applied and extended in a number of important ways. The "asymptotic simplicity" model for isolated gravitational systems proposed by Penrose [343] has been very influential. This model assumes existence of smooth conformal completions to study global properties of asymptotically flat spacetimes. The question of existence of such vacuum spacetimes was open until Chruściel and Delay [131], and subsequently Corvino [162], used this type of gluing construction to demonstrate the existence of infinite dimensional families of vacuum initial data sets which evolve to asymptotically simple spacetimes.

The extension of Corvino's theorem 3.5.11 to non-time-symmetric data was done in [132, 165]. This allowed for the construction of spacetimes which are exactly Kerr outside of a compact set, as well as showing that one can specify other types of controlled and physically relevant asymptotic behavior.

See also Section 3.5.8 for a list of further generalisations. □

3.5.6 Initial data engineering

The gluing constructions of [248] and [249] discussed in Section 3.5.2 are performed using a determined elliptic system provided by the conformal method, which necessarily leads to a global deformation of the initial data set, small away from the gluing site. Now, the ability of the Corvino gluing technique to establish compactly supported deformations invited the question of whether these conformal gluings could be localized. This was answered in the affirmative in [132] for CMC initial data under the additional, generically satisfied [46], assumption that there are no KIDs in a neighborhood of the gluing site.

In [143, 144], this was substantially improved upon by combining the gluing construction of [248] together with the Corvino gluing technique of [131, 161], to obtain a localized gluing construction in which the only assumption is the absence of KIDs near points. For a given n-manifold M (which may or may not be connected) and two points $p_a \in M$, $a = 1, 2$, we let $\tilde{M}$ denote the manifold obtained by replacing small geodesic balls around these points by a neck $S^{n-1} \times I$. When M is connected this corresponds to performing codimension n surgery on the manifold. When the points p_a lie in different connected components of M, this corresponds to taking the connected sum of those components.

Theorem 3.5.13 ([143, 144]) *Let (M, g, K) be a smooth vacuum initial data set, with M not necessarily connected, and consider two open sets $\Omega_a \subset M$, $a = 1, 2$, with compact closure and smooth boundary such that*

$$\textit{the set of KIDs, } \mathscr{K}(\Omega_a)\textit{, is trivial}$$

(see Definition 3.5.8, p. 138). Then for all $p_a \in \Omega_a$, $\epsilon > 0$, and $k \in \mathbb{N}$ there exists a smooth vacuum initial data set $(\tilde{M}, g(\epsilon), K(\epsilon))$ on the glued manifold $\tilde{M}$ such that $(g(\epsilon), K(\epsilon))$ is ϵ-close to (g, K) in a $C^k \times C^k$ topology away from $B(p_1, \epsilon) \cup B(p_2, \epsilon)$. Moreover $(g(\epsilon), K(\epsilon))$ coincides with (g, K) away from $\Omega_1 \cup \Omega_2$.

This result is sharp in the following sense: first note that, by the positive mass theorem, initial data for Minkowski spacetime cannot locally be glued to anything which is non-singular and vacuum. This meshes with the fact that for Minkowskian initial data, we have $\mathscr{K}(\Omega) \neq \{0\}$ for any open set Ω. Next, recall that by the results in [46], the no-KID hypothesis in Theorem 3.5.13 is generically satisfied. Thus, the result can be interpreted as the statement that for generic vacuum initial data sets the local gluing can be performed around arbitrarily chosen points p_a. In particular the collection of initial data with generic regions Ω_a satisfying the hypotheses of Theorem 3.5.13 is not empty.

The proof of Theorem 3.5.13 is a mixture of gluing techniques developed in [246, 248] and those of [132, 161, 165]. In fact, the proof proceeds initially via a generalization of the analysis in [248] to compact manifolds with boundary. In order to have CMC initial data near the gluing points, which the analysis based on [248] requires, one makes use of the work of Bartnik [32] on the plateau problem for prescribed mean curvature spacelike hypersurfaces in a Lorentzian manifold.

Arguments in the spirit of those of the proof of Theorem 3.5.13 lead to the construction of *many-body initial data* [129, 130]: starting from initial data for N gravitating isolated systems, one can construct a new initial data set which comprises isometrically compact subsets of each of the original systems, as large as desired, in a distant configuration; compare Section 3.6.1 below.

An application of the gluing techniques concerns the question of the existence of CMC slices in spacetimes with compact Cauchy surfaces. In [33], Bartnik showed that there exist maximally extended, globally hyperbolic solutions of the Einstein equations *with dust* which admit no CMC slices. Later, Eardley and Witt (unpublished) proposed a scheme for showing that similar vacuum solutions exist, but their argument was incomplete. It turns out that these ideas can be implemented using Theorem 3.5.13, which leads to:

COROLLARY 3.5.14 *[143, 144] There exist maximal globally hyperbolic vacuum spacetimes with compact Cauchy surfaces which contain no compact spacelike hypersurfaces with constant mean curvature.*

Compact Cauchy surfaces with constant mean curvature are useful objects, as the existence of one such surface gives rise to a unique foliation by such surfaces [77], and hence a canonical choice of time function (often referred to as CMC or York time). Foliations by CMC Cauchy surfaces have also been extensively used in numerical analysis to explore the nature of cosmological singularities. Thus the demonstration that there exist spacetimes with no such surfaces has a negative impact on such studies.

One natural question is the extent to which spacetimes with no CMC slices are common among solutions to the vacuum Einstein equations with a fixed spatial topology. It is expected that the examples constructed in [143, 144] are not isolated. In general, there is a great deal of flexibility (in the way of free parameters) in the local gluing construction. This can be used to produce one parameter families of distinct sets of vacuum initial data which lead to spacetimes as in Corollary 3.5.14. What is less obvious is how to prove that all members of these families give rise to *distinct* maximally extended, globally hyperbolic vacuum spacetimes.

A deeper question is whether a sequence of spacetimes which admit constant mean curvature Cauchy surfaces may converge, in a strong topology, to one which admits no such Cauchy surface. (See [30, 33, 217] for general criteria leading to the existence of CMC Cauchy surfaces.)

3.5.7 Non-zero cosmological constant

Gluing constructions have also been carried out with a non-zero cosmological constant [134, 152, 154]. In these papers one constructs spacetimes which coincide, in the asymptotic region, with the corresponding black hole models. One thus obtains constant negative scalar curvature metrics with exact Schwarzschild-anti de Sitter behaviour outside of a compact set. In such spacetimes one has complete control of the geometry in the domain of dependence of the asymptotic region, described there by the Birmingham-Kottler metrics (see Appendix B.3, p. 262). For time-symmetric slices of these spacetimes, the constraint equations reduce to the equation for constant scalar curvature $R = 2\Lambda$. In [134, 152, 154] the emphasis is on gluing with compact support, in the spirit of Corvino's thesis and its extensions already discussed.

The time-symmetric slices of the $\Lambda > 0$ Kottler spacetimes provide "Delaunay" metrics (see [154] and references therein), and the main result of [152, 154] is the construction of large families of metrics with exactly Delaunay ends. When $\Lambda < 0$ the focus is on asymptotically hyperbolic metrics with constant negative scalar curvature. With hindsight, within the family of Kottler metrics with $\Lambda \in \mathbb{R}$ (with $\Lambda = 0$ corresponding to the Schwarzschild metric), the gluing in the $\Lambda > 0$ setting is technically easiest, while that with $\Lambda < 0$ is the most difficult. This is due to the fact that for $\Lambda > 0$ one deals with one linearized operator with a one-dimensional kernel; in the case $\Lambda = 0$ the kernel is $(n+1)$–dimensional; while for $\Lambda < 0$ the construction involves a one-parameter family of operators with $(n+1)$–dimensional kernels.

> INCIDENTALLY: In [160] it has been shown how to use the gluing method to construct non-time-symmetric vacuum initial data containing periodic asymptotic ends, with the evolved metric coinciding with the Kerr-de Sitter metric outside of a spatially-compact region.

3.5.8 Further generalisations

Let us list some further generalisations or applications of the Corvino-Schoen gluing technique:

1. In [289] the method is used to construct non-trivial asymptotically flat black hole spacetimes with smooth asymptotic structure.

2. In [132] it has been shown how to reduce the gluing problem to the verification of a few properties of the weight functions φ and ψ, together with the verification of the Poincaré and Korn inequalities. It has also been shown there how to use the technique to control the asymptotics of solutions in asymptotically flat regions.

3. The differentiability thresholds for the applicability of the method have been lowered in [133]. It has also been shown there how to construct a Banach manifold structure for the set of vacuum initial data under various asymptotic conditions using the general ideas developed in the process of gluing.

4. In [187] gluings are done by interpolating scalar curvature.

5. In [188] the gluing method has been used to construct compactly supported solutions for a wide class of underdetermined elliptic systems. As a particular case, for any open set $\mathscr{U}$ one obtains an infinite-dimensional space of solutions of the vector constraint equation which are compactly supported in $\mathscr{U}$.

6. In [190] the gluing is used to make local "generalised connected-sum" gluings along submanifolds.

7. In [163] the gluing method is used to deform initial data satisfying the dominant energy condition to ones where the condition is strict.

8. In [64, 128] the gluing is used to construct null hypersurfaces with generators complete to the future in maximal globally hyperbolic developments of asymptotically flat initial data sets, without any smallness conditions on the data.

9. The gluing methods have been extended in [85] to k-Yamabe metrics, i.e. metrics for which the k-th symmetric polynomial of the Schouten curvature tensor is constant.

10. In [237] the gluing method is used to exhibit instabilities of a class of black-hole spacetimes.

3.6 Gravity shielding a la Carlotto-Schoen

In [84] Carlotto and Schoen show that gravitational fields can be used to shield gravitational fields. That is to say, one can produce spacetime regions extending to infinity where no gravitational forces are felt whatsoever, by manipulating the gravitational field around these regions. A concise version of their result reads:

THEOREM 3.6.1 (Carlotto & Schoen [84]) *Given an asymptotically flat initial data set for vacuum Einstein equations there exist cones and asymptotically flat vacuum initial data which coincide with the original ones inside the cones and are Minkowskian outside slightly larger cones, see Figure 3.6.1.*

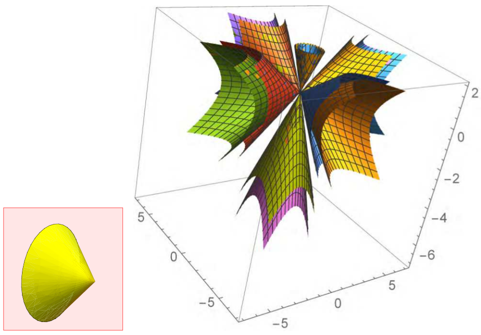

Figure 3.6.1: Left picture: The new initial data are Minkowskian outside the larger cone, and coincide with the original ones inside the smaller one. The construction can also be carried out the other way round, with Minkowskian data inside the smaller cone and the original ones outside the larger cone. Both cones extend to infinity, and their tips are located very far in the asymptotically flat region. Right picture: Iterating the construction, one can embed any finite number of distinct initial data sets into Minkowskian data, or paste-in Minkowskian data inside several cones into a given data set.

Actually, the result is true for all cones with preassigned axis and pair of apertures provided the vertex is shifted sufficiently far away in the asymptotically flat regions.

In the associated spacetimes $(\mathscr{M}, \mathbf{g})$ the metric coincides with the Minkowski metric within the domain of dependence, which we will denote by $\mathscr{D}$, of the complement of the larger cones, which forms an open subset of $\mathscr{M}$; we return to this in Section 3.6.1 below. Physical objects in $\mathscr{D}$ do not feel any gravitational fields. The Carlotto-Schoen gluing has effectively switched-off any gravitational effects in this region. This has been achieved by manipulating vacuum gravitational fields only.

In this section we will highlight some key elements of the proof of Theorem 3.6.1, and discuss selected further developments. Our presentation follows closely [127].

Newtonian gravity

To put the Carlotto-Schoen Theorem in its proper context, let us recall that Newtonian gravity can be viewed as the theory of a gravitational potential field ϕ which solves the equation

$$\Delta\phi = -4\pi G\rho \,, \tag{3.6.1}$$

where Δ is the Laplace operator in an Euclidean $\mathbb{R}^3$ and G is Newton's constant. Up to conventions on signs, proportionality factors, and units, ρ is the matter density, which is not allowed to be negative. Isolated systems are defined by the requirement that both ρ and ϕ decay to zero as one recedes to infinity.

Freely falling bodies experience an acceleration proportional to the gradient of ϕ. So no gravitational forces exist in those regions where ϕ is constant.

Suppose that ρ has support contained in a compact set K, and that ϕ is constant on an open set Ω. Since solutions of (3.6.1) are analytic on $\mathbb{R}^3 \setminus K$, ϕ is constant on any connected component of $\mathbb{R}^3 \setminus K$ which meets Ω. We conclude that if Ω extends to infinity, then ϕ vanishes at all large distances. This implies, for all sufficiently large spheres $S(R)$,

$$0 = \int_{S(R)} \nabla\phi \cdot n \, d^2S = \int_{B(R)} \Delta\phi \, d^3V = -4\pi G \int_{B(R)} \rho \, d^3V \,.$$

Since ρ is non-negative, we conclude that $\rho \equiv 0$. Equivalently, for isolated systems with compact sources, *Newtonian gravity cannot be screened away on open sets extending to large distances.*

The striking discovery of Carlotto and Schoen is, that such a screening is possible in Einsteinian gravity.

The Newtonian argument above fails if matter with negative density is allowed. It should therefore be emphasised that the Carlotto-Schoen construction is done by manipulating vacuum initial data, without involvement of matter fields.

Asymptotic flatness

Since the choice of decay rate of the metric towards the flat one plays an essential role in the Carlotto-Schoen construction, for the convenience of the reader we review the relevant definitions.

Initial data for general relativistic isolated systems are typically modelled by *asymptotically flat* data with vanishing cosmological constant Λ. Actually, astrophysical observations indicate that Λ is positive. However, for the purpose of observing nearby stars, or for our stellar system, the corrections arising from Λ are negligible, they only become important at cosmological scales.

The class of asymptotically flat systems should obviously include the Schwarzschild black holes. In those, on the usual slicing by $t = \text{const}$ hypersurfaces it holds that $K_{ij} \equiv 0$ and

$$g_{ij} = \Big(1 + \frac{m}{2r}\Big)^4 \delta_{ij} + O(r^{-2})\,, \tag{3.6.2}$$

in spacetime dimension four, or

$$g_{ij} = \Big(1 + \frac{m}{2r^{n-2}}\Big)^{\frac{4}{n-2}} \delta_{ij} + O(r^{-(n-1)})\,, \tag{3.6.3}$$

in general spacetime dimension $n+1$. Here δ_{ij} denotes the Euclidean metric in manifestly flat coordinates. The asymptotics (3.6.2) is often referred to as *Schwarzschildean*, and the parameter m is called the ADM mass of the metric. Now, one can obtain initial data with non-vanishing total momentum by taking Lorentz-transformed slices in the Schwarzschild spacetime. This leads to initial data sets satisfying

$$\partial_{i_1} \cdots \partial_{i_\ell} (g_{ij} - \delta_{ij}) = O(r^{-\alpha-\ell})\,, \tag{3.6.4}$$

$$\partial_{i_1} \cdots \partial_{i_k} K_{ij} = O(r^{-\alpha-k-1})\,, \tag{3.6.5}$$

with $\alpha = n-2$, for any $k, \ell \in \mathbb{N}$. Metrics g satisfying (3.6.4) will be called *asymptotically Euclidean*.

The flexibility of choosing $\alpha \in (0, n-2)$ in the definition of asymptotic flatness (3.6.4)-(3.6.5), as well as k, ℓ smaller than some threshold, is necessary in Theorem 3.6.1. Indeed, the new initial data constructed there are *not expected* to satisfy (3.6.4) with $\alpha = n-2$. It would be of interest to settle the question, whether or not this is really case.

There does not appear to be any justification for the Schwarzschildean threshold $\alpha = n-2$ other than historical. On the other hand, the threshold

$$\alpha = (n-2)/2 \tag{3.6.6}$$

appears naturally as the optimal threshold for a well-defined total energy-momentum of the initial data set. This has been first discussed in [116–118, 192], compare [31].

Time-symmetric initial data and the Riemannian problem

Recall that initial data are called *time-symmetric* when $K_{ij} \equiv 0$. In this case, and assuming vacuum, the *vector constraint equation* is trivially satisfied, while the *scalar constraint equation* becomes the requirement that $(\mathscr{S}, g)$ has constant scalar curvature R:

$$R = 2\Lambda\,. \tag{3.6.7}$$

In particular $(\mathscr{S}, g)$ should be scalar-flat when the cosmological constant Λ vanishes. So all statements about vacuum initial data translate immediately into statements concerning scalar-flat Riemannian manifolds. For example, the following statement is a special case of Theorem 3.6.1:

THEOREM 3.6.2 (Carlotto & Schoen) *Given a scalar-flat asymptotically Euclidean metric g there exist cones and scalar-flat asymptotically Euclidean metrics which coincide with g inside of the cones and are flat outside slightly larger cones.*

This theorem was certainly one of the motivations for the proof of Theorem 3.6.1. Indeed, the question of existence of non-trivial, scalar-flat, asymptotically flat metrics $\hat{g}$ which are exactly flat in a half-space arises when studying complete, non-compact minimal hypersurfaces. Indeed, if $\hat{g}$ is such a metric, then all hyperplanes lying in the flat half-space minimize area under compactly supported deformations which do not extend into the non-flat region. So Theorem 3.6.2 shows that such metrics $\hat{g}$ actually exist. This should be contrasted with the following beautiful result of Chodosh and Eichmair [83], which shows that *minimality under all compactly supported perturbations* implies flatness:

THEOREM 3.6.3 (Chodosh, Eichmair) *The only asymptotically Euclidean three-dimensional manifold with non-negative scalar curvature that contains a complete non-compact embedded surface S which is a (component of the) boundary of some properly embedded full-dimensional submanifold of (M, g) and is area-minimizing under compactly supported deformations is flat $\mathbb{R}^3$, and S is a flat plane.*

The above was preceded by a related rigidity result of Carlotto [82]:

THEOREM 3.6.4 (Carlotto) *Let (M, g) be a complete, three-dimensional, asymptotically Schwarzschildean Riemannian manifold with non-negative scalar curvature. If M contains a complete, properly embedded, stable minimal surface S, then (M, g) is the Euclidean space and S is a flat plane.*

Such results immediately imply non-compactness for sequences of solutions of the Plateau problem with a diverging sequence of boundaries. We note that compactness results in this spirit play a key role in the Schoen & Yau proof of the positive energy theorem. One could likewise imagine that convergence of such sequences of solutions of the Plateau problem could provide a tool to study stationary black hole solutions, but no arguments in such a spirit have been successfully implemented so far.

3.6.1 Localised scalar curvature

It is an immediate consequence of the positive energy theorem that, for complete asymptotically Euclidean manifolds with non-negative scalar curvature,

curvature cannot be localised in a compact set.

In other words, a flat region cannot enclose a non-flat one. A similar statement applies for general relativistic initial data sets satisfying the dominant energy condition. Indeed a metric which is flat outside of a compact set would have zero total mass and hence would be flat everywhere by the rigidity-part of the Positive Energy Theorem

One would then like to know *how much flatness can a non-trivial initial data set carry?* This question provided another motivation for Theorem 3.6.1, which shows that non-flatness can be localised within cones.

A previous family of non-trivial asymptotically Euclidean scalar-flat metrics containing flat regions is provided by the *quasi-spherical metrics* of Bartnik [36]. In Bartnik's examples flatness can be localised within balls.

In [84] it is noticed that *non-flat* regions *cannot* be sandwiched between parallel planes. This follows immediately from the formula for ADM mass, due to Beig [42] (compare [22, 28, 116, 233]),

$$m = \lim_{R\to\infty} \frac{1}{16\pi} \int_{r=R} G_{ij} x^i n^j dS \,, \tag{3.6.8}$$

where $G_{ij} := R_{ij} - \frac{1}{2} R g_{ij}$ is the Einstein tensor, and x^i is the coordinate vector in the asymptotically Euclidean coordinate system; note that the R term usually decays fast enough so that it can be dropped in this equation. Indeed, (3.6.8) together with a non-zero mass and asymptotic flatness imply that the region where the Ricci tensor has to have non-trivial angular extent as one recedes to infinity. But this is not the case for a region sandwiched between two parallel planes.

Time evolution

Consider a non-trivial asymptotically flat Carlotto-Schoen initial data set, at $t = 0$, with the "non-Minkowskianity" localised in a cone with vertex at $\vec{a}$, axis $\vec{i}$ and aperture $\theta > 0$, which we will denote by $C(\vec{a}, \vec{i}, \theta)$. It follows from [63] that the associated vacuum spacetime will exist globally when the data are small enough, in a norm compatible with the Carlotto-Schoen setting.

One can think of the associated vacuum spacetime as describing a gravitational wave localised, at $t = 0$, in an angular sector of opening angle 2θ and direction defined by the vector $\vec{i}$. We have seen that θ can be made as small as desired but cannot be zero, so that all such solutions must have non-trivial angular extent.

It is of interest to enquire how Carlotto-Schoen solutions evolve in time. In what follows we assume that $\theta < \pi/2$. Standard results on Einstein equations show that the boundary enclosing the non-trivial region travels outwards no faster than the speed of light c. This, together with elementary geometry shows that at time t the spacetime metric will certainly be flat outside of a cone

$$C\Big(\vec{a} - \frac{c}{\sin(\theta)} t\, \vec{i}, \vec{i}, \theta\Big) \,. \tag{3.6.9}$$

The reader will note that the tip of the cone (3.6.9) travels faster than light, which is an artefact of the rough estimate. A more careful inspection near the tip of the cone shows that the domain of dependence at time t consists of a cone of aperture θ spanned tangentally on the boundary of a sphere of radius t as shown in Figure 3.6.2. In any case the angular opening of the wave remains constant on slices of constant time. However, the wave is likely to spread and meet all generators of null infinity, but not before the intersection of $\mathscr{S}$ with the light-cone of the origin $(t = 0, \vec{x} = 0)$ is reached.

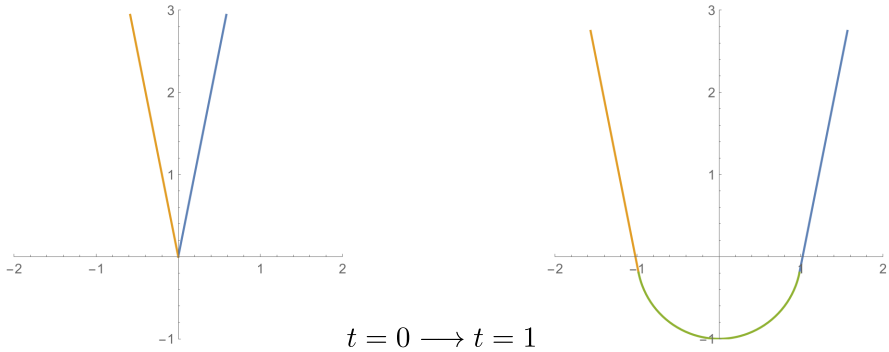

Figure 3.6.2: Left graph: The cone in the figure represents the exterior boundary of Ω at $t = 0$, with initial data Minkowskian below the graph. Right graph: At $t = 1$, the evolved spacetime metric is Minkowskian below the graph. The three-dimensional picture is obtained by rotating the graphs around the vertical axis.

Many-body problem

Given two initial data sets (Ω_a, g_a, K_a), $a = 1, 2$, the question arises, whether one can find a new initial data set which contains both? An answer to this is not known in full generality. However, the Carlotto-Schoen construction gives a positive answer to this question when the original initial data are part of asymptotically flat initial data $(\mathscr{S}_a, g_a, K_a)$, provided that the sets $\Omega_a \subset \mathscr{S}_a$ can be enclosed in cones which do not intersect after "small angular fattenings". Indeed, one can then apply the deformation of Theorem 3.6.1 to each original data set to new initial data $(\mathscr{S}_a, \hat{g}_a, \hat{K}_a)$ which coincide with the original ones on Ω_a and are Minkowskian outside the fattened cones. But then one can superpose the resulting initial data sets in the Minkowskian region, as shown in Figure 3.6.3.

The construction can be iterated to produce many-body initial data sets.

An alternative gluing construction with *bounded* sets $\Omega_a \subset \mathscr{S}_a$ has been previously carried-out in [129, 130].

3.6.2 Elements of the proof

As already mentioned, the main idea of the proof of Theorem 3.6.1 is essentially identical to that of the Corvino-Schoen gluing described in Section 3.5.3. There is, however, a significant amount of new work involved.

As such, the first extraordinary insight is to imagine that the result can be true at all when Ω is the difference of two cones with different apertures, smoothed out at the vertex, see Figure 3.6.4. This is the geometry that we are going to assume in the remainder of this section.

Next, all generalisations of [161, 165] listed in Section 3.5.8 involve gluing across a compact boundary. In the current case $\partial\Omega$ is not compact, and so some analytical aspects have to be revisited. In addition to weights governing decay at $\partial\Omega$, radial weights need to be introduced in order to account for the infinite extent of the cone.

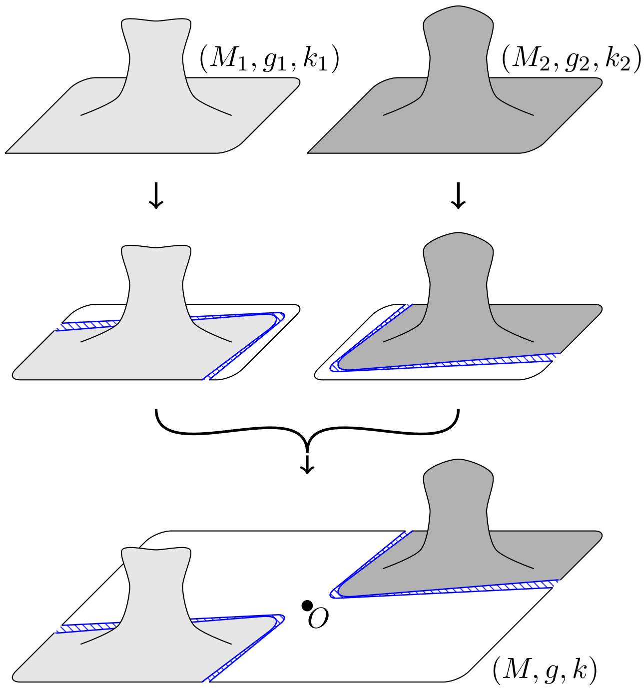

Figure 3.6.3: Theorem 3.6.1 allows to merge an assigned collection of data into an exotic $N-$body solution of the Einstein constraint equations. From [84], with kind permission of the authors.

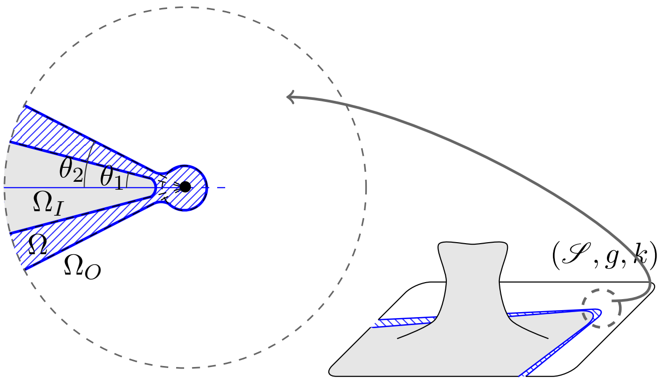

Figure 3.6.4: Regularized cones and the gluing region Ω, from [84] with kind permission of the authors. $\Omega_I \subset \Omega$ is the interior cone, Ω_O is the region outside the larger cone.

Let $d(p)$ denote the distance from $p \in \Omega$ to $\partial\Omega$. Let $\theta_1 < \theta_2$ be the respective apertures of the inner and outer cones and let θ denote the angle away from the axis of the cones. A substantial part of the paper consists in establishing inequalities in the spirit of (3.5.33) and (3.5.37) with $\varphi = r$ and with weight functions ψ which are smooth everywhere, behave like d^σ for small d, and are equal to

$$\psi = r^{n/2-q}(\theta - \theta_1)^\sigma(\theta_2 - \theta)^\sigma\,, \tag{3.6.10}$$

for large distances, with $q, \sigma > 0$. More precisely, let $\sigma > 0$ be large enough, and assume that $0 < q < (n-2)/2$, with $q \neq (n-4)/2$ for $n \geq 5$. Suppose that g is the Euclidean metric and let Ω be as above. Let ϕ be a positive function which for large distances equals $\theta - \theta_1$ and $\theta_2 - \theta$ close to the inner and outer cones, respectively, and which behaves as the distance from $\partial\Omega$ otherwise. Then there exists a constant C such that for all differentiable functions u and vector fields X, both with bounded support in Ω (no conditions at $\partial\Omega$), the following inequalities are true:

$$\int_\Omega |u|^2 r^{-n+2q}\phi^\sigma \leq C\int_\Omega |\nabla u|^2 r^{2-n+2q}\phi^\sigma\,, \tag{3.6.11}$$

$$\int_\Omega |Y|^2 r^{-n+2q}\phi^\sigma \leq C\int_\Omega |S(Y)|^2 r^{2-n+2q}\phi^\sigma\,. \tag{3.6.12}$$

A clever lemma relying on the coarea formula, ([84, Lemma 4.1]), reduces the proof of the inequalities (3.6.11)-(3.6.12) to the case $\phi \equiv 1$.

A key point in the proof of (3.6.12) is the inequality established in [84, Proposition 4.5] (also known in [84] as "Basic Estimate II"), which takes the form

$$\int_\Omega |\nabla Y|^2 r^{2-n+2q}\phi^\sigma \leq C\int_\Omega |S(Y)|^2 r^{2-n+2q}\phi^\sigma\,. \tag{3.6.13}$$

The justification of (3.6.13) requires considerable ingenuity.

It is simple to show that (3.6.11)-(3.6.12) continue to hold for asymptotically Euclidean metrics which are close enough to the Euclidean one, with uniform constants. As explained in Section 3.5.4, these inequalities provide the stepping stones for the analysis of the linear equations.

Note that the radial weights in (3.6.11) guarantee that affine functions are not in the space obtained by completing $C^1_c(\overline{\Omega})$ with respect to the norm defined by the right-hand side. A similar remark concerning vectors with components which are affine functions applies in the context of (3.6.12). This guarantees that neither KIDs, nor asymptotic KIDs, interfere with the construction, which would otherwise have introduced a serious obstruction to the argument.

Once these *decoupled* functional inequalities are gained, a perturbation argument ensures coercivity of the adjoint linearised constraint operator (in suitable doubly-weighted Sobolev spaces). This allows one to use direct methods to obtain existence of a unique global minimum for the functional whose Euler-Lagrange equations are the linearized constraints. We refer the reader to Propositions 4.6 and 4.7 of [84] for precise statements.

The argument of [84] continues with a Picard iteration scheme, which allows one to use the analysis of the linear operator to obtain solutions to the nonlinear problem under a smallness condition. This is not an off-the-shelf argument: it involves some delicate choices of functional spaces for the iteration, where one takes a combination of weighted-Sobolev and weighted-Schauder norms. Alternatively one could use [131, Appendix G] at this stage of the proof, after establishing somewhat different estimates, compare [135].

To end the proof it suffices to start moving the cones to larger and larger distances in the asymptotic region, so that the metric on Ω approaches the flat one. When the tips of the cones are far enough the smallness conditions needed to make the whole machinery work are met, and Theorem 3.6.1 follows.

An interesting, and somehow surprising, aspect of the result is the fact that *no matter how small the cone angles are*, the ADM energy-momentum of the glued data provides an arbitrarily good approximation of the ADM energy-momentum of the given data when the vertex of the cones is chosen far enough in the asymptotic region. This is proven in Section 5.6 of [84] and is then exploited in the construction of N-body Carlotto-Schoen solutions, already presented in Section 3.6.1. This is the object of Section 6 of their paper.

3.6.3 Beyond Theorem 3.6.1

The results of Carlotto and Schoen have meanwhile been extended in a few directions.

In the initial-data context, gluings in the same spirit have been done in [136] for asymptotically hyperbolic initial data sets. In terms of the half-space model for hyperbolic space, the analogues of cones are half-annuli extending to the conformal boundary at infinity. As a result one obtains e.g. non-trivial constant scalar curvature metrics which are exactly hyperbolic in half-balls centered at the conformal boundary. We provide more details in Section 3.6.4 below.

In a Riemannian asymptotically Euclidean setting, with $K_{ij} \equiv 0$ so that only the scalar curvature matters, the following generalisations are straightforward:

1. Rather than gluing an asymptotically Euclidean metric to a flat one, any two asymptotically metrics g_1 and g_2 are glued together.

2. In the spirit of [187], the gluings at zero-scalar curvature can be replaced by gluings where the scalar curvature of the final metric equals

$$\chi R(g_1) + (1 - \chi) R(g_2)$$

   where, as before, χ is a cut-off function varying between zero and one in the gluing region. Thus, the scalar curvature of the final metric is sandwiched between the scalar curvatures of the original ones. This reduces of course to a zero-scalar-curvature gluing if both g_1 and g_2 are scalar-flat.

3. The geometry of the gluing region can be allowed to be somewhat more general than the interface between two cones [135].

A few more details about this can be found in Section 3.6.5 below.

3.6.4 Asymptotically hyperbolic gluings

Let us outline here one of the gluing constructions in [136], the reader is referred to that reference for some more general "exotic hyperbolic gluings". The underlying manifold is taken to be the "half-space model" of hyperbolic space:

$$\mathscr{H} = \{(z, \theta) |\ z > 0, \theta \in \mathbb{R}^{n-1}\} \subset \mathbb{R}^n .$$

One wishes to glue together metrics asymptotic to each other while interpolating their respective scalar curvatures. The first metric is assumed to take the form, in suitable local coordinates,

$$g = \frac{1}{z^2}((1+O(z))dz^2 + \underbrace{h_{AB}(z,\theta^C)d\theta^A d\theta^B}_{=:h(z)} + O(z)_A dz\, d\theta^A)\,, \qquad (3.6.14)$$

where $h(z)$ is a continuous family of Riemannian metrics on $\mathbb{R}^{n-1}$.

We define

$$B_\lambda := \{z>0\,,\ \underbrace{\sum_i(\theta^i)^2}_{=:|\theta|^2} + z^2 < \lambda^2\}\,,\quad A_{\epsilon,\lambda} = B_\lambda \setminus \overline{B_\epsilon}\,.$$

The gluing construction will take place in the region

$$\Omega = A_{1,4}\,. \qquad (3.6.15)$$

Let $\hat g$ be a second metric on B_5 which is close to g in $C^{k+4}_{1,z^{-\sigma}}(A_{1,4})$. Here, for ϕ and φ — smooth strictly positive functions on M, and for $k\in\mathbb{N}$, we define $C^k_{\phi,\varphi}$ to be the space of C^k functions or tensor fields for which the norm

$$\|u\|_{C^k_{\phi,\varphi}(g)} = \sup_{x\in M}\textstyle\sum_{i=0}^k \|\varphi\phi^i\nabla^{(i)}u(x)\|_g$$

is finite.

Let χ be a smooth non-negative function on $\mathscr{H}$, equal to 1 on $\mathscr{H}\setminus B_3$, equal to zero on B_2, and positive on $\mathscr{H}\setminus\overline{B_2}$. We set

$$g_\chi := \chi\hat g + (1-\chi)g\,. \qquad (3.6.16)$$

In [136] a gluing-by-interpolation of the constraint equations is carried out. In the time-symmetric case, the main interest is that of constant scalar curvature metrics, which then continue to have constant scalar curvature, or for metrics with positive scalar curvature, which then remains positive. Since the current problem is related to the construction of initial data sets for Einstein equations, in general-relativistic matter models such as Vlasov or dust, an interpolation of scalar curvature is of direct interest.

One has [136]:

THEOREM 3.6.5 *Let $n/2<k<\infty$, $b\in[0,\frac{n+1}{2}]$, $\sigma>\frac{n-1}{2}+b$, suppose that $g-\mathring{g}\in C^{k+4}_{1,z^{-1}}$. For all $\hat g$ close enough to g in $C^{k+4}_{1,z^{-\sigma}}(A_{1,4})$ there exists a two-covariant symmetric tensor field h in $C^{k+2-\lfloor n/2\rfloor}(\mathscr{H})$, vanishing outside of $A_{1,4}$, such that the tensor field $g_\chi+h$ defines an asymptotically hyperbolic metric satisfying*

$$R(g_\chi+h) = \chi R(\hat g) + (1-\chi)R(g)\,. \qquad (3.6.17)$$

A similar result is established in [136] for the full constraint equations.

The proof involves “triply weighted Sobolev spaces” on $A_{1,4}$ with

$$\varphi = x/\rho\,,\quad \psi = x^a z^b \rho^c\,,$$

where a and c are chosen large as determined by k, n and σ. Here z is the coordinate of (3.6.14), the function x is taken to be any smooth function on Ω which equals the $z^2\mathring{g}$-distance to

$$\{|\theta|^2 + z^2 = 1\} \cup \{|\theta|^2 + z^2 = 4\}$$

near this last set, while $\rho := \sqrt{x^2+z^2}$. The heart of the proof is the establishing of the relevant Poincaré and Korn inequalities. Once this is done, the scheme of the proof follows closely the arguments described so far.

One would like to have a version of Theorem 3.6.5 with weights which exponentially decay as x tends to zero. However, the triply-weighted Korn inequality needed for this has not been established so far.

3.6.5 Asymptotically Euclidean scalar curvature gluings by interpolation

We finish this *séminaire* by describing a straightforward generalisation of Theorem 3.6.1 in the time-symmetric asymptotically Euclidean setting.

Let $S(p,R) \subset \mathbb{R}^n$ denote a sphere of radius R centred at p. Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary (thus Ω is open and connected, and $\partial\Omega = \overline{\Omega}\setminus\Omega$ is a smooth manifold). We further assume that $\partial\Omega$ has *exactly two* connected components, and that there exists $R_0 \geq 1$ such that

$$\Omega_S := \Omega \cap S(0,R_0) \tag{3.6.18}$$

also has exactly two connected components, with

$$\Omega \setminus B(0,R_0) = \{\lambda p \mid p \in \Omega_S\,, \lambda \geq 1\}\,. \tag{3.6.19}$$

The regularised differences of cones in the Carlotto-Schoen gluings provide examples of such sets. Another example is displayed in Figure 3.6.5. Since the construction can be iterated, the requirement that Ω be connected is irrelevant.

We let $x : \overline{\Omega} \to \mathbb{R}$ be any smooth defining function for $\partial\Omega$ which has been chosen so that

$$x(\lambda p) = \lambda x(p) \text{ for } \lambda \geq 1 \text{ and for } p, \lambda p \in \Omega \setminus B(0,R_0). \tag{3.6.20}$$

Equivalently, for $p \in \Omega_S$ and λ larger than one, we require $x(\lambda p) = \lambda x_S(p)$, where x_S is a defining function for $\partial\Omega_S$ within $S(0,R_0)$.

We will denote by r a smooth positive function which coincides with $|\vec{x}|$ for $|\vec{x}| \geq 1$.

By definition of Ω there exists a constant c such that the distance function $d(p)$ from a point $p \in \Omega$ to $\partial\Omega$ is smooth for all $d(p) \leq cr(p)$. The function x can be chosen to be equal to d in that region.

For $\beta, s, \mu \in \mathbb{R}$ we define

$$\varphi = \left(\frac{x}{r}\right)^2 r = \frac{x^2}{r}\,, \quad \psi = r^{-n/2-\beta}\left(\frac{x}{r}\right)^\sigma e^{-sr/x} =: r^\mu x^\sigma e^{-sr/x} \tag{3.6.21}$$

on Ω. One can show that the weighted Poincaré ineqality (3.5.33) holds with these weights, modulo a supplementary integral of $|u|^2$ on a compact subset

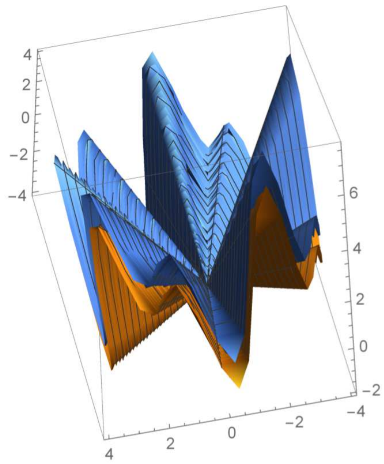

Figure 3.6.5: A possible set Ω, located between the two surfaces. The glued metric coincides with g_1 above the higher surface, coincides with g_2 below the lower surface, and is scalar-flat if both g_1 and g_2 were.

of Ω, for all tensor fields u compactly supported in Ω as long as $s \neq 0$ and $\beta \equiv \sigma + \mu + n/2 \neq 0$. The supplementary integral does not lead to any new difficulties in the proof, which proceeds as described above.

In order to carry out the scalar-curvature interpolation, recall that $\partial\Omega$ has exactly two connected components. We denote by χ a smooth function with the following properties:

1. $0 \leq \chi \leq 1$;
2. χ equals one in a neighborhood of one of the components and equals zero in a neighborhood of the other component;
3. on $\Omega \setminus B(0, R_0)$ the function χ is a function of x/r wherever it is not constant.

The metric g_χ is then defined as in (3.6.16).

Letting Ω, x and r be as just described, with ψ, φ given by (3.6.21), in [135] the following is proved:

THEOREM 3.6.6 *Let $\epsilon > 0$, $k > n/2$, $\beta \in [-(n-2), 0)$, and $\tilde\beta < \min(\beta, -\epsilon)$. Suppose that $g - \delta \in C^{k+4}_{r,r^\epsilon} \cap C^\infty$, where δ is the Euclidean metric. For all real numbers σ and $s > 0$ and*

$$\textit{all smooth metrics } \hat g \textit{ close enough to } g \textit{ in } C^{k+4}_{r,r^{-\tilde\beta}}(\Omega)$$

there exists on Ω a unique smooth two-covariant symmetric tensor field h such that the metric $g_\chi + h$ solves

$$R\,[g_\chi + h] = \chi R(\hat g) + (1-\chi) R(g)\,. \tag{3.6.22}$$

The tensor field h vanishes at $\partial\Omega$ and can be C^∞-extended by zero across $\partial\Omega$, leading to a smooth asymptotically Euclidean metric $g_\chi + h$.

There is little doubt that there is an equivalent of Theorem 3.6.6 in the full initial-data context. In fact, the only missing element of the proof at the time of writing of this review is a doubly-weighted Korn inequality with weights as in (3.6.21).

The smallness assumptions needed in the theorem can be realised by moving the set Ω to large distances, as in the Carlotto-Schoen theorem. When Ω does not meet $S(0,1)$, an alternative is provided by "scaling Ω up" by a large factor. This is equivalent to keeping Ω fixed and scaling-down the metrics from large to smaller distances. It should be clear that the metrics will approach each other, as well as the flat metric, when the scaling factor becomes large.

Part II

Appendices

Appendix A

Introduction to pseudo-Riemannian geometry

A.1 Manifolds

It is convenient to start with the definition of a manifold:

Definition A.1.1 *An n–dimensional manifold is a set M equipped with the following:*

1. *topology: a "connected Hausdorff paracompact topological space" (think of nicely looking subsets of $\mathbb{R}^{1+n}$, like spheres, hyperboloids, and such), together with*

2. *local charts: a collection of coordinate patches $(\mathscr{U}, x^i)$ covering M, where $\mathscr{U}$ is an open subset of M, with the functions $x^i : \mathscr{U} \to \mathbb{R}^n$ being continuous. One further requires that the maps*

$$M \supset \mathscr{U} \ni p \mapsto (x^1(p), \ldots, x^n(p)) \in \mathscr{V} \subset \mathbb{R}^n$$

are homeomorphisms.

3. *compatibility: given two overlapping coordinate patches, $(\mathscr{U}, x^i)$ and $(\widetilde{\mathscr{U}}, \tilde{x}^i)$, with corresponding sets $\mathscr{V}, \widetilde{\mathscr{V}} \subset \mathbb{R}^n$, the maps $\tilde{x}^j \mapsto x^i(\tilde{x}^j)$ are smooth diffeomorphisms wherever defined: this means that they are bijections differentiable as many times as one wishes, with*

$$\det\left[\frac{\partial x^i}{\partial \tilde{x}^j}\right] \text{ nowhere vanishing}.$$

Definition of differentiability: A function on M is smooth if it is smooth when expressed in terms of local coordinates. Similarly for tensors.

Examples:
1. $\mathbb{R}^n$ with the usual topology, one single global coordinate patch.

2. A sphere: use stereographic projection to obtain two overlapping coordinate systems (or use spherical angles, but then one must avoid borderline angles, so they don't cover the whole manifold!).

3. We will use several coordinate patches (in fact, five), to describe the Schwarzschild black hole, though one spherical coordinate system would suffice.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$, and define $N := f^{-1}(0)$. If ∇f has no zeros on N, then every connected component of N is a smooth $(n-1)$–dimensional manifold. This construction leads to a plethora of examples. For example, if $f = \sqrt{(x^1)^2 + \ldots + (x^n)^2} - R$, with $R > 0$, then N is a sphere of radius R.

In this context a useful example is provided by the function $f = t^2 - x^2$ on $\mathbb{R}^2$: its zero-level-set is the light-cone $t = \pm x$, which is a manifold except at the origin; note that $\nabla f = 0$ there, which shows that the criterion is sharp.

A.2 Scalar functions

Let M be an n-dimensional manifold. Since manifolds are defined using coordinate charts, we need to understand how things behave under coordinate changes. For instance, under a change of coordinates $x^i \to y^j(x^i)$, to a function $f(x)$ we can associate a new function $\bar{f}(y)$, using the rule

$$\bar{f}(y) = f(x(y)) \quad \Longleftrightarrow \quad f(x) = \bar{f}(y(x)) \,.$$

In general relativity it is a common abuse of notation to write the same symbol f for what we wrote $\bar{f}$, when we think that this is the same function but expressed in a different coordinate system. We then say that a real- or complex-valued f is a *scalar function* when, under a change of coordinates $x \to y(x)$, the function f transforms as $f \to f(x(y))$.

In this section, to make things clearer, we will write $\bar{f}$ for $f(x(y))$ even when f is a scalar, but this will almost never be done in the remainder of these notes. For example we will systematically use the same symbol $g_{\mu\nu}$ for the metric components, whatever the coordinate system used.

A.3 Vector fields

Physicists often think of vector fields in terms of coordinate systems: a vector field X is an object which in a coordinate system $\{x^i\}$ is represented by a collection of functions X^i. In a new coordinate system $\{y^j\}$ the field X is represented by a new set of functions:

$$X^i(x) \to X^j(y) := X^j(x(y)) \frac{\partial y^i}{\partial x^j}(x(y)) \,. \tag{A.3.1}$$

(The summation convention is used throughout, so that the index j has to be summed over.)

The notion of a vector field finds its roots in the notion of the tangent to a curve, say $s \to \gamma(s)$. If we use local coordinates to write $\gamma(s)$ as $(\gamma^1(s), \gamma^2(s), \ldots, \gamma^n(s))$, the tangent to that curve at the point $\gamma(s)$ is defined as the set of numbers

$$(\dot{\gamma}^1(s), \dot{\gamma}^2(s), \ldots, \dot{\gamma}^n(s)) \,.$$

Consider, then, a curve $\gamma(s)$ given in a coordinate system x^i and let us perform a change of coordinates $x^i \to y^j(x^i)$. In the new coordinates y^j the curve γ is represented by the functions $y^j(\gamma^i(s))$, with new tangent

$$\frac{dy^j}{ds}(y(\gamma(s))) = \frac{\partial y^j}{\partial x^i}(\gamma(s))\dot{\gamma}^i(s)\,.$$

This motivates (A.3.1).

In modern differential geometry a different approach is taken: one identifies vector fields with homogeneous first order differential operators acting on real valued functions $f : M \to \mathbb{R}$. In local coordinates $\{x^i\}$ a vector field X will be written as $X^i\partial_i$, where the X^i's are the "physicists's functions" just mentioned. This means that the action of X on functions is given by the formula

$$\boxed{X(f) := X^i\partial_i f} \tag{A.3.2}$$

(recall that ∂_i is the partial derivative with respect to the coordinate x^i). Conversely, given some abstract first order homogeneous derivative operator X, the (perhaps locally defined) functions X^i in (A.3.2) can be found by acting on the coordinate functions:

$$X(x^i) = X^i\,. \tag{A.3.3}$$

One justification for the differential operator approach is the fact that the tangent $\dot{\gamma}$ to a curve γ can be calculated — in a way independent of the coordinate system $\{x^i\}$ chosen to represent γ — using the equation

$$\dot{\gamma}(f) := \frac{d(f\circ\gamma)}{dt}\,.$$

Indeed, if γ is represented as $\gamma(t) = \{x^i = \gamma^i(t)\}$ within a coordinate patch, then we have

$$\frac{d(f\circ\gamma)(t)}{dt} = \frac{d(f(\gamma(t)))}{dt} = \frac{d\gamma^i(t)}{dt}(\partial_i f)(\gamma(t))\,,$$

recovering the previous coordinate formula $\dot{\gamma} = (d\gamma^i/dt)$.

An even better justification is that *the transformation rule (A.3.1) becomes implicit in the formalism.* Indeed, consider a (scalar) function f, so that the differential operator X acts on f by differentiation:

$$X(f)(x) := \sum_i X^i \frac{\partial f(x)}{\partial x^i}\,. \tag{A.3.4}$$

If we make a coordinate change so that

$$x^j = x^j(y^k) \quad\Longleftrightarrow\quad y^k = y^k(x^j)\,,$$

keeping in mind that

$$\bar{f}(y) = f(x(y)) \quad\Longleftrightarrow\quad f(x) = \bar{f}(y(x))\,,$$

then

$$\begin{aligned} X(f)(x) &:= \sum_i X^i(x)\frac{\partial f(x)}{\partial x^i} \\ &= \sum_i X^i(x)\frac{\partial \bar{f}(y(x))}{\partial x^i} \\ &= \sum_{i,k} X^i(x)\frac{\partial \bar{f}(y(x))}{\partial y^k}\frac{\partial y^k}{\partial x^i}(x) \\ &= \sum_k \bar{X}^k(y(x))\frac{\partial \bar{f}(y(x))}{\partial y^k} \\ &= \left(\sum_k \bar{X}^k \frac{\partial \bar{f}}{\partial y^k}\right)(y(x))\,, \end{aligned}$$

with $\bar{X}^k$ given by the right-hand side of (A.3.1). So

$X(f)$ is a scalar iff the coefficients X^i satisfy the transformation law of a vector.

EXERCICE A.3.1 Check that this is a necessary and sufficient condition.

One often uses the middle formula in the above calculation in the form

$$\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}\,. \tag{A.3.5}$$

Note that the tangent to the curve $s \to (s, x^2, x^3, \dots x^n)$, where $(x^2, x^3, \dots x^n)$ are constants, is identified with the differential operator

$$\partial_1 \equiv \frac{\partial}{\partial x^1}\,.$$

Similarly the tangent to the curve $s \to (x^1, s, x^3, \dots x^n)$, where $(x^1, x^3, \dots x^n)$ are constants, is identified with

$$\partial_2 \equiv \frac{\partial}{\partial x^2}\,,$$

etc. Thus, $\dot\gamma$ is identified with

$$\dot\gamma(s) = \dot\gamma^i \partial_i\,.$$

At any given point $p \in M$ the set of vectors forms a vector space, denoted by T_pM. The collection of all the tangent spaces is called the tangent bundle to M, denoted by TM.

A.3.1 Lie bracket

Vector fields can be added and multiplied by functions in the obvious way. Another useful operation is the *Lie bracket*, or *commutator*, defined as

$$\boxed{[X,Y](f) := X(Y(f)) - Y(X(f))}\,. \tag{A.3.6}$$

One needs to check that this does indeed define a new vector field: the simplest way is to use local coordinates,

$$\begin{aligned}
[X,Y](f) &= X^j\partial_j(Y^i\partial_i f) - Y^j\partial_j(X^i\partial_i f)\\
&= X^j(\partial_j(Y^i)\partial_i f + Y^i\partial_j\partial_i f) - Y^j(\partial_j(X^i)\partial_i f + X^i\partial_j\partial_i f)\\
&= (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i f + \underbrace{X^jY^i\partial_j\partial_i f - Y^jX^i\partial_j\partial_i f}_{=X^jY^i\underbrace{(\partial_j\partial_i f - \partial_i\partial_j f)}_{0}}\\
&= (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i f\,,
\end{aligned} \tag{A.3.7}$$

which is indeed a homogeneous first order differential operator. Here we have used the symmetry of the matrix of second derivatives of twice differentiable functions. We note that the last line of (A.3.7) also gives an explicit coordinate expression for the commutator of two differentiable vector fields.

The Lie bracket satisfies the *Jacobi identity*:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\,.$$

Indeed, if we write $S_{X,Y,Z}$ for a cyclic sum, then

$$\begin{aligned}
&([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]])(f) = S_{X,Y,Z}[X,[Y,Z]](f)\\
&\quad = S_{X,Y,Z}\{X([Y,Z](f)) - [Y,Z](X(f))\}\\
&\quad = S_{X,Y,Z}\{X(Y(Z(f))) - X(Z(Y(f))) - Y(Z(X(f))) + Z(Y(X(f)))\}\,.
\end{aligned}$$

The third term is a cyclic permutation of the first, and the fourth a cyclic permutation of the second, so the sum gives zero.

A.4 Covectors

Covectors are *maps from the space of vectors to functions which are linear under addition and multiplication by functions.*

The basic object is the *coordinate differential* dx^i, defined by its action on vectors as follows:

$$dx^i(X^j\partial_j) := X^i\,. \tag{A.4.1}$$

Equivalently,

$$dx^i(\partial_j) := \delta^i_j := \begin{cases} 1, & i = j;\\ 0, & \text{otherwise.}\end{cases}$$

The dx^i's form a basis for the space of covectors: indeed, let φ be a linear map on the space of vectors, then

$$\varphi(\underbrace{X}_{X^i\partial_i}) = \varphi(X^i\partial_i) \underbrace{=}_{\text{linearity}} X^i \underbrace{\varphi(\partial_i)}_{\text{call this }\varphi_i} = \varphi_i dx^i(X) \underbrace{=}_{\text{def. of sum of functions}} (\varphi_i dx^i)(X)\,,$$

hence

$$\varphi = \varphi_i dx^i ,$$

and every φ can indeed be written as a linear combination of the dx^i's. Under a change of coordinates we have

$$\bar\varphi_i \bar X^i = \bar\varphi_i \frac{\partial y^i}{\partial x^k} X^k = \varphi_k X^k ,$$

leading to the following transformation law for components of covectors:

$$\varphi_k = \bar\varphi_i \frac{\partial y^i}{\partial x^k} . \qquad \text{(A.4.2)}$$

Given a scalar f, we define its *differential* df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n .$$

With this definition, dx^i *is* the differential of the coordinate function x^i.

As presented above, the differential of a function is a covector by definition. As an exercice, you should check directly that the collection of functions $\varphi_i := \partial_i f$ satisfies the transformation rule (A.4.2).

We have a formula which is often used in calculations

$$dy^j = \frac{\partial y^j}{\partial x^k} dx^k .$$

INCIDENTALLY: An elegant approach to the definition of differentials proceeds as follows: Given any function f, we define:

$$df(X) := X(f) . \qquad \text{(A.4.3)}$$

(Recall that here we are viewing a vector field X as a differential operator on functions, defined by (A.3.4).) The map $X \mapsto df(X)$ is linear under addition of vectors, and multiplication of vectors by numbers: if λ, μ are real numbers, and X and Y are vector fields, then

$$\begin{aligned} df(\lambda X + \mu Y) &\underbrace{=}_{\text{by definition (A.4.3)}} (\lambda X + \mu Y)(f) \\ &\underbrace{=}_{\text{by definition (A.3.4)}} \lambda X^i \partial_i f + \mu Y^i \partial_i f \\ &\underbrace{=}_{\text{by definition (A.4.3)}} \lambda df(X) + \mu df(Y) . \end{aligned}$$

Applying (A.4.3) to the function $f = x^i$ we obtain

$$dx^i(\partial_j) = \frac{\partial x^i}{\partial x^j} = \delta^i_j ,$$

recovering (A.4.1). □

EXAMPLE A.4.2 Let (ρ, φ) be polar coordinates on $\mathbb{R}^2$, thus $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, and then

$$dx = d(\rho\cos\varphi) = \cos\varphi d\rho - \rho\sin\varphi d\varphi\,,$$
$$dy = d(\rho\sin\varphi) = \sin\varphi d\rho + \rho\cos\varphi d\varphi\,.$$

At any given point $p \in M$, the set of covectors forms a vector space, denoted by T_p^*M. The collection of all the tangent spaces is called the cotangent bundle to M, denoted by T^*M.

Summarising, covectors are dual to vectors. It is convenient to define

$$\boxed{dx^i(X) := X^i}\,,$$

where X^i is as in (A.3.2). With this definition the (locally defined) bases $\{\partial_i\}_{i=1,\ldots,\dim M}$ of TM and $\{dx^j\}_{i=1,\ldots,\dim M}$ of T^*M are dual to each other:

$$\langle dx^i, \partial_j\rangle := dx^i(\partial_j) = \delta^i_j\,,$$

where δ^i_j is the Kronecker delta, equal to one when $i = j$ and zero otherwise.

A.5 Bilinear maps, two-covariant tensors

A map is said to be multi-linear if it is linear in every entry; e.g. g is bilinear if

$$g(aX + bY, Z) = ag(X, Z) + bg(Y, Z)\,,$$

and

$$g(X, aZ + bW) = ag(X, Z) + bg(X, W)\,.$$

Here, as elsewhere when talking about *tensors*, bilinearity is meant with respect to addition and to multiplication by functions.

A map g which is bilinear on the space of vectors can be represented by a matrix with two indices down:

$$g(X, Y) = g(X^i\partial_i, Y^j\partial_j) = X^iY^j \underbrace{g(\partial_i, \partial_j)}_{=:g_{ij}} = g_{ij}X^iY^j = g_{ij}dx^i(X)dx^j(Y)\,.$$

We say that g is a *covariant tensor of valence two.*

We say that g is *symmetric* if $g(X, Y) = g(Y, X)$ for all X, Y; equivalently, $g_{ij} = g_{ji}$.

A symmetric bilinear tensor field is said to be *non-degenerate* if $\det g_{ij}$ has no zeros.

By Sylvester's inertia theorem, there exists a basis θ^i of the space of covectors so that a symmetric bilinear map g can be written as

$$g(X, Y) = -\theta^1(X)\theta^1(Y) - \ldots - \theta^s(X)\theta^s(Y) + \theta^{s+1}(X)\theta^{s+1}(Y) + \ldots + \theta^{s+r}(X)\theta^{s+r}(Y)$$

(s, r) is called the signature of g; in geometry, unless specifically said otherwise, one always assumes that the signature does not change from point to point.

If $r = n$, in dimension n, then g is said to be a Riemannian metric tensor.

A canonical example is provided by the flat Riemannian metric on $\mathbb{R}^n$,

$$g = (dx^1)^2 + \ldots + (dx^n)^2 .$$

By definition, a *Riemannian metric* is a field of symmetric two-covariant tensors with signature $(+, \ldots, +)$ and with $\det g_{ij}$ without zeros.

INCIDENTALLY: A Riemannian metric can be used to define the length of curves: if $\gamma : [a, b] \ni s \to \gamma(s)$, then

$$\ell_g(\gamma) = \int_a^b \sqrt{g(\dot\gamma, \dot\gamma)} ds .$$

One can then define the distance between points by minimizing the length of the curves connecting them. □

If $s = 1$ and $r = N - 1$, in dimension N, then g is said to be a *Lorentzian metric tensor*.

For example, the *Minkowski metric* on $\mathbb{R}^{1+n}$ is

$$\eta = (dx^0)^2 - (dx^1)^2 - \ldots - (dx^n)^2 .$$

A.6 Tensor products

If φ and θ are covectors we can define a bilinear map using the formula

$$(\varphi \otimes \theta)(X, Y) = \varphi(X)\theta(Y) . \tag{A.6.1}$$

For example

$$(dx^1 \otimes dx^2)(X, Y) = X^1 Y^2 .$$

Using this notation we have

$$g(X, Y) = g(X^i \partial_i, Y^j \partial_j) = \underbrace{g(\partial_j, \partial_j)}_{=:g_{ij}} \underbrace{\underbrace{X^i}_{dx^i(X)} \underbrace{Y^j}_{dx^j(Y)}}_{(dx^i \otimes dx^j(X,Y)} = (g_{ij} dx^i \otimes dx^j)(X, Y)$$

We will write $dx^i dx^j$ for the symmetric product,

$$dx^i dx^j := \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i) ,$$

and $dx^i \wedge dx^j$ for twice the anti-symmetric one (compare Section A.15):

$$dx^i \wedge dx^j := dx^i \otimes dx^j - dx^j \otimes dx^i .$$

It should be clear how this generalises: the tensors $dx^i \otimes dx^j \otimes dx^k$, defined as

$$(dx^i \otimes dx^j \otimes dx^k)(X, Y, Z) = X^i Y^j Z^k ,$$

form a basis of three-linear maps on the space of vectors, which are objects of the form

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k \,.$$

Here X is a called *tensor of valence* $(0,3)$. Each index transforms as for a covector:

$$X = X_{ijk} dx^i \otimes dx^j \otimes dx^k = X_{ijk} \frac{\partial x^i}{\partial y^m} \frac{\partial x^j}{\partial y^\ell} \frac{\partial x^k}{\partial y^n} dy^m \otimes dy^\ell \otimes dy^n \,.$$

It is sometimes useful to think of vectors as linear maps on co-vectors, using a formula which looks funny when first met: if θ is a covector, and X is a vector, then

$$X(\theta) := \theta(X) \,.$$

So if $\theta = \theta_i dx^i$ and $X = X^i \partial_i$ then

$$\theta(X) = \theta_i X^i = X^i \theta_i = X(\theta) \,.$$

It then makes sense to define e.g. $\partial_i \otimes \partial_j$ as a bilinear map on covectors:

$$(\partial_i \otimes \partial_j)(\theta, \psi) := \theta_i \psi_j \,.$$

And one can define a map $\partial_i \otimes dx^j$ which is linear on forms in the first slot, and linear in vectors in the second slot as

$$(\partial_i \otimes dx^j)(\theta, X) := \partial_i(\theta) dx^j(X) = \theta_i X^j \,. \tag{A.6.2}$$

The $\partial_i \otimes dx^j$'s form the basis of the space of *tensors of rank* $(1,1)$:

$$T = T^i{}_j \partial_i \otimes dx^j \,.$$

Generally, a *tensor of valence, or rank,* (r,s) can be defined as an object which has r vector indices and s covector indices, so that it transforms as

$$S^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \to S^{m_1 \ldots m_r}{}_{\ell_1 \ldots \ell_s} \frac{\partial y^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial y^{i_r}}{\partial x^{m_r}} \frac{\partial x^{\ell_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\ell_s}}{\partial y^{j_s}}$$

For example, if $X = X^i \partial_i$ and $Y = Y^j \partial_j$ are vectors, then $X \otimes Y = X^i Y^j \partial_i \otimes \partial_j$ forms a contravariant tensor of valence two.

Tensors of same valence can be added in the obvious way: *e.g.*

$$(A+B)(X,Y) := A(X,Y) + B(X,Y) \quad \Longleftrightarrow \quad (A+B)_{ij} = A_{ij} + B_{ij} \,.$$

Tensors can be multiplied by scalars: *e.g.*

$$(fA)(X,Y,Z) := fA(X,Y,Z) \quad \Longleftrightarrow \quad f(A_{ijk}) := (fA_{ijk}) \,.$$

Finally, we have seen in (A.6.1) how to take tensor products for one-forms, and in (A.6.2) how to take a tensor product of a vector and a one-form, but this can also be done for higher order tensor; e.g., if S is of valence (a,b) and T is a multilinear map of valence (c,d), then $S \otimes T$ is a multilinear map of valence $(a+c, b+d)$, defined as

$$(S \otimes T)(\underbrace{\theta, \ldots}_{a \text{ covectors and } b \text{ vectors}}, \underbrace{\psi, \ldots}_{c \text{ covectors and } d \text{ vectors}}) := S(\theta, \ldots) T(\psi, \ldots) \,.$$

A.6.1 Contractions

Given a tensor field $S^i{}_j$ with one index down and one index up one can perform the sum

$$S^i{}_i\,.$$

This defines a scalar, i.e., a function on the manifold. Indeed, using the transformation rule

$$S^i{}_j \to \bar S^\ell{}_k = S^i{}_j \frac{\partial x^j}{\partial y^k}\frac{\partial y^\ell}{\partial x^i}\,,$$

one finds

$$\bar S^\ell{}_\ell = S^i{}_j \underbrace{\frac{\partial x^j}{\partial y^\ell}\frac{\partial y^\ell}{\partial x^i}}_{\delta^j_i} = S^i{}_i\,,$$

as desired.

One can similarly do contractions on higher valence tensors, e.g.

$$S^{i_1 i_2\ldots i_r}{}_{j_1 j_2 j_3 \ldots j_s} \to S^{\ell i_2\ldots i_r}{}_{j_1 \ell j_3 \ldots j_s}\,.$$

After contraction, a tensor of rank $(r+1, s+1)$ becomes of rank (r, s).

A.7 Raising and lowering of indices

Let g be a symmetric two-covariant tensor field on M, by definition such an object is the assignment to each point $p \in M$ of a bilinear map $g(p)$ from $T_pM \times T_pM$ to $\mathbb{R}$, with the additional property

$$g(X, Y) = g(Y, X)\,.$$

In this work the symbol g will be reserved to *non-degenerate* symmetric two-covariant tensor fields. It is usual to simply write g for $g(p)$, the point p being implicitly understood. We will sometimes write g_p for $g(p)$ when referencing p will be useful.

The usual Sylvester's inertia theorem tells us that at each p the map g will have a well defined signature; clearly this signature will be point-independent on a connected manifold when g is non-degenerate. A pair (M, g) is said to be a *Riemannian manifold* when the signature of g is $(\dim M, 0)$; equivalently, when g is a positive definite bilinear form on every product $T_pM \times T_pM$. A pair (M, g) is said to be a *Lorentzian manifold* when the signature of g is $(\dim M - 1, 1)$. One talks about *pseudo-Riemannian* manifolds whatever the signature of g, as long as g is non-degenerate, but we will only encounter Riemannian and Lorentzian metrics in this work.

Since g is non-degenerate it induces an isomorphism

$$\flat : T_pM \to T_p^*M$$

by the formula

$$\boxed{X_\flat(Y) = g(X, Y)}\,.$$

In local coordinates this gives

$$X_\flat = g_{ij}X^i dx^j =: X_j dx^j \,. \tag{A.7.1}$$

This last equality defines X_j — "the vector X^j with the index j lowered":

$$\boxed{X_i := g_{ij}X^j}\,. \tag{A.7.2}$$

The operation (A.7.2) is called the *lowering of indices* in the physics literature and, again in the physics literature, one does not make a distinction between the one-form $X_\flat$ and the vector X.

The inverse map will be denoted by $\sharp$ and is called the *raising of indices*; from (A.7.1) we obviously have

$$\alpha^\sharp = g^{ij}\alpha_i\partial_j =: \alpha^i\partial_i \quad\Longleftrightarrow\quad dx^i(\alpha^\sharp) = \boxed{\alpha^i = g^{ij}\alpha_j}\,,$$

where g^{ij} is the matrix inverse to g_{ij}. For example,

$$(dx^i)^\sharp = g^{ik}\partial_k \,.$$

Clearly g^{ij}, understood as the matrix of a bilinear form on T^*_pM, has the same signature as g, and can be used to define a scalar product $g^\sharp$ on $T^*_p(M)$:

$$g^\sharp(\alpha,\beta) := g(\alpha^\sharp,\beta^\sharp) \quad\Longleftrightarrow\quad g^\sharp(dx^i,dx^j) = g^{ij} \,.$$

This last equality is justified as follows:

$$g^\sharp(dx^i,dx^j) = g((dx^i)^\sharp,(dx^j)^\sharp) = g(g^{ik}\partial_k, g^{j\ell}\partial_\ell) = \underbrace{g^{ik}g_{k\ell}}_{=\delta^i_\ell}g^{j\ell} = g^{ji} = g^{ij} \,.$$

It is convenient to use the same letter g for $g^\sharp$ — physicists do it all the time — or for scalar products induced by g on all the remaining tensor bundles, and we will sometimes do so.

INCIDENTALLY: One might wish to check by direct calculations that $g_{\mu\nu}X^\nu$ transforms as a one-form if X^μ transforms as a vector. The simplest way is to notice that $g_{\mu\nu}X^\nu$ is a contraction, over the last two indices, of the three-index tensor $g_{\mu\nu}X^\alpha$. Hence it is a one-form by the analysis at the end of the previous section. Alternatively, if we write $\bar g_{\mu\nu}$ for the transformed $g_{\mu\nu}$'s, and $\bar X^\alpha$ for the transformed X^α's, then

$$\underbrace{\bar g_{\alpha\beta}}_{g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}}\bar X^\beta = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^\alpha}\underbrace{\frac{\partial x^\nu}{\partial y^\beta}\bar X^\beta}_{X^\nu} = g_{\mu\nu}X^\nu\frac{\partial x^\mu}{\partial y^\alpha}\,,$$

which is indeed the transformation law of a covector. □

The *gradient* ∇f of a function f is a vector field obtained by raising the indices on the differential df:

$$g(\nabla f, Y) := df(Y) \qquad\Longleftrightarrow\qquad \nabla f := g^{ij}\partial_i f\partial_j \,. \tag{A.7.3}$$

A.8 The Lie derivative

A.8.1 A pedestrian approach

We start with a pedestrian approach to the definition of Lie derivative; the elegant geometric definition will be given in the next section.

Given a vector field X, the *Lie derivative* $\mathscr{L}_X$ is an operation on tensor fields, defined as follows:

For a function f, one sets

$$\mathscr{L}_X f := X(f)\,. \tag{A.8.1}$$

For a vector field Y, the Lie derivative coincides with the Lie bracket:

$$\mathscr{L}_X Y := [X,Y]\,. \tag{A.8.2}$$

For a one-form α, $\mathscr{L}_X\alpha$ is defined by imposing the Leibniz rule written the wrong-way round:

$$(\mathscr{L}_X\alpha)(Y) := \mathscr{L}_X(\alpha(Y)) - \alpha(\mathscr{L}_X Y)\,. \tag{A.8.3}$$

(Indeed, the Leibniz rule applied to the contraction $\alpha_i X^i$ would read

$$\mathscr{L}_X(\alpha_i Y^i) = (\mathscr{L}_X\alpha)_i Y^i + \alpha_i(\mathscr{L}_X Y)^i\,,$$

which can be rewritten as (A.8.3).)

Let us check that (A.8.3) defines a one-form. Clearly, the right-hand side transforms in the desired way when Y is replaced by Y_1+Y_2. Now, if we replace Y by fY, where f is a function, then

$$\begin{aligned}
(\mathscr{L}_X\alpha)(fY) &= \mathscr{L}_X(\alpha(fY)) - \alpha(\underbrace{\mathscr{L}_X(fY)}_{X(f)Y+f\mathscr{L}_X Y})\\
&= X(f\alpha(Y)) - \alpha(X(f)Y + f\mathscr{L}_X Y))\\
&= X(f)\alpha(Y) + fX(\alpha(Y)) - \alpha(X(f)Y) - \alpha(f\mathscr{L}_X Y))\\
&= fX(\alpha(Y)) - f\alpha(\mathscr{L}_X Y))\\
&= f((\mathscr{L}_X\alpha)(Y))\,.
\end{aligned}$$

So $\mathscr{L}_X\alpha$ is a C^∞-linear map on vector fields, hence a covector field.

In coordinate-components notation we have

$$(\mathscr{L}_X\alpha)_a = X^b\partial_b\alpha_a + \alpha_b\partial_a X^b\,. \tag{A.8.4}$$

Indeed,

$$\begin{aligned}
(\mathscr{L}_X\alpha)_i Y^i &:= \mathscr{L}_X(\alpha_i Y^i) - \alpha_i(\mathscr{L}_X Y)^i\\
&= X^k\partial_k(\alpha_i Y^i) - \alpha_i(X^k\partial_k Y^i - Y^k\partial_k X^i)\\
&= X^k(\partial_k\alpha_i)Y^i + \alpha_i Y^k\partial_k X^i\\
&= \left(X^k\partial_k\alpha_i + \alpha_k\partial_i X^k\right)Y^i\,,
\end{aligned}$$

as desired

For tensor products, the Lie derivative is defined by imposing linearity under addition together with the Leibniz rule:

$$\mathscr{L}_X(\alpha\otimes\beta) = (\mathscr{L}_X\alpha)\otimes\beta + \alpha\otimes\mathscr{L}_X\beta\,.$$

Since a general tensor A is a sum of tensor products,

$$A = A^{a_1\ldots a_p}{}_{b_1\ldots b_q}\partial_{a_1}\otimes\ldots\partial_{a_p}\otimes dx^{b_1}\otimes\ldots\otimes dx^{a_p}\,,$$

requiring linearity with respect to addition of tensors gives thus a definition of Lie derivative for any tensor.

For example, we claim that

$$\mathscr{L}_X T^a{}_b = X^c\partial_c T^a{}_b - T^c{}_b\partial_c X^a + T^a{}_c\partial_b X^c\,, \tag{A.8.5}$$

To see this, call a tensor $T^a{}_b$ *simple* if it is of the form $Y\otimes\alpha$, where Y is a vector and α is a covector. Using indices, this corresponds to $Y^a\alpha_b$ and so, by the Leibniz rule,

$$\begin{aligned}\mathscr{L}_X(Y\otimes\alpha)^a{}_b &= \mathscr{L}_X(Y^a\alpha_b)\\ &= (\mathscr{L}_XY)^a\alpha_b + Y^a(\mathscr{L}_X\alpha)_b\\ &= (X^c\partial_cY^a - Y^c\partial_cX^a)\alpha_b + Y^a(X^c\partial_c\alpha_b + \alpha_c\partial_bX^c)\\ &= X^c\partial_c(Y^a\alpha_b) - Y^c\alpha_b\partial_cX^a + Y^a\alpha_c\partial_bX^c\,,\end{aligned}$$

which coincides with (A.8.5) if $T^a{}_b = Y^b\alpha_b$. But a general $T^a{}_b$ can be written as a linear combination with constant coefficients of simple tensors,

$$T = \sum_{a,b}\underbrace{T^a{}_b\partial_a\otimes dx^b}_{\text{no summation, so simple}}\,,$$

and the result follows.

Similarly, one has, e.g.,

$$\mathscr{L}_X R^{ab} = X^c\partial_c R^{ab} - R^{ac}\partial_c X^b - R^{bc}\partial_c X^a\,, \tag{A.8.6}$$

$$\mathscr{L}_X S_{ab} = X^c\partial_c S_{ab} + S_{ac}\partial_b X^c + S_{bc}\partial_a X^c\,. \tag{A.8.7}$$

etc. Those are all special cases of the general formula for the Lie derivative $\mathscr{L}_X A^{a_1\ldots a_p}{}_{b_1\ldots b_q}$:

$$\begin{aligned}\mathscr{L}_X A^{a_1\ldots a_p}{}_{b_1\ldots b_q} = {}& X^c\partial_c A^{a_1\ldots a_p}{}_{b_1\ldots b_q} - A^{ca_2\ldots a_p}{}_{b_1\ldots b_q}\partial_c X^{a_1} - \ldots\\ &+A^{a_1\ldots a_p}{}_{cb_1\ldots b_q}\partial_{b_1}X^c + \ldots\,.\end{aligned}$$

A useful property of Lie derivatives is

$$\mathscr{L}_{[X,Y]} = [\mathscr{L}_X, \mathscr{L}_Y]\,, \tag{A.8.8}$$

where, for a tensor T, the commutator $[\mathscr{L}_X, \mathscr{L}_Y]T$ is defined in the usual way:

$$[\mathscr{L}_X, \mathscr{L}_Y]T := \mathscr{L}_X(\mathscr{L}_Y T) - \mathscr{L}_Y(\mathscr{L}_X T)\,. \tag{A.8.9}$$

To see this, we first note that if $T = f$ is a function, then the right-hand side of (A.8.9) is the definition of $[X,Y](f)$, which in turn coincides with the definition of $\mathscr{L}_{[X,Y]}(f)$.

Next, for a vector field $T = Z$, (A.8.8) reads

$$\mathscr{L}_{[X,Y]}Z = \mathscr{L}_X(\mathscr{L}_Y Z) - \mathscr{L}_Y(\mathscr{L}_X Z)\,, \tag{A.8.10}$$

which is the same as

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]\,, \tag{A.8.11}$$

which is the same as

$$[Z,[Y,X]] + [X,[Z,Y]] + [Y,[X,Z]] = 0\,, \tag{A.8.12}$$

which is the Jacobi identity. Hence (A.8.8) holds for vector fields.

We continue with a one-form α, exploiting the fact that we have already established the result for functions and vectors: For any vector field Z we have, by definition

$$\begin{aligned}([\mathscr{L}_X,\mathscr{L}_Y]\alpha)(Z) &= [\mathscr{L}_X,\mathscr{L}_Y](\alpha(Z)) - \alpha([\mathscr{L}_X,\mathscr{L}_Y](Z))\\ &= \mathscr{L}_{[X,Y]}(\alpha(Z)) - \alpha(\mathscr{L}_{[X,Y]}(Z))\\ &= (\mathscr{L}_{[X,Y]}\alpha)(Z)\,.\end{aligned}$$

INCIDENTALLY: A direct calculation for one-forms, using the definitions, proceed as follows: Let Z be any vector field,

$$\begin{aligned}(\mathscr{L}_X\mathscr{L}_Y\alpha)(Z) &= X(\underbrace{(\mathscr{L}_Y\alpha)(Z)}_{Y(\alpha(Z))-\alpha(\mathscr{L}_Y Z))}) - \underbrace{(\mathscr{L}_Y\alpha)(\mathscr{L}_X Z)}_{Y(\alpha(\mathscr{L}_X Z))-\alpha(\mathscr{L}_Y\mathscr{L}_X Z)}\\ &= X\big(Y(\alpha(Z))\big) - X\big(\alpha(\mathscr{L}_Y Z)\big)\big) - Y\left(\alpha(\mathscr{L}_X Z)\right) + \alpha(\mathscr{L}_Y\mathscr{L}_X Z)\,.\end{aligned}$$

Antisymmetrizing over X and Y, the second and third term above cancel out, so that

$$\begin{aligned}\big((\mathscr{L}_X\mathscr{L}_Y\alpha - \mathscr{L}_Y\mathscr{L}_X)\alpha\big)(Z) &= X\big(Y(\alpha(Z))\big) + \alpha(\mathscr{L}_Y\mathscr{L}_X Z) - (X \longleftrightarrow Y)\\ &= [X,Y]\big(\alpha(Z)\big) - \alpha(\mathscr{L}_X\mathscr{L}_Y Z - \mathscr{L}_Y\mathscr{L}_X Z)\\ &= \mathscr{L}_{[X,Y]}\big(\alpha(Z)\big) - \alpha(\mathscr{L}_{[X,Y]}Z)\\ &= \big(\mathscr{L}_{[X,Y]}\alpha\big)(Z)\,.\end{aligned}$$

Since Z is arbitrary, (A.8.8) for covectors follows. □

To conclude that (A.8.8) holds for arbitrary tensor fields, we note that by construction we have

$$\mathscr{L}_{[X,Y]}(A\otimes B) = \mathscr{L}_{[X,Y]}A\otimes B + A\otimes\mathscr{L}_{[X,Y]}B\,. \tag{A.8.13}$$

Similarly

$$\begin{aligned}\mathscr{L}_X\mathscr{L}_Y(A\otimes B) &= \mathscr{L}_X(\mathscr{L}_Y A\otimes B + A\otimes\mathscr{L}_Y B)\\ &= \mathscr{L}_X\mathscr{L}_Y A\otimes B + \mathscr{L}_X A\otimes\mathscr{L}_Y B + \mathscr{L}_Y A\otimes\mathscr{L}_X B\\ &\quad + A\otimes\mathscr{L}_X\mathscr{L}_Y B\,.\end{aligned} \tag{A.8.14}$$

Exchanging X with Y and subtracting, the middle terms drop out:

$$[\mathscr{L}_X,\mathscr{L}_Y](A\otimes B)=[\mathscr{L}_X,\mathscr{L}_Y]A\otimes B+A\otimes[\mathscr{L}_X,\mathscr{L}_Y]B\,. \qquad \text{(A.8.15)}$$

Basing on what has been said, the reader should have no difficulties finishing the proof of (A.8.8).

EXAMPLE A.8.2 As an example of application of the formalism, suppose that there exists a coordinate system in which $(X^a)=(1,0,0,0)$ and $\partial_0 g_{bc}=0$. Then

$$\mathscr{L}_X g_{ab}=\partial_0 g_{ab}=0\,.$$

But the Lie derivative of a tensor field is a tensor field, and we conclude that $\mathscr{L}_X g_{ab}=0$ holds in every coordinate system.

Vector fields for which $\mathscr{L}_X g_{ab}=0$ are called *Killing vectors*: they arise from symmetries of spacetime. We have the useful formula

$$\boxed{\mathscr{L}_X g_{ab}=\nabla_a X_b+\nabla_b X_a\,.} \qquad \text{(A.8.16)}$$

An effortless proof of this proceeds as follows: in adapted coordinates in which the derivatives of the metric vanish at a point p, one immediately checks that equality holds at p. But both sides are tensor fields, therefore the result holds at p for all coordinate systems, and hence also everywhere.

The brute-force proof of (A.8.16) proceeds as follows:

$$\begin{aligned}\mathscr{L}_X g_{ab} &= X^c\partial_c g_{ab}+\partial_a X^c g_{cb}+\partial_b X^c g_{ca}\\ &= X^c\partial_c g_{ab}+\partial_a(X^c g_{cb})-X^c\partial_a g_{cb}+\partial_b(X^c g_{ca})-X^c\partial_b g_{ca}\\ &= \partial_a X_b+\partial_b X_a+X^c\underbrace{(\partial_c g_{ab}-\partial_a g_{cb}-\partial_b g_{ca})}_{-2g_{cd}\Gamma^d_{ab}}\\ &= \nabla_a X_b+\nabla_b X_a\,.\end{aligned}$$

□

A.8.2 The geometric approach

We pass now to a geometric definition of Lie derivative. This requires, first, an excursion through the land of push-forwards and pull-backs.

Transporting tensor fields

We start by noting that, given a point p_0 in a manifold M, every vector $X\in T_{p_0}M$ is tangent to some curve. To see this, let $\{x^i\}$ be any local coordinates near p_0, with $x^i(p_0)=x^i_0$, then X can be written as $X^i(p_0)\partial_i$. If we set $\gamma^i(s)=x^i_0+sX^i(p_0)$, then $\dot\gamma^i(0)=X^i(p_0)$, which establishes the claim. This observation shows that studies of vectors can be reduced to studies of curves.

Let, now, M and N be two manifolds, and let $\phi:M\to N$ be a differentiable map between them. Given a vector $X\in T_pM$, the *push-forward* ϕ_*X of X is a vector in $T_{\phi(p)}N$ defined as follows: let γ be any curve for which $X=\dot\gamma(0)$, then

$$\phi_*X:=\left.\frac{d(\phi\circ\gamma)}{ds}\right|_{s=0}. \qquad \text{(A.8.17)}$$

In local coordinates y^A on N and x^i on M, so that $\phi(x) = (\phi^A(x^i))$, we find

$$\begin{aligned}(\phi_*X)^A &= \left.\frac{d\phi^A(\gamma^i(s))}{ds}\right|_{s=0} = \frac{\partial\phi^A(\gamma^i(s))}{\partial x^i}\dot\gamma^i(s)|_{s=0} \\ &= \frac{\partial\phi^A(x^i)}{\partial x^i}X^i\,. \end{aligned} \tag{A.8.18}$$

The formula makes it clear that the definition is independent of the choice of the curve γ satisfying $X = \dot\gamma(0)$.

Equivalently, and more directly, if X is a vector at p and h is a function on h, then ϕ^*X acts on h as

$$\boxed{\phi^*X(h) := X(h\circ\phi)\,.} \tag{A.8.19}$$

Applying (A.8.18) to a *vector field* X defined on M we obtain

$$(\phi_*X)^A(\phi(x)) = \frac{\partial\phi^A}{\partial x^i}(x)\,X^i(x)\,. \tag{A.8.20}$$

The equation shows that if a point $y\in N$ has more than one pre-image, say $y = \phi(x_1) = \phi(x_2)$ with $x_1 \neq x_2$, then (A.8.20) might will define more than one tangent vector at y in general. This leads to an important caveat: we will be certain that the push-forward *of a vector field* on M defines a vector field on N only when ϕ is a diffeomorphism. More generally, ϕ_*X defines locally a vector field on $\phi(M)$ if and only if ϕ is a local diffeomorphism. In such cases we can invert ϕ (perhaps locally) and write (A.8.20) as

$$(\phi_*X)^j(x) = \left(\frac{\partial\phi^j}{\partial x^i}X^i\right)(\phi^{-1}(x))\,. \tag{A.8.21}$$

When ϕ is understood as a coordinate change rather than a diffeomorphism between two manifolds, this is simply the standard transformation law of a vector field under coordinate transformations.

The push-forward operation can be extended to *contravariant* tensors by defining it on tensor products in the obvious way, and extending by linearity: for example, if X, Y and Z are vectors, then

$$\phi_*(X\otimes Y\otimes Z) := \phi_*X\otimes\phi_*Y\otimes\phi_*Z\,.$$

Consider, next, a k-multilinear map α from $T_{\phi(p_0)}M$ to $\mathbb{R}$. The *pull-back* $\phi^*\alpha$ of α is a multilinear map on $T_{p_0}M$ defined as

$$T_pM \ni (X_1,\dots X_k)\mapsto \phi^*(\alpha)(X_1,\dots,X_k) := \alpha(\phi^*X_1,\dots,\phi_*X_k)\,.$$

As an example, let $\alpha = \alpha_A dy^A$ be a one-form. If $X = X^i\partial_i$ then

$$\begin{aligned}&\boxed{(\phi^*\alpha)(X) = \alpha(\phi_*X)} \\ &= \alpha(\frac{\partial\phi^A}{\partial x^i}X^i\partial_A) = \alpha_A\frac{\partial\phi^A}{\partial x^i}X^i = \alpha_A\frac{\partial\phi^A}{\partial x^i}dx^i(X)\,.\end{aligned} \tag{A.8.22}$$

Equivalently,

$$(\phi^*\alpha)_i = \alpha_A \frac{\partial \phi^A}{\partial x^i} \,. \tag{A.8.23}$$

If α is a *one-form field* on N, this reads

$$(\phi^*\alpha)_i(x) = \alpha_A(\phi(x)) \frac{\partial \phi^A(x)}{\partial x^i} \,. \tag{A.8.24}$$

It follows that $\phi^*\alpha$ is a field of one-forms on M, *irrespective of injectivity or surjectivity* properties of ϕ. Similarly, pull-backs of covariant tensor fields of higher rank are smooth tensor fields.

For a function f equation (A.8.24) reads

$$(\phi^* df)_i(x) = \frac{\partial f}{\partial y^A}(\phi(x)) \frac{\partial \phi^A(x)}{\partial x^i} = \frac{\partial (f \circ \phi)}{\partial x^i}(x) \,, \tag{A.8.25}$$

which can be succinctly written as

$$\phi^* df = d(f \circ \phi) \,. \tag{A.8.26}$$

Using the notation

$$\phi^* f := f \circ \phi \,, \tag{A.8.27}$$

we can write (A.8.26) as

$$\phi^* d = d\phi^* \quad \text{for functions.} \tag{A.8.28}$$

Summarising:

1. Pull-backs of covariant tensor fields define covariant tensor fields. In particular the metric can *always* be pulled back.

2. Push-forwards of contravariant tensor fields can be used to define contravariant tensor fields when ϕ is a diffeomorphism.

In this context it is thus clearly of interest to consider diffeomorphisms ϕ, as then tensor products can now be transported in the following way; we will denote by $\hat\phi$ the associated map: We define $\hat\phi f := f \circ \phi$ for functions, $\hat\phi := \phi_*$ for covariant fields, $\hat\phi := (\phi^{-1})_*$ for contravariant tensor fields. We use the rule

$$\hat\phi(A \otimes B) = \hat\phi A \otimes \hat\phi B$$

for tensor products.

So, for example, if X is a vector field and α is a field of one-forms, one has

$$\hat\phi(X \otimes \alpha) := (\phi^{-1})_* X \otimes \phi^*\alpha \,. \tag{A.8.29}$$

The definition is extended by linearity under addition and multiplication by functions to any tensor fields. Thus, if f is a function and T and S are tensor fields, then

$$\hat\phi(f\,T + S) = \hat\phi f \,\hat\phi T + \hat\phi S \equiv f \circ \phi \,\hat\phi T + \hat\phi S \,.$$

Since everything was fairly natural so far, one would expect that contractions transform in a natural way under transport. To make this clear, we start by rewriting (A.8.22) with the base-points made explicit:

$$((\hat{\phi}\alpha)(X))(x) = (\alpha(\phi^* X))(\phi(x)) . \qquad \text{(A.8.30)}$$

Replacing X by $(\phi^*)^{-1}Y$ this becomes

$$((\hat{\phi}\alpha)(\hat{\phi}Y))(x) = (\alpha(Y))(\phi(x)) . \qquad \text{(A.8.31)}$$

Equivalently

$$(\hat{\phi}\alpha)(\hat{\phi}Y) = \hat{\phi}(\alpha(Y)) . \qquad \text{(A.8.32)}$$

Flows of vector fields

Let X be a vector field on M. For every $p_0 \in M$ consider the solution to the problem

$$\frac{dx^i}{dt} = X^i(x(t)) , \quad x^i(0) = x^i_0 . \qquad \text{(A.8.33)}$$

(Recall that there always exists a *maximal* interval I containing the origin on which (A.8.33) has a solution. Both the interval and the solution are unique. This will always be the solution $I \ni t \mapsto x(t)$ that we will have in mind.) The map

$$(t, x_0) \mapsto \phi_t[X](x_0) := x(t)$$

where $x^i(t)$ is the solution of (A.8.33), is called *the local flow of* X. We say that X generates $\phi_t[X]$. We will write ϕ_t for $\phi_t[X]$ when X is unambiguous in the context.

The interval of existence of solutions of (A.8.33) depends upon x_0 in general.

EXAMPLE A.8.3 As an example, let $M = \mathbb{R}$ and $X = x^2\partial_x$. We then have to solve

$$\frac{dx}{dt} = x^2 , \ x(0) = x_0 \quad \Longrightarrow \quad x(t) = \begin{cases} 0, & x_0 = 0; \\ \frac{x_0}{1-x_0 t}, & x_0 \neq 0 , \ 1 - x_0 t > 0. \end{cases}$$

Hence

$$\phi_t(x) = \frac{x}{1 - xt} ,$$

with $t \in \mathbb{R}$ when $x = 0$, with $t \in (-\infty, 1/x)$ when $x > 0$ and with $t \in (1/x, \infty)$ when $x < 0$. □

We say that X is *complete* if $\phi_t[X](p)$ is defined for all $(t, p) \in \mathbb{R} \times M$.

The following standard facts are left as exercices to the reader:

1. ϕ_0 is the identity map.

2. $\phi_t \circ \phi_s = \phi_{t+s}$.

   In particular, $\phi_t^{-1} = \phi_{-t}$, and thus:

3. The maps $x \mapsto \phi_t(x)$ are local diffeomorphisms; global if for all $x \in M$ the maps ϕ_t are defined for all $t \in \mathbb{R}$.

4. $\phi_{-t}[X]$ is generated by $-X$:

$$\phi_{-t}[X] = \phi_t[-X]\,.$$

A family of diffeomorphisms satisfying property 2. above is called a *one parameter group of diffeomorphisms.* Thus, *complete* vector fields generate one-parameter families of diffeomorphisms via (A.8.33).

Reciprocally, suppose that a local or global one-parameter group ϕ_t is given, then the formula

$$X = \frac{d\phi_t}{dt}\Big|_{t=0}$$

defines a vector field, said to be *generated by* ϕ_t.

The Lie derivative revisited

The idea of the *Lie transport*, and hence of the *Lie derivative*, is to be able to compare objects along integral curves of a vector field X. This is pretty obvious for scalars: we just compare the values of $f(\phi_t(x))$ with $f(x)$, leading to a derivative

$$\mathscr{L}_X f := \lim_{t\to 0} \frac{f\circ\phi_t - f}{t} \equiv \lim_{t\to 0} \frac{\phi_t^* f - f}{t} \equiv \lim_{t\to 0} \frac{\hat\phi_t f - f}{t} \equiv \frac{d(\hat\phi_t f)}{dt}\Big|_{t=0}. \quad \text{(A.8.34)}$$

We wish, next, to compare the value of a vector field Y at $\phi_t(x)$ with the value at x. For this, we move from x to $\phi_t(x)$ following the integral curve of X, and produce a new vector at x by applying $(\phi_t^{-1})_*$ to $Y|_{\phi_t(x)}$. This makes it perhaps clearer why we introduced the transport map $\hat\phi$, since $(\hat\phi Y)(x)$ is precisely the value at x of $(\phi_t^{-1})_*Y$. We can then calculate

$$\mathscr{L}_X Y(x) := \lim_{t\to 0} \frac{((\phi_t^{-1}) * Y)(\phi_t(x)) - Y(x)}{t} \equiv \lim_{t\to 0} \frac{(\hat\phi_t Y)(x) - Y(x)}{t} \equiv \frac{d(\hat\phi_t Y(x))}{dt}\Big|_{t=0}. \quad \text{(A.8.35)}$$

In general, let X be a vector field and let ϕ_t be the associated local one-parameter family of diffeomorphisms. Let $\hat\phi_t$ be the associated family of transport maps for tensor fields. For any tensor field T one sets

$$\mathscr{L}_X T := \lim_{t\to 0} \frac{\hat\phi_t T - T}{t} \equiv \frac{d(\hat\phi_t T)}{dt}\Big|_{t=0}. \quad \text{(A.8.36)}$$

We want to show that this operation coincides with that defined in Section A.8.1.

The equality of the two operations for functions should be clear, since (A.8.34) easily implies:

$$\mathscr{L}_X f = X(f)\,.$$

Consider, next, a vector field Y. From (A.8.21), setting $\psi_t := \phi_{-t} \equiv (\phi_t)^{-1}$ we have

$$\hat\phi_t Y^j(x) := ((\phi_t^{-1})_* Y)^j(x) = (\frac{\partial\psi_t^j}{\partial x^i}\, Y^i)(\phi_t(x))\,. \quad \text{(A.8.37)}$$

Since ϕ_{-t} is generated by $-X$, we have

$$\psi^i_0(x) = x^i\,, \quad \frac{\partial\psi^j_t}{\partial x^i}\Big|_{t=0} = \delta^j_i\,,$$
$$\dot\psi^j_t|_{t=0} := \frac{d\psi^j_t}{dt}\Big|_{t=0} = -X^j\,, \quad \frac{\partial\dot\psi^j_t}{\partial x^i}\Big|_{t=0} = -\partial_i X^j\,. \tag{A.8.38}$$

Hence

$$\begin{aligned}\frac{d(\hat\phi_t Y^j)}{dt}(x)|_{t=0} &= \frac{\partial\dot\psi^j_0}{\partial x^i}(x)\,Y^i(x) + \partial_k(\underbrace{\frac{\partial\psi^j_0}{\partial x^i}\,Y^i}_{Y^j})(x)\dot\phi^k(x)\\ &= -\partial_i X^j(x)Y^i(x) + \partial_j Y^i(x)X^j(x)\\ &= [X,Y]^j(x)\,,\end{aligned}$$

and we have obtained (A.8.2), p. 174.

For a covector field α, it seems simplest to calculate directly from (A.8.24):

$$(\hat\phi_t\alpha)_i(x) = (\phi^*_t\alpha)_i(x) = \alpha_k(\phi_t(x))\frac{\partial\phi^k_t(x)}{\partial x^i}\,.$$

Hence

$$\mathscr{L}_X\alpha_i = \frac{d(\phi^*_t\alpha)_i(x)}{dt}\Big|_{t=0} = \partial_j\alpha_i(x)X^j(x) + \alpha_k(x)\frac{\partial X^k(x)}{\partial x^i}(x)\,, \tag{A.8.39}$$

as in (A.8.4).

The formulae just derived show that the *Leibniz rule under duality* holds by inspection:

$$\mathscr{L}_X(\alpha(Y)) = \mathscr{L}_X\alpha(Y) + \alpha(\mathscr{L}_X(Y))\,. \tag{A.8.40}$$

INCIDENTALLY: Alternatively, one can start by showing that the Leibniz rule under duality holds for (A.8.36), and then use the calculations in Section A.8.1 to derive (A.8.39): Indeed, by definition we have

$$\phi^*_t\alpha(Y) = \alpha((\phi_t)_*Y)\,,$$

hence

$$\alpha(Y)|_{\phi_t(x)} = \alpha((\phi_t)_*(\phi_t^{-1})_*Y)|_{\phi_t(x)} = \phi^*_t\alpha|_x((\phi_t^{-1})_*Y|_{\phi_t(x)}) = \hat\phi_t\alpha(\hat\phi_t Y)|_x\,.$$

Equivalently,

$$\hat\phi_t(\alpha(Y)) = (\hat\phi_t\alpha)(\hat\phi_t Y)\,,$$

from which the Leibniz rule under duality immediately follows.

A similar calculation leads to the Leibniz rule under tensor products. □

The reader should have no difficulties checking that the remaining requirements set forth in Section A.8.1 are satisfied.

The following formula of Cartan provides a convenient tool for calculating the Lie derivative of a differential form α:

$$\mathscr{L}_X\alpha = X\rfloor d\alpha + d(X\rfloor\alpha)\,. \tag{A.8.41}$$

The commuting of d and $\mathscr{L}_X$ is an immediate consequence of (A.8.41) and of the identity $d^2 = 0$:

$$\mathscr{L}_X d\alpha = d(\mathscr{L}_X\alpha)\,. \tag{A.8.42}$$

A.9 Covariant derivatives

When dealing with $\mathbb{R}^n$, or subsets thereof, there exists an obvious prescription for how to differentiate tensor fields: in this case we have at our disposal the canonical "trivialization $\{\partial_i\}_{i=1,\ldots,n}$ of $T\mathbb{R}^n$" (this means: a globally defined set of vectors which, at every point, form a basis of the tangent space), together with its dual trivialization $\{dx^j\}_{j=1,\ldots,n}$ of $T^*\mathbb{R}^n$. We can expand a tensor field T of valence (k,ℓ) in terms of those bases,

$$\begin{aligned} T &= T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell} \\ &\Longleftrightarrow \quad T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell} = T(dx^{i_1},\ldots,dx^{i_k},\partial_{j_1},\ldots,\partial_{j_\ell})\,, \end{aligned} \tag{A.9.1}$$

and differentiate each component $T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}$ of T separately:

$$X(T)_{\text{in the coordinate system } x^i} := X^i\frac{\partial T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell}}{\partial x^i}\partial_{x^{i_1}}\otimes\ldots\otimes\partial_{x^{i_k}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell}\,. \tag{A.9.2}$$

The resulting object does, however, *not* behave as a tensor under coordinate transformations, in the sense that the above form of the right-hand side will *not* be preserved under coordinate transformations: as an example, consider the one-form $T = dx$ on $\mathbb{R}^n$, which has vanishing derivative as defined by (A.9.2). When expressed in spherical coordinates we have

$$T = d(\rho\cos\varphi) = -\rho\sin\varphi d\varphi + \cos\varphi d\rho\,,$$

the partial derivatives of which are non-zero (both with respect to the original cartesian coordinates (x,y) and to the new spherical ones (ρ,φ)).

The Lie derivative $\mathscr{L}_X$ of Section A.8 maps tensors to tensors but does not resolve this question, because it is *not* linear under multiplication of X by a function.

The notion of *covariant derivative*, sometimes also referred to as *connection*, is introduced precisely to obtain a notion of derivative which has tensorial properties. By definition, a covariant derivative is a map which to a vector field X and a tensor field T assigns a tensor field of the same type as T, denoted by $\nabla_X T$, with the following properties:

1. $\nabla_X T$ is linear with respect to addition both with respect to X and T:

$$\nabla_{X+Y}T = \nabla_X T + \nabla_Y T\,, \qquad \nabla_X(T+Y) = \nabla_X T + \nabla_X Y\,; \tag{A.9.3}$$

2. $\nabla_X T$ is linear with respect to multiplication of X by functions f,

$$\nabla_{fX}T = f\nabla_X T\,; \tag{A.9.4}$$

3. and, finally, $\nabla_X T$ satisfies the *Leibniz rule* under multiplication of T by a differentiable function f:

$$\nabla_X(fT) = f\nabla_X T + X(f)T\,. \tag{A.9.5}$$

By definition, if T is a tensor field of rank (p,q), then for any vector field X the field $\nabla_X T$ is again a tensor of type (p,q). Since $\nabla_X T$ is linear in X, the field ∇T can naturally be viewed as a tensor field of rank $(p,q+1)$.

It is natural to ask whether covariant derivatives do exist at all in general and, if so, how many of them can there be. First, it immediately follows from the axioms above that if D and ∇ are two covariant derivatives, then

$$\Delta(X,T) := D_X T - \nabla_X T$$

is multi-linear both with respect to addition and multiplication by functions — the non-homogeneous terms $X(f)T$ in (A.9.5) cancel — and is thus a tensor field. Reciprocally, if ∇ is a covariant derivative and $\Delta(X,T)$ is bilinear with respect to addition and multiplication by functions, then

$$D_X T := \nabla_X T + \Delta(X,T) \tag{A.9.6}$$

is a new covariant derivative. So, at least locally, on tensors of valence (r,s) there are as many covariant derivatives as tensors of valence $(r+s, r+s+1)$.

We note that the sum of two covariant derivatives is *not* a covariant derivative. However, *convex* combinations of covariant derivatives, with coefficients which may vary from point to point, are again covariant derivatives. This remark allows one to construct covariant derivatives using partitions of unity: Let, indeed, $\{\mathscr{O}_i\}_{i\in\mathbb{N}}$ be an open covering of M by coordinate patches and let φ_i be an associated partition of unity. In each of those coordinate patches we can decompose a tensor field T as in (A.9.1), and define

$$D_X T := \sum_i \varphi_i X^j \partial_j (T^{i_1\ldots i_k}{}_{j_1\ldots j_\ell})\partial_{i_1}\otimes\ldots\otimes\partial_{i_k}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_\ell}\,. \tag{A.9.7}$$

This procedure, which depends upon the choice of the coordinate patches and the choice of the partition of unity, defines *one* covariant derivative; all other covariant derivatives are then obtained from D using (A.9.6). Note that (A.9.2) is a special case of (A.9.7) when there exists a global coordinate system on M. Thus (A.9.2) *does* define a covariant derivative. However, the associated operation on tensor fields will *not* take the simple form (A.9.2) when we go to a different coordinate system $\{y^i\}$ in general.

A.9.1 Functions

The *canonical covariant derivative on functions* is defined as

$$\nabla_X(f) = X(f)\,,$$

and we will always use the above. This has all the right properties, so obviously covariant derivatives of functions exist. From what has been said, any covariant derivative on functions is of the form

$$\nabla_X f = X(f) + \alpha(X) f\,, \tag{A.9.8}$$

where α is a one-form. Conversely, given any one-form α, (A.9.8) defines a covariant derivative on functions. The addition of the lower-order term $\alpha(X)f$

(A.9.8) does not appear to be very useful here, but it turns out to be useful in geometric formulation of electrodynamics, or in *geometric quantization*. In any case such lower-order terms play an essential role when defining covariant derivatives of tensor fields.

A.9.2 Vectors

The simplest next possibility is that of a covariant derivative of vector fields. Let us not worry about existence at this stage, but assume that a covariant derivative exists, and work from there. (Anticipating, we will show shortly that a metric defines a covariant derivative, called the *Levi-Civita* covariant derivative, which is the unique covariant derivative operator satisfying a natural set of conditions, to be discussed below.)

We will first assume that we are working on a set $\Omega \subset M$ over which we have a *global trivialization* of the tangent bundle TM; by definition, this means that there exist vector fields e_a, $a = 1, \ldots, \dim M$, such that at every point $p \in \Omega$ the fields $e_a(p) \in T_pM$ form a basis of T_pM.[1]

Let θ^a denote the dual trivialization of T^*M — by definition the θ^a's satisfy

$$\boxed{\theta^a(e_b) = \delta^a_b}\,.$$

Given a covariant derivative ∇ on vector fields we set

$$\Gamma^a{}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_b(X) e_a\,, \quad (A.9.9a)$$

$$\boxed{\Gamma^a{}_{bc} := \Gamma^a{}_b(e_c) = \theta^a(\nabla_{e_c} e_b)} \quad \Longleftrightarrow \quad \nabla_X e_b = \Gamma^a{}_{bc} X^c e_a\,. \quad (A.9.9b)$$

The (locally defined) functions $\Gamma^a{}_{bc}$ are called *connection coefficients*. If $\{e_a\}$ is the coordinate basis $\{\partial_\mu\}$ we shall write

$$\Gamma^\mu{}_{\alpha\beta} := dx^\mu(\nabla_{\partial_\beta}\partial_\alpha) \quad \Big(\Longleftrightarrow \quad \nabla_{\partial_\mu}\partial_\nu = \Gamma^\sigma{}_{\nu\mu}\partial_\sigma\Big)\,, \quad (A.9.10)$$

etc. In this particular case the connection coefficients are usually called *Christoffel symbols*. We will sometimes write $\Gamma^\sigma_{\nu\mu}$ instead of $\Gamma^\sigma{}_{\nu\mu}$; note that the former convention is more common. By using the Leibniz rule (A.9.5) we find

$$\begin{aligned} \nabla_X Y &= \nabla_X (Y^a e_a) \\ &= X(Y^a) e_a + Y^a \nabla_X e_a \\ &= X(Y^a) e_a + Y^a \Gamma^b{}_a(X) e_b \\ &= (X(Y^a) + \Gamma^a{}_b(X) Y^b) e_a \\ &= (X(Y^a) + \Gamma^a{}_{bc} Y^b X^c) e_a\,, \end{aligned} \quad (A.9.11)$$

which gives various equivalent ways of writing $\nabla_X Y$. The (perhaps only locally defined) $\Gamma^a{}_b$'s are linear in X, and the collection $(\Gamma^a{}_b)_{a,b=1,\ldots,\dim M}$ is sometimes

[1] This is the case when Ω is a coordinate patch with coordinates (x^i), then the $\{e_a\}_{a=1,\ldots,\dim M}$ can be chosen to be equal to $\{\partial_i\}_{a=1,\ldots,\dim M}$. Recall that a manifold is said to be parallelizable if a basis of TM can be chosen globally over M — in such a case Ω can be taken equal to M. We emphasize that we are *not* assuming that M is parallelizable, so that equations such as (A.9.9) have only a local character in general.

referred to as the *connection one-form.* The one-covariant, one-contravariant tensor field ∇Y is defined as

$$\nabla Y := \nabla_a Y^b \theta^a \otimes e_b \Longleftrightarrow \nabla_a Y^b := \theta^b(\nabla_{e_a} Y) \Longleftrightarrow \boxed{\nabla_a Y^b = e_a(Y^b) + \Gamma^b{}_{ca} Y^c}. \tag{A.9.12}$$

We will often write ∇_a for ∇_{e_a}. Further, $\nabla_a Y^b$ will sometimes be written as $Y^b{}_{;a}$.

A.9.3 Transformation law

Consider a coordinate basis ∂_{x^i}, it is natural to enquire about the transformation law of the connection coefficients $\Gamma^i{}_{jk}$ under a change of coordinates $x^i \to y^k(x^i)$. To make things clear, let us write $\Gamma^i{}_{jk}$ for the connection coefficients in the x–coordinates, and $\hat\Gamma^i{}_{jk}$ for the ones in the y–cordinates. We calculate:

$$\begin{aligned}
\Gamma^i{}_{jk} &:= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\Big) \\
&= dx^i\Big(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial y^\ell}{\partial x^j}\frac{\partial}{\partial y^\ell}\Big) \\
&= dx^i\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s} dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\nabla_{\frac{\partial y^r}{\partial x^k}\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s} dy^s\Big(\frac{\partial^2 y^\ell}{\partial x^k \partial x^j}\frac{\partial}{\partial y^\ell} + \frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\nabla_{\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^\ell}\Big) \\
&= \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k \partial x^j} + \frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k}\hat\Gamma^s{}_{\ell r}\,.
\end{aligned} \tag{A.9.13}$$

Summarising,

$$\boxed{\Gamma^i{}_{jk} = \hat\Gamma^s{}_{\ell r}\frac{\partial x^i}{\partial y^s}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^r}{\partial x^k} + \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^k \partial x^j}}\,. \tag{A.9.14}$$

Thus, the $\Gamma^i{}_{jk}$'s do *not* form a tensor; instead they transform as a tensor *plus* a non-homogeneous term containing second derivatives, as seen above.

EXERCICE A.9.1 Let $\Gamma^i{}_{jk}$ transform as in (A.9.14) under coordinate transformations. If X and Y are vector fields, define in local coordinates

$$\nabla_X Y := \Big(X(Y^i) + \Gamma^i{}_{jk} X^k Y^k\Big)\,\partial_i\,. \tag{A.9.15}$$

Show that $\nabla_X Y$ transforms as a vector field under coordinate transformations (and thus is a vector field). Hence, a collection of fields $\{\Gamma^i{}_{jk}\}$ satisfying the transformation law (A.9.14) can be used to define a covariant derivative using (A.9.15).

A.9.4 Torsion

Because the inhomogeneous term in (A.9.14) is symmetric under the interchange of i and j, it follows from (A.9.14) that

$$T^i_{jk} := \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$$

does transform as a tensor, called *the torsion tensor* of ∇.

An index-free definition of torsion proceeds as follows: Let ∇ be a covariant derivative defined for vector fields, the *torsion tensor* T is defined by the formula

$$\boxed{T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]}\,, \tag{A.9.16}$$

where $[X,Y]$ is the Lie bracket. We obviously have

$$T(X,Y) = -T(Y,X)\,. \tag{A.9.17}$$

Let us check that T is actually a tensor field: multi-linearity with respect to addition is obvious. To check what happens under multiplication by functions, in view of (A.9.17) it is sufficient to do the calculation for the first slot of T. We then have

$$\begin{aligned} T(fX,Y) &= \nabla_{fX}Y - \nabla_Y(fX) - [fX,Y] \\ &= f\Big(\nabla_X Y - \nabla_Y X\Big) - Y(f)X - [fX,Y]\,. \end{aligned} \tag{A.9.18}$$

To work out the last commutator term we compute, for any function φ,

$$[fX,Y](\varphi) = fX(Y(\varphi)) - \underbrace{Y(fX(\varphi))}_{=Y(f)X(\varphi)+fY(X(\varphi))} = f[X,Y](\varphi) - Y(f)X(\varphi)\,,$$

hence

$$[fX,Y] = f[X,Y] - Y(f)X\,, \tag{A.9.19}$$

and the last term here cancels the undesirable second-to-last term in (A.9.18), as required.

In a coordinate basis ∂_μ we have $[\partial_\mu, \partial_\nu] = 0$ and one finds from (A.9.10)

$$\boxed{T(\partial_\mu, \partial_\nu) = (\Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\mu\nu})\partial_\sigma}\,, \tag{A.9.20}$$

which shows that T is determined by twice the antisymmetrization of the $\Gamma^\sigma{}_{\mu\nu}$'s over the lower indices. In particular that last antisymmetrization produces a tensor field.

A.9.5 Covectors

Suppose that we are given a covariant derivative on vector fields, there is a natural way of inducing a covariant derivative on one-forms by imposing the condition that *the duality operation be compatible with the Leibniz rule*: given two vector fields X and Y together with a field of one-forms α, one sets

$$\boxed{(\nabla_X \alpha)(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y)}\,. \tag{A.9.21}$$

Let us, first, check that (A.9.21) indeed defines a field of one-forms. The linearity, in the Y variable, with respect to addition is obvious. Next, for any function f we have

$$\begin{aligned}(\nabla_X\alpha)(fY) &= X(\alpha(fY)) - \alpha(\nabla_X(fY))\\ &= X(f)\alpha(Y) + fX(\alpha(Y)) - \alpha(X(f)Y + f\nabla_X Y)\\ &= f(\nabla_X\alpha)(Y)\,,\end{aligned}$$

as should be the case for one-forms. Next, we need to check that ∇ defined by (A.9.21) does satisfy the remaining axioms imposed on covariant derivatives. Again multi-linearity with respect to addition is obvious, as well as linearity with respect to multiplication of X by a function. Finally,

$$\begin{aligned}\nabla_X(f\alpha)(Y) &= X(f\alpha(Y)) - f\alpha(\nabla_X Y)\\ &= X(f)\alpha(Y) + f(\nabla_X\alpha)(Y)\,,\end{aligned}$$

as desired.

The duality pairing

$$T_p^*M \times T_pM \ni (\alpha, X) \to \alpha(X) \in \mathbb{R}$$

is sometimes called *contraction*. As already pointed out, the operation ∇ on one-forms has been defined in (A.9.21) so as to satisfy the *Leibniz rule under duality pairing*:

$$X(\alpha(Y)) = (\nabla_X\alpha)(Y) + \alpha(\nabla_X Y)\,; \tag{A.9.22}$$

this follows directly from (A.9.21). This should not be confused with the Leibniz rule under multiplication by functions, which is part of the definition of a covariant derivative, and therefore always holds. It should be kept in mind that (A.9.22) does not necessarily hold for all covariant derivatives: if ${}^v\nabla$ is some covariant derivative on vectors, and ${}^f\nabla$ is some covariant derivative on one-forms, in general one will have

$$X(\alpha(Y)) \neq ({}^f\nabla_X)\alpha(Y) + \alpha({}^v\nabla_X Y)\,.$$

Using the basis-expression (A.9.11) of $\nabla_X Y$ and the definition (A.9.21) we have

$$\nabla_X\alpha = X^a\nabla_a\alpha_b\,\theta^b\,,$$

with

$$\begin{aligned}\boxed{\nabla_a\alpha_b} &:= (\nabla_{e_a}\alpha)(e_b)\\ &= e_a(\alpha(e_b)) - \alpha(\nabla_{e_a}e_b)\\ &= \boxed{e_a(\alpha_b) - \Gamma^c{}_{ba}\alpha_c}\,.\end{aligned}$$

A.9.6 Higher order tensors

It should now be clear how to extend ∇ to tensors of arbitrary valence: if T is r covariant and s contravariant one sets

$$\begin{aligned}(\nabla_X T)(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s) &:= X\Big(T(X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s)\Big)\\ &-T(\nabla_X X_1,\ldots,X_r,\alpha_1,\ldots\alpha_s)-\ldots-T(X_1,\ldots,\nabla_X X_r,\alpha_1,\ldots\alpha_s)\\ &-T(X_1,\ldots,X_r,\nabla_X\alpha_1,\ldots\alpha_s)-\ldots-T(X_1,\ldots,X_r,\alpha_1,\ldots\nabla_X\alpha_s)\,.\end{aligned} \tag{A.9.23}$$

The verification that this defines a covariant derivative proceeds in a way identical to that for one-forms. In a basis we have

$$\nabla_X T = X^a \nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s}\theta^{a_1}\otimes\ldots\otimes\theta^{a_r}\otimes e_{b_1}\otimes\ldots\otimes e_{b_s}\,,$$

and (A.9.23) gives

$$\begin{aligned}\nabla_a T_{a_1\ldots a_r}{}^{b_1\ldots b_s} &:= (\nabla_{e_a}T)(e_{a_1},\ldots,e_{a_r},\theta^{b_1},\ldots,\theta^{b_s})\\ &= e_a(T_{a_1\ldots a_r}{}^{b_1\ldots b_s}) - \Gamma^c{}_{a_1 a}T_{c\ldots a_r}{}^{b_1\ldots b_s} - \ldots - \Gamma^c{}_{a_r a}T_{a_1\ldots c}{}^{b_1\ldots b_s}\\ &\quad +\Gamma^{b_1}{}_{ca}T_{a_1\ldots a_r}{}^{c\ldots b_s}+\ldots+\Gamma^{b_s}{}_{ca}T_{a_1\ldots a_r}{}^{b_1\ldots c}\,.\end{aligned} \tag{A.9.24}$$

Carrying over the last two lines of (A.9.23) to the left-hand side of that equation one obtains the Leibniz rule for ∇ under pairings of tensors with vectors or forms. It should be clear from (A.9.23) that ∇ defined by that equation is the *only covariant derivative which agrees with the original one on vectors, and which satisfies the Leibniz rule under the pairing operation.* We will only consider such covariant derivatives in this work.

A.10 The Levi-Civita connection

One of the fundamental results in pseudo-Riemannian geometry is that of the existence of a torsion-free connection which preserves the metric:

THEOREM A.10.1 *Let g be a two-covariant symmetric non-degenerate tensor field on a manifold M. Then there exists a unique connection ∇ such that*

1. $\nabla g = 0$,
2. *the torsion tensor T of ∇ vanishes.*

PROOF: Suppose, first, that a connection satisfying the above is given. By the Leibniz rule we then have, for any vector fields X, Y and Z,

$$0 = (\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z)\,. \tag{A.10.1}$$

We rewrite the same equation applying cyclic permutations to X, Y, and Z, with a minus sign for the last equation:

$$\begin{aligned}g(\nabla_X Y,Z)+g(Y,\nabla_X Z) &= X(g(Y,Z))\,,\\ g(\nabla_Y Z,X)+g(Z,\nabla_Y X) &= Y(g(Z,X))\,,\\ -g(\nabla_Z X,Y)-g(X,\nabla_Z Y) &= -Z(g(X,Y))\,.\end{aligned} \tag{A.10.2}$$

As the torsion tensor vanishes, the sum of the left-hand sides of these equations can be manipulated as follows:

$$\begin{aligned}
&g(\nabla_X Y,Z)+g(Y,\nabla_X Z)+g(\nabla_Y Z,X)+g(Z,\nabla_Y X)-g(\nabla_Z X,Y)-g(X,\nabla_Z Y)\\
&=g(\nabla_X Y+\nabla_Y X,Z)+g(Y,\nabla_X Z-\nabla_Z X)+g(X,\nabla_Y Z-\nabla_Z Y)\\
&=g(2\nabla_X Y-[X,Y],Z)+g(Y,[X,Z])+g(X,[Y,Z])\\
&=2g(\nabla_X Y,Z)-g([X,Y],Z)+g(Y,[X,Z])+g(X,[Y,Z])\,.
\end{aligned}$$

This shows that the sum of the three equations (A.10.2) can be rewritten as

$$\begin{aligned}
2g(\nabla_X Y,Z) &= g([X,Y],Z)-g(Y,[X,Z])-g(X,[Y,Z])\\
&\quad +X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\,. \qquad \text{(A.10.3)}
\end{aligned}$$

Since Z is arbitrary and g is non-degenerate, the left-hand side of this equation determines the vector field $\nabla_X Y$ uniquely, and uniqueness of ∇ follows.

To prove existence, let $S(X,Y)(Z)$ be defined as one half of the right-hand side of (A.10.3),

$$\begin{aligned}
S(X,Y)(Z) &= \frac{1}{2}\Big(X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\\
&\quad +g(Z,[X,Y])-g(Y,[X,Z])-g(X,[Y,Z])\Big)\,. \qquad \text{(A.10.4)}
\end{aligned}$$

Clearly S is linear with respect to addition in all fields involved. Let us check that it is also linear with respect to multiplication of Z by a function:

$$\begin{aligned}
S(X,Y)(fZ) &= \frac{f}{2}\Big(X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\\
&\quad +g(Z,[X,Y])-g(Y,[X,Z])-g(X,[Y,Z])\Big)\\
&\quad +\frac{1}{2}\Big(X(f)g(Y,Z)+Y(f)g(Z,X)-g(Y,X(f)Z)-g(X,Y(f)Z)\Big)\\
&= fS(X,Y)(Z)\,. \qquad \text{(A.10.5)}
\end{aligned}$$

Since g is non-degenerate, we conclude that there exists a unique vector field $W(X,Y)$ such that

$$S(X,Y)(Z)=g(W(X,Y),Z)\,.$$

One readily checks that the assignment

$$(X,Y)\to W(X,Y)=:\nabla_X Y$$

satisfies all the requirements imposed on a covariant derivative $\nabla_X Y$.

It is immediate from (A.10.3), which is equivalent to (A.10.4), that the connection ∇ so defined is torsion free: Indeed, the sum of all-but-first terms at the right-hand side of (A.10.3) is symmetric in (X,Y), and the first term is what is needed to produce the torsion tensor when removing from (A.10.3) its counterpart with X and Y interchanged.

Finally, one checks that ∇ is metric-compatible by inserting $\nabla_X Y$ and $\nabla_X Z$, as defined by (A.10.3), into (A.10.1). This concludes the proof. □

INCIDENTALLY: Let us give an index-notation version of the above. Using the definition of $\nabla_i g_{jk}$ we have

$$0 = \nabla_i g_{jk} \equiv \partial_i g_{jk} - \Gamma^\ell{}_{ji} g_{\ell k} - \Gamma^\ell{}_{ki} g_{\ell j}\,; \tag{A.10.6}$$

here we have written $\Gamma^i{}_{jk}$ instead of Γ^i_{jk}, as is standard in the literature. We rewrite this equation making cyclic permutations of indices, and changing the overall sign:

$$0 = -\nabla_j g_{ki} \equiv -\partial_j g_{ki} + \Gamma^\ell{}_{kj} g_{\ell i} + \Gamma^\ell{}_{ij} g_{\ell k}\,.$$

$$0 = -\nabla_k g_{ij} \equiv -\partial_k g_{ij} + \Gamma^\ell{}_{ik} g_{\ell j} + \Gamma^\ell{}_{jk} g_{\ell i}\,.$$

Adding the three equations and using symmetry of Γ^k_{ji} in ij one obtains

$$0 = \partial_i g_{jk} - \partial_j g_{ki} - \partial_k g_{ij} + 2\Gamma^\ell{}_{jk} g_{\ell i}\,,$$

Multiplying by g^{im} we obtain

$$\Gamma^m{}_{jk} = g^{mi}\Gamma^\ell{}_{jk} g_{\ell i} = \frac{1}{2} g^{mi}(\partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk})\,. \tag{A.10.7}$$

This proves uniqueness.

A straightforward, though somewhat lengthy, calculation shows that the $\Gamma^m{}_{jk}$'s defined by (A.10.7) satisfy the transformation law (A.9.14). Exercice A.9.1 shows that the formula (A.9.15) defines a torsion-free connection. It then remains to check that the insertion of the $\Gamma^m{}_{jk}$'s, as given by (A.10.7), into the right-hand side of (A.10.6), indeed gives zero, proving existence. □

Let us check that (A.10.3) reproduces (A.10.7): Consider (A.10.3) with $X = \partial_\gamma$, $Y = \partial_\beta$ and $Z = \partial_\sigma$,

$$\begin{aligned} 2g(\nabla_\gamma \partial_\beta, \partial_\sigma) &= 2g(\Gamma^\rho{}_{\beta\gamma}\partial_\rho, \partial_\sigma) \\ &= 2g_{\rho\sigma}\Gamma^\rho{}_{\beta\gamma} \\ &= \partial_\gamma g_{\beta\sigma} + \partial_\beta g_{\gamma\sigma} - \partial_\sigma g_{\beta\gamma} \end{aligned} \tag{A.10.8}$$

Multiplying this equation by $g^{\alpha\sigma}/2$ we then obtain

$$\boxed{\Gamma^\alpha{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}\{\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}\}}\,. \tag{A.10.9}$$

□

A.10.1 Geodesics and Christoffel symbols

A geodesic can be defined as the stationary point of the action

$$I(\gamma) = \int_a^b \underbrace{\frac{1}{2} g(\dot\gamma, \dot\gamma)(s)}_{=:\mathscr{L}(\gamma,\dot\gamma)} ds\,, \tag{A.10.10}$$

where $\gamma : [a, b] \to M$ is a differentiable curve. Thus,

$$\mathscr{L}(x^\mu, \dot x^\nu) = \frac{1}{2} g_{\alpha\beta}(x^\mu)\dot x^\alpha \dot x^\beta\,.$$

One readily finds the Euler-Lagrange equations for this Lagrange function:

$$\frac{d}{ds}\left(\frac{\partial \mathscr{L}}{\partial \dot x^\mu}\right) = \frac{\partial \mathscr{L}}{\partial x^\mu} \quad \Longleftrightarrow \quad \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0\,. \tag{A.10.11}$$

This provides a very convenient way of calculating the Christoffel symbols: given a metric g, write down $\mathscr{L}$, work out the Euler-Lagrange equations, and identify the Christoffels as the coefficients of the first derivative terms in those equations.

EXERCICE A.10.3 Prove (A.10.11). □

(The Euler-Lagrange equations for (A.10.10) are identical with those of

$$\tilde{I}(\gamma) = \int_a^b \sqrt{|g(\dot\gamma,\dot\gamma)(s)|}ds\,, \tag{A.10.12}$$

but (A.10.10) is more convenient to work with. For example, $\mathscr{L}$ is differentiable at points where $\dot\gamma$ vanishes, while $\sqrt{|g(\dot\gamma,\dot\gamma)(s)|}$ is not. The aesthetic advantage of (A.10.12), of being reparameterization-invariant, is more than compensated by the calculational convenience of $\mathscr{L}$.)

INCIDENTALLY: EXAMPLE A.10.5 As an example, consider a metric of the form

$$g = dr^2 + f(r)d\varphi^2\,.$$

Special cases of this metric include the Euclidean metric on $\mathbb{R}^2$ (then $f(r) = r^2$), and the canonical metric on a sphere (then $f(r) = \sin^2 r$, with r actually being the polar angle θ). The Lagrangian (A.10.12) is thus

$$L = \frac{1}{2}\left(\dot r^2 + f(r)\dot\varphi^2\right)\,.$$

The Euler-Lagrange equations read

$$\underbrace{\frac{\partial L}{\partial \varphi}}_{0} = \frac{d}{ds}\left(\frac{\partial L}{\partial \dot\varphi}\right) = \frac{d}{ds}(f(r)\dot\varphi)\,,$$

so that

$$0 = f\ddot\varphi + f'\dot r\dot\varphi = f\left(\ddot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi^2 + 2\Gamma^\varphi_{r\varphi}\dot r\dot\varphi + \Gamma^\varphi_{rr}\dot r^2\right) \quad\Longrightarrow\quad \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{rr} = 0\,,\quad \Gamma^\varphi_{r\varphi} = \frac{f'}{2f}\,.$$

Similarly

$$\underbrace{\frac{\partial L}{\partial r}}_{f'\dot\varphi^2/2} = \frac{d}{ds}\left(\frac{\partial L}{\partial \dot r}\right) = \ddot r\,,$$

so that

$$\Gamma^r_{r\varphi} = \Gamma^r_{rr} = 0\,,\quad \Gamma^r_{\varphi\varphi} = -\frac{f'}{2}\,.$$

□

A.11 "Local inertial coordinates"

PROPOSITION A.11.1 *1. Let g be a Lorentzian metric, for every $p \in M$ there exists a neighborhood thereof with a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu} =$* $\mathrm{diag}(1, -1, \cdots, -1)$ *at p.*

2. If g is differentiable, then the coordinates can be further chosen so that

$$\partial_\sigma g_{\alpha\beta} = 0 \tag{A.11.1}$$

at p.

The coordinates above will be referred to as *local inertial coordinates near* p.

REMARK A.11.2 An analogous result holds for any pseudo-Riemannian metric. Note that *normal coordinates*, constructed by shooting geodesics from p, satisfy the above. However, for metrics of finite differentiability, the introduction of normal coordinates leads to a loss of differentiability of the metric components, while the construction below preserves the order of differentiability.

PROOF: 1. Let y^μ be any coordinate system around p, shifting by a constant vector we can assume that p corresponds to $y^\mu = 0$. Let $e_a = {e_a}^\mu \partial/\partial y^\mu$ be any frame at p such that $g(e_a, e_b) = \eta_{ab}$ — such frames can be found by, *e.g.*, a Gram-Schmidt orthogonalisation. Calculating the determinant of both sides of the equation

$$g_{\mu\nu}{e_a}^\mu {e_b}^\nu = \eta_{ab}$$

we obtain, at p,

$$\det(g_{\mu\nu})\det({e_a}^\mu)^2 = -1\,,$$

which shows that $\det({e_a}^\mu)$ is non-vanishing. It follows that the formula

$$y^\mu = {e^\mu}_a x^a$$

defines a (linear) diffeomorphism. In the new coordinates we have, again at p,

$$g\Big(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\Big) = {e^\mu}_a {e^\nu}_b g\Big(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\Big) = \eta_{ab}\,. \tag{A.11.2}$$

2. We will use (A.9.14), which uses latin indices, so let us switch to that notation. Let x^i be the coordinates described in point 1., recall that p lies at the origin of those coordinates. The new coordinates $\hat x^j$ will be implicitly defined by the equations

$$x^i = \hat x^i + \frac{1}{2}{A^i}_{jk}\hat x^j \hat x^k\,,$$

where ${A^i}_{jk}$ is a set of constants, symmetric with respect to the interchange of j and k. Recall (A.9.14),

$${\hat\Gamma^i}_{jk} = {\Gamma^s}_{\ell r}\frac{\partial \hat x^i}{\partial x^s}\frac{\partial x^\ell}{\partial \hat x^j}\frac{\partial x^r}{\partial \hat x^k} + \frac{\partial \hat x^i}{\partial x^s}\frac{\partial^2 x^s}{\partial \hat x^k \partial \hat x^j}\,; \tag{A.11.3}$$

here we use ${\hat\Gamma^s}_{\ell r}$ to denote the Christoffel symbols of the metric in the hatted coordinates. Then, at $x^i = 0$, this equation reads

$$\begin{aligned} {\hat\Gamma^i}_{jk} &= {\Gamma^s}_{\ell r}\underbrace{\frac{\partial \hat x^i}{\partial x^s}}_{\delta^i_s}\underbrace{\frac{\partial x^\ell}{\partial \hat x^j}}_{\delta^\ell_j}\underbrace{\frac{\partial x^r}{\partial \hat x^k}}_{\delta^r_k} + \underbrace{\frac{\partial x^i}{\partial x^s}}_{\delta^i_s}\underbrace{\frac{\partial^2 x^s}{\partial \hat x^k \partial \hat x^j}}_{A^s_{kj}} \\ &= {\Gamma^i}_{jk} + {A^i}_{kj}\,. \end{aligned}$$

Choosing A^i_{jk} as $-{\Gamma^i}_{jk}(0)$, the result follows.

INCIDENTALLY: If you do not like to remember formulae such as (A.9.14), proceed as follows: Let x^μ be the coordinates described in point 1. The new coordinates $\hat x^\alpha$ will be implicitly defined by the equations

$$x^\mu = \hat x^\mu + \frac{1}{2} A^\mu{}_{\alpha\beta}\hat x^\alpha \hat x^\beta \,,$$

where $A^\mu{}_{\alpha\beta}$ is a set of constants, symmetric with respect to the interchange of α and β. Set

$$\hat g_{\alpha\beta} := g\Big(\frac{\partial}{\partial \hat x^\alpha}, \frac{\partial}{\partial \hat x^\beta}\Big)\,, \qquad g_{\alpha\beta} := g\Big(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\Big)\,.$$

Recall the transformation law

$$\hat g_{\mu\nu}(\hat x^\sigma) = g_{\alpha\beta}(x^\rho(\hat x^\sigma))\frac{\partial x^\alpha}{\partial \hat x^\mu}\frac{\partial x^\beta}{\partial \hat x^\nu}\,.$$

By differentiation one obtains at $x^\mu = \hat x^\mu = 0$,

$$\begin{aligned}\frac{\partial \hat g_{\mu\nu}}{\partial \hat x^\rho}(0) &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + g_{\alpha\beta}(0)\Big(A^\alpha{}_{\mu\rho}\delta^\beta_\nu + \delta^\alpha_\mu A^\beta{}_{\nu\rho}\Big)\\ &= \frac{\partial g_{\mu\nu}}{\partial x^\rho}(0) + A_{\nu\mu\rho} + A_{\mu\nu\rho}\,,\end{aligned} \tag{A.11.4}$$

where

$$A_{\alpha\beta\gamma} := g_{\alpha\sigma}(0)A^\sigma{}_{\beta\gamma}\,.$$

It remains to show that we can choose $A^\sigma{}_{\beta\gamma}$ so that the left-hand side can be made to vanish at p. An explicit formula for $A_{\sigma\beta\gamma}$ can be obtained from (A.11.4) by a cyclic permutation calculation similar to that in (A.10.2). After raising the first index, the final result is

$$A^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho}\left\{\frac{\partial g_{\beta\gamma}}{\partial x^\rho} - \frac{\partial g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial g_{\rho\gamma}}{\partial x^\beta}\right\}(0)\,;$$

the reader may wish to check directly that this does indeed lead to a vanishing right-hand side of (A.11.4).

□

A.12 Curvature

Let ∇ be a covariant derivative defined for vector fields, the curvature tensor is defined by the formula

$$\boxed{R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z}\,, \tag{A.12.1}$$

where, as elsewhere, $[X,Y]$ is the Lie bracket defined in (A.3.6). We note the anti-symmetry

$$R(X,Y)Z = -R(Y,X)Z\,. \tag{A.12.2}$$

It turns out this defines a tensor. Multi-linearity with respect to addition is obvious, but multiplication by functions require more work.

First, we have (see (A.9.19))

$$\begin{aligned}R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z\\ &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \underbrace{\nabla_{f[X,Y]-Y(f)X}Z}_{=f\nabla_{[X,Y]}Z - Y(f)\nabla_X Z}\\ &= fR(X,Y)Z\,.\end{aligned}$$

INCIDENTALLY: The simplest proof of linearity in the last slot proceeds via an index calculation in adapted coordinates; so while we will do the elegant, index-free version shortly, let us do the ugly one first. We use the coordinate system of Proposition A.11.1 below, in which the first derivatives of the metric vanish at the prescribed point p:

$$\begin{aligned}\nabla_i \nabla_j Z^k &= \partial_i(\partial_j Z^k - \Gamma^k{}_{\ell j} Z^\ell) + \underbrace{0 \times \nabla Z}_{\text{at } p} \\ &= \partial_i \partial_j Z^k - \partial_i \Gamma^k{}_{\ell j} Z^\ell \quad \text{at } p\,. \end{aligned} \tag{A.12.3}$$

Antisymmetrising in i and j, the terms involving the second derivatives of Z drop out, so the result is indeed linear in Z. So $\nabla_i \nabla_j Z^k - \nabla_j \nabla_i Z^k$ is a tensor field linear in Z, and therefore can be written as $R^k{}_{\ell ij} Z^\ell$.

Note that $\nabla_i \nabla_j Z^k$ is, by definition, the tensor field of first covariant derivatives of the tensor field $\nabla_j Z^k$, while (A.12.1) involves covariant derivatives of vector fields only, so the equivalence of both approaches requires a further argument. This is provided in the calculation below leading to (A.12.7). □

We continue with

$$\begin{aligned}R(X,Y)(fZ) &= \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ) \\ &= \Big\{\nabla_X \Big(Y(f)Z + f\nabla_Y Z\Big)\Big\} - \Big\{\cdots\Big\}_{X\leftrightarrow Y} \\ &\quad -[X,Y](f)Z - f\nabla_{[X,Y]}Z \\ &= \Big\{\underbrace{X(Y(f))Z}_{a} + \underbrace{Y(f)\nabla_X Z + X(f)\nabla_Y Z}_{b} + f\nabla_X\nabla_Y Z\Big\} - \Big\{\cdots\Big\}_{X\leftrightarrow Y} \\ &\quad - \underbrace{[X,Y](f)Z}_{c} - f\nabla_{[X,Y]}Z\,.\end{aligned}$$

Now, a together with its counterpart with X and Y interchanged cancel out with c, while b is symmetric with respect to X and Y and therefore cancels out with its counterpart with X and Y interchanged, leading to the desired equality

$$R(X,Y)(fZ) = fR(X,Y)Z\,.$$

In a coordinate basis $\{e_a\} = \{\partial_\mu\}$ we find[2] (recall that $[\partial_\mu, \partial_\nu] = 0$)

$$\begin{aligned}R^\alpha{}_{\beta\gamma\delta} &:= \langle dx^\alpha, R(\partial_\gamma, \partial_\delta)\partial_\beta\rangle \\ &= \langle dx^\alpha, \nabla_\gamma \nabla_\delta \partial_\beta\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \langle dx^\alpha, \nabla_\gamma(\Gamma^\sigma{}_{\beta\delta}\partial_\sigma)\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \langle dx^\alpha, \partial_\gamma(\Gamma^\sigma{}_{\beta\delta})\partial_\sigma + \Gamma^\rho{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\partial_\rho\rangle - \langle\cdots\rangle_{\delta\leftrightarrow\gamma} \\ &= \{\partial_\gamma \Gamma^\alpha{}_{\beta\delta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}\} - \{\cdots\}_{\delta\leftrightarrow\gamma}\,,\end{aligned}$$

leading finally to

$$\boxed{R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}}\,. \tag{A.12.4}$$

[2]The reader is warned that certain authors use other sign conventions either for $R(X,Y)Z$, or for $R^\alpha{}_{\beta\gamma\delta}$, or both. A useful table that lists the sign conventions for a series of standard GR references can be found on the backside of the front cover of [326].

In a general frame some supplementary commutator terms will appear in the formula for $R^a{}_{bcd}$.

INCIDENTALLY: An alternative way of introducing the Riemann tensor proceeds as in [414]; here we assume for simplicity that ∇ is torsion-free, but a similar calculation applies in general:

PROPOSITION A.12.3 *Let ∇ be torsion-free. There exists a tensor field $R^d{}_{abc}$ of type $(1,3)$ such that*

$$\nabla_a\nabla_b X^d - \nabla_b\nabla_a X^d = R^d{}_{cab}X^c\,. \tag{A.12.5}$$

PROOF: We need to check that the derivatives of X cancel. Now,

$$\begin{aligned}\nabla_a\nabla_b X^d &= \partial_a(\underbrace{\nabla_b X^d}_{\partial_b X^d+\Gamma^d{}_{be}X^e}) + \Gamma^d_{ac}\underbrace{\nabla_b X^c}_{\partial_b X^c+\Gamma^c_{be}X^e} - \Gamma^e_{ab}\nabla_e X^d \\ &= \underbrace{\partial_a\partial_b X^d}_{=:1_{ab}} + \partial_a\Gamma^d{}_{be}X^e + \underbrace{\Gamma^d_{be}\partial_a X^e}_{=:2_{ab}} + \underbrace{\Gamma^d_{ac}\partial_b X^c}_{=:3_{ab}} + \Gamma^d_{ac}\Gamma^c_{be}X^e - \underbrace{\Gamma^e_{ab}\nabla_e X^d}_{=:4_{ab}}\,.\end{aligned}$$

If we subtract $\nabla_b\nabla_a X^d$, then

1. 1_{ab} is symmetric in a and b, so will cancel out; similarly for 4_{ab} because ∇ has been assumed to have no torsion;
2. 2_{ab} will cancel out with 3_{ba}; similarly 3_{ab} will cancel out with 2_{ba}.

So the left-hand side of (A.12.5) is indeed linear in X^e. Since it is a tensor, the right-hand side also is. Since X^e is arbitrary, we conclude that $R^d{}_{cab}$ is a tensor of the desired type. □

We note the following:

THEOREM A.12.4 *There exists a coordinate system in which the metric tensor field has vanishing second derivatives at p if and only if its Riemann tensor vanishes at p. Furthermore, there exists a coordinate system in which the metric tensor field has constant entries near p if and only if the Riemann tensor vanishes near p.*

PROOF: The condition is necessary, since Riem is a tensor. The sufficiency will be admitted. □

The calculation of the curvature tensor may be a very traumatic experience. There is one obvious case where things are painless, when all $g_{\mu\nu}$'s are constants: in this case the Christoffels vanish, and so does the curvature tensor. Metrics with the last property are called *flat*.

For more general metrics, one way out is to use symbolic computer algebra. This can, e.g., be done online on `http://grtensor.phy.queensu.ca/NewDemo`. MATHEMATICA packages to do this can be found at URL's `http://www.math.washington.edu/~lee/Ricci`, or `http://grtensor.phy.queensu.ca/NewDemo`, or `http://luth.obspm.fr/~luthier/Martin-Garcia/xAct`. This last package is least-user-friendly as of today, but is the most flexible, especially for more involved computations.

We also note an algorithm of Benenti [51] to calculate the curvature tensor, starting from the variational principle for geodesics, which avoids writing-out explicitly all the Christoffel coefficients.

INCIDENTALLY: EXAMPLE A.12.6 As an example less trivial than a metric with constant coefficients, consider the round two sphere, which we write in the form

$$g = d\theta^2 + e^{2f} d\varphi^2 \,, \qquad e^{2f} = \sin^2\theta \,.$$

As seen in Example A.10.5, the Christoffel symbols are easily founds from the Lagrangian for geodesics:

$$\mathscr{L} = \frac{1}{2}(\dot\theta^2 + e^{2f}\dot\varphi^2)\,.$$

The Euler-Lagrange equations give

$$\Gamma^\theta{}_{\varphi\varphi} = -f' e^{2f} \,, \quad \Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} = f' \,,$$

with the remaining Christoffel symbols vanishing. Using the definition of the Riemann tensor we then immediately find

$$R^\varphi{}_{\theta\varphi\theta} = -f'' - (f')^2 = -e^{-f}(e^f)'' = 1 \,. \qquad \text{(A.12.6)}$$

All remaining components of the Riemann tensor can be obtained from this one by raising and lowering of indices, together with the symmetry operations which we are about to describe. This leads to

$$R_{ab} = g_{ab} \,, \qquad R = 2 \,.$$

□

Equation (A.12.1) is most frequently used "upside-down", not as a definition of the Riemann tensor, but as a tool for calculating what happens when one changes the order of covariant derivatives. Recall that for partial derivatives we have

$$\partial_\mu \partial_\nu Z^\sigma = \partial_\nu \partial_\mu Z^\sigma \,,$$

but this is not true in general if partial derivatives are replaced by covariant ones:

$$\nabla_\mu \nabla_\nu Z^\sigma \neq \nabla_\nu \nabla_\mu Z^\sigma \,.$$

To find the correct formula let us consider the tensor field S defined as

$$Y \longrightarrow S(Y) := \nabla_Y Z \,.$$

In local coordinates, S takes the form

$$S = \nabla_\mu Z^\nu \, dx^\mu \otimes \partial_\nu \,.$$

It follows from the Leibniz rule — or, equivalently, from the definitions in Section A.9 — that we have

$$\begin{aligned}(\nabla_X S)(Y) &= \nabla_X(S(Y)) - S(\nabla_X Y) \\ &= \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z \,.\end{aligned}$$

The commutator of the derivatives can then be calculated as

$$\begin{aligned}(\nabla_X S)(Y) - (\nabla_Y S)(X) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &\quad + \nabla_{[X,Y]} Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z \\ &= R(X,Y)Z - \nabla_{T(X,Y)} Z \,. \qquad \text{(A.12.7)}\end{aligned}$$

Writing ∇S in the usual form

$$\nabla S = \nabla_\sigma S_\mu{}^\nu \, dx^\sigma \otimes dx^\mu \otimes \partial_\nu = \nabla_\sigma \nabla_\mu Z^\nu \, dx^\sigma \otimes dx^\mu \otimes \partial_\nu \,,$$

we are thus led to

$$\nabla_\mu \nabla_\nu Z^\alpha - \nabla_\nu \nabla_\mu Z^\alpha = R^\alpha{}_{\sigma\mu\nu} Z^\sigma - T^\sigma{}_{\mu\nu} \nabla_\sigma Z^\alpha \,. \tag{A.12.8}$$

In the important case of vanishing torsion, the coordinate-component equivalent of (A.12.1) is thus

$$\boxed{\nabla_\mu \nabla_\nu X^\alpha - \nabla_\nu \nabla_\mu X^\alpha = R^\alpha{}_{\sigma\mu\nu} X^\sigma}\,. \tag{A.12.9}$$

An identical calculation gives, still for torsionless connections,

$$\nabla_\mu \nabla_\nu a_\alpha - \nabla_\nu \nabla_\mu a_\alpha = -R^\sigma{}_{\alpha\mu\nu} a_\sigma \,. \tag{A.12.10}$$

For a general tensor t and torsion-free connection each tensor index comes with a corresponding Riemann tensor term:

$$\begin{aligned} &\nabla_\mu \nabla_\nu t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} - \nabla_\nu \nabla_\mu t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} = \\ &\quad -R^\sigma{}_{\alpha_1 \mu\nu} t_{\sigma \ldots \alpha_r}{}^{\beta_1 \ldots \beta_s} - \ldots - R^\sigma{}_{\alpha_r \mu\nu} t_{\alpha_1 \ldots \sigma}{}^{\beta_1 \ldots \beta_s} \\ &\quad + R^{\beta_1}{}_{\sigma\mu\nu} t_{\alpha_1 \ldots \alpha_r}{}^{\sigma \ldots \beta_s} + \ldots + R^{\beta_s}{}_{\sigma\mu\nu} t_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \sigma} \,. \end{aligned} \tag{A.12.11}$$

A.12.1 Bianchi identities

We have already seen the anti-symmetry property of the Riemann tensor, which in the index notation corresponds to the equation

$$R^\alpha{}_{\beta\gamma\delta} = -R^\alpha{}_{\beta\delta\gamma} \,. \tag{A.12.12}$$

There are a few other identities satisfied by the Riemann tensor, we start with the *first Bianchi identity*. Let $A(X,Y,Z)$ be any expression depending upon three vector fields X,Y,Z which is antisymmetric in X and Y, we set

$$\sum_{[XYZ]} A(X,Y,Z) := A(X,Y,Z) + A(Y,Z,X) + A(Z,X,Y) \,, \tag{A.12.13}$$

thus $\sum_{[XYZ]}$ is a sum over cyclic permutations of the vectors X,Y,Z. Clearly,

$$\sum_{[XYZ]} A(X,Y,Z) = \sum_{[XYZ]} A(Y,Z,X) = \sum_{[XYZ]} A(Z,X,Y) \,. \tag{A.12.14}$$

Suppose, first, that X, Y and Z commute. Using (A.12.14) together with the definition (A.9.16) of the torsion tensor T we calculate

$$\begin{aligned} \sum_{[XYZ]} R(X,Y)Z &= \sum_{[XYZ]} \Big(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \Big) \\ &= \sum_{[XYZ]} \Big(\nabla_X \nabla_Y Z - \nabla_Y \underbrace{(\nabla_Z X + T(X,Z))}_{\text{we have used } [X,Z]=0, \text{ see (A.9.16)}} \Big) \end{aligned}$$

$$= \underbrace{\sum_{[XYZ]} \nabla_X \nabla_Y Z - \sum_{[XYZ]} \nabla_Y \nabla_Z X}_{=0 \text{ (see (A.12.14))}} - \sum_{[XYZ]} \nabla_Y (\underbrace{T(X,Z)}_{=-T(Z,X)})$$

$$= \sum_{[XYZ]} \nabla_X (T(Y,Z)),$$

and in the last step we have again used (A.12.14). This can be somewhat rearranged by using the definition of the covariant derivative of a higher order tensor (compare (A.9.23)) — equivalently, using the Leibniz rule rewritten upside-down:

$$(\nabla_X T)(Y,Z) = \nabla_X (T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z).$$

This leads to

$$\begin{aligned}
\sum_{[XYZ]} \nabla_X (T(Y,Z)) &= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(\nabla_X Y, Z) + T(Y, \underbrace{\nabla_X Z}_{=T(X,Z)+\nabla_Z X}) \Big) \\
&= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) - T(\underbrace{T(X,Z)}_{=-T(Z,X)}, Y) \Big) \\
&\quad \underbrace{+ \sum_{[XYZ]} T(\nabla_X Y, Z) + \sum_{[XYZ]} \underbrace{T(Y, \nabla_Z X)}_{=-T(\nabla_Z X, Y)}}_{=0 \text{ (see (A.12.14))}} \\
&= \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(T(X,Y),Z) \Big).
\end{aligned}$$

Summarizing, we have obtained the first Bianchi identity:

$$\sum_{[XYZ]} R(X,Y)Z = \sum_{[XYZ]} \Big((\nabla_X T)(Y,Z) + T(T(X,Y),Z) \Big), \tag{A.12.15}$$

under the hypothesis that X, Y and Z commute. However, both sides of this equation are tensorial with respect to X, Y and Z, so that they remain correct without the commutation hypothesis.

We are mostly interested in connections with vanishing torsion, in which case (A.12.15) can be rewritten as

$$\boxed{R^\alpha{}_{\beta\gamma\delta} + R^\alpha{}_{\gamma\delta\beta} + R^\alpha{}_{\delta\beta\gamma} = 0}. \tag{A.12.16}$$

Equivalently,

$$R^\alpha{}_{[\beta\gamma\delta]} = 0, \tag{A.12.17}$$

where brackets over indices denote complete antisymmetrisation, e.g.

$$A_{[\alpha\beta]} = \tfrac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha}),$$

$$A_{[\alpha\beta\gamma]} = \tfrac{1}{6}(A_{\alpha\beta\gamma} - A_{\beta\alpha\gamma} + A_{\gamma\alpha\beta} - A_{\gamma\beta\alpha} + A_{\alpha\gamma\beta} - A_{\beta\gamma\alpha}),$$

etc.

Our next goal is the *second Bianchi identity*. We consider four vector fields X, Y, Z and W and we assume again that everybody commutes with everybody else. We calculate

$$\begin{aligned}\sum_{[XYZ]} \nabla_X(R(Y,Z)W) &= \sum_{[XYZ]} \Big(\underbrace{\nabla_X\nabla_Y\nabla_Z W}_{=R(X,Y)\nabla_Z W+\nabla_Y\nabla_X\nabla_Z W} -\nabla_X\nabla_Z\nabla_Y W\Big) \\ &= \sum_{[XYZ]} R(X,Y)\nabla_Z W \\ &\quad \underbrace{+\sum_{[XYZ]} \nabla_Y\nabla_X\nabla_Z W - \sum_{[XYZ]} \nabla_X\nabla_Z\nabla_Y W}_{=0} \\ &\quad . \end{aligned} \tag{A.12.18}$$

Next,

$$\begin{aligned}\sum_{[XYZ]} (\nabla_X R)(Y,Z)W &= \sum_{[XYZ]} \Big(\nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W \\ &\quad -R(Y, \underbrace{\nabla_X Z}_{=\nabla_Z X+T(X,Z)})W - R(Y,Z)\nabla_X W\Big) \\ &= \sum_{[XYZ]} \nabla_X(R(Y,Z)W) \\ &\quad \underbrace{- \sum_{[XYZ]} R(\nabla_X Y,Z)W - \sum_{[XYZ]} \underbrace{R(Y,\nabla_Z X)W}_{=-R(\nabla_Z X,Y)W}}_{=0} \\ &\quad - \sum_{[XYZ]} \Big(R(Y,T(X,Z))W + R(Y,Z)\nabla_X W\Big) \\ &= \sum_{[XYZ]} \Big(\nabla_X(R(Y,Z)W) - R(T(X,Y),Z)W - R(Y,Z)\nabla_X W\Big)\,. \end{aligned}$$

It follows now from (A.12.18) that the first term cancels out the third one, leading to

$$\sum_{[XYZ]} (\nabla_X R)(Y,Z)W = - \sum_{[XYZ]} R(T(X,Y),Z)W\,, \tag{A.12.19}$$

which is the desired second Bianchi identity for commuting vector fields. As before, because both sides are multi-linear with respect to addition and multiplication by functions, the result remains valid for arbitrary vector fields.

For torsionless connections the components equivalent of (A.12.19) reads

$$\boxed{R^\alpha{}_{\mu\beta\gamma;\delta} + R^\alpha{}_{\mu\gamma\delta;\beta} + R^\alpha{}_{\mu\delta\beta;\gamma} = 0}\,. \tag{A.12.20}$$

INCIDENTALLY: In the case of the Levi-Civita connection, the proof of the second Bianchi identity is simplest in coordinates in which the derivatives of the metric vanish at p: Indeed, a calculation very similar to the one leading to (A.12.25) below gives

$$\nabla_\delta R_{\alpha\mu\beta\gamma}(0) = \partial_\delta R_{\alpha\mu\beta\gamma}(0) = \frac{1}{2}\Big\{\partial_\delta\partial_\beta\partial_\mu g_{\alpha\gamma} - \partial_\delta\partial_\beta\partial_\alpha g_{\mu\gamma} - \partial_\delta\partial_\gamma\partial_\mu g_{\alpha\beta} + \partial_\delta\partial_\gamma\partial_\alpha g_{\mu\beta}\Big\}(0)\,. \quad (A.12.21)$$

and (A.12.20) follows by inspection □

A.12.2 Pair interchange symmetry

There is one more identity satisfied by the curvature tensor which is specific to the curvature tensor associated with the Levi-Civita connection, namely

$$g(X, R(Y,Z)W) = g(Y, R(X,W)Z)\,. \quad (A.12.22)$$

If one sets

$$\boxed{R_{abcd} := g_{ae}R^e{}_{bcd}}\,, \quad (A.12.23)$$

then (A.12.22) is equivalent to

$$\boxed{R_{abcd} = R_{cdab}}\,. \quad (A.12.24)$$

We will present two proofs of (A.12.22). The first is direct, but not very elegant. The second is prettier, but less insightful.

For the ugly proof, we suppose that the metric is twice-differentiable. By point 2. of Proposition A.11.1, in a neighborhood of any point $p \in M$ there exists a coordinate system in which the connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ vanish at p. Equation (A.12.4) evaluated at p therefore reads

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &= \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma} \\ &= \frac{1}{2}\Big\{g^{\alpha\sigma}\partial_\gamma(\partial_\delta g_{\sigma\beta} + \partial_\beta g_{\sigma\delta} - \partial_\sigma g_{\beta\delta}) \\ &\quad -g^{\alpha\sigma}\partial_\delta(\partial_\gamma g_{\sigma\beta} + \partial_\beta g_{\sigma\gamma} - \partial_\sigma g_{\beta\gamma})\Big\} \\ &= \frac{1}{2}g^{\alpha\sigma}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}\,. \end{aligned}$$

Equivalently,

$$R_{\sigma\beta\gamma\delta}(0) = \frac{1}{2}\Big\{\partial_\gamma\partial_\beta g_{\sigma\delta} - \partial_\gamma\partial_\sigma g_{\beta\delta} - \partial_\delta\partial_\beta g_{\sigma\gamma} + \partial_\delta\partial_\sigma g_{\beta\gamma}\Big\}(0)\,. \quad (A.12.25)$$

This last expression is obviously symmetric under the exchange of $\sigma\beta$ with $\gamma\delta$, leading to (A.12.24).

The above calculation traces back the pair-interchange symmetry to the definition of the Levi-Civita connection in terms of the metric tensor. As already mentioned, there exists a more elegant proof, where the origin of the symmetry is perhaps somewhat less apparent, which proceeds as follows: We start by noting that

$$0 = \nabla_a\nabla_b g_{cd} - \nabla_b\nabla_a g_{cd} = -R^e{}_{cab}g_{ed} - R^e{}_{dab}g_{ce}\,, \quad (A.12.26)$$

leading to anti-symmetry in the first two indices:

$$R_{abcd} = -R_{bacd}\,.$$

Next, using the cyclic symmetry for a torsion-free connection, we have

$$\begin{aligned}
R_{abcd} + R_{cabd} + R_{bcad} &= 0\,,\\
R_{bcda} + R_{dbca} + R_{cdba} &= 0\,,\\
R_{cdab} + R_{acdb} + R_{dacb} &= 0\,,\\
R_{dabc} + R_{bdac} + R_{abdc} &= 0\,.
\end{aligned} \tag{A.12.27}$$

The desired equation (A.12.24) follows now by adding the first two and subtracting the last two equations, using (A.12.26).

Remark A.12.8 In dimension two, the pair-interchange symmetry and the anti-symmetry in the last two indices immediately imply that the only non-zero components of the Riemann tensor are

$$R_{1212} = -R_{2112} = R_{2121} = -R_{2112}\,.$$

This is equivalent to the formula

$$R_{abcd} = \frac{R}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})\,,$$

as easily checked at a point p in a coordinate system where g_{ab} is diagonal at p.

In dimension three, a similar argument gives

$$R_{abcd} = (P_{ac}g_{bd} - P_{ad}g_{bc} + g_{ac}P_{bd} - g_{ad}P_{bc})\,, \tag{A.12.28}$$

where

$$P_{ab} := R_{ab} - \frac{R}{2}g_{ab}\,.$$

□

Incidentally: It is natural to enquire about the number of independent components of a tensor with the symmetries of a metric Riemann tensor in dimension n, the calculation proceeds as follows: as R_{abcd} is symmetric under the exchange of ab with cd, and anti-symmetric in each of these pairs, we can view it as a symmetric map from the space of anti-symmetric tensor with two indices. Now, the space of anti-symmetric tensors is $N = n(n-1)/2$ dimensional, while the space of symmetric maps in dimension N is $N(N+1)/2$ dimensional, so we obtain at most

$$\frac{n(n-1)(n^2-n+2)}{8}$$

free parameters. However, we need to take into account the cyclic identity:

$$R_{dabc} + R_{dbca} + R_{dcab} = 0\,. \tag{A.12.29}$$

If $a = b$ this reads

$$R_{daac} + R_{daca} + R_{dcaa} = 0\,,$$

which has already been accounted for. Similarly if $a = d$ we obtain

$$R_{abca} + R_{bcaa} + R_{caba} = 0\,,$$

which holds in view of the previous identities. We conclude that the only new identities which could possibly arise are those where $abcd$ are all distinct. (Another way to see this is to note the identity

$$R_{a[bcd]} = R_{[abcd]}\,, \tag{A.12.30}$$

which holds for any tensor satisfying

$$R_{abcd} = R_{[ab]cd} = R_{ab[cd]} = R_{cdab}\,, \tag{A.12.31}$$

and which can be proved by writing explicitly all the terms in $R_{[abcd]}$; this is the same as adding the left-hand sides of the first and third equations in (A.12.27), and removing those of the second and fourth.)

Clearly no identity involving four distinct components of the Riemann tensor can be obtained using (A.12.31), so for each distinct set of four indices the Bianchi identity provides a constraint which is independent of (A.12.31). In dimension four (A.12.29) provides thus four candidate equations for another constraint, labeled by d, but it is easily checked that they all coincide either directly, or using (A.12.30). This leads to 20 free parameters at each space point. (Strictly speaking, to prove this one would still need to show that there are no further algebraic identities satisfied by the Riemann tensor, which is indeed the case.

Note that (A.12.30) shows that in dimension $n \geq 4$ the Bianchi identity introduces $\begin{pmatrix} n \\ 4 \end{pmatrix}$ new constraints, leading to

$$\frac{n(n-1)(n^2-n+2)}{8} - \frac{n(n-1)(n-2)(n-3)}{12} = \frac{n^2(n^2-1)}{12} \tag{A.12.32}$$

independent components at each point. □

A.12.3 Summmary for the Levi-Civita connection

Here is a full list of algebraic symmetries of the curvature tensor of the Levi-Civita connection:

1. directly from the definition, we obtain

$$R^\delta{}_{\gamma\alpha\beta} = -R^\delta{}_{\gamma\beta\alpha}\,; \tag{A.12.33}$$

2. the next symmetry, known as the *first Bianchi identity*, is less obvious:

$$R^\delta{}_{\gamma\alpha\beta} + R^\delta{}_{\alpha\beta\gamma} + R^\delta{}_{\beta\gamma\alpha} = 0 \quad \Longleftrightarrow \quad R^\delta{}_{[\gamma\alpha\beta]} = 0\,; \tag{A.12.34}$$

3. and finally we have the pair-interchange symmetry:

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}\,. \tag{A.12.35}$$

Here, of course, $R_{\gamma\delta\alpha\beta} = g_{\gamma\sigma}R^\sigma{}_{\delta\alpha\beta}$.

It is not obvious, but true, that the above exhaust the list of all independent algebraic identities satisfied by $R_{\alpha\beta\gamma\delta}$.

As a consequence of (A.12.33) and (A.12.35) we find

$$R_{\alpha\beta\delta\gamma} = R_{\delta\gamma\alpha\beta} = -R_{\delta\gamma\beta\alpha} = -R_{\beta\alpha\gamma\delta}\,,$$

and so the Riemann tensor is also anti-symmetric in its first two indices:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}\,. \tag{A.12.36}$$

The *Ricci tensor* is defined as

$$R_{\alpha\beta} := R^{\sigma}{}_{\alpha\sigma\beta}\,.$$

The pair-interchange symmetry implies that the Ricci tensor is symmetric:

$$R_{\alpha\beta} = g^{\sigma\rho}R_{\sigma\alpha\rho\beta} = g^{\sigma\rho}R_{\rho\beta\sigma\alpha} = R_{\beta\alpha}\,.$$

Finally we have the differential *second Bianchi identity*:

$$\nabla_\alpha R^{\sigma}{}_{\delta\beta\gamma} + \nabla_\beta R^{\sigma}{}_{\delta\gamma\alpha} + \nabla_\gamma R^{\sigma}{}_{\delta\alpha\beta} = 0 \quad\Longleftrightarrow\quad \nabla_{[\alpha}R_{\beta\gamma]\mu\nu} = 0\,. \tag{A.12.37}$$

A.12.4 Curvature of product metrics

Let (M,g) and (N,h) be two pseudo-Riemannian manifolds, on the product manifold $M\times N$ we define a metric $g\oplus h$ as follows: Every element of $T(M\times N)$ can be uniquely written as $X\oplus Y$ for some $X\in TM$ and $Y\in TN$. We set

$$(g\oplus h)(X\oplus Y,\hat X\oplus\hat Y) = g(X,\hat X) + h(Y,\hat Y)\,.$$

Let ∇ be the Levi-Civita connection associated with g, D that associated with h, and $\mathscr{D}$ the one associated with $g\oplus h$. To understand the structure of ∇, we note that sections of $T(M\times N)$ are linear combinations, with coefficients in $C^\infty(M\times N)$, of elements of the form $X\oplus Y$, where $X\in\Gamma(TM)$ and $Y\in\Gamma(TN)$. (Thus, X does not depend upon $q\in N$ and Y does not depend upon $p\in M$.) We claim that for such fields $X\oplus Y$ and $W\oplus Z$ we have

$$\mathscr{D}_{X\oplus Y}(W\oplus Z) = \nabla_X W\oplus D_Y Z\,. \tag{A.12.38}$$

(If true, (A.12.38) together with the Leibniz rule characterises $\mathscr{D}$ uniquely.) To verify (A.12.38), we check first that $\mathscr{D}$ has no torsion:

$$\begin{aligned}
\mathscr{D}_{X\oplus Y}(W\oplus Z) - \mathscr{D}_{W\oplus Z}(X\oplus Y) &= \nabla_X W\oplus D_Y Z - \nabla_W X\oplus D_Z Y\\
&= (\nabla_X W - \nabla_W X)\oplus(D_Y Z - D_Z Y)\\
&= [X,W]\oplus[Y,Z]\\
&= [X\oplus Y, W\oplus Z]\,.
\end{aligned}$$

(In the last step we have used $[X\oplus 0, 0\oplus Z] = [0\oplus Y, W\oplus 0] = 0$.) Next, we check metric compatibility:

$$\begin{aligned}
&X\oplus Y\Big((g\oplus h)(W\oplus Z,\hat W\oplus\hat Z)\Big)\\
&= X\oplus Y\Big(g(W,\hat W) + h(Z,\hat Z)\Big)\\
&= \underbrace{X\Big(g(W,\hat W)\Big)}_{g(\nabla_X W,\hat W)+g(W,\nabla_X\hat W)} + \underbrace{Y\Big(h(Z,\hat Z)\Big)}_{h(D_Y Z,\hat Z)+h(Z,D_Y\hat Z)}\\
&= \underbrace{g(\nabla_X W,\hat W) + h(D_Y Z,\hat Z)}_{(g\oplus h)(\nabla_X W\oplus D_Y Z,\hat W\oplus\hat Z)} + \underbrace{g(W,\nabla_X\hat W) + h(Z,D_Y\hat Z)}_{(g\oplus h)(W\oplus Z,\nabla_X\hat W\oplus D_Y\hat Z)}\\
&= (g\oplus h)(\mathscr{D}_{X\oplus Y}W\oplus Z,\hat W\oplus\hat Z) + (g\oplus h)(W\oplus Z,\mathscr{D}_{X\oplus Y}\hat W\oplus\hat Z)\,.
\end{aligned}$$

Uniqueness of Levi-Civita connections proves (A.12.38).

Let Riem(k) denote the Riemann tensor of the metric k. It should be clear from (A.12.38) that the Riemann tensor of $g \oplus h$ has a sum structure,

$$\mathrm{Riem}(g \oplus h) = \mathrm{Riem}(g) \oplus \mathrm{Riem}(h)\,. \tag{A.12.39}$$

More precisely,

$$\mathrm{Riem}(g \oplus h)(X \oplus Y, \hat{X} \oplus \hat{Y})W \oplus Z = \mathrm{Riem}(g)(X, \hat{X})W \oplus \mathrm{Riem}(h)(Y, \hat{Y})Z\,. \tag{A.12.40}$$

This implies

$$\mathrm{Ric}(g \oplus h) = \mathrm{Ric}(g) \oplus \mathrm{Ric}(h)\,, \tag{A.12.41}$$

in the sense that

$$\mathrm{Ric}(g \oplus h)(X \oplus Y, \hat{X} \oplus \hat{Y}) = \mathrm{Ric}(g)(X, \hat{X}) \oplus \mathrm{Ric}(h)(Y, \hat{Y})\,, \tag{A.12.42}$$

and

$$\mathrm{tr}_{g\oplus h}\mathrm{Ric}(g \oplus h) \quad = \quad \mathrm{tr}_g\mathrm{Ric}(g) + \mathrm{tr}_h\mathrm{Ric}(h)\,. \tag{A.12.43}$$

A.12.5 An identity for the Riemann tensor

We write $\delta^{\alpha\beta}_{\gamma\delta}$ for $\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta} \equiv \frac{1}{2}(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} - \delta^{\beta}_{\gamma}\delta^{\alpha}_{\delta})$, etc.

For completeness we prove the following identity satisfied by the Riemann tensor, which is valid in any dimension, is clear in dimensions two and three, implies the double-dual identity for the Weyl tensor in dimension four, and is probably well known in higher dimensions as well:

$$\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\gamma\delta} = \frac{1}{3!}\left(R^{\alpha\beta}{}_{\mu\nu} + \delta^{\alpha\beta}_{\mu\nu}R - 4\delta^{[\alpha}_{[\mu}R^{\beta]}{}_{\nu]}\right)\,. \tag{A.12.44}$$

The above holds for any tensor field satisfying

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\beta\alpha\delta\gamma}\,. \tag{A.12.45}$$

To prove (A.12.44) one can calculate as follows:

$$\begin{aligned}
4!\,\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\gamma\delta} \quad = \quad & 2[\delta^{\alpha}_{\mu}\left(\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\gamma}_{\nu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\nu}\delta^{\delta}_{\rho}\right) \\
& -\delta^{\alpha}_{\nu}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\gamma}_{\mu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\rho}\right) \\
& +\delta^{\alpha}_{\rho}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\delta^{\delta}_{\sigma} - \delta^{\beta}_{\nu}\delta^{\gamma}_{\mu}\delta^{\delta}_{\sigma} + \delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right) \\
& -\delta^{\alpha}_{\sigma}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}\delta^{\delta}_{\rho} - \delta^{\beta}_{\nu}\delta^{\gamma}_{\mu}\delta^{\delta}_{\rho} + \delta^{\beta}_{\rho}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right)]R^{\rho\sigma}{}_{\gamma\delta} \\
= \quad & 2\left(2\delta^{\alpha\beta}_{\mu\nu}\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - 4\delta^{\alpha\gamma}_{\mu\nu}\delta^{\beta\delta}_{\rho\sigma} + 4\delta^{\beta\gamma}_{\mu\nu}\delta^{\alpha\delta}_{\rho\sigma} + 2\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}\delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\right)R^{\rho\sigma}{}_{\gamma\delta} \\
= \quad & 4\left(\delta^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}{}_{\gamma\delta} - 2\delta^{\alpha\gamma}_{\mu\nu}R^{\beta\sigma}{}_{\gamma\sigma} + 2\delta^{\beta\gamma}_{\mu\nu}R^{\alpha\sigma}{}_{\gamma\sigma} + R^{\alpha\beta}{}_{\mu\nu}\right) \\
= \quad & 4\left(R^{\alpha\beta}{}_{\mu\nu} + \delta^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}{}_{\gamma\delta} - 4\delta^{[\alpha}_{[\mu}R^{\beta]\gamma}{}_{\nu]\gamma}\right)\,.
\end{aligned} \tag{A.12.46}$$

If the sums are over all indices we obtain (A.12.44). The reader is warned, however, that in some of our calculations the sums will be only over a subset of all possible indices, in which case the last equation remains valid but the last two terms in (A.12.46) *cannot* be replaced by the Ricci scalar and the Ricci tensor.

Let us show that the *double-dual identity* for the Weyl tensor does indeed follow from (A.12.44). For this, note that in spacetime dimension four we have

$$4!\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma}\,, \tag{A.12.47}$$

since both sides are completely anti-symmetric in the upper and lower indices, and coincide when both pairs equal 0123. Hence, since the Weyl tensor $W^{\rho\sigma}{}_{\gamma\delta}$ has all the required symmetries and vanishing traces, we find

$$4W^{\alpha\beta}{}_{\mu\nu} \underbrace{=}_{\text{by (A.12.46)}} 4!\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}W^{\rho\sigma}{}_{\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}{}_{\gamma\delta}\,. \tag{A.12.48}$$

This is equivalent to the desired identity

$$\epsilon_{\mu\nu\rho\sigma}W^{\rho\sigma}{}_{\gamma\delta} = W^{\alpha\beta}{}_{\mu\nu}\,\epsilon_{\alpha\beta\gamma\delta}\,. \tag{A.12.49}$$

A.13 Geodesics

An *affinely parameterised geodesic* γ is a maximally extended solution of the equation

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

(compare (A.10.11).) It is a fundamental postulate of general relativity that physical observers move on timelike geodesics. This motivates the following definition: an *observer* is a maximally extended future directed timelike geodesics.

INCIDENTALLY: It is sometimes convenient to consider geodesics which are not necessarily affinely parameterised. Those are solutions of

$$\nabla_{\frac{d\gamma}{d\lambda}}\frac{d\gamma}{d\lambda} = \chi\frac{d\gamma}{d\lambda}\,. \tag{A.13.1}$$

Indeed, let us show that a change of parameter obtained by solving the equation

$$\frac{d^2\lambda}{ds^2} + \chi\left(\frac{d\lambda}{ds}\right)^2 = 0 \tag{A.13.2}$$

brings (A.13.2) to the form (A.13.2): under a change of parameter $\lambda = \lambda(s)$ we have

$$\frac{d\gamma^\mu}{ds} = \frac{d\lambda}{ds}\frac{d\gamma^\mu}{d\lambda}\,,$$

and

$$\begin{aligned}\frac{D}{ds}\frac{d\gamma^\nu}{ds} &= \frac{D}{ds}(\frac{d\lambda}{ds}\frac{d\gamma^\nu}{d\lambda})\\ &= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + \frac{d\lambda}{ds}\frac{D}{ds}\frac{d\gamma^\nu}{d\lambda}\\ &= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + (\frac{d\lambda}{ds})^2\frac{D}{d\lambda}\frac{d\gamma^\nu}{d\lambda}\\ &= \frac{d^2\lambda}{ds^2}\frac{d\gamma^\nu}{d\lambda} + (\frac{d\lambda}{ds})^2\chi\frac{d\gamma^\nu}{d\lambda}\,,\end{aligned}$$

and the choice indicated above gives zero, as desired. □

Let f be a smooth function and let $\lambda \mapsto \gamma(\lambda)$ be any integral curve of ∇f; by definition, this means that $d\gamma^\mu/d\lambda = \nabla^\mu f$. The following provides a convenient tool for finding geodesics:

PROPOSITION A.13.2 (Integral curves of gradients) *Let f be a function satisfying*

$$g(\nabla f, \nabla f) = \psi(f)\,,$$

for some function ψ. Then the integral curves of ∇f are geodesics, affinely parameterised if $\psi' = 0$.

PROOF: We have

$$\dot\gamma^\alpha \nabla_\alpha \dot\gamma^\beta = \nabla^\alpha f \nabla_\alpha \nabla^\beta f = \nabla^\alpha f \nabla^\beta \nabla_\alpha f = \frac{1}{2}\nabla^\beta(\nabla^\alpha f \nabla_\alpha f) = \frac{1}{2}\nabla^\beta \psi(f) = \frac{1}{2}\psi' \nabla^\beta f\,. \tag{A.13.3}$$

Let λ the natural parameter on the integral curves of ∇f,

$$\frac{d\gamma^\mu}{d\lambda} = \nabla^\mu f\,,$$

then (A.13.3) can be rewritten as

$$\frac{D}{d\lambda}\frac{d\gamma^\mu}{d\lambda} = \frac{1}{2}\psi' \frac{d\gamma^\mu}{d\lambda}\,.$$

□

A significant special case is that of a coordinate function $f = x^i$. Then

$$g(\nabla f, \nabla f) = g(\nabla x^i, \nabla x^i) = g^{ii} \text{ (no summation)}\,.$$

For example, in Minkowski spacetime, all $g^{\mu\nu}$'s are constant, which shows that the integral curves of the gradient of any coordinate, and hence also of any linear combination of coordinates, are affinely parameterized geodesics. An other example is provided by the coordinate r in Schwarzschild spacetime, where $g^{rr} = 1 - 2m/r$; this is indeed a function of r, so the integral curves of $\nabla r = (1 - 2m/r)\partial_r$ are (non-affinely parameterized) geodesics.

Similarly one shows:

PROPOSITION A.13.3 *Suppose that $d(g(X,X)) = 0$ along an orbit γ of a Killing vector field X. Then γ is a geodesic.*

EXERCICE A.13.4 Consider the Killing vector field $X = \partial_t + \Omega\partial_\varphi$, where Ω is a constant, in the Schwarzschild spacetime. Find all geodesic orbits of X by studying the equation $d(g(X,X)) = 0$. □

A.14 Geodesic deviation (Jacobi equation)

Suppose that we have a one parameter family of geodesics

$$\gamma(s,\lambda) \text{ (in local coordinates, } (\gamma^\alpha(s,\lambda))),$$

where s is an affine parameter along the geodesic, and λ is a parameter which labels the geodesics. Set

$$Z(s,\lambda) := \frac{\partial\gamma(s,\lambda)}{\partial\lambda} \equiv \frac{\partial\gamma^\alpha(s,\lambda)}{\partial\lambda}\partial_\alpha\,;$$

for each λ this defines a vector field Z along $\gamma(s,\lambda)$, which measures how nearby geodesics deviate from each other, since, to first order, using a Taylor expansion,

$$\gamma^\alpha(s,\lambda) = \gamma^\alpha(s,\lambda_0) + Z^\alpha(\lambda-\lambda_0) + O((\lambda-\lambda_0)^2)\,.$$

To measure how a vector field W changes along $s \mapsto \gamma(s,\lambda)$, one introduces the differential operator D/ds, defined as

$$\frac{DW^\mu}{ds} := \frac{\partial(W^\mu\circ\gamma)}{\partial s} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha \tag{A.14.1}$$
$$= \dot\gamma^\beta\frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta W^\alpha \tag{A.14.2}$$
$$= \dot\gamma^\beta\nabla_\beta W^\mu\,. \tag{A.14.3}$$

(It would perhaps be more logical to write $\frac{DW^\mu}{\partial s}$ in the current context, but this is rarely done. Another notation for $\frac{D}{ds}$ often used in the mathematical literature is $\gamma_*\partial_s$.) The last two lines only make sense if W is defined in a whole neighbourhood of γ, but for the first it suffices that $W(s)$ be defined along $s \mapsto \gamma(s,\lambda)$. (One possible way of making sense of the last two lines is to extend, whenever possible, W^μ to any smooth vector field defined in a neighorhood of $\gamma^\mu(s,\lambda)$, and note that the result is independent of the particular choice of extension because the equation involves only derivatives tangential to $s\mapsto\gamma^\mu(s,\lambda)$.)

Analogously one sets

$$\frac{DW^\mu}{d\lambda} := \frac{\partial(W^\mu\circ\gamma)}{\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha \tag{A.14.4}$$
$$= \partial_\lambda\gamma^\beta\frac{\partial W^\mu}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}\partial_\lambda\gamma^\beta W^\alpha \tag{A.14.5}$$
$$= Z^\beta\nabla_\beta W^\mu\,. \tag{A.14.6}$$

Note that since $s\to\gamma(s,\lambda)$ is a geodesic we have from (A.14.1) and (A.14.3)

$$\frac{D^2\gamma^\mu}{ds^2} := \frac{D\dot\gamma^\mu}{ds} = \frac{\partial^2\gamma^\mu}{\partial s^2} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\dot\gamma^\alpha = 0\,. \tag{A.14.7}$$

(This is sometimes written as $\dot\gamma^\alpha\nabla_\alpha\dot\gamma^\mu = 0$, which is again an abuse of notation since typically we will only know $\dot\gamma^\mu$ as a function of s, and so there is no such thing as $\nabla_\alpha\dot\gamma^\mu$.) Furthermore,

$$\frac{DZ^\mu}{ds} \underbrace{=}_{\text{(A.14.1)}} \frac{\partial^2\gamma^\mu}{\partial s\partial\lambda} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta\partial_\lambda\gamma^\alpha \underbrace{=}_{\text{(A.14.4)}} \frac{D\dot\gamma^\mu}{d\lambda}\,, \tag{A.14.8}$$

(The abuse-of-notation derivation of the same formula proceeds as:

$$\begin{aligned}\nabla_{\dot\gamma} Z^\mu &= \dot\gamma^\nu \nabla_\nu Z^\mu = \dot\gamma^\nu \nabla_\nu \partial_\lambda \gamma^\mu \underbrace{=}_{\text{(A.14.3)}} \frac{\partial^2 \gamma^\mu}{\partial s \partial \lambda} + \Gamma^\mu{}_{\alpha\beta}\dot\gamma^\beta \partial_\lambda \gamma^\alpha \underbrace{=}_{\text{(A.14.6)}} Z^\beta \nabla_\beta \dot\gamma^\mu \\ &= \nabla_Z \dot\gamma^\mu\,, \end{aligned} \tag{A.14.9}$$

which can then be written as

$$\nabla_{\dot\gamma} Z = \nabla_Z \dot\gamma\,.) \tag{A.14.10}$$

We have the following identity for any vector field W defined along $\gamma^\mu(s,\lambda)$, which can be proved by e.g. repeating the calculation leading to (A.12.9):

$$\frac{D}{ds}\frac{D}{d\lambda}W^\mu - \frac{D}{d\lambda}\frac{D}{ds}W^\mu = R^\mu{}_{\delta\alpha\beta}\dot\gamma^\alpha Z^\beta W^\delta\,. \tag{A.14.11}$$

If $W^\mu = \dot\gamma^\mu$ the second term at the left-hand side of (A.14.11) vanishes, and from $\frac{D}{d\lambda}\dot\gamma = \frac{D}{ds}Z$ we obtain

$$\frac{D^2 Z^\mu}{ds^2}(s) = R^\mu{}_{\sigma\alpha\beta}\dot\gamma^\alpha Z^\beta \dot\gamma^\sigma\,. \tag{A.14.12}$$

This is an equation known as the *Jacobi equation*, or as the *geodesic deviation equation*; in index-free notation:

$$\boxed{\frac{D^2 Z}{ds^2} = R(\dot\gamma, Z)\dot\gamma}\,. \tag{A.14.13}$$

Solutions of (A.14.13) are called *Jacobi fields* along γ.

INCIDENTALLY: The advantage of the abuse-of-notation equations above is that, instead of adapting the calculation, one can directly invoke the result of Proposition A.12.3to obtain (A.14.11):

$$\begin{aligned}\frac{D^2 Z^\mu}{ds^2}(s) &= \dot\gamma^\alpha \nabla_\alpha(\dot\gamma^\beta \nabla_\beta Z^\mu) \\ &= \dot\gamma^\alpha \nabla_\alpha (Z^\beta \nabla_\beta \dot\gamma^\mu) \\ &= (\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha \nabla_\alpha \nabla_\beta \dot\gamma^\mu \\ &= (\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha (\nabla_\alpha \nabla_\beta - \nabla_\beta\nabla_\alpha)\dot\gamma^\mu + Z^\beta\dot\gamma^\alpha \nabla_\beta\nabla_\alpha \dot\gamma^\mu \\ &= (\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha R^\mu{}_{\sigma\alpha\beta}\dot\gamma^\sigma + Z^\beta \dot\gamma^\alpha \nabla_\beta \nabla_\alpha \dot\gamma^\mu \\ &= (\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu + Z^\beta \dot\gamma^\alpha R^\mu{}_{\sigma\alpha\beta}\dot\gamma^\sigma \\ &\quad + Z^\beta \nabla_\beta(\underbrace{\dot\gamma^\alpha \nabla_\alpha \dot\gamma^\mu}_{0}) - (Z^\beta \nabla_\beta \dot\gamma^\alpha)\nabla_\alpha \dot\gamma^\mu\,. \end{aligned} \tag{A.14.14}$$

A renaming of indices in the first and the last term gives

$$(\dot\gamma^\alpha \nabla_\alpha Z^\beta)\nabla_\beta \dot\gamma^\mu - (Z^\beta \nabla_\beta \dot\gamma^\alpha)\nabla_\alpha \dot\gamma^\mu = (\dot\gamma^\alpha \nabla_\alpha Z^\beta - Z^\alpha \nabla_\alpha \dot\gamma^\beta)\nabla_\beta \dot\gamma^\mu\,,$$

which is zero by (A.14.10). This leads again to (A.14.12). □

A.15 Exterior algebra

A preferred class of tensors is provided by those that are totally antisymmetric in all indices. Such k-covariant tensors are called *k-forms*. They are of special interest because they can naturally be used for integration. Furthermore, on such tensors one can introduce a differentiation operation, called *exterior derivative*, which does not require a connection.

By definition, functions are *zero-forms*, and covectors are *one-forms*.

Let α_i, $i = 1, \ldots, k$, be a collection of one-forms, the *exterior product* of the α_i's is a k-form defined as

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(X_1, \ldots, X_k) = \det(\alpha_i(X_j)) , \qquad \text{(A.15.1)}$$

where $\det(\alpha_i(X_j))$ denotes the determinant of the matrix obtained by applying all the α_i's to all the vectors X_j. For example

$$(dx^a \wedge dx^b)(X, Y) = X^a Y^b - Y^a X^b .$$

Note that

$$dx^a \wedge dx^b = dx^a \otimes dx^b - dx^b \otimes dx^a ,$$

which is twice the antisymmetrisation $dx^{[a} \otimes dx^{b]}$.

Quite generally, if α is a totally anti-symmetric k-covariant tensor with coordinate coefficients $\alpha_{a_1 \ldots a_k}$, then

$$\begin{aligned} \alpha &= \alpha_{a_1 \ldots a_k} dx^{a_1} \otimes \cdots \otimes dx^{a_k} \\ &= \alpha_{a_1 \ldots a_k} dx^{[a_1} \otimes \cdots \otimes dx^{a_k]} \\ &= \frac{1}{k!} \alpha_{a_1 \ldots a_k} dx^{a_1} \wedge \cdots \wedge dx^{a_k} \\ &= \sum_{a_1 < \cdots < a_k} \alpha_{a_1 \ldots a_k} dx^{a_1} \wedge \cdots \wedge dx^{a_k} . \end{aligned} \qquad \text{(A.15.2)}$$

The middle formulae exhibits the factorial coefficients needed to go from tensor components to the components in the $dx^{a_1} \wedge \cdots \wedge dx^{a_k}$ basis.

Equation (A.15.2) makes it clear that in dimension n for any non-trivial k-form we have $k \leq n$. It also shows that the dimension of the space of k-forms, with $0 \leq k \leq n$, equals

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} .$$

A differential form is defined as a linear combination of k-forms, with k possibly taking different values for different summands.

Let Y be a vector and α a k-form. The *contraction* $Y \rfloor \alpha$, also called the *interior product* of Y and α, is a $(k-1)$-form defined as

$$(Y \rfloor \alpha)(X_1, \ldots, X_{k-1}) := \alpha(Y, X_1, \ldots, X_{k-1}) . \qquad \text{(A.15.3)}$$

The operation $Y \rfloor$ is often denoted by i_Y.

Let α be a k-form and β an ℓ-form, the exterior product $\alpha\wedge\beta$ of α and β, also called *wedge product*, is defined using bilinearity:

$$\begin{aligned}\alpha\wedge\beta \equiv & \\ & \left(\sum_{a_1<\cdots<a_k}\alpha_{a_1\ldots a_k}dx^{a_1}\wedge\cdots\wedge dx^{a_k}\right)\wedge\left(\sum_{b_1<\cdots<b_\ell}\beta_{b_1\ldots b_\ell}dx^{b_1}\wedge\cdots\wedge dx^{b_\ell}\right)\\ & := \sum_{a_1<\cdots<a_k\,,\ b_1<\cdots<b_\ell}\alpha_{a_1\ldots a_k}\beta_{b_1\ldots b_\ell}\times \\ & \qquad\qquad dx^{a_1}\wedge\cdots\wedge dx^{a_k}\wedge dx^{b_1}\wedge\cdots\wedge dx^{b_\ell}\,. \end{aligned} \tag{A.15.4}$$

The product so-defined is associative:

$$\alpha\wedge(\beta\wedge\gamma)=(\alpha\wedge\beta)\wedge\gamma =: \alpha\wedge\beta\wedge\gamma\,. \tag{A.15.5}$$

INCIDENTALLY: In order to establish (A.15.5), we start by rewriting the definition of the wedge product of a k-form α and l-form β as

$$(\alpha\wedge\beta)\,(X_1\ldots,X_{k+l}) := \frac{1}{k!}\frac{1}{l!}\sum_{\pi\in S_{k+l}}\mathrm{sgn}(\pi)\,(\alpha\otimes\beta)\,(X_{\pi(1)},\ldots,X_{\pi(k+l)})\,, \tag{A.15.6}$$

where $X_i\in\Gamma(TM)$ for $i=1,\ldots,k+l$.

Let S_p denote the group of permutations of p elements and let $\Omega^\ell(M)$ denote the space of ℓ-forms. For $\alpha\in\Omega^k(M)$, $\beta\in\Omega^l(M)$ and $\gamma\in\Omega^m(M)$ we have

$$\begin{aligned}&((\alpha\wedge\beta)\wedge\gamma)(X_1,\ldots,X_{k+l+m})\\ &= \frac{1}{(k+l)!m!}\sum_{\pi\in S_{k+l+m}}\mathrm{sgn}(\pi)\,((\alpha\wedge\beta)\otimes\gamma)\,(X_{\pi(1)},\ldots,X_{\pi(k+l+m)})\\ &= \frac{1}{(k+l)!m!}\sum_{\pi\in S_{k+l+m}}\mathrm{sgn}(\pi)(\alpha\wedge\beta)(X_{\pi(1)},\ldots,X_{\pi(k+l)})\cdot\gamma(X_{\pi(k+l+1)},\ldots,X_{\pi(k+l+m)})\\ &= \frac{1}{(k+l)!k!l!m!}\sum_{\pi\in S_{k+l+m}}\mathrm{sgn}(\pi)\sum_{\pi'\in S_{k+l}}\mathrm{sgn}(\pi')(\alpha\otimes\beta)(X_{\pi'(\pi(1))},\ldots,X_{\pi'(\pi(k+l))})\cdot\\ &\qquad\qquad \gamma(X_{\pi(k+l+1)},\ldots,X_{\pi(k+l+m)})\,.\end{aligned} \tag{A.15.7}$$

We introduce a new permutation $\pi''\in S_{k+l+m}$ such that

$$\pi''(\pi(i)) = \begin{cases}\pi'(\pi(i)) & \text{for}\quad 1\le i\le k+l\,,\\ \pi(i) & \text{for}\quad i>k+l\,,\end{cases}$$

which implies $\mathrm{sgn}(\pi'')=\mathrm{sgn}(\pi')$. One then obtains

$$\begin{aligned}&((\alpha\wedge\beta)\wedge\gamma)(X_1,\ldots,X_{k+l+m})\\ &= \frac{1}{(k+l)!k!l!m!}\sum_{\pi'\in S_{k+l}}\mathrm{sgn}(\pi')\sum_{\pi\in S_{k+l+m}}\mathrm{sgn}(\pi)((\alpha\otimes\beta)\otimes\gamma)(X_{\pi''(\pi(1))},\ldots,X_{\pi''(\pi(k+l+m))})\,.\end{aligned}$$

Set $\sigma:=\pi''\circ\pi$. Then $\mathrm{sgn}(\sigma)=\mathrm{sgn}(\pi'')\mathrm{sgn}(\pi)$, thus

$$\mathrm{sgn}(\pi)=\mathrm{sgn}(\sigma)\mathrm{sgn}(\pi'')=\mathrm{sgn}(\sigma)\mathrm{sgn}(\pi')$$

and we get

$$\begin{aligned}
&((\alpha\wedge\beta)\wedge\gamma)(X_1,\ldots,X_{k+l+m}) \\
&= \frac{1}{(k+l)!k!l!m!}\underbrace{\sum_{\pi'\in S_{k+l}}(\mathrm{sgn}(\pi'))^2}_{=(k+l)!} \\
&\qquad \sum_{\sigma\in S_{k+l+m}}\mathrm{sgn}(\sigma)((\alpha\otimes\beta)\otimes\gamma)(X_{\sigma(1)},\ldots,X_{\sigma(k+l+m)}) \\
&= \frac{1}{k!l!m!}\sum_{\sigma\in S_{k+l+m}}\mathrm{sgn}(\sigma)((\alpha\otimes\beta)\otimes\gamma)(X_{\sigma(1)},\ldots,X_{\sigma(k+l+m)})\,. \qquad \text{(A.15.8)}
\end{aligned}$$

A similar calculation gives

$$\begin{aligned}
&(\alpha\wedge(\beta\wedge\gamma))(X_1,\ldots,X_{k+l+m}) \\
&= \frac{1}{k!l!m!}\sum_{\sigma\in S_{k+l+m}}\mathrm{sgn}(\sigma)(\alpha\otimes(\beta\otimes\gamma))(X_{\sigma(1)},\ldots,X_{\sigma(k+l+m)})\,, \qquad \text{(A.15.9)}
\end{aligned}$$

and the associativity of the wedge product follows.

The above calculations lead to the following form of the wedge product of n forms, where associativity is hidden in the notation:

$$\begin{aligned}
&(\alpha_1\wedge\cdots\wedge\alpha_n)(X_1,\ldots,X_{k_1+\ldots+k_n}) \\
&= \frac{1}{k_1!\cdots k_n!}\sum_{\pi\in S_{k_1+\ldots+k_n}}\mathrm{sgn}(\pi)(\alpha_1\otimes\cdots\otimes\alpha_n)(X_{\pi(1)},\ldots,X_{\pi(k_1+\ldots+k_n)})\,,
\end{aligned}$$

(A.15.10)

where $\alpha_i\in\Omega^{k_i}(M)$ for $i=1,\ldots,n$ and $X_j\in\Gamma(TM)$ for $j=1,\ldots,k_1+\ldots+k_n$.

Let us apply the last formula to one-forms: if $\alpha_i\in\Omega^1(M)$ we have

$$\begin{aligned}
(\alpha_1\wedge\cdots\wedge\alpha_n)(X_1,\ldots,X_n) &= \sum_{\pi\in S_n}\mathrm{sgn}(\pi)(\alpha_1\otimes\cdots\otimes\alpha_n)(X_{\pi(1)},\ldots,X_{\pi(n)}) \\
&= \sum_{\pi\in S_n}\mathrm{sgn}(\pi)\prod_{i=1}^n\alpha_i(X_{\pi(i)}) \\
&= \det(\alpha_i(X_j))\,, \qquad \text{(A.15.11)}
\end{aligned}$$

where we have used the Leibniz formula for determinants. □

The *exterior derivative* of a differential form is defined as follows:

1. For a zero form f, the exterior derivative of f is its *usual differential* df.
2. For a k-form α, its *exterior derivative* $d\alpha$ is a $(k+1)$-form defined as

$$d\alpha\equiv d\left(\frac{1}{k!}\alpha_{\mu_1\ldots\mu_k}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\right):=\frac{1}{k!}d\alpha_{\mu_1\ldots\mu_k}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\,. \qquad \text{(A.15.12)}$$

Equivalently

$$\begin{aligned}
d\alpha &= \frac{1}{k!}\partial_\beta\alpha_{\mu_1\ldots\mu_k}dx^\beta\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k} \\
&= \frac{k+1}{(k+1)!}\partial_{[\beta}\alpha_{\mu_1\ldots\mu_k]}dx^\beta\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\,, \qquad \text{(A.15.13)}
\end{aligned}$$

which can also be written as

$$(d\alpha)_{\mu_1\ldots\mu_{k+1}} = (k+1)\,\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]}\,. \qquad \text{(A.15.14)}$$

One easily checks, using $\partial_\alpha\partial_\beta y^\gamma = \partial_\beta\partial_\alpha y^\gamma$, that the exterior derivative behaves as a tensor under coordinate transformations. An "active way" of saying this is

$$d(\phi^*\alpha) = \phi^*(d\alpha)\,, \qquad \text{(A.15.15)}$$

for any differentiable map ϕ. The tensorial character of d is also made clear by noting that for any torsion-free connection ∇ we have

$$\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]} = \nabla_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]}\,. \qquad \text{(A.15.16)}$$

Again by symmetry of second derivatives, it immediately follows from (A.15.12) that $d(df) = 0$ for any function, and subsequently also for any differential form:

$$d^2\alpha := d(d\alpha) = 0\,. \qquad \text{(A.15.17)}$$

A coordinate-free definition of $d\alpha$ is

$$\begin{aligned} d\alpha(X_0, X_1, \ldots, X_k) &= \sum_{0\le j\le k} (-1)^j X_j\left(\alpha(X_0,\ldots,\widehat{X_j},\ldots X_k)\right) \\ &\quad + \sum_{0\le i<j\le k} (-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\widehat{X_j},\ldots X_k)\,, \end{aligned} \qquad \text{(A.15.18)}$$

where $\widehat{X_\ell}$ denotes the omission of the vector X_ℓ.

It is not too difficult to prove that if $\overset{k}{\alpha}$ is a k-form and $\overset{\ell}{\beta}$ is an ℓ-form, then the following version of the Leibniz rule holds:

$$d(\overset{k}{\alpha}\wedge\overset{\ell}{\beta}) = (d\overset{k}{\alpha})\wedge\overset{\ell}{\beta} + (-1)^k\overset{k}{\alpha}\wedge(d\overset{\ell}{\beta})\,. \qquad \text{(A.15.19)}$$

In dimension n, let $\sigma\in\{\pm1\}$ denote the parity of a permutation, set

$$\epsilon_{\mu_1\ldots\mu_n} = \begin{cases} \sqrt{|\det g_{\alpha\beta}|}\,\sigma(\mu_1\ldots\mu_n) & \text{if } (\mu_1\ldots\mu_n) \text{ is a permutation of } (1\ldots n); \\ 0 & \text{otherwise.}\end{cases}$$

The *Hodge dual* $\star\alpha$ of a k-form $\alpha = \alpha_{\mu_1\ldots\mu_k}dx^{\mu_1}\otimes\cdots\otimes dx^{\mu_k}$ is a $(n-k)$-form defined as

$$\star\alpha = \frac{1}{k!(n-k)!}\epsilon_{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n}\alpha^{\mu_1\ldots\mu_k}dx^{\mu_{k+1}}\otimes\cdots\otimes dx^{\mu_n}\,. \qquad \text{(A.15.20)}$$

Equivalently,

$$\star\alpha_{\mu_{k+1}\ldots\mu_n} = \frac{1}{k!(n-k)!}\epsilon_{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n}\alpha^{\mu_1\ldots\mu_k}\,. \qquad \text{(A.15.21)}$$

For example, in Euclidean three-dimensional space,

$$\star1 = dx\wedge dy\wedge dz\,,\ \star dx = dy\wedge dz\,,\ \star(dy\wedge dz) = dx\,,\ \star(dx\wedge dy\wedge dz) = 1\,,$$

etc. In Minkowski four-dimensional spacetime we have, e.g.,

$$\star dt = -dx\wedge dy\wedge dz\,,\quad \star dx = -dy\wedge dz\wedge dt\,,$$
$$\star(dt\wedge dx) = -dy\wedge dz\,,\quad \star(dx\wedge dy) = -dz\wedge dt\,,\quad \star(dx\wedge dy\wedge dz) = -dt\,.$$

A.16 Submanifolds, integration, and Stokes' theorem

When integrating on manifolds, the starting observation is that the integral of a *scalar function* f with respect to the coordinate measure

$$d^n x := dx^1 \cdots dx^n$$

is *not* a coordinate-independent operation. This is due to the fact that, under a change of variables $x \mapsto \bar{x}(x)$, one has

$$\int_{\mathbb{R}^n} \bar{f}(\bar{x}) d^n \bar{x} = \int_{\mathbb{R}^n} \underbrace{\bar{f}(\bar{x}(x))}_{f(x)} |J_{x\mapsto\bar{x}}(x)| d^n x \,, \tag{A.16.1}$$

where the *Jacobian* $J_{x\mapsto\bar{x}}$ is the determinant of the Jacobi matrix,

$$J_{x\mapsto\bar{x}} = \left| \frac{\partial(\bar{x}^1,\ldots,\bar{x}^n)}{\partial(x^1,\ldots,x^n)} \right| .$$

Supposing that we have a metric

$$g = g_{ij}(x) dx^i dx^j = g_{ij}(x) \frac{\partial x^i}{\partial \bar{x}^k}(\bar{x}(x)) \frac{\partial x^j}{\partial \bar{x}^\ell}(\bar{x}(x))\, d\bar{x}^k d\bar{x}^\ell = \bar{g}_{k\ell}(\bar{x}(x)) d\bar{x}^k d\bar{x}^\ell \tag{A.16.2}$$

at our disposal, the problem can be cured by introducing the *metric measure*

$$d\mu_g := \sqrt{\det g_{ij}}\, d^n x \,. \tag{A.16.3}$$

Indeed, using

$$x(\bar{x}(x)) = x \implies \frac{\partial x^k}{\partial \bar{x}^\ell}(\bar{x}(x)) \frac{\partial \bar{x}^\ell}{\partial x^i}(x) = \delta^k_i \implies J_{\bar{x}\mapsto x}(\bar{x}(x))\, J_{x\mapsto\bar{x}}(x) = 1 \,,$$

it follows from (A.16.2) that

$$\sqrt{\det \bar{g}_{ij}(\bar{x}(x))} = \sqrt{\det g_{ij}(x)} |J_{\bar{x}\mapsto x}(\bar{x}(x))| = \frac{\sqrt{\det g_{ij}(x)}}{|J_{x\mapsto\bar{x}}(x)|} \,,$$

hence

$$d\mu_g \equiv \sqrt{\det g_{ij}(x)}\, d^n x = \sqrt{\det \bar{g}_{ij}(x(\bar{x}))} |J_{x\mapsto\bar{x}}(x)| d^n x \,. \tag{A.16.4}$$

This shows that

$$\int_{\mathbb{R}^n} f(x) \sqrt{\det g_{ij}}\, d^n x = \int_{\mathbb{R}^n} f(x) \sqrt{\det \bar{g}_{ij}} |J_{x\mapsto\bar{x}}(x)| d^n x \,.$$

Comparing with (A.16.1), this is equal to

$$\int_{\mathbb{R}^n} f(x) d\mu_g = \int_{\mathbb{R}^n} f(x(\bar{x})) \sqrt{\det \bar{g}_{ij}}\, d^n \bar{x} = \int_{\mathbb{R}^n} \bar{f}(\bar{x}) d\mu_{\bar{g}} \,.$$

A similar formula holds for subsets of $\mathbb{R}^n$. We conclude that the metric measure $d\mu_g$ is the right thing to use when integrating scalars over a manifold.

Now, when defining conserved charges we have been integrating on submanifolds. The first naive thought would be to use the spacetime metric determinant as above for that, e.g., in spacetime dimension $n+1$,

$$\int_{\{x^0=0\}} f = \int_{\mathbb{R}^n} f(0, x^1, \ldots, x^n)\sqrt{\det g_{\mu\nu}}dx^1 \ldots dx^n \,.$$

This does not work because if we take g to be the Minkowski metric on $\mathbb{R}^n$, and replace x^0 by $\bar{x}^0$ using $x^0 = 2\bar{x}^0$, the only thing that will change in the last integral is the determinant $\sqrt{\det g_{\mu\nu}}$, giving a different value for the answer.

So, to proceed, it is useful to make first a short excursion into hypersurfaces, induced metrics and measures.

A.16.1 Hypersurfaces

A subset $\mathscr{S} \subset \mathscr{M}$ is called a *hypersurface* if near every point $p \in \mathscr{S}$ there exists a coordinate system $\{x^1, \ldots, x^n\}$ on a neighborhood $\mathscr{U}$ of p in $\mathscr{M}$ and a constant C such that

$$\mathscr{S} \cap \mathscr{U} = \{x^1 = C\}\,.$$

For example, any hyperplane $\{x^1 = \text{const}\}$ in $\mathbb{R}^n$ is a hypersurface. Similarly, a sphere $\{r = R\}$ in $\mathbb{R}^n$ is a hypersurface if $R > 0$.

Further examples include graphs,

$$x^1 = f(x^2, \ldots, x^{n-1})\,,$$

which is seen by considering new coordinates $(\bar{x}^i) = (x^1 - f, x^2, \ldots x^n)$.

A standard result in analysis asserts that if φ is a differentiable function on an open set Ω such that $d\varphi$ nowhere zero on $\Omega \cap \{\varphi = c\}$ for some constant c, then

$$\Omega \cap \{\varphi = c\}$$

forms a hypersurface in Ω.

A vector $X \in T_p\mathscr{M}$, $p \in \mathscr{S}$, is said to be *tangent to* $\mathscr{S}$ if there exists a differentiable curve γ with image lying on $\mathscr{S}$, with $\gamma(0) = p$, such that $X = \dot{\gamma}(0)$. One denotes by $T\mathscr{S}$ the set of such vectors. Clearly, the bundle $T\mathscr{S}$ of all vectors tangent to $\mathscr{S}$, defined when $\mathscr{S}$ is viewed as a manifold on its own, is naturally diffeomorphic with the bundle $T\mathscr{S} \subset T\mathscr{M}$ just defined.

As an example, suppose that $\mathscr{S} = \{x^1 = C\}$ for some constant C, then $T\mathscr{S}$ is the collection of vectors defined along $\mathscr{S}$ for which $X^1 = 0$.

As another example, suppose that

$$\mathscr{S} = \{x^0 = f(x^i)\} \tag{A.16.5}$$

for some differentiable function f. Then a curve γ lies on $\mathscr{S}$ if and only if

$$\gamma^0 = f(\gamma^1, \ldots, \gamma^n)\,,$$

and so its tangent satisfies

$$\dot\gamma^0 = \partial_1 f \dot\gamma^1 + \ldots + \partial_n f \dot\gamma^n \,.$$

We conclude that X is tangent to $\mathscr{S}$ if and only if

$$X^0 = X^1\partial_1 f + \ldots + X^n \partial_n f = X^i \partial_i f \quad \Longleftrightarrow \quad X = X^i\partial_i f \partial_0 + X^i \partial_i \,. \tag{A.16.6}$$

Equivalently, the vectors

$$\partial_i f \partial_0 + \partial_i$$

form a basis of the tangent space $T\mathscr{S}$.

Finally, if

$$\mathscr{S} = \Omega \cap \{\varphi = c\} \tag{A.16.7}$$

then for any curve lying on $\mathscr{S}$ we have

$$\varphi(\gamma(s)) = c \quad \Longleftrightarrow \quad \dot\gamma^\mu \partial_\mu \varphi = 0 \;\text{ and }\; \varphi(\gamma(0)) = c \,.$$

Hence, a vector $X \in T_p\mathscr{M}$ is tangent to $\mathscr{S}$ if and only if $\varphi(p) = c$ and

$$X^\mu \partial_\mu \varphi = 0 \iff X(\varphi) = 0 \iff d\varphi(X) = 0 \,. \tag{A.16.8}$$

A one-form α is said to *annihilate* $T\mathscr{S}$ if

$$\forall X \in T\mathscr{S} \qquad \alpha(X) = 0 \,. \tag{A.16.9}$$

The set of such one-forms is called *the annihilator* of $T\mathscr{S}$, and denoted as $(T\mathscr{S})^o$. By elementary algebra, $(T\mathscr{S})^o$ is a one-dimensional subset of $T^*\mathscr{M}$. So, (A.16.8) can be rephrased as the statement that $d\varphi$ annihilates $T\mathscr{S}$.

A vector $Y \in T_p\mathscr{M}$ is said to be normal to $\mathscr{S}$ if Y is orthogonal to every vector in $X \in T_p\mathscr{S}$, where $T_p\mathscr{S}$ is viewed as a subset of $T_p\mathscr{M}$. Equivalently, the one form $g(Y,\cdot)$ annihilates $T_p\mathscr{S}$. If N has unit length, $g(N,N) \in \{-1,+1\}$, then N is said to be the unit normal. Thus,

$$\forall X \in T\mathscr{S} \qquad g(X,N) = 0 \,, \quad g(N,N) = \epsilon \in \{\pm 1\} \,. \tag{A.16.10}$$

In Riemannian geometry only the plus sign is possible, and a unit normal vector always exists. This might not be the case in Lorentzian geometry: Indeed, consider the hypersurface

$$\mathscr{S} = \{t = x\} \subset \mathbb{R}^{1,1} \tag{A.16.11}$$

in two-dimensional Minkowski spacetime. A curve lying on $\mathscr{S}$ satisfies $\gamma^0(s) = \gamma^1(s)$, hence X is tangent to $\mathscr{S}$ if and only if $X^0 = X^1$. Let Y be orthogonal to $X \neq 0$, then

$$0 = \eta(X,Y) = X^0(-Y^0 + Y^1) \,,$$

whence

$$Y^0 = Y^1 \,. \tag{A.16.12}$$

We conclude that, for non-zero X,

$$0 = \eta(X,Y) \quad \Rightarrow \quad Y \in T\mathscr{S}, \text{ in particular } 0 = \eta(Y,Y),$$

and so no such vector Y can have length one or minus one.

Since vectors of the form (A.16.12) are tangent to $\mathscr{S}$ as given by (A.16.11), we also reach the surprising conclusion that vectors normal to $\mathscr{S}$ coincide with vectors tangent to $\mathscr{S}$ in this case.

Suppose that the direction normal to $\mathscr{S}$ is timelike or spacelike. Then the *metric h induced by g on $\mathscr{S}$* is defined as

$$\forall\, X, Y \in T\mathscr{S} \quad h(X,Y) = g(X,Y)\,. \tag{A.16.13}$$

Hence, h coincides with g whenever both are defined, but we are only allowed to consider vectors tangent to $\mathscr{S}$ when using h.

Some comments are in order: If g is Riemannian, then normals to $\mathscr{S}$ are spacelike, and (A.16.13) defines a Riemannian metric on $\mathscr{S}$. For Lorentzian g's, it is easy to see that h is Riemannian if and only if vectors orthogonal to $\mathscr{S}$ are timelike, and then $\mathscr{S}$ is called *spacelike*. Similarly, h is Lorentzian if and only if vectors orthogonal to $\mathscr{S}$ are spacelike, and then $\mathscr{S}$ is called *timelike*. When the normal direction to $\mathscr{S}$ is null, then (A.16.13) defines a symmetric tensor on $\mathscr{S}$ with signature $(0,+,\cdots,+)$, which is degenerate and therefore not a metric; such hypersurfaces are called *null*, or *degenerate*.

If $\mathscr{S}$ is *not* degenerate, it comes equipped with a Riemannian or Lorentzian metric h. This metric defines a measure $d\mu_h$ which can be used to integrate over $\mathscr{S}$.

We are ready now to formulate the Stokes theorem for open bounded sets: Let Ω be a bounded open set with piecewise differentiable boundary and assume that there exists a well-defined field of exterior-pointing conormals $N = N_\mu dx^\mu$ to Ω. Then for any differentiable vector field X it holds that

$$\int_\Omega \nabla_\alpha X^\alpha d\mu_g = \int_{\partial\Omega} X^\mu N_\mu dS\,. \tag{A.16.14}$$

If $\partial\Omega$ is non-degenerate, N_μ can be normalised to have unit length, and then dS is the measure $d\mu_h$ associated with the metric h induced on $\partial\Omega$ by g.

The definition of dS for null hypersurfaces is somewhat more complicated. The key point is that (A.16.14) remains valid for a suitable measure dS on null components of the boundary. This measure is not uniquely defined by the geometry of the problem, but the product $N_\mu dS$ is.

INCIDENTALLY: In order to prove (A.16.14) on a smooth null hypersurface $\mathscr{N}$ one can proceed as follows. Let use denote by N any smooth field of null normals to $\mathscr{N}$. The field N is defined up to multiplication by a nowhere-vanishing smooth function. We can find an ON-frame $\{e_\mu\}$ so that the vector fields $e_2, \dots e_n$ are tangent to $\mathscr{N}$ and orthogonal to N, with

$$N = e_0 + e_1\,. \tag{A.16.15}$$

Note that $\{e_0, e_1\}$ form an ON-basis of the space $\{e_2, \dots, e_n\}^\perp$, and are thus defined up to changes of signs $(e_0, e_1) \mapsto (\pm e_0, \pm e_1)$ and two-dimensional Lorentz transformations. If $\mathscr{N} = \partial\Omega$ we choose e_0 to be outwards directed; then (A.16.15) determines the orientation of e_1.

Let $\{\theta^\mu\}$ be the dual basis, thus the volume form $d\mu_g$ is

$$d\mu_g = \theta^0 \wedge \cdots \wedge \theta^n\,.$$

Set

$$dS := -\theta^1 \wedge \cdots \wedge \theta^n|_{\mathscr{N}} \,,$$

where $(\cdots)|_{\mathscr{N}}$ denotes the pull-back to $\mathscr{N}$. It holds that

$$dS = -\theta^0 \wedge \theta^2 \wedge \cdots \wedge \theta^n|_{\mathscr{N}} \,. \tag{A.16.16}$$

Indeed, we have $X \rfloor d\mu_g|_{\mathscr{N}} = 0$ for any vector field tangent to $\mathscr{N}$, in particular

$$0 = N \rfloor d\mu_g|_{\mathscr{N}} = \big(\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n - \theta^0 \wedge \theta^2 \wedge \cdots \wedge \theta^n\big)\big|_{\mathscr{N}} \,,$$

which is (A.16.16).

In the formalism of differential forms Stokes' theorem on oriented manifolds reads

$$\int_\Omega \nabla_\mu X^\mu d\mu_g = \int_{\partial\Omega} X \rfloor d\mu_g \,. \tag{A.16.17}$$

If $\partial\Omega$ is null, in the adapted frame just described we have $X^\mu N_\mu = -X^0 + X^1$ and

$$\begin{aligned} X \rfloor d\mu_g|_{\partial\Omega} &= \big(X^0\theta^1 \wedge \cdots \wedge \theta^n - X^1\theta^0 \wedge \theta^2 \wedge \cdots \wedge \theta^n\big)|_{\partial\Omega} = (-X^0 + X^1)dS \\ &= X^\mu N_\mu dS \,, \end{aligned} \tag{A.16.18}$$

as desired.

Since the left-hand side of (A.16.18) is independent of any choices made, so is the right-hand side. □

Remark A.16.2 The reader might wonder how (A.16.14) fits with the usual version of the divergence theorem

$$\int_\Omega \partial_\alpha X^\alpha d\mu_g = \int_{\partial\Omega} X^\mu dS_\mu \,, \tag{A.16.19}$$

which holds for sets Ω which can be covered by a single coordinate chart. For this we note the identity

$$\nabla_\mu X^\mu = \frac{1}{\sqrt{|\det g|}} \partial_\mu\big(\sqrt{|\det g|} X^\mu\big) \,, \tag{A.16.20}$$

which gives

$$\int_\Omega \nabla_\alpha X^\alpha d\mu_g = \int_\Omega \frac{1}{\sqrt{|\det g|}} \partial_\alpha\big(\sqrt{|\det g|} X^\alpha\big) \sqrt{|\det g|} d^n x = \int_\Omega \partial_\alpha\big(\sqrt{|\det g|} X^\alpha\big) d^n x \,. \tag{A.16.21}$$

This should make clear the relation between (A.16.19) and (A.16.14). □

A.17 Odd forms (densities)

In this section we review the notion of an *odd* differential n-form on a manifold M; we follow the very clear approach of [406].

Locally, in a vicinity of a point x_0, an odd form may be defined as an equivalence class $[(\alpha_n, \mathcal{O})]$, where α_n is a differential n-form defined in a neighbourhood U and $\mathcal{O}$ is an orientation of U; the equivalence relation is given by:

$$(\alpha_n, \mathcal{O}) \sim (-\alpha_n, -\mathcal{O}) \,,$$

where $-\mathcal{O}$ denotes the orientation opposite to $\mathcal{O}$. Using a partition of unity, we may define odd forms globally, even if the manifold is non-orientable.

Odd differential n-forms on an m-dimensional manifold can be described using antisymmetric contravariant tensor densities of rank $r = (m-n)$ (see [392]). Indeed, if $f^{i_1 \dots i_r}$ are components of such a tensor density with respect to a coordinate system (x^i), then we may assign to f an odd n-form defined by the representative $(\alpha_n, \mathcal{O})$, where $\mathcal{O}$ is the local orientation carried by $(x^1, \dots, x^m)$ and

$$\alpha_n := f^{i_1 \dots i_r} \left(\frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_r}} \right) \lrcorner (\, dx^1 \wedge \dots \wedge \, dx^m) \; .$$

In particular, within this description scalar densities (*i.e.*, densities of rank $m-n=0$) are odd forms of maximal rank, whereas vector densities are odd $(m-1)$-forms.

Odd n-forms are designed to be integrated over *externally oriented* n-dimensional submanifolds. An exterior orientation of a submanifold Σ is an orientation of a bundle of tangent vectors transversal with respect to Σ. The integral of an odd form $\tilde{\alpha}_n = [(\alpha_n, \mathcal{O})]$ over a n-dimensional submanifold D with exterior orientation $\mathcal{O}_{\text{ext}}$ is defined as follows:

$$\int_{(D, \mathcal{O}_{\text{ext}})} \tilde{\alpha}_n := \int_{(D, \mathcal{O}_{\text{int}})} \alpha_n \; ,$$

where $\mathcal{O}_{\text{int}}$ is an internal orientation of D, such that $(\mathcal{O}_{\text{ext}}, \mathcal{O}_{\text{int}}) = \mathcal{O}$; it should be obvious that the result does not depend upon the choice of a representative. For example, a flow through a hypersurface depends usually upon its *exterior* orientation (given by a transversal vector) and does not feel any *interior* orientation. Similarly, the canonical formalism in field theory uses structures, which are defined in terms of flows through Cauchy hypersurfaces in spacetime. This is why canonical momenta are described by odd $(m-1)$-forms. The integrals of such forms are insensitive to any *internal* orientation of the hypersurfaces they are integrated upon, but are sensitive to a choice of the time arrow (*i.e.*, to its *exterior* orientation).

The Stokes theorem generalizes to odd forms in a straightforward way:

$$\int_{(D, \mathcal{O}_{\text{ext}})} d\tilde{\alpha}_{n-1} = \int_{\partial (D, \mathcal{O}_{\text{ext}})} \tilde{\alpha}_{n-1} \; ,$$

where $d[(\alpha_n, \mathcal{O})] := [(\, d\alpha_n, \mathcal{O})]$ and $\partial(D, \mathcal{O}_{\text{ext}})$ is the boundary of D, equipped with an exterior orientation inherited in the canonical way from $(D, \mathcal{O}_{\text{ext}})$. This means that if $(e_1, \dots, e_{m-n})$ is an oriented basis of vectors transversal to D and if f is a vector tangent to D, transversal to ∂D and pointing outwards of D, then the exterior orientation of $\partial(D, \mathcal{O}_{\text{ext}})$ is given by $(e_1, \dots, e_{m-n}, f)$.

A.18 Moving frames

A formalism which is very convenient for practical calculations is that of *moving frames*; it also plays a key role when considering spinors. By definition, a moving frame is a (locally defined) field of bases $\{e_a\}$ of TM such that the scalar products

$$g_{ab} := g(e_a, e_b) \tag{A.18.1}$$

are point independent. In most standard applications one assumes that the e_a's form an orthonormal basis, so that g_{ab} is a diagonal matrix with plus and minus ones on the diagonal. However, it is sometimes convenient to allow other such frames, *e.g.* with isotropic vectors being members of the frame.

It is customary to denote by $\omega^a{}_{bc}$ the associated connection coefficients:

$$\omega^a{}_{bc} := \theta^a(\nabla_{e_c} e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \omega^a{}_{bc} X^c e_a \,, \tag{A.18.2}$$

where, as elsewhere, $\{\theta^a(p)\}$ is a basis of T_p^*M dual to $\{e_a(p)\} \subset T_pM$; we will refer to θ^a as a *coframe*. The *connection one forms* $\omega^a{}_b$ are defined as

$$\omega^a{}_b(X) := \theta^a(\nabla_X e_b) \quad \Longleftrightarrow \quad \nabla_X e_b = \omega^a{}_b(X) e_a \;;. \tag{A.18.3}$$

As always we use the metric to raise and lower indices, even though the $\omega^a{}_{bc}$'s do not form a tensor, so that

$$\omega_{abc} := g_{ad}\omega^e{}_{bc} \,, \quad \omega_{ab} := g_{ae}\omega^e{}_b \,. \tag{A.18.4}$$

When ∇ is metric compatible, the ω_{ab}'s are anti-antisymmetric: indeed, as the g_{ab}'s are point independent, for any vector field X we have

$$\begin{aligned} 0 = X(g_{ab}) = X(g(e_a, e_b)) &= g(\nabla_X e_a, e_b) + g(e_a, \nabla_X e_b) \\ &= g(\omega^c{}_a(X) e_c, e_b) + g(e_a, \omega^d{}_b(X) e_d) \\ &= g_{cb}\omega^c{}_a(X) + g_{ad}\omega^d{}_b(X) \\ &= \omega_{ba}(X) + \omega_{ab}(X) \,. \end{aligned}$$

Hence

$$\boxed{\omega_{ab} = -\omega_{ba} \quad \Longleftrightarrow \quad \omega_{abc} = -\omega_{bac}} \,. \tag{A.18.5}$$

One can obtain a formula for the ω_{ab}'s in terms of Christoffels, the frame vectors and their derivatives: In order to see this, we note that

$$g(e_a, \nabla_{e_c} e_b) = g(e_a, \omega^d{}_{bc} e_d) = g_{ad}\omega^d{}_{bc} = \omega_{abc} \,. \tag{A.18.6}$$

Rewritten the other way round this gives an alternative equation for the ω's with all indices down:

$$\omega_{abc} = g(e_a, \nabla_{e_c} e_b) \quad \Longleftrightarrow \quad \omega_{ab}(X) = g(e_a, \nabla_X e_b) \,. \tag{A.18.7}$$

Then, writing

$$e_a = e_a{}^\mu \partial_\mu \,,$$

we find

$$\begin{aligned} \omega_{abc} &= g(e_a{}^\mu \partial_\mu, e_c{}^\lambda \nabla_\lambda e_b) \\ &= g_{\mu\sigma} e_a{}^\mu e_c{}^\lambda (\partial_\lambda e_b{}^\sigma + \Gamma^\sigma_{\lambda\nu} e_b{}^\nu) \,. \end{aligned} \tag{A.18.8}$$

Next, it turns out that we can calculate the ω_{ab}'s in terms of the Lie brackets of the vector fields e_a, without having to calculate the Christoffel symbols. This shouldn't be too surprising, since an ON frame defines the metric uniquely. If ∇ has no torsion, from (A.18.7) we find

$$\omega_{abc} - \omega_{acb} = g(e_a, \nabla_{e_c} e_b - \nabla_{e_b} e_c) = g(e_a, [e_c, e_b])\,.$$

We can now carry out the usual cyclic-permutations calculation to obtain

$$\begin{aligned} \omega_{abc} - \omega_{acb} &= g(e_a, [e_c, e_b])\,, \\ -(\omega_{bca} - \omega_{bac}) &= -g(e_b, [e_a, e_c])\,, \\ -(\omega_{cab} - \omega_{cba}) &= -g(e_c, [e_b, e_a])\,. \end{aligned}$$

So, if the connection is the Levi-Civita connection, summing the three equations and using (A.18.5) leads to

$$\boxed{\omega_{cba} = \frac{1}{2}\Big(g(e_a, [e_c, e_b]) - g(e_b, [e_a, e_c]) - g(e_c, [e_b, e_a])\Big)}\,. \tag{A.18.9}$$

Equations (A.18.8)-(A.18.9) provide explicit expressions for the ω's; yet another formula can be found in (A.18.11) below. While it is useful to know that there are such expressions, and while those expressions are useful to estimate things for PDE purposes, they are rarely used for practical calculations; see Example A.18.3 for more comments about that last issue.

It turns out that one can obtain a simple expression for the torsion of ω using exterior differentiation. Recall that if α is a one-form, then its exterior derivative $d\alpha$ can be calculated using the formula

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])\,. \tag{A.18.10}$$

Exercice A.18.1 Use (A.18.9) and (A.18.10) to show that

$$\omega_{cba} = \frac{1}{2}\Big(- \eta_{ad}\, d\theta^d(e_c, e_b) + \eta_{bd}\, d\theta^d(e_a, e_c) + \eta_{cd}\, d\theta^d(e_b, e_a)\Big)\,. \tag{A.18.11}$$

□

We set

$$T^a(X, Y) := \theta^a(T(X, Y))\,,$$

and using (A.18.10) together with the definition (A.9.16) of the torsion tensor T we calculate as follows:

$$\begin{aligned} T^a(X, Y) &= \theta^a(\nabla_X Y - \nabla_Y X - [X, Y]) \\ &= X(Y^a) + \omega^a{}_b(X) Y^b - Y(X^a) - \omega^a{}_b(Y) X^b - \theta^a([X, Y]) \\ &= X(\theta^a(Y)) - Y(\theta^a(X)) - \theta^a([X, Y]) + \omega^a{}_b(X)\theta^b(Y) - \omega^a{}_b(Y)\theta^b(X) \\ &= d\theta^a(X, Y) + (\omega^a{}_b \wedge \theta^b)(X, Y)\,. \end{aligned}$$

It follows that

$$T^a = d\theta^a + \omega^a{}_b \wedge \theta^b\,. \tag{A.18.12}$$

In particular when the torsion vanishes we obtain the so-called *Cartan's first structure equation*

$$\boxed{d\theta^a + \omega^a{}_b \wedge \theta^b = 0}\,. \tag{A.18.13}$$

EXAMPLE A.18.2 As a simple example, we consider a two-dimensional metric of the form

$$g = dx^2 + e^{2f}dy^2 \,, \tag{A.18.14}$$

where f could possibly depend upon x and y. A natural frame is given by

$$\theta^1 = dx \,, \quad \theta^2 = e^f dy \,.$$

The first Cartan structure equations read

$$0 = \underbrace{d\theta^1}_{0} + \omega^1{}_b \wedge \theta^b = \omega^1{}_2 \wedge \theta^2 \,,$$

since $\omega^1{}_1 = \omega_{11} = 0$ by antisymmetry, and

$$0 = \underbrace{d\theta^2}_{e^f \partial_x f dx \wedge dy} + \omega^2{}_b \wedge \theta^b = \partial_x f \theta^1 \wedge \theta^2 + \omega^2{}_1 \wedge \theta^1 \,.$$

It should then be clear that both equations can be solved by choosing ω_{12} proportional to θ^2, and such an ansatz leads to

$$\omega_{12} = -\omega_{21} = -\partial_x f \,\theta^2 = -\partial_x(e^f)\,dy \,. \tag{A.18.15}$$

We continue this example in Example A.18.5, p. 224. □

EXAMPLE A.18.3 As another example of the moving frame technique we consider (the most general) three-dimensional spherically symmetric metric

$$g = e^{2\beta(r)}dr^2 + e^{2\gamma(r)}d\theta^2 + e^{2\gamma(r)}\sin^2\theta d\varphi^2 \,. \tag{A.18.16}$$

There is an obvious choice of ON coframe for g given by

$$\theta^1 = e^{\beta(r)}dr \,, \ \theta^2 = e^{\gamma(r)}d\theta \,, \ \theta^3 = e^{\gamma(r)}\sin\theta d\varphi \,, \tag{A.18.17}$$

leading to

$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3 \,,$$

so that the frame e_a dual to the θ^a's will be ON, as desired:

$$g_{ab} = g(e_a, e_b) = \mathrm{diag}(1,1,1) \,.$$

The idea of the calculation which we are about to do is the following: there is only one connection which is compatible with the metric, and which is torsion free. If we find a set of one forms ω_{ab} which exhibit the properties just mentioned, then they have to be the connection forms of the Levi-Civita connection. As shown in the calculation leading to (A.18.5), the compatibility with the metric will be ensured if we require

$$\omega_{11} = \omega_{22} = \omega_{33} = 0 \,,$$

$$\omega_{12} = -\omega_{21} \,, \quad \omega_{13} = -\omega_{31} \,, \quad \omega_{23} = -\omega_{32} \,.$$

Next, we have the equations for the vanishing of torsion:

$$\begin{aligned} 0 = d\theta^1 &= -\underbrace{\omega^1{}_1}_{=0}\theta^1 - \omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \\ &= -\omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \,, \\ d\theta^2 &= \gamma' e^\gamma dr \wedge d\theta = \gamma' e^{-\beta}\theta^1 \wedge \theta^2 \end{aligned}$$

$$
\begin{aligned}
&= -\underbrace{\omega^2{}_1}_{=-\omega^1{}_2}\theta^1 - \underbrace{\omega^2{}_2}_{=0}\theta^2 - \omega^2{}_3\theta^3 \\
&= \omega^1{}_2\theta^1 - \omega^2{}_3\theta^3\,, \\
d\theta^3 &= \gamma' e^\gamma \sin\theta\, dr\wedge d\varphi + e^\gamma\cos\theta\, d\theta\wedge d\varphi = \gamma' e^{-\beta}\theta^1\wedge\theta^3 + e^{-\gamma}\cot\theta\,\theta^2\wedge\theta^3 \\
&= -\underbrace{\omega^3{}_1}_{=-\omega^1{}_3}\theta^1 - \underbrace{\omega^3{}_2}_{=-\omega^2{}_3}\theta^2 - \underbrace{\omega^3{}_3}_{=0}\theta^3 \\
&= \omega^1{}_3\theta^1 + \omega^2{}_3\theta^2\,.
\end{aligned}
$$

Summarising,

$$
\begin{aligned}
-\omega^1{}_2\theta^2 - \omega^1{}_3\theta^3 \qquad\qquad &= 0\,, \\
\omega^1{}_2\theta^1 \qquad - \omega^2{}_3\theta^3 &= \gamma' e^{-\beta}\theta^1\wedge\theta^2\,, \\
\omega^1{}_3\theta^1 + \omega^2{}_3\theta^2 &= \gamma' e^{-\beta}\theta^1\wedge\theta^3 + e^{-\gamma}\cot\theta\,\theta^2\wedge\theta^3\,.
\end{aligned}
$$

It should be clear from the first and second line that an $\omega^1{}_2$ proportional to θ^2 should do the job; similarly from the first and third line one sees that an $\omega^1{}_3$ proportional to θ^3 should work. It is then easy to find the relevant coefficient, as well as to find $\omega^2{}_3$:

$$
\begin{aligned}
\omega^1{}_2 &= -\gamma' e^{-\beta}\theta^2 = -\gamma' e^{-\beta+\gamma}d\theta\,, && (A.18.18a)\\
\omega^1{}_3 &= -\gamma' e^{-\beta}\theta^3 = -\gamma' e^{-\beta+\gamma}\sin\theta\, d\varphi\,, && (A.18.18b)\\
\omega^2{}_3 &= -e^{-\gamma}\cot\theta\,\theta^3 = -\cos\theta\, d\varphi\,. && (A.18.18c)
\end{aligned}
$$

We continue this example on p. 224. □

It is convenient to define *curvature two-forms*:

$$
\Omega^a{}_b = R^a{}_{bcd}\theta^c\otimes\theta^d = \frac{1}{2}R^a{}_{bcd}\theta^c\wedge\theta^d\,. \qquad (A.18.19)
$$

The *second Cartan structure equation* reads

$$
\boxed{\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b}\,. \qquad (A.18.20)
$$

This identity is easily verified using (A.18.10):

$$
\begin{aligned}
\Omega^a{}_b(X,Y) &= \frac{1}{2}R^a{}_{bcd}\underbrace{\theta^c\wedge\theta^d(X,Y)}_{=X^cY^d-X^dY^c} \\
&= R^a{}_{bcd}X^cY^d \\
&= \theta^a(\nabla_X\nabla_Y e_b - \nabla_Y\nabla_X e_b - \nabla_{[X,Y]}e_b) \\
&= \theta^a(\nabla_X(\omega^c{}_b(Y)e_c) - \nabla_Y(\omega^c{}_b(X)e_c) - \omega^c{}_b([X,Y])e_c) \\
&= \theta^a\Big(X(\omega^c{}_b(Y))e_c + \omega^c{}_b(Y)\nabla_X e_c \\
&\qquad -Y(\omega^c{}_b(X))e_c - \omega^c{}_b(X)\nabla_Y e_c - \omega^c{}_b([X,Y])e_c\Big) \\
&= X(\omega^a{}_b(Y)) + \omega^c{}_b(Y)\omega^a{}_c(X) \\
&\qquad -Y(\omega^a{}_b(X)) - \omega^c{}_b(X)\omega^a{}_c(Y) - \omega^a{}_b([X,Y]) \\
&= \underbrace{X(\omega^a{}_b(Y)) - Y(\omega^a{}_b(X)) - \omega^a{}_b([X,Y])}_{=d\omega^a{}_b(X,Y)} \\
&\qquad +\omega^a{}_c(X)\omega^c{}_b(Y) - \omega^a{}_c(Y)\omega^c{}_b(X) \\
&= (d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b)(X,Y)\,.
\end{aligned}
$$

Equation (A.18.20) provides an efficient way of calculating the curvature tensor of any metric.

EXAMPLE A.18.4 In dimension two the only non-vanishing components of $\omega^a{}_b$ are $\omega^1{}_2 = -\omega^2{}_1$, and it follows from (A.18.20) that

$$\Omega^1{}_2 = d\omega^1{}_2 + \omega^1{}_a \wedge \omega^a{}_2 = d\omega^1{}_2 \,. \tag{A.18.21}$$

In particular (assuming that θ^2 is dual to a spacelike vector, whatever the signature of the metric)

$$\begin{aligned} R\, d\mu_g &= R\,\theta^1 \wedge \theta^2 = 2R^{12}{}_{12}\theta^1 \wedge \theta^2 = R^1{}_{2ab}\theta^a \wedge \theta^b = 2\Omega^1{}_2 \\ &= 2d\omega^1{}_2 \,, \end{aligned} \tag{A.18.22}$$

where $d\mu_g$ is the volume two-form. □

EXAMPLE A.18.5 (Example A.18.2 continued) We have seen that the connection one-forms for the metric

$$g = dx^2 + e^{2f} dy^2 \tag{A.18.23}$$

read

$$\omega_{12} = -\omega_{21} = -\partial_x f\, \theta^2 = -\partial_x(e^f)\, dy \,.$$

By symmetry the only non-vanishing curvature two-forms are $\Omega_{12} = -\Omega_{21}$. From (A.18.20) we find

$$\Omega_{12} = d\omega_{12} + \underbrace{\omega_{1b} \wedge \omega^b{}_2}_{=\omega_{12}\wedge\omega^2{}_2=0} = -\partial_x^2(e^f)\, dx \wedge dy = -e^{-f}\partial_x^2(e^f)\, \theta^1 \wedge \theta^2 \,.$$

We conclude that

$$R_{1212} = -e^{-f}\partial_x^2(e^f) \,. \tag{A.18.24}$$

(Compare Example A.12.6, p. 197.) For instance, if g is the unit round metric on the two-sphere, then $e^f = \sin x$, and $R_{1212} = 1$. If $e^f = \sinh x$, then g is the canonical metric on hyperbolic space, and $R_{1212} = -1$. Finally, the function $e^f = \cosh x$ defines a *hyperbolic wormhole*, with again $R_{1212} = -1$. □

EXAMPLE A.18.6 (Example A.18.3 continued): From (A.18.18) we find:

$$\begin{aligned} \Omega^1{}_2 &= d\omega^1{}_2 + \underbrace{\omega^1{}_1}_{=0} \wedge \omega^1{}_2 + \omega^1{}_2 \wedge \underbrace{\omega^2{}_2}_{=0} + \underbrace{\omega^1{}_3 \wedge \omega^3{}_2}_{\sim\theta^3\wedge\theta^3=0} \\ &= -d(\gamma' e^{-\beta+\gamma} d\theta) \\ &= -(\gamma' e^{-\beta+\gamma})' dr \wedge d\theta \\ &= -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma}\theta^1 \wedge \theta^2 \\ &= \sum_{a<b} R^1{}_{2ab}\theta^a \wedge \theta^b \,, \end{aligned}$$

which shows that the only non-trivial coefficient (up to permutations) with the pair 12 in the first two slots is

$$R^1{}_{212} = -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} \,. \tag{A.18.25}$$

A similar calculation, or arguing by symmetry, leads to

$$R^1{}_{313} = -(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} \,. \tag{A.18.26}$$

Finally,

$$\begin{aligned}\Omega^2{}_3 &= d\omega^2{}_3 + \omega^2{}_1 \wedge \omega^1{}_3 + \underbrace{\omega^2{}_2}_{=0} \wedge \omega^2{}_3 + \omega^2{}_3 \wedge \underbrace{\omega^3{}_3}_{=0} \\ &= -d(\cos\theta\, d\varphi) + (\gamma' e^{-\beta}\theta^2) \wedge (-\gamma' e^{-\beta}\theta^3) \\ &= (e^{-2\gamma} - (\gamma')^2 e^{-2\beta})\theta^2 \wedge \theta^3\,,\end{aligned}$$

yielding

$$R^2{}_{323} = e^{-2\gamma} - (\gamma')^2 e^{-2\beta}\,. \tag{A.18.27}$$

The curvature scalar can easily be calculated now to be

$$\begin{aligned}R = R^{ij}{}_{ij} &= 2(R^{12}{}_{12} + R^{13}{}_{13} + R^{23}{}_{23}) \\ &= -4(\gamma' e^{-\beta+\gamma})' e^{-\beta-\gamma} + 2(e^{-2\gamma} - (\gamma')^2 e^{-2\beta})\,. \end{aligned} \tag{A.18.28}$$

□

Example A.18.7 Consider an n-dimensional Riemannian metric of the form

$$g = e^{2h(r)}dr^2 + e^{2f(r)} \underbrace{\mathring{h}_{AB}(x^C)dx^A dx^B}_{=:\mathring{h}}\,. \tag{A.18.29}$$

Let $\mathring{\theta}^A$ be an ON-frame for $\mathring{h}$, with corresponding connection coefficients $\mathring{\omega}^A{}_B$:

$$d\mathring{\theta}^A + \mathring{\omega}^A{}_B \wedge \mathring{\theta}^B = 0\,.$$

Set

$$\theta^1 = e^h dr\,, \quad \theta^A = e^f \mathring{\theta}^A\,.$$

Then the first structure equations,

$$\underbrace{d\theta^1}_{0} + \omega^1{}_B \wedge \theta^B = 0\,,$$

$$d(e^f \mathring{\theta}^A) + e^h \omega^A{}_1 \wedge dr + e^f \omega^A{}_B \wedge \mathring{\theta}^B = 0\,,$$

are solved by

$$\omega^A{}_1 = e^{-h}(e^f)'\, \mathring{\theta}^A\,, \quad \omega^A{}_B = \mathring{\omega}^A{}_B\,. \tag{A.18.30}$$

This leads to

$$\begin{aligned}\Omega^A{}_1 &= d\omega^A{}_1 + \omega^A{}_B \wedge \omega^B{}_1 \\ &= e^{-h-f}(e^{-h}(e^f)')'\, \theta^1 \wedge \theta^A\,, &\text{(A.18.31)}\\ \Omega^A{}_B &= \mathring{\Omega}^A{}_B - e^{-2h-2f}((e^f)')^2\, \theta^A \wedge \theta_B\,, &\text{(A.18.32)}\end{aligned}$$

where $\mathring{\Omega}^A{}_B$ are the curvature two-forms of the metric $\mathring{h}$,

$$\mathring{\Omega}^A{}_B = \frac{1}{2}\mathring{R}^A{}_{BCD}\, \mathring{\theta}^A \wedge \mathring{\theta}^B = \frac{e^{-2f}}{2}\mathring{R}^A{}_{BCD}\, \theta^A \wedge \theta^B\,. \tag{A.18.33}$$

Hence

$$\begin{aligned}R^A{}_{1B1} &= -e^{-h-f}(e^{-h}(e^f)')'\, \delta^A_B\,, &\text{(A.18.34)}\\ R^A{}_{1BC} &= 0 \;=\; R^1{}_B\,, &\text{(A.18.35)}\\ R^A{}_{BCD} &= e^{-2f}\mathring{R}^A{}_{BCD} - e^{-2h-2f}((e^f)')^2 \delta^A_{[C}g_{D]B}\,, &\text{(A.18.36)}\\ R^A{}_C &= -e^{-h-f}\big((e^{-h}(e^f)')' + (n-2)e^{-h-f}((e^f)')^2\big)\delta^A_C \\ &\quad + e^{-2f}\mathring{R}^A{}_C\,, &\text{(A.18.37)}\\ R^1{}_1 &= -(n-1)e^{-h-f}(e^{-h}(e^f)')'\,, &\text{(A.18.38)}\\ R &= -(n-1)e^{-h-f}\big(2(e^{-h}(e^f)')' + (n-2)e^{-h-f}((e^f)')^2\big) \\ &\quad + e^{-2f}\mathring{R}\,. &\text{(A.18.39)}\end{aligned}$$

Let g be the space-part of the Birmingham metrics thus g takes the form (A.18.29) with

$$e^f = r\,, \quad e^{-2h} = \beta - \frac{2m}{r^{n-2}} - \epsilon\frac{r^2}{\ell^2}\,, \qquad \epsilon \in \{0, \pm 1\}\,, \tag{A.18.40}$$

where β, m and ℓ are real constants. Then

$$\begin{aligned}
R^A{}_{1B1} &= \big(\frac{\epsilon}{\ell^2} - m(n-2)r^{-n}\big)\delta^A_B\,, && \text{(A.18.41)}\\
R^A{}_{1BC} &= 0 \;=\; R^1{}_B\,, && \text{(A.18.42)}\\
R^A{}_{BCD} &= \mathring{R}^A{}_{BCD} - \big(\frac{\beta}{r^2} - \frac{\epsilon}{\ell^2} + 2mr^{-n}\big)\delta^A_{[C}g_{D]B}\,, && \text{(A.18.43)}\\
R^A{}_B &= \frac{\ell^2(n-2)\left(mr^2 - \beta r^n\right) + \epsilon(n-1)r^{n+2}}{\ell^2 r^{n+2}}\delta^A_B \\
&\quad + r^{-2}\mathring{R}^A{}_B\,, && \text{(A.18.44)}\\
R^1{}_1 &= (n-1)\big(\frac{\epsilon}{\ell^2} - m(n-2)r^{-n}\big)\,, && \text{(A.18.45)}\\
R &= \frac{(n-1)\left(\epsilon n r^2 - \beta\ell^2(n-2)\right)}{\ell^2 r^2} + r^{-2}\mathring{R}\,. && \text{(A.18.46)}
\end{aligned}$$

If $\mathring{h}$ is Einstein, with

$$\mathring{R}_{AB} = (n-2)\beta\mathring{h}_{AB}\,, \tag{A.18.47}$$

the last formulae above simplify to

$$\begin{aligned}
R^A{}_B &= \frac{\ell^2(n-2)r^{-n-2}\left(\beta(n-2)r^n + mr^2\right) + \epsilon(n-1)}{\ell^2}\delta^A_B\,, && \text{(A.18.48)}\\
R &= \frac{\epsilon(n-1)n}{\ell^2}\,. && \text{(A.18.49)}
\end{aligned}$$

□

Example A.18.8 We can use (A.18.11),

$$\omega_{cba} = \frac{1}{2}\Big(- \eta_{ad}\, d\theta^d(e_c, e_b) + \eta_{bd}\, d\theta^d(e_a, e_c) + \eta_{cd}\, d\theta^d(e_b, e_a)\Big)\,, \tag{A.18.50}$$

to determine how the curvature tensor transforms under conformal rescalings. For this let $g = \eta_{ab}\theta^a\theta^b$ with $d\eta_{ab} = 0$, and let

$$\overline{g} = e^{2f}g = \eta_{ab}\underbrace{e^f\theta^a}_{=:\overline{\theta}a}\otimes e^f\theta^b \equiv \eta_{ab}\overline{\theta}^a\overline{\theta}^b\,. \tag{A.18.51}$$

If the vector fields $\{e_a\}$ form a basis dual to the basis $\{\theta^a\}$, then the vector fields $\overline{e}_a = e^{-f}e_a$ provide a basis dual to $\{\overline{\theta}^b\}$,

$$\begin{aligned}
\overline{\omega}_{cba} &= \frac{1}{2}\Big(- \eta_{ad}\, d(e^f\theta^d)(e^{-f}e_c, e^{-f}e_b) + \eta_{bd}\, d(e^f\theta^d)(e^{-f}e_a, e^{-f}e_c)\\
&\qquad + \eta_{cd}\, d(e^f\theta^d)(e^{-f}e_b, e^{-f}e_a)\Big)\\
&= e^{-f}\left(\omega_{cba} + \frac{1}{2}\Big(- \eta_{ad}\,(df\wedge\theta^d)(e_c, e_b) + \eta_{bd}\,(df\wedge\theta^d)(e_a, e_c) + \eta_{cd}\,(df\wedge\theta^d)(e_b, e_a)\Big)\right)\\
&= e^{-f}\left(\omega_{cba} - \eta_{a[b}e_{c]}(f) + \eta_{b[c}e_{a]}(f) + \eta_{c[a}e_{b]}(f)\right)\\
&= e^{-f}\left(\omega_{cba} - \eta_{ab}e_c(f) + \eta_{ac}e_b(f)\right)\,.
\end{aligned} \tag{A.18.52}$$

Equivalently,

$$\overline{\omega}_{cb} = \overline{\omega}_{cba}\overline{\theta}^a = e^f\overline{\omega}_{cba}\theta^a = \omega_{cb} + \big(\eta_{ac}\nabla_b f - \eta_{ab}\nabla_c f\big)\theta^a\,. \tag{A.18.53}$$

Taking the exterior derivative one finds

$$\begin{aligned}\overline{\Omega}_{cb} &= \Omega_{cb} + \Big(\eta_{ab}\nabla_d\nabla_c f - \eta_{ac}\nabla_d\nabla_b f \\ &\quad +\eta_{ac}\nabla_d f\nabla_b f + \eta_{db}\nabla_c f\nabla_a f - \eta_{ac}\eta_{db}|df|^2_g\Big)\theta^a\wedge\theta^d\,. \end{aligned}\tag{A.18.54}$$

Reexpressed in terms of the Riemann tensor, this reads

$$\begin{aligned}e^{2f}\overline{R}_{cbad} &= R_{cbad} + 2\Big(\eta_{b[a}\nabla_{d]}\nabla_c f - \eta_{c[a}\nabla_{d]}\nabla_b f \\ &\quad +\eta_{c[a}\nabla_{d]}f\nabla_b f + \eta_{b[d}\nabla_{a]}f\nabla_c f - \eta_{c[a}\eta_{d]b}|df|^2_g\Big)\,, \end{aligned}\tag{A.18.55}$$

where the components $\overline{R}_{cbad}$ are taken with respect to a $\overline{g}$-ON frame, and all components of the right-hand side are taken with respect to a g-ON frame. Taking traces we obtain, in dimension d,

$$e^{2f}\overline{R}_{ac} = R_{ac} + (2-d)\Big(\nabla_a\nabla_c f - \nabla_a f\nabla_c f + |df|^2_g\eta_{ac}\Big) - \Delta_g f\eta_{ac}\,, \tag{A.18.56}$$

$$e^{2f}\overline{R} = R + (1-d)\big(2\Delta_g f + (d-2)|df|^2_g\big)\,. \tag{A.18.57}$$

□

The Bianchi identities have a particularly simple proof in the moving frame formalism. For this, let ψ^a be any vector-valued differential form, and define

$$D\psi^a = d\psi^a + \omega^a{}_b\wedge\psi^b\,. \tag{A.18.58}$$

Thus, in this notation the vanishing of torsion reads

$$D\theta^a = 0\,. \tag{A.18.59}$$

Whether or not the torsion vanishes, we find

$$\begin{aligned} D\tau^a &= d\tau^a + \omega^a{}_b\wedge\tau^b = d(d\theta^a + \omega^a{}_b\wedge\theta^b) + \omega^a{}_c\wedge(d\theta^c + \omega^c{}_b\wedge\theta^b) \\ &= d\omega^a{}_b\wedge\theta^b - \omega^a{}_b\wedge d\theta^b + \omega^a{}_c\wedge(d\theta^c + \omega^c{}_b\wedge\theta^b) \\ &= \Omega^a{}_b\wedge\theta^b\,. \end{aligned}$$

If the torsion vanishes the left-hand side is zero, and we find

$$\Omega^a{}_b\wedge\theta^b = 0\,. \tag{A.18.60}$$

This is equivalent to the first Bianchi identity:

$$0 = \Omega^a{}_b\wedge\theta^b = \frac{1}{2}R^a{}_{bcd}\theta^c\wedge\theta^d\wedge\theta^b = R^a{}_{[bcd]}\theta^c\wedge\theta^d\wedge\theta^b \quad\Longleftrightarrow\quad R^a{}_{[bcd]} = 0\,. \tag{A.18.61}$$

Next, for any differential form α_b with two-frame indices, such as the curvature two-form, we define

$$D\alpha^a{}_b := d\alpha^a{}_b + \omega^a{}_c\wedge\alpha^c{}_b - \omega^c{}_b\wedge\alpha^a{}_c\,. \tag{A.18.62}$$

(The reader will easily work-out the obvious generalisation of this definition to differential forms with any number of frame indices.) For the curvature two-form we find

$$\begin{aligned}
D\Omega^a{}_b &= d(d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c \\
&= d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c \\
&= (\Omega^a{}_c - \omega^a{}_e \wedge \omega^e{}_c) \wedge \omega^c{}_b - \omega^a{}_c \wedge (\Omega^c{}_b - \omega^c{}_e \wedge \omega^e{}_b) \\
&\quad + \omega^a{}_c \wedge \Omega^c{}_b - \omega^c{}_b \wedge \Omega^a{}_c = 0\,.
\end{aligned}$$

Thus

$$D\Omega^a{}_b = 0\,, \tag{A.18.63}$$

Let us show that this is equivalent to the second Bianchi identity:

$$\begin{aligned}
0 &= D\Omega^a{}_b = \frac{1}{2} R^a{}_{bc;d}\theta^d \wedge \theta^b \wedge \theta^c = \frac{1}{2} R^a{}_{[bc;d]}\theta^d \wedge \theta^b \wedge \theta^c \\
&\Longleftrightarrow \quad R^a{}_{[bc;d]} = 0\,.
\end{aligned} \tag{A.18.64}$$

Here the only not-obviously-apparent fact is, if any, the second equality in the first line of(A.18.64):

$$\begin{aligned}
D\Omega^a{}_b &= \frac{1}{2}\Big(d(R^a{}_{bef}\theta^e \wedge \theta^f) + \omega^a{}_c \wedge R^c{}_{bef}\,\theta^e \wedge \theta^f - \omega^c{}_b \wedge R^a{}_{cef}\,\theta^e \wedge \theta^f \Big) \\
&= \frac{1}{2}\Big(\underbrace{dR^a{}_{bef}}_{e_k(R^a{}_{bef})\,\theta^k} \wedge \theta^e \wedge \theta^f + R^a{}_{bef} \underbrace{d\theta^e}_{-\omega^e{}_k \wedge \theta^k} \wedge \theta^f + R^a{}_{bef}\theta^e \wedge \underbrace{d\theta^f}_{-\omega^f{}_k \wedge \theta^k} \\
&\quad + R^c{}_{bef}\,\omega^a{}_c \wedge \theta^e \wedge \theta^f - R^a{}_{cef}\,\omega^c{}_b \wedge \theta^e \wedge \theta^f \Big) \\
&= \frac{1}{2} \nabla_{e_k} R^a{}_{bef}\,\theta^k \wedge \theta^e \wedge \theta^f\,,
\end{aligned} \tag{A.18.65}$$

as desired.

A.19 Arnowitt-Deser-Misner (ADM) decomposition

In the study of the general relativistic Cauchy problem it is sometimes convenient to write the spacetime metric $\mathbf{g}$ in the form

$$\mathbf{g} = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)\,. \tag{A.19.1}$$

so that

$$\mathbf{g}_{00} = -\alpha^2 + g_{ij}\beta^i\beta^j\,, \quad \mathbf{g}_{0i} = g_{ij}\beta^j\,, \quad \mathbf{g}_{ij} = g_{ij}\,. \tag{A.19.2}$$

It is straightforward to check that the relations $\mathbf{g}^{\alpha\beta}\mathbf{g}_{\beta\gamma} = \delta^\alpha_\gamma$ are satisfied by the following tensor:

$$\mathbf{g}^{00} = -\alpha^{-2}\,, \quad \mathbf{g}^{0i} = \alpha^{-2}\beta^j\,, \quad \mathbf{g}^{ij} = g^{ij} - \alpha^{-2}\beta^i\beta^j\,, \tag{A.19.3}$$

where g^{ij} is the matrix inverse to g_{ij}. Equivalently, the inverse metric $\mathbf{g}^\sharp = \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$ takes the form

$$\mathbf{g}^\sharp = -\alpha^{-2}(\partial_t - \beta^i\partial_i)^2 + g^{ij}\partial_i\partial_j\,. \tag{A.19.4}$$

Here we have implicitly assumed that the level sets of t are spacelike, so that ∇t is timelike.

If t is increasing to the future, then the field N of future-directed unit normals to the hypersurfaces

$$\mathscr{S}_\tau := \{t = \tau\}$$

is

$$N = \alpha^{-1}(\partial_t - \beta^i\partial_i)\,, \tag{A.19.5}$$

while $g := g_{ij}dx^i dx^j$ is the metric induced by $\mathbf{g}$ on $\mathscr{S}_\tau$. The $\mathscr{S}_\tau$'s are spacelike if and only if g_{ij} is Riemannian everywhere if and only if N is timelike everywhere.

The following formula is often used:

$$\sqrt{|\det \mathbf{g}_{\mu\nu}|} = \alpha\sqrt{|\det g_{ij}|}\,. \tag{A.19.6}$$

The covariant counterpart of N takes a simple form,

$$N^\flat := \mathbf{g}_{\mu\nu}n^\mu dx^\nu = -\alpha dt\,. \tag{A.19.7}$$

The *extrinsic curvature tensor* K of $\mathscr{S}_\tau$, also called the *second fundamental form* of $\mathscr{S}_\tau$, as defined in (1.4.6), takes therefore the form[3]

$$\begin{aligned} K_{ij} &= \nabla_i N_j = \partial_i N_j - \Gamma^\mu_{ij}N_\mu = \alpha\Gamma^0_{ij} \\ &= \frac{1}{2\alpha}\Big(\frac{\partial g_{ij}}{\partial t} - D_i\beta_j - D_j\beta_i\Big)\,, \end{aligned} \tag{A.19.8}$$

where D denotes the covariant derivative operator of the metric g_{ij}, and $\beta_i = g_{ij}\beta^j$. It holds that:

$$\Gamma^i_{j0} = \alpha K^i{}_j + D_j\beta^i - \alpha^{-1}\beta^i D_j\alpha - \alpha^{-1}\beta^i\beta^k K_{kj}\,, \tag{A.19.9}$$

$$\Gamma^m_{k\ell}(\mathbf{g}) = \Gamma^m_{k\ell}(g) - \alpha^{-1}\beta^m K_{k\ell}\,, \tag{A.19.10}$$

$$\Gamma^0_{0\ell} = \frac{1}{\alpha}\Big(D_\ell\alpha + \beta^k K_{k\ell}\Big)\,, \tag{A.19.11}$$

$$\Gamma^0_{00} = \frac{1}{\alpha}\partial_t\alpha + \frac{1}{2\alpha}\beta^\ell D_\ell(\alpha^2 - \beta^k\beta_k)\,. \tag{A.19.12}$$

The space-components of the spacetime curvature tensor can be related to the curvature tensor of g by (1.4.16). Assuming that the $\mathscr{S}_\tau$'s are spacelike we have

$$R_{ijk\ell}(\mathbf{g}) = R_{ijk\ell}(g) + K_{ik}K_{j\ell} - K_{i\ell}K_{jk}\,. \tag{A.19.13}$$

We also have (1.4.23),

$$D_k K_{ij} - D_i K_{kj} = \frac{1}{\alpha}(R_{j0ki}(\mathbf{g}) - R_{j\ell ki}(\mathbf{g})\beta^\ell)\,, \tag{A.19.14}$$

[3]The reader is warned that some authors use an opposite sign in the definition of K.

which together with (A.19.13) provides a formula for $R_{j0ki}(\mathbf{g})$.

Similarly to (A.19.13), the *constraint equations* (1.4.19) and (1.4.24) do not involve neither the lapse function α nor the shift vector β in vacuum:

$$R(g) = 16\pi T_{\mu\nu}N^\mu N^\nu + 2\Lambda + |K|_h^2 - (\mathrm{tr}_h K)^2 \,, \qquad \text{(A.19.15)}$$

$$D_j K^j{}_k - D_k K^j{}_j = 8\pi T_{k\nu}N^\mu \,. \qquad \text{(A.19.16)}$$

The vacuum Einstein equations imply the following evolution equation for K_{ij}:

$$\partial_t K_{ij} \;=\; -\mathscr{L}_\beta K_{ij} - \alpha(R(g)_{ij} - 2K_i{}^k K_{kj} + K^k{}_k K_{ij}) + D_i D_j \alpha \,.$$

(Not unexpectedly, the vacuum KID equation (3.5.16), p. 139 is obtained by setting $\partial_t K_{ij} = 0$.)

It turns out that *the constraint equations propagate causally*, in the following sense: If the vacuum constraint equations are satisfied on a spacelike hypersurface $\mathscr{O}$, and if the space-components $R(\mathbf{g})_{ij} = 0$ of the vacuum Einstein equations hold on the domain of dependence $\mathscr{U}$ of $\mathscr{O}$, then the constraint equations (and hence all Einstein equations) are satisfied on $\mathscr{U}$. This can be proved by noting that the Bianchi identities imply a homogeneous symmetric-hyperbolic system of equations for the constraint functions [417].

A.20 Extrinsic curvature vector

Let N be a smooth n-dimensional submanifold of an m-dimensional pseudo-Riemannian manifold (M,g), $n < m$. The normal bundle $T^\perp N$ is defined as the bundle of all vectors $\xi \in T_pM$, $p \in N$, such that ξ is orthogonal to all vectors tangent to N at p. Here we view T_pN as a subspace of T_pM.

We will assume that the tensor field induced by g on N is non-degenerate, hence defines a pseudo-Riemannian metric on N. Equivalently,

$$T_p^\perp N \cap T_pN = \{0\}$$

for every $p \in N$.

For $p \in N$ let P_p denote the orthogonal projection from T_pM to T_pN. Let ξ, X and Y be vector fields defined near N such that X and Y are tangent to N along N, while ξ is normal to N along N. On N one sets

$$D_XY := P(\nabla_X Y)\,, \quad \sigma(X,Y) := \nabla_X Y - D_X Y\,, \qquad \text{(A.20.1)}$$

$$A_\xi(X) := P(\nabla_X \xi)\,, \quad D_X^\perp \xi := \nabla_X \xi - A_\xi(X)\,. \qquad \text{(A.20.2)}$$

The tensor field σ is called the *shape operator* of N. We thus have

$$\nabla_X Y = D_X Y + \sigma(X,Y)\,, \quad \nabla_X \xi = D_X^\perp \xi + A_\xi(X)\,. \qquad \text{(A.20.3)}$$

Calculations essentially identical to those of Section 1.4 (see, e.g., [87]) show that D_XY is the covariant derivative operator associated with the metric induced by g on M, that σ is symmetric, that $D^\perp$ defines a connection on the normal bundle of N, and that

$$g(A_\xi(X),Y) = g(\sigma(X,Y),\xi)\,. \qquad \text{(A.20.4)}$$

For each pair X, Y as above the field $\sigma(X,Y)$ is a vector field defined along N and normal to N. Let e_i, $i = 1, \dots, n$ be vector fields defined in a neighborhood of N which form an ON-basis of TN near a point $p \in N$. The *mean extrinsic curvature vector* of N at p is defined as

$$\kappa = \sum_i \sigma(e_i, e_i)\,. \tag{A.20.5}$$

It is simple to check that κ does not depend upon the choice of the vector fields e_i, and therefore is globally defined on N.

A.21 Null hyperplanes

One of the objects that occur in Lorentzian geometry and which posses rather disturbing properties are *null hyperplanes* and *null hypersurfaces*, and it appears useful to include a short discussion of those. Perhaps the most unusual feature of such objects is that the direction normal is actually tangential as well. Furthermore, because the normal has no natural normalization, there is no natural measure induced on a null hypersurface by the ambient metric.

In this section we present some algebraic preliminaries concerning null hyperplanes, null hypersurfaces will be discussed in Section **??** below.

Let W be a real vector space, and recall that its dual W^* is defined as the set of all linear maps from W to $\mathbb{R}$ in the applications (in this work only vector spaces over the reals are relevant, but the field makes no difference for the discussion below). To avoid unnecessary complications we assume that W is finite dimensional. It is then standard that W^* has the same dimension as W.

We suppose that W is equipped with a *a) bilinear, b) symmetric, and c) non-degenerate form* q. Thus

$$q : W \times W \to \mathbb{R}$$

satisfies

$$a) \quad q(\lambda X + \mu Y, Z) = \lambda q(X,Z) + \mu q(Y,Z)\,, \qquad b) \quad q(X,Y) = q(Y,X)\,,$$

and we also have the implication

$$c) \quad \forall Y \in W \;\; q(X,Y) = 0 \implies X = 0\,. \tag{A.21.1}$$

(Strictly speaking, we should have indicated linearity with respect to the second variable in a) as well, but this property follows from a) and b) as above). By an abuse of terminology, we will call q a *scalar product*; note that standard algebra textbooks often add the condition of positive-definiteness to the definition of scalar product, which we do not include here.

Let $V \subset W$ be a vector subspace of W. The *annihilator* V^0 of W is defined as the set of linear forms on W which vanish on V:

$$V^0 := \{\alpha \in W^* : \; \forall Y \in V \quad \alpha(Y) = 0\} \subset W^*\,.$$

V^0 is obviously a linear subspace of W^*.

Because q non-degenerate, it defines a linear isomorphism, denoted by $\flat$, between W and W^* by the formula:

$$X^\flat(Y) = q(X,Y)\,.$$

Indeed, the map $X \mapsto X^\flat$ is clearly linear. Next, it has no kernel by (A.21.1). Since the dimensions of W and W^* are the same, it must be an isomorphism. The inverse map is denoted by $\sharp$. Thus, by definition we have

$$q(\alpha^\sharp, Y) = \alpha(Y)\,.$$

The map $\flat$ is nothing but "the lowering of the index on a vector using the metric q", while $\sharp$ is the "raising of the index on a one-form using the inverse metric".

For further purposes it is useful to recall the standard fact:

PROPOSITION A.21.1

$$\dim V + \dim V^0 = \dim W\,.$$

PROOF: Let $\{e_i\}_{i=1,\ldots,\dim V}$ be any basis of V, we can complete $\{e_i\}$ to a basis $\{e_i, f_a\}$, with $a = 1,\ldots,\dim W - \dim V$, of W. Let $\{e_i^*, f_a^*\}$ be the dual basis of W^*. It is straightforward to check that V^0 is spanned by $\{f_a^*\}$, which gives the result. □

The quadratic form q defines the notion of orthogonality:

$$V^\perp := \{Y \in W : \forall X \in V \;\; q(X,Y) = 0\}\,.$$

A chase through the definitions above shows that

$$V^\perp = (V^0)^\sharp\,.$$

Proposition A.21.1 implies:

PROPOSITION A.21.2

$$\dim V + \dim V^\perp = \dim W\,.$$

This implies, again regardless of signature:

PROPOSITION A.21.3

$$(V^\perp)^\perp = V\,.$$

PROOF: The inclusion $(V^\perp)^\perp \supset V$ is obvious from the definitions. The equality follows now because both spaces have the same dimension, as a consequence of Proposition (A.21.2). □

Now,

$$X \in V \cap V^\perp \implies q(X,X) = 0\,, \tag{A.21.2}$$

so that X vanishes if q is positive- or negative-definite, leading to $\dim V \cap \dim V^\perp = \{0\}$ in those cases. However, this does not have to be the case anymore for non-definite scalar products q.

A vector subspace V of W is called a *hyperplane* if

$$\dim V = \dim W - 1\,.$$

Proposition A.21.2 implies then

$$\dim V^{\perp} = 1\,,$$

regardless of the signature of q. Thus, given a hyperplane V there exists a vector w such that

$$V^{\perp} = \mathbb{R}w\,.$$

If q is Lorentzian, we say that

$$V \text{ is } \begin{cases} \text{spacelike} & \text{if } w \text{ is timelike;} \\ \text{timelike} & \text{if } w \text{ is spacelike;} \\ \text{null} & \text{if } w \text{ is null.} \end{cases}$$

An argument based e.g. on Gram-Schmidt orthonormalization shows that if V is spacelike, then the scalar product defined on V by restriction is positive-definite; similarly if V is timelike, then the resulting scalar product is Lorentzian. The last case, of a null V, leads to a degenerate induced scalar product. In fact, we claim that

$$V \text{ is null if and only if } V \text{ contains its normal.}\,. \qquad \text{(A.21.3)}$$

To see (A.21.3), suppose that $V^{\perp} = \mathbb{R}w$, with w null. Since $q(w,w) = 0$ we have $w \in (\mathbb{R}w)^{\perp}$, and from Proposition A.21.3

$$w \in (\mathbb{R}w)^{\perp} = (V^{\perp})^{\perp} = V\,.$$

Since V does not contain its normal in the remaining cases, the equivalence is established.

As discussed in more detail in the next section, a hypersurface $\mathscr{N} \subset \mathscr{M}$ is called *null* if at every $p \in \mathscr{N}$ the scalar product restricted to $T_p\mathscr{N}$ is degenerate. Equivalently, the tangent space $T_p\mathscr{N}$ is a null subspace of $T_p\mathscr{M}$. So (A.21.2) shows that vectors normal to a null hypersurface $\mathscr{N}$ are also tangent to $\mathscr{N}$.

A.22 Elements of causality theory

We collect here some definitions from causality theory. Given a manifold $\mathscr{M}$ equipped with a Lorentzian metric g, at each point $p \in \mathscr{M}$ the set of timelike vectors in T_pM has precisely two components. A *time-orientation of* $T_p\mathscr{M}$ is the assignment of the name "future pointing vectors" to one of those components; vectors in the remaining component are then called "past pointing". A Lorentzian manifold is said to be *time-orientable* if such locally defined time-orientations can be defined globally in a consistent way. A *spacetime* is a time-orientable Lorentzian manifold on which a time-orientation has been chosen.

A differentiable path γ will be said to be *timelike* if at each point the tangent vector $\dot\gamma$ is timelike; it will be said *future directed* if $\dot\gamma$ is future directed. There is an obvious extension of this definition to *null, causal* or *spacelike* curves. We define an *observer* to be an *inextendible, future directed timelike path.* In these notes the names "path" and "curve" will be used interchangeably.

Let $\mathscr{U} \subset \mathscr{O} \subset \mathscr{M}$. One sets

$$\begin{aligned} I^+(\mathscr{U};\mathscr{O}) &:= \{q \in \mathscr{O} : \text{there exists a timelike future directed path} \\ &\qquad \text{from } \mathscr{U} \text{ to } q \text{ contained in } \mathscr{O}\}\,, \\ J^+(\mathscr{U};\mathscr{O}) &:= \{q \in \mathscr{O} : \text{there exists a causal future directed path} \\ &\qquad \text{from } \mathscr{U} \text{ to } q \text{ contained in } \mathscr{O}\} \cup \mathscr{U}\,. \end{aligned}$$

$I^-(\mathscr{U};\mathscr{O})$ and $J^-(\mathscr{U};\mathscr{O})$ are defined by replacing "future" by "past" in the definitions above. The set $I^+(\mathscr{U};\mathscr{O})$ is called the *timelike future of $\mathscr{U}$ in $\mathscr{O}$*, while $J^+(\mathscr{U};\mathscr{O})$ is called the *causal future of $\mathscr{U}$ in $\mathscr{O}$*, with similar terminology for the timelike past and the causal past. We will write $I^\pm(\mathscr{U})$ for $I^\pm(\mathscr{U};\mathscr{M})$, similarly for $J^\pm(\mathscr{U})$, and one then omits the qualification "in $\mathscr{M}$" when talking about the causal or timelike futures and pasts of $\mathscr{U}$. We will write $I^\pm(p;\mathscr{O})$ for $I^\pm(\{p\};\mathscr{O})$, $I^\pm(p)$ for $I^\pm(\{p\};\mathscr{M})$, *etc.*

A function f will be called a *time function* if its gradient is *timelike, past pointing.* Similarly a function f will be said to be a *causal function* if its gradient is *causal, past pointing.* The choice "past-pointing" here has to do with our choice $(-,+,\ldots,+)$ of the signature of the metric. This is easily understood on the example of Minkowski spacetime $(\mathbb{R}^{n+1},\eta)$, where the gradient of the usual time coordinate t is $-\partial_t$, since $\eta^{00}=-1$. Had we chosen to work with the signature $(+,-,\ldots,-)$, time functions would have been defined to have future pointing gradients.

A differentiable hypersurface $\mathscr{S} \subset \mathscr{M}$ is called a *Cauchy surface* if every inextendible causal curve intersects $\mathscr{S}$ precisely once. A spacetime is called *globally hyperbolic* if it contains a Cauchy hypersurface. A key property of globally hyperbolic spacetimes is, that they possess a time-function t (in fact, many) with the property that each level set of t is a Cauchy surface.

A spacetime $(\mathscr{M},g)$ is called *maximal globally hyperbolic* if it is globally hyperbolic and if there exists no spacetime $(\widetilde{\mathscr{M}},\widetilde{g})$ such that $(\mathscr{M},g)$ is a proper subset of $(\widetilde{\mathscr{M}},\widetilde{g})$.

The reader is referred to [125, 139, 198, 222, 283, 284, 323, 403] for extensive modern treatments of causality theory, including applications to incompleteness theorems (also known as "singularity theorems").

Appendix B

Some interesting spacetimes

B.1 Taub-NUT spacetimes

The Taub–NUT metrics are solutions of vacuum Einstein equations on space-time manifolds $\mathscr{M}_I$ of the form

$$\mathscr{M}_I := I \times S^3 \,,$$

where I an interval. They take the form [324]

$$-U^{-1}dt^2 + (2\ell)^2 U\sigma_1^2 + (t^2+\ell^2)(\sigma_2^2+\sigma_3^2) \,, \tag{B.1.1}$$

$$U(t) = -1 + \frac{2(mt+\ell^2)}{t^2+\ell^2} \,. \tag{B.1.2}$$

Here ℓ and m are real numbers with $\ell > 0$. Further, the one-forms σ_1, σ_2 and σ_3 form a basis of the set of left-invariant one-forms on $SU(2) \approx S^3$: If

$$i_{S^3} : S^3 \to \mathbb{R}^4$$

is the standard embedding of S^3 into $\mathbb{R}^4$, then (see Section B.1.4 below)

$$\begin{aligned} \sigma_1 &= 2i^*_{S^3}(x\,dw - w\,dx + y\,dz - z\,dy) \,, \\ \sigma_2 &= 2i^*_{S^3}(z\,dx - x\,dz + y\,dw - w\,dy) \,, \\ \sigma_3 &= 2i^*_{S^3}(x\,dy - y\,dx + z\,dw - w\,dz) \,. \end{aligned} \tag{B.1.3}$$

The function U always has two zeros,

$$U(t) = \frac{(t_+ - t)(t - t_-)}{t^2+\ell^2} \,,$$

where

$$t_\pm := m \pm \sqrt{m^2+\ell^2} \,.$$

It follows that I has to be chosen so that $t_\pm \notin I$. The spacetime $(\mathscr{M}_{(t_-,t_+)}, g)$ will be referred to as *the Taub spacetime* [401].

It is convenient to parameterize S^3 with Euler angles

$$(\zeta, \theta, \varphi) \in [0, 4\pi] \times [0, \pi] \times [0, 2\pi] \,,$$

normalised so that

$$x + iy = \sin\left(\frac{\theta}{2}\right) e^{i(\varphi-\zeta)/2}\,, \quad z + iw = \cos\left(\frac{\theta}{2}\right) e^{i(\varphi+\zeta)/2}\,. \tag{B.1.4}$$

This leads to the following form of the metric

$$g = -U^{-1}dt^2 + (2\ell)^2 U(d\zeta + \cos\theta d\varphi)^2 + (t^2+\ell^2)(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{B.1.5}$$

INCIDENTALLY: REMARK B.1.2 There is a natural action of S^1 on the circles obtained by varying ζ at fixed t, θ and φ:

$$\zeta \to \zeta + 2\alpha\,.$$

Equation (B.1.4) shows that this corresponds to the following action of $e^{i\alpha} \in S^1$ on $\mathbb{R}^4 \approx \mathbb{C}^2$:

$$(x+iy, z+iw) \to \Big(e^{-i\alpha}(x+iy), e^{i\alpha}(z+iw)\Big)\,. \tag{B.1.6}$$

The action (B.1.6) is clearly free (this means that no point is left invariant by elements of the group different from the identity), so that each orbit is a circle.

The coordinates (θ,φ) in (B.1.4) relate to the usual spherical coordinates on S^2, as follows: writing momentarily z and w for two complex numbers (not to be confused with the real numbers appearing in (B.1.3) or (B.1.6)), consider the map

$$\mathbb{C}^2 \supset S^3 \ni (z,w) \longrightarrow (2zw, |z|^2 - |w|^2) \in \mathbb{C}\times\mathbb{R} \approx \mathbb{R}^3\,. \tag{B.1.7}$$

This is invariant under (B.1.6), and maps S^3 into S^2:

$$|2zw|^2 + (|z|^2 - |w|^2)^2 = (|z|^2 + |w|^2)^2 = 1\,.$$

Inserting (B.1.4) into (B.1.7) gives immediately

$$\left(\sin\left(\frac{\theta}{2}\right) e^{i(\varphi-\zeta)/2}, \cos\left(\frac{\theta}{2}\right) e^{i(\varphi+\zeta)/2}\right) \longrightarrow \Big(\sin(\theta)e^{i\varphi}, \cos(\theta)\Big)\,.$$

The above structure is known as the *Hopf fibration* of S^3 by S^1's, and provides a non-trivial circle bundle over S^2.

□

From (B.1.5) we have

$$g^\#(dt,dt) = g^{tt} = -U\,, \tag{B.1.8}$$

which shows that the level sets of t are

- spacelike for $t \in (t_-, t_+)$, and
- timelike for $t < t_-$ or $t > t_+$.

Equation (B.1.8) further shows that

$$\nabla t = g^{\mu t}\partial_\mu = -U\partial_t\,,$$

so that

$$g(\nabla t, \nabla t) = -U(t) < 0 \quad \text{for} \quad t \in (t_-, t_+)\,.$$

Equivalently, t is a time function in this range of t's. Thus, $(\mathscr{M}_{(t_-,t_+)}, g)$ is stable causal, with all level sets of t compact, which easily implies that

$$(\mathscr{M}_{(t_-,t_+)}, g) \text{ is globally hyperbolic.} \tag{B.1.9}$$

From (B.1.5) we further find

$$g(\partial_\zeta, \partial_\zeta) = 2\ell^2 U\,,$$

so that the Hopf circles — to which ∂_ζ is tangent — are

- spacelike for $t \in (t_-, t_+)$, and
- timelike for $t < t_-$ or $t > t_+$.

In particular

$$(\mathscr{M}_{(-\infty,t_-)}, g) \text{ and } (\mathscr{M}_{(t_+,\infty)}, g) \text{ contain closed timelike curves.}$$

Let γ be the metric induced by g on the level sets of t,

$$\gamma = (2\ell)^2 U \sigma_1^2 + (t^2+\ell^2)(\sigma_2^2+\sigma_3^2)\,.$$

Again in the t-range (t_-, t_+), the volume $|\mathscr{S}_\tau|$ of the level sets $\mathscr{S}_\tau$ of t equals

$$|\mathscr{S}_\tau| = \int_{\mathscr{S}_\tau} d\mu_\gamma = \sqrt{\frac{U(t)(t^2+\ell^2)}{U(0)\ell^2}}|\mathscr{S}_0|\,.$$

Here $d\mu_\gamma$ is the volume element of the metric γ — in local coordinates

$$d\mu_\gamma = \sqrt{\det \gamma_{ij}}\; d^3x\,.$$

This is a typical "big-bang — big-crunch" behaviour, where the volume of the space-slices of the universe "starts" at zero, expands to a maximum, and collapses again to a zero value.

The standard way of performing extensions across the Cauchy horizons $t_\pm$ is to introduce new coordinates

$$(t,\zeta,\theta,\varphi) \to (t, \zeta \pm \int_{t_0}^{t} [2\ell U(s)]^{-1} ds, \theta, \varphi)\,, \tag{B.1.10}$$

which gives

$$\begin{aligned} g_\pm &= \pm 4\ell(d\zeta + \cos\theta d\varphi)dt \\ &\quad + (2\ell)^2 U (d\zeta+\cos\theta d\varphi)^2 + (t^2+\ell^2)(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \tag{B.1.11}$$

Somewhat surprisingly, the metrics $g_\pm$ are non-singular for all $t \in \mathbb{R}$. In order to see that let, as before, $\mathscr{M}_\mathbb{R} := \mathbb{R} \times S^3$, and on $\mathscr{M}_\mathbb{R}$ consider the one-forms θ^a, $a = 0, \ldots, 3$, defined as

$$\begin{aligned} \theta^0 &= dt\,, \\ \theta^1 &= 2\epsilon\ell(d\zeta + \cos\theta d\phi) = -2\ell\epsilon\sigma_3\,, \\ \theta^2 &= \sqrt{t^2+\ell^2}\; d\theta\,, \\ \theta^3 &= \sqrt{t^2+\ell^2}\; \sin\theta d\varphi\,. \end{aligned}$$

θ^0 and θ^1 are smooth everywhere on $\mathscr{M}_{\mathbb{R}}$, while θ^2 and θ^3 are smooth except for the usual harmless spherical coordinates singularity at the south and north poles of S^3. In this frame the metrics g_ϵ, $\epsilon = \pm$, take the form

$$\begin{aligned} g_\epsilon &= \theta^0 \otimes \theta^1 + \theta^1 \otimes \theta^0 + U\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3 \\ &=: g_{ab}\theta^a \otimes \theta^b , \end{aligned} \tag{B.1.12}$$

which manifestly shows their regular — in fact analytic — character on $\mathscr{M}_{\mathbb{R}}$.

Each of the spacetimes $(\mathscr{M}_{\mathbb{R}}, g_\pm)$ will be referred to as the *Taub-NUT* spacetime.

B.1.1 Geodesics

The geodesics in Taub-NUT spacetime possess many interesting features. This can be studied in detail because of the large number of Killing vectors. Our presentation is based on [226, 322, 325].

By definition, a Killing vector field is a solution of the equation

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0 . \tag{B.1.13}$$

The interest of Killing vectors here arises from the fact that they provide constants of motion along geodesics: Indeed, if X is a solution of the Killing equation, and γ is an affinely parameterised geodesic, we have

$$\begin{aligned} \frac{dg(X,\dot\gamma)}{ds} &= g(\nabla_{\dot\gamma} X, \dot\gamma) + g(X, \underbrace{\nabla_{\dot\gamma}\dot\gamma}_{=0}) \\ &= \nabla_\alpha X_\beta \dot\gamma^\alpha \dot\gamma^\beta \\ &= \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)\dot\gamma^\alpha\dot\gamma^\beta = 0 . \end{aligned}$$

It is shown in Section B.1.4 (see (B.1.40)) that the following vector fields are Killing vectors[1] of g (recall that $\csc\theta = 1/\sin\theta$):

$$\begin{aligned} \xi_1 &:= \frac{1}{2}(y\partial_w - z\partial_x - w\partial_y + x\partial_z) \\ &= -\sin\varphi\partial_\theta - \cos\varphi(\cot\theta\partial_\varphi - \csc\theta\partial_\zeta) , \end{aligned} \tag{B.1.14a}$$

$$\begin{aligned} \xi_2 &:= \frac{1}{2}(-x\partial_w + w\partial_x - z\partial_y + y\partial_z) \\ &= \cos\varphi\partial_\theta - \sin\varphi(\cot\theta\partial_\varphi - \csc\theta\partial_\zeta) , \end{aligned} \tag{B.1.14b}$$

$$\begin{aligned} \xi_3 &:= \frac{1}{2}(-z\partial_w - y\partial_x + x\partial_y + w\partial_z) \\ &= \partial_\varphi , \end{aligned} \tag{B.1.14c}$$

$$\begin{aligned} \eta &:= \frac{1}{2}(-x\partial_w + y\partial_x - x\partial_y + w\partial_z) \\ &= \partial_\zeta . \end{aligned} \tag{B.1.14d}$$

We set

$$p_a := g(\dot\gamma, \xi_a) , \quad p_\| = g(\dot\gamma, \eta) . \tag{B.1.15}$$

[1] By analyticity, these are also Killing vectors of g_ϵ.

Inserting (B.1.14) into (B.1.1) gives

$$\begin{aligned} p_3 &= (t^2+\ell^2)\sin^2\theta\dot\varphi\,, && \text{(B.1.16a)}\\ p_\| &= (2\ell)^2U(\dot\zeta+\cos\theta\dot\varphi)\\ &= (2\ell)^2U(\dot\zeta+\frac{p_3\cos\theta}{(t^2+\ell^2)\sin^2\theta})\,, && \text{(B.1.16b)}\\ p_1 &= \csc\theta\cos\varphi(2\ell)^2U(\dot\zeta+\cos\theta\dot\varphi)\\ &\quad +(t^2+\ell^2)(-\sin\varphi\dot\theta-\sin\theta\cos\theta\cos\varphi\dot\varphi)\\ &= \csc\theta\cos\varphi\,p_\| -(t^2+\ell^2)\sin\varphi\dot\theta-\cot\theta\cos\varphi p_3\,, && \text{(B.1.16c)}\\ p_2 &= \csc\theta\sin\varphi(2\ell)^2U(\dot\zeta+\cos\theta\dot\varphi)\\ &\quad +(t^2+\ell^2)(\cos\varphi\dot\theta-\sin\theta\cos\theta\sin\varphi\dot\varphi)\\ &= \csc\theta\sin\varphi\,p_\| +(t^2+\ell^2)\cos\varphi\dot\theta-\cot\theta\sin\varphi p_3\,. && \text{(B.1.16d)} \end{aligned}$$

Equations (B.1.16a)-(B.1.16b) provide equations for $\dot\varphi$ and $\dot\zeta$. Multiplying (B.1.16c) by $-\sin\varphi$, (B.1.16d) by $\cos\varphi$, and adding, one obtains an equation for $\dot\theta$. This leads to the following first order equations, assuming $\sin\theta\neq 0$,

$$\frac{d\theta}{ds}=\frac{p_2\cos\varphi-p_1\sin\varphi}{t^2+\ell^2}\,, \tag{B.1.17a}$$

$$\frac{d\varphi}{ds}=\frac{p_3}{(t^2+\ell^2)\sin^2\theta}\,, \tag{B.1.17b}$$

$$\frac{d\zeta}{ds}=\frac{p_\|}{(2\ell)^2U}-\frac{\cos\theta p_3}{(t^2+\ell^2)\sin^2\theta}\,. \tag{B.1.17c}$$

The missing first-order equation for t is obtained from the fact that the Lorentzian length

$$\varepsilon := g(\dot\gamma,\dot\gamma)\in\{0,\pm1\}$$

is constant along an affinely parameterised geodesic:

$$\Big(\frac{dt}{ds}\Big)^2=\Big(\frac{p_\|}{2\ell}\Big)^2+U\Big(-\varepsilon+\frac{(p_2\cos\varphi-p_1\sin\varphi)^2}{t^2+\ell^2}+\frac{p_3^2}{(t^2+\ell^2)\sin^2\theta}\Big)\,. \tag{B.1.17d}$$

There is a useful equation linking the constants of motion which can be obtained as follows: Multiplying (B.1.16c) by $\cos\varphi$, (B.1.16d) by $\sin\varphi$, and adding, the terms $\dot\theta$ drop out, leading to

$$\sin\theta\cos\varphi p_1+\sin\theta\sin\varphi p_2+\cos\theta p_3=p_\|\,. \tag{B.1.18}$$

(This can also be seen directly from the relation

$$\sin\theta\cos\varphi\xi_1+\sin\theta\sin\varphi\xi_2+\cos\theta\xi_3=\eta \tag{B.1.19}$$

which follows from (B.1.14).)

Now, the left-hand side of (B.1.18) is the scalar product, with respect to the Euclidean metric δ on $\mathbb{R}^3$, of the δ-unit vector $(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$ and of (p_1,p_2,p_3). The Cauchy-Schwarz inequality gives

$$|p_\||\le\sqrt{p_1^2+p_2^2+p_3^2}\,,$$

with equality if and only if the vectors $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and (p_1, p_2, p_3) are aligned. If we denote by α the angle between those vectors, then (B.1.18) shows that

$$p_\| = \sqrt{p_1^2 + p_2^2 + p_3^2}\cos\alpha\,.$$

This further implies that α is constant along geodesics. If we write $\gamma(s)$ as

$$\gamma(s) = (t(s), \zeta(s), q(s))\,, \quad q(s) \in S^2\,, \tag{B.1.20}$$

that property is equivalent to the statement that $q(s)$ lies on a circle corresponding to the intersection of S^2, viewed as a subset of $\mathbb{R}^3$, with a plane orthogonal to (p_1, p_2, p_3). It follows that (regardless of the character of γ):

1. If $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and (p_1, p_2, p_3) are initially aligned, they remain so, thus $q(s)$ is constant. Equivalently, if $p_\| = \pm\sqrt{p_1^2 + p_2^2 + p_3^2}$ then $\dot\varphi = \dot\theta = 0$;

2. Performing a rotation if necessary, one can choose the Euler coordinates (ζ, θ, φ) so that the *entire* orbit $q(s)$ avoids the singular points $\sin\theta = 0$. In what follows this choice will always be made.

We start by noting

Proposition B.1.3 *All inextendible causal geodesics in Taub spacetime are future and past incomplete.*

Proof: For causal geodesics we have $\varepsilon \in \{0, -1\}$. It follows from (B.1.17d) that dt/ds has no zeros, hence t is strictly monotonic on γ and can be used as a parameter. Without loss of generality we can assume $dt/ds > 0$.

Equation (B.1.17d) can be rewritten as

$$s(t_2)-s(t_1) = \int_{t_1}^{t_2} \Big((\frac{p_\|}{2\ell})^2 + U(-\varepsilon + \frac{(p_2\cos\varphi - p_1\sin\varphi)^2}{t^2+\ell^2} + \frac{p_3^2}{(t^2+\ell^2)\sin^2\theta}) \Big)^{-\frac{1}{2}} dt\,. \tag{B.1.21}$$

The proof is simplest for timelike geodesics, for then $\varepsilon = -1$ and we can estimate

$$|s(t_2) - s(t_1)| \le \int_{t_1}^{t_2} \frac{dt}{\sqrt{U}} \le \int_{t_-}^{t_+} \sqrt{\frac{t^2+\ell^2}{(t_+ - t)(t - t_-)}} dt < \infty \tag{B.1.22}$$

(recall that $x^{-1/2}$ is integrable near zero). This provides the required bound on s, independently of γ.

For $\epsilon = 0$ some more work is needed. If $p_\| \neq 0$ (B.1.21) gives

$$|s(t_2) - s(t_1)| \le \frac{2\ell|t - t_0|}{|p_\||} \le \frac{2\ell|t_+ - t_-|}{|p_\||}\,,$$

providing again the desired bound on s. Similarly for $p_3 \neq 0$

$$|s(t_2) - s(t_1)| \le \int_{t_1}^{t_2} \frac{|\sin\theta|\sqrt{t^2+\ell^2}}{\sqrt{U}|p_3|} \le \frac{1}{|p_3|}\int_{t_-}^{t_+} \frac{t^2+\ell^2}{\sqrt{(t_+ - t)(t - t_-)}} dt < \infty\,.$$

The only case which has not been covered so far is $\varepsilon = p_\| = p_3 = 0$. Then φ is constant by (B.1.17b), with $p_2\cos\varphi - p_1\sin\varphi \neq 0$ (otherwise $dt/ds = 0$ by (B.1.17d), which is not possible on a non-trivial geodesic). Equation (B.1.21) gives

$$|s(t_2)-s(t_1)| \le \frac{1}{|p_2\cos\varphi - p_1\sin\varphi|}\int_{t_-}^{t_+}\frac{\sqrt{t^2+\ell^2}}{\sqrt{U}}dt < \infty\,,$$

and the proof is complete. □

We continue with

PROPOSITION B.1.4 *Let $q(t)$ be as in (B.1.20). For any maximally extended geodesic γ the limits*

$$\lim_{t\to t_\pm} q(t)$$

exist, and are continuous functions of the initial values for γ.

PROOF: From (B.1.17a) and (B.1.17b) we have

$$\begin{aligned}\left|\frac{d\theta}{dt}\right| &= \left|\frac{d\theta}{ds}\frac{ds}{dt}\right| \le \frac{1}{\sqrt{U(t^2+\ell^2)}}\,,\\ \left|\frac{d\varphi}{dt}\right| &\le \frac{1}{\sqrt{U(t^2+\ell^2)}}\,.\end{aligned}$$

Note that the right-hand side is in $L^1([t_-,t_+])$. The existence of the limits follows by integration, while the continuity of the limits upon the initial values is a consequence of the Lebesgue dominated convergence theorem. □

It remains to analyse the behavior of ζ near the end points. For definiteness we assume $dt/ds > 0$ and study $t\to t_+$, the analysis near t_- is essentially identical. It is again best to use directly the t parameterisation, rewriting (B.1.17c) as

$$\frac{d\zeta}{dt} = \frac{d\zeta}{ds}\frac{ds}{dt} = \frac{\frac{p_\|}{(2\ell)^2U} - \frac{\cos\theta p_3}{(t^2+\ell^2)\sin^2\theta}}{\sqrt{\left(\frac{p_\|}{2\ell}\right)^2 + U\Big(-\varepsilon + \frac{(p_2\cos\varphi - p_1\sin\varphi)^2}{t^2+\ell^2} + \frac{p_3^2}{(t^2+\ell^2)\sin^2\theta}\Big)}}\,. \tag{B.1.23}$$

As elsewhere, we assume that coordinates are chosen so that θ is uniformly bounded away from 0 and $\pi/2$. If $p_\| \neq 0$ we obtain

$$\frac{d\zeta}{dt} = \frac{p_\|}{|p_\||}\frac{1}{(2\ell)U} + O(t_+ - t)\,, \tag{B.1.24}$$

leading to a logarithmic divergence of ζ:

$$\zeta(t) = -\frac{p_\|}{|p_\||}\frac{\ln(t_+ - t)}{2\ell} + \zeta_+ + O(t_+ - t)\,, \tag{B.1.25}$$

for some constant ζ_+.

For $p_\| = 0$ we obtain completely different behavior:

$$\frac{d\zeta}{dt} = -\frac{\cos\theta p_3}{\sqrt{U\Big(-\varepsilon + \frac{(p_2\cos\varphi - p_1\sin\varphi)^2}{t^2+\ell^2} + \frac{p_3^2}{(t^2+\ell^2)\sin^2\theta}\Big)}(t^2+\ell^2)\sin^2\theta}\,. \tag{B.1.26}$$

An analysis as in the proof of Proposition B.1.3 shows that we have

$$\zeta(t) = \zeta_+ + O(t_+ - t)\,, \tag{B.1.27}$$

for some constant ζ_+.

Comparing (B.1.25) with (B.1.10) we have obtained:

THEOREM B.1.5 (Hajiček [226]) *Let γ be a maximally extended causal geodesic in the Taub spacetime $(\mathscr{M}_{(t_-,t_+)}, g)$. Then:*

1. *γ can be smoothly extended to $\{t = t_+\}$ in the Taub-NUT spacetime $(\mathscr{M}_{\mathbb{R}}, g_+)$ if and only if $p_{\|} < 0$.*

2. *Similarly, γ can be smoothly extended to $\{t = t_+\}$ in the Taub-NUT spacetime $(\mathscr{M}_{\mathbb{R}}, g_-)$ if and only if $p_{\|} > 0$.*

3. *For geodesics with $p_{\|} = 0$, and only for those geodesics, the limits*

$$\lim_{t\to t_+} (\zeta(t), \theta(t), \varphi(t))$$

exist.

PROOF: Necessary conditions are obtained from Equation (B.1.10): extendibility in $(\mathscr{M}_{\mathbb{R}}, g_+)$ requires that

$$(\zeta + \int_{t_0}^{t} [2\ell U(s)]^{-1} ds, \theta, \varphi)$$

admits a limit along γ as t approaches t_+. It follows from (B.1.25), (B.1.27) and Proposition B.1.4 that this limit exists if and only if $p_{\|} < 0$. If that last condition holds, then γ acquires an end point on $\{t = t_+\}$, and smooth extendibility readily follows. The proof for $(\mathscr{M}_{\mathbb{R}}, g_-)$ is similar. □

B.1.2 Inequivalent extensions of the maximal globally hyperbolic region

Let us summarise what we learned so far about the Taub-NUT spacetime:

1. $(\mathscr{M}_{(t_-,t_+)}, g)$ is globally hyperbolic.

2. $(\mathscr{M}_{\mathbb{R}}, g_\pm)$ are extensions of $(\mathscr{M}_{(t_-,t_+)}, g)$, in the sense that there exists an isometric diffeomorphism from $(\mathscr{M}_{(t_-,t_+)}, g)$ into its image in $(\mathscr{M}_{\mathbb{R}}, g_+)$, similarly for $(\mathscr{M}_{\mathbb{R}}, g_-)$.

3. Viewed as a subset of each of $(\mathscr{M}_{\mathbb{R}}, g_\pm)$, $(\mathscr{M}_{(t_-,t_+)}, g)$ is maximal globally hyperbolic, in the sense that any larger subset of $(\mathscr{M}_{\mathbb{R}}, g_\pm)$ containing $(\mathscr{M}_{(t_-,t_+)}, g)$ is not globally hyperbolic. This follows from the fact that the hypersurfaces $\{t = t_\pm\}$, both in $(\mathscr{M}_{\mathbb{R}}, g_+)$ and in $(\mathscr{M}_{\mathbb{R}}, g_-)$, are threaded by null geodesics, which therefore do not enter the globally hyperbolic region $\{t \in (t_-, t_+)\}$.

4. By Theorem B.1.5 there exist null geodesics starting in $(\mathscr{M}_{(t_-,t_+)}, g)$ which extend into $(\mathscr{M}_{\mathbb{R}}, g_+)$ but which do not extend into $(\mathscr{M}_{\mathbb{R}}, g_-)$, and vice-versa.

This last property gives a precise sense in which $(\mathscr{M}_{\mathbb{R}}, g_+)$ and $(\mathscr{M}_{\mathbb{R}}, g_-)$ are different extensions of $(\mathscr{M}_{(t_-,t_+)}, g)$. One is therefore tempted to draw the following conclusion, essentially due to Misner [324] (compare [124, 140]):

THEOREM B.1.6 *The Taub-NUT spacetime $(\mathscr{M}_{(t_-,t_+)}, g)$ provides an example of maximal globally hyperbolic vacuum spacetime which possesses smooth non-isometric extensions.*

While this theorem is correct, the conclusion is somewhat hasty as reached because $(\mathscr{M}_{\mathbb{R}}, g_+)$ and $(\mathscr{M}_{\mathbb{R}}, g_-)$ are actually isometric. Indeed, it is clear from (B.1.11) that the map

$$t \mapsto -t \tag{B.1.28}$$

maps $g_\pm$ to $g_\mp$. Another isometry, which offers the advantage of preserving time-orientation, is provided by the map

$$(\zeta, \varphi) \mapsto (-\zeta, -\varphi)\,. \tag{B.1.29}$$

We see from (B.1.4) that (B.1.29) is the restriction to $S^3 \subset \mathbb{R}^4$ of the real-analytic map

$$(x, y, z, w) \mapsto (x, -y, z, -w)\,,$$

and thus clearly a smooth map, in fact real-analytic, from S^3 to S^3. Note, however, that neither of the maps (B.1.28) and (B.1.29) belongs to the connected component of the identity of the group of diffeomorphisms of $\mathscr{M}_{\mathbb{R}}$.

As such, a correct proof of Theorem B.1.6 proceeds as follows: Consider the four spacetimes

$$(\mathscr{M}_{(t_-,\infty)}, g_\pm) \text{ and } (\mathscr{M}_{(-\infty,t_+)}, g_\pm)$$

These are all extensions of $(\mathscr{M}_{(t_-,t_+)}, g)$. We can glue together $(\mathscr{M}_{(t_-,\infty)}, g_\pm)$ with $(\mathscr{M}_{(-\infty,t_+)}, g_\pm)$ using the identity map on the overlap $(\mathscr{M}_{(t_-,t_+)}, g_\pm)$, which results in the spacetimes $(\mathscr{M}_{\mathbb{R}}, g_\pm)$. But another possibility is to glue $(\mathscr{M}_{(t_-,\infty)}, g_+)$ with $(\mathscr{M}_{(-\infty,t_+)}, g_-)$ by using the map (B.1.29) on the overlap; let us denote the resulting spacetime by $(\mathscr{M}_{\mathbb{R}}, \breve{g}_+)$. One can likewise glue $(\mathscr{M}_{(t_-,\infty)}, g_-)$ with $(\mathscr{M}_{(-\infty,t_+)}, g_+)$ using (B.1.29), which results in a spacetime, denoted by $(\mathscr{M}_{\mathbb{R}}, \breve{g}_-)$, isometric to $(\mathscr{M}_{\mathbb{R}}, \breve{g}_+)$. A straightforward argument based on the properties of maximally extended geodesics pointed-out in Theorem B.1.5 shows that $(\mathscr{M}_{\mathbb{R}}, g_+)$ and $(\mathscr{M}_{\mathbb{R}}, \breve{g}_+)$ are *not* isometric, and thus provide the example needed for Theorem B.1.6.

REMARK B.1.7 The argument just given provides two non-isometric analytic vacuum extensions, say $(\mathscr{M}_{\mathbb{R}}, g_+)$ and $(\mathscr{M}_{\mathbb{R}}, \breve{g}_+)$. These are actually maximal but, were they not, one could take any maximal analytic extension of each, obtaining two non-isometric maximal analytic extensions. An uncountable number of further distinct maximal analytic vacuum extensions can be obtained by removing sets, say $\mathscr{U}$, from $\mathscr{M}_{\mathbb{R}}$ such that 1) $\mathscr{U}$ does not intersect the globally hyperbolic region $\mathscr{M}_{(t_-,t_+)}$ and, e.g., 2) such that $\mathscr{M}_{\mathbb{R}} \setminus \mathscr{U}$ is *not* simply connected. Taking any maximal analytic vacum extension of the universal covering space of $(\mathscr{M}_{\mathbb{R}} \setminus \mathscr{U}, g_+)$ leads to the an uncountable plethora of non-isometric maximal analytic vacuum extensions. □

Further related arguments can be found in [140].

B.1.3 Conformal completions at infinity

Each of the metrics $g_\pm$ can be smoothly conformally extended to the boundary at infinity "$t = \infty$" by introducing

$$x = 1/t\,,$$

so that (B.1.11) becomes

$$\begin{aligned} g_\pm \;\; &= \;\; x^{-2}\Big(\mp 4\ell(d\zeta + \cos\theta d\varphi)dx \\ &\quad +(2\ell)^2x^2U(d\zeta + \cos\theta d\varphi)^2 + (1+\ell^2x^2)(d\theta^2 + \sin^2\theta d\varphi^2)\Big)\,. \end{aligned} \tag{B.1.30}$$

In each case this leads to a conformal boundary at infinity

$$\mathscr{I} := \{x = 0\}\,,$$

called "Scri", diffeomorphic to S^3 with a metric, obtained by crossing-out the x^{-2} factor in (B.1.30), which smoothly extends to $\mathscr{I}$. Now, the isometry (B.1.29) between g_+ and g_- given by

$$(x, \zeta, \theta, \varphi) \to (x, -\zeta, \theta, -\varphi)\,,$$

shows that the two conformal completions so obtained are isometric. However, in addition to the two ways of attaching Scri to the region $t \in (t_+, \infty)$ there are the two corresponding ways of extending this region across the Cauchy horizon $t = t_+$ pointed-out in Section B.1.2, leading to four manifolds with boundary $(\mathscr{M}_{\mathbb{R}}, g_\pm)$ and $(\mathscr{M}_{\mathbb{R}}, \breve{g}_\pm)$. As already seen, each of the manifolds from one pair is *not* isometric to one from the other. Taking into account the corresponding completion at "$t = -\infty$", and the two extensions across the Cauchy horizon $t = t_-$, one is led to four inequivalent conformal completions of each of the two inequivalent maximally analytically-extended standard Taub-NUT spacetimes.

B.1.4 Taub-NUT metrics and quaternions

Isometries of Taub-NUT metrics are best understood using quaternions, we start by recalling some elementary facts about those. Consider $\mathbb{R} \times \mathbb{R}^3$, where $\mathbb{R}^3$ is equipped with an orientation and with the product Euclidean metric. We will write

$$w + \vec{x}\,, \qquad w \in \mathbb{R}\,,\ \vec{x} \in \mathbb{R}^3\,,$$

for the element $(w, \vec{x})$. The set $\mathbb{R} \times \mathbb{R}^3$ is equipped with the product

$$(w + \vec{x})(v + \vec{y}) = wv - (\vec{x}, \vec{y}) + w\vec{y} + v\vec{x} + \vec{x} \times \vec{y}\,, \tag{B.1.31}$$

where $(\cdot, \cdot)$ is the metric on $\mathbb{R}^3$ and $\times$ is the associated vector product: if e_i is an oriented ON-basis in $\mathbb{R}^3$, then

$$\vec{x} \times \vec{y} = \epsilon_{ijk} x^i y^j e_k\,. \tag{B.1.32}$$

Here, as elsewhere, ϵ_{ijk} is completely symmetric and satisfies $\epsilon_{123} = 1$. It is well known, though somewhat tedious to show, that $\mathbb{R} \times \mathbb{R}^3$ equipped with the usual vector space-structure, and with the above product, is a field, called *the field of quaternions.*

If $p = w + \vec{x}$ one defines its *conjugate* $\overline{p}$ by the formula

$$\overline{p} = w - \vec{x}\,.$$

Note that

$$\overline{\overline{p}} = p\,, \qquad \overline{p\,q} = \overline{q}\,\overline{p}\,. \tag{B.1.33}$$

A quaternion is called *real* if

$$\overline{p} = p \iff p = w + i\vec{0} \quad (\text{we then write } p \in \mathbb{R})\,,$$

and *pure* if $\overline{p} = -p$. The quaternionic product is non-commutative in general, however the following property is often used:

$$p = \overline{p} \implies \forall q \;\; pq = qp\,.$$

The norm $|p|$ of $p = w + \vec{x}$ is defined as

$$|p| = \sqrt{N(p)}\,, \qquad \text{where} \;\; N(p) := p\overline{p} = w^2 + |\vec{x}|^2 = \overline{p}p\,. \tag{B.1.34}$$

The set of unit quaternions (*i.e.*, quaternions satisfying $|p| = 1$) is thus naturally diffeomorphic to a three-dimensional sphere.

Let R_q, respectively L_q, denote the *right translations*, respectively *left translations*:

$$R_q(p) = pq^{-1}\,, \qquad L_q(p) = qp\,.$$

Both maps preserve the unit sphere provided q is unit:

$$|L_q(p)|^2 = qp\overline{qp} = q\underbrace{p\overline{p}}_{\in\mathbb{R}}\overline{q} = p\overline{p}q\overline{q} = p\overline{p} = |p|^2 \;\text{ if }\; q\overline{q} = 1\,,$$

$$|R_q(p)|^2 = pq^{-1}\overline{pq^{-1}} = p\overline{q}\overline{p\overline{q}} = p\underbrace{\overline{q}q}_{=1}\overline{p} = |p|^2 \;\text{ if }\; q\overline{q} = 1\,.$$

One easily checks the composition rules

$$R_{q_1}R_{q_2} = R_{q_1q_2}\,, \quad L_{q_1}L_{q_2} = L_{q_1q_2}\,.$$

It follows that R_q and L_q each define a homomorphism from the sphere of unit quaternions to $SO(4)$. Note that

$$L_{q_1}R_{q_2}(p) = q_1pq_2^{-1} = R_{q_2}L_{q_1}(p)\,,$$

hence the left and right translations commute.

It can be shown (*cf., e.g.* [58, Vol. I, Section 8.9]) that

$$R\times L : S^3\times S^3 \to SO(4)$$

is surjective, with kernel $\{(1,1),(-1,-1)\}$, but this is irrelevant for our purposes.

The maps $(R_q)^*$ and $(L_q)^*$ preserve the Euclidean metric δ on $\mathbb{R}^4$; by restriction, they also preserve the round metric on S^3. In order to see that, let us write (x^α) for (w,x^i). The map $p\to qp$ is linear and therefore there exists a matrix $Q^\alpha{}_\beta$ such that, if we represent p by x^μ, then qp is represented by $Q^\mu{}_\nu x^\nu$. Let $X=X^\mu\partial_\mu$ be a vector in $T_p\mathbb{R}^4$, by definition we have

$$\Big(\Big((L_q)_*X\Big)f\Big)(x^\beta) = X^\alpha\partial_\alpha\Big(f(Q^\mu{}_\nu x^\nu)\Big) = \Big(Q^\sigma{}_\alpha X^\alpha\partial_\sigma f\Big)(Q^\mu{}_\nu x^\nu)\,.$$

This shows that if we identify the vector $X^w\partial_w + X^i\partial_i$ with the quaternion $X^w+X^ie_i$, where e_i is the canonical (ON, oriented) basis of $\mathbb{R}^3$, then

$$\Big((L_q)_*X\Big)_{qp} = qX_p\,. \tag{B.1.35}$$

In this formula we have used a subscript to indicate the point at which the vector is attached.

(A proper way of handling the identification buisiness above would be to define a map, say ι, by the formula $\iota(X^w\partial_w+X^i\partial_i)=X^w+X^ie_i$, then (B.1.35) would read $(L_q)_*X = \iota^{-1}(q\iota(X))$, a rather heavy notation. We hope that equations such as (B.1.35) will not introduce an undue amount of confusion.)

One similarly finds

$$\Big((R_q)_*X\Big)_{pq^{-1}} = X_pq^{-1}\,. \tag{B.1.36}$$

Now, still using the identification above, the Euclidean scalar product δ of two vectors $X,Y\in T_p\mathbb{R}^4$ can be written as

$$\delta(X,Y) = \frac{1}{2}(\overline{X}Y+\overline{Y}X) \tag{B.1.37}$$

(this follows immediately by polarisation from (B.1.34)). Equation (B.1.35) then gives, assuming again $|q|=1$

$$\begin{aligned}\Big((L_q)^*\delta\Big)(X,Y) &= \delta((L_q)_*X,(L_q)_*Y) = \frac{1}{2}(\overline{qX}Y+\overline{qY}qX)\\ &= \frac{1}{2}(\overline{X}\,\overline{q}qY+\overline{Y}\,\overline{q}qX) = \frac{1}{2}(\overline{X}Y+\overline{Y}X) = \delta(X,Y)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\Big((R_q)^*\delta\Big)(X,Y) &= \delta((R_q)_*X,(R_q)_*Y) = \frac{1}{2}(\overline{X\overline{q}}Y\overline{q}+\overline{Y\overline{q}}X\overline{q}) \\ &= \frac{1}{2}q\underbrace{(\overline{X}Y+\overline{Y}X)}_{\in\mathbb{R}}\overline{q} = \frac{1}{2}\underbrace{q\overline{q}}_{=1}(\overline{X}Y+\overline{Y}X) = \delta(X,Y)\,.\end{aligned}$$

A vector field X is called *left-invariant* if $(L_q)_*X = X$ for all unit quaternions q; similarly, X is called *right-invariant* if $(R_q)_*X = X$ for all unit quaternions q. Setting $p = 1$ in (B.1.35) and (B.1.36) one obtains necessary conditions for invariance:

$$(L_q)_*X = X \implies X_q = qX_1\,. \tag{B.1.38}$$

$$(R_q)_*X = X \implies X_q = X_1 q\,. \tag{B.1.39}$$

One easily checked that these are also sufficient, thus any $X_1 \in T_{(1,0,0,0)}S^3$ (recall that we order the coordinates as w,x,y,z, so that the unity is the quaternion $(1,0,0,0)$) defines left- and right-invariant vector fields by the formulae above. For example, if (B.1.39) holds, then from (B.1.36) we find

$$\Big((R_q)_*X\Big)_{pq^{-1}} = X_p q^{-1} = X_1 p q^{-1} = X_{pq^{-1}}\,.$$

We let ξ_1 be the the right-invariant vector field associated with $-e_2/2$, and ξ_2 the right-invariant vector field associated with $e_1/2$, and ξ_3 the right-invariant vector field associated with $e_3/2$. From (B.1.39) with $q = (w,x,y,z)$ we obtain[2]

$$\xi_1 = \frac{1}{2}(y\partial_w - z\partial_x - w\partial_y + x\partial_z)\,, \tag{B.1.40a}$$

$$\xi_2 = \frac{1}{2}(-x\partial_w + w\partial_x - z\partial_y + y\partial_z)\,, \tag{B.1.40b}$$

$$\xi_3 = \frac{1}{2}(-z\partial_w - y\partial_x + x\partial_y + w\partial_z)\,. \tag{B.1.40c}$$

(The right-invariant vector fields on $\mathbb{R}^4$ with $X_1 = \partial_w$ lead to a vector field over S^3 which is not tangent to the sphere, and therefore will not be of interest to us.)

The left-invariant vector fields tangent to S^3 can likewise be calculated from (B.1.38). Letting η_i be the vector field with $X_1 = e_i/2$ one obtains

$$\eta_1 = -\frac{1}{2}(x\partial_w - w\partial_x - z\partial_y + y\partial_z)\,, \tag{B.1.41a}$$

$$\eta_2 = -\frac{1}{2}(y\partial_w + z\partial_x - w\partial_y - x\partial_z)\,, \tag{B.1.41b}$$

$$\eta_3 = -\frac{1}{2}(z\partial_w - y\partial_x + x\partial_y - w\partial_z)\,. \tag{B.1.41c}$$

Let $\mathring{h}$ be the usual round metric on S^3 induced by the flat metric δ on $\mathbb{R}^4$. Set

$$\sigma_i = -4\mathring{h}(\eta_i,\cdot) \in \Gamma T_*S^3\,. \tag{B.1.42}$$

[2]The somewhat funny ordering of the ξ_i's is chosen for consistency of notation with [325].

We obtain

$$\sigma_1 = 2i^*_{S^3}(xdw - wdx - zdy + ydz)\,, \tag{B.1.43a}$$
$$\sigma_2 = 2i^*_{S^3}(ydw + zdx - wdy - xdz)\,, \tag{B.1.43b}$$
$$\sigma_3 = 2i^*_{S^3}(zdw - ydx + xdy - wdz)\,, \tag{B.1.43c}$$

and we recognise the one-forms (B.1.3). This leads to:

PROPOSITION B.1.8 *The metric (B.1.1) is invariant under left translations. Further, the connected component of the identity of the group of isometries of g is $SO(3)\times SO(2)$, with the vector fields (B.1.40) and (B.1.41c) spanning the set of Killing vectors of g.*

PROOF: We have shown that the usual round metric on S^3 is invariant under left translations, and so are the vector fields η_i. This implies that the forms σ_i are left-invariant,

$$(L_q)_*\sigma_i = \sigma_i\,.$$

Hence, any metric constructed out of the σ_i's with coefficients which are constant over S^3 will be invariant under left translations, proving our first claim.

For $q\in S^3$ consider the one-parameter group of isometries of $\mathbb{R}^4$ defined as $L_{\exp(\alpha q/2)}$. Denote by X the associated generator,

$$X = \frac{dL_{\exp(\alpha q/2)}}{d\alpha}\Big|_{\alpha=0}\,.$$

We have

$$\Big(\Big(R_{\hat q}\Big)_*X\Big)f(p) = X(f(p\hat q)) = \Big(\frac{d}{d\alpha}f(\exp(\alpha q/2)p\hat q)\Big)\Big|_{\alpha=0} = \Big(X(f)\Big)(p\hat q)\,,$$

which shows that X is right-invariant. (Similarly, generators of one parameter groups of right translations provide left-invariant vector fields.) But we have shown that right-invariant vector fields tangent to S^3 are necessarily linear combinations of (B.1.40). As S^3 is three-dimensional, we conclude that all three vector fields (B.1.40) must be Killing.

The metric g has one more continuous symmetry, which can be seen as follows. Note, first, that every vector field X is invariant under its own flow:

$$\mathscr{L}_X X = [X,X] = 0\,.$$

Since $\mathring{h}$ is right-invariant, we then have, using the Leibniz rule,

$$\mathscr{L}_{\eta_3}\sigma_3 = -4\mathscr{L}_{\eta_3}\left(\mathring{h}(\eta_3,\cdot)\right) = 0\,. \tag{B.1.44}$$

Next, the set of quaternions $\{1,e_i\}$, viewed as vector fields on $\mathbb{R}^4$ using our identifications described above, forms at each point $p\in\mathbb{R}^4$ an orthonormal basis of the tangent space $T_p\mathbb{R}^4$. At $(1,0,0,0)$ this basis has the property that the first vector $1=\partial_w$ is normal to S^3, hence $\{e_i\}$ forms an orthornomal basis of $T_{(1,0,0,0)}S^3$. But, up to permutations and signs, the collection of vector fields

$\{2\xi_i\}$ coincides with $\{e_i\}$ at $T_{(1,0,0,0)}S^3$. Right-invariance and (the obvious) transitivity imply that $\{2\xi_i\}$ forms an $\mathring{h}$-orthonormal basis of T_pS^3 at every $p \in S^3$. From the definition (B.1.42) we conclude that

$$\mathring{h} = 4(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)\,.$$

Since $\mathscr{L}_{\eta_3}\mathring{h} = 0$ we obtain

$$\mathscr{L}_{\eta_3}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) = \mathscr{L}_{\eta_3}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) - \mathscr{L}_{\eta_3}(\sigma_3 \otimes \sigma_3) = 0\,.$$

It now directly follows from the form (B.1.1) of g that η_3 is a Killing vector, as desired. Note that η_3 is left-invariant, therefore it commutes with the ξ_a's. This shows the product structure of the $SO(3) \times SO(2)$ isometries obtained so far. Here the first factor is the left translations, while the second factor is the right translations generated by η_3.

To see that g does not have any further Killing vectors we use the facts that a) every Killing vector of g is necessarily a Killing vector of each of the metrics $g|_{t=\mathrm{const}}$; b) the dimension of an isometry group of a metric on S^3 cannot be five [281, Theorem 8.17], and c) if that dimension is six, then $g|_{t=\mathrm{const}}$ is proportional to the canonical metric on S^3, which is not the case for the metrics (B.1.1). □

INCIDENTALLY: For completeness we give a direct calculation of the generators of left translations. Suppose that q is a *pure* unit quaternion q. We then have $\overline{q} = -q$ hence $q^2 = -q\overline{q} = -|q|^2 = -1$, so that

$$q^{2n} = (-1)^n\,, \qquad q^{2n+1} = (-1)^n q\,.$$

This gives

$$\begin{aligned}
\exp\left(\frac{\alpha q}{2}\right) &= \sum_{n=0}^{\infty} \frac{(\alpha q)^n}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(\alpha q)^{2n}}{2^{2n}(2n)!} + \sum_{n=0}^{\infty} \frac{(\alpha q)^{2n+1}}{2^{2n+1}(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{2^{2n}(2n)!} + q\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{2^{2n+1}(2n+1)!} \\
&= \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) q\,.
\end{aligned}$$

It is usual to write i for e_1, j for e_2 and k for e_3, this leads to the multiplication rules

$$i^2 = j^2 = k^2 = -1\,, \quad ij = -ji = k\,, \quad jk = -kj = i\,, \quad ki = -ik = j\,. \tag{B.1.45}$$

Let

$$Y_1 = \frac{dL_{\exp(\alpha i/2)}}{d\alpha}\Big|_{\alpha=0}$$

be the vector field generated by $L_{\exp(\alpha i/2)}$, thus $Y_1(p)$ is tangent, at $\alpha = 0$, to the curve

$$\alpha \to \left(\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) i\right) p\,.$$

The cos terms give no contribution after differentiating and setting $\alpha = 0$. If we write $p = w + xi + yj + zk$, we obtain, using (B.1.45),

$$\sin\Big(\frac{\alpha}{2}\Big)i\Big(w + xi + yj + zk\Big) = \sin\Big(\frac{\alpha}{2}\Big)\Big(-x + wi - zj + yk\Big)$$

Taking a derivative at $\alpha = 0$ leads to

$$Y_1 = \frac{1}{2}(-x\partial_w + w\partial_x - z\partial_y + y\partial_z)\,. \tag{B.1.46a}$$

One recognises the field ξ_2 of (B.1.40b). Similar calculations show that

$$\begin{aligned} Y_2 &:= \frac{dL_{\exp(\alpha j/2)}}{d\alpha}\Big|_{\alpha=0} \\ &= \frac{1}{2}(-y\partial_w + z\partial_x + w\partial_y - x\partial_z) = -\xi_1\,, \qquad \text{(B.1.46b)} \\ Y_3 &:= \frac{dL_{\exp(\alpha k/2)}}{d\alpha}\Big|_{\alpha=0} \\ &= \frac{1}{2}(-z\partial_w - y\partial_x + x\partial_y + w\partial_z) = \xi_3\,. \qquad \text{(B.1.46c)} \end{aligned}$$

□

We close this section by an analysis of the transformation properties of the Killing vector fields of g under left translations. First, η_3 is left-invariant by construction. Next, consider any right-invariant vector field X on S^3. From (B.1.35) and (B.1.38) we have

$$\Big((L_q)_*X\Big)_{qp} = qX_p = qX_1p = qX_1q^{-1}qp = \Big((L_q)_*X\Big)_1 qp\,.$$

This shows that X_1 transforms according to the rule

$$\Big((L_q)_*X\Big)_1 = \rho_q X_1\,, \qquad \text{where} \quad \rho_q p := qpq^{-1}\,. \tag{B.1.47}$$

We have

PROPOSITION B.1.10 *The action of S^3 in (B.1.47) maps pure quaternions $p \in \mathbb{R}^3$ to pure quaternions by $SO(3)$-rotations of $\mathbb{R}^3$. The map*

$$S^3 \ni q \to \rho_q \in SO(3)$$

is a surjective homomorphism, with kernel $\{\pm 1\}$.

REMARK B.1.11 The last property shows that $SO(3)$ is diffeomorphic to the real projective space $\mathbb{R}P^3 = S^3/\{\pm 1\}$.

PROOF: We follow [58, Section 8.9], and start by noting that a quaternion p is pure if and only if p^2 is a negative real number. So, assuming p is pure, we have

$$(\rho_q p)^2 = (qpq^{-1})(qpq^{-1}) = q\underbrace{p^2}_{\in\mathbb{R}}q^{-1} = p^2qq^{-1} = p^2 \in \mathbb{R}^-\,.$$

It follows that the action (B.1.47) leaves the set of pure quaternions invariant, as desired.

We have $\rho_q = L_q \circ R_q$, and since each of those maps preserves $|p|$, ρ_q also does, showing that ρ_q is an isometry of $(\mathbb{R}^3, \delta)$.

To show surjectivity, suppose that q is pure, we will show that $-\rho_q$ is a reflection in the plane orthogonal to q. Since $O(3)$ is generated by reflections [58, Theorem 8.2.12], surjectivity will follow. First, if $p = q$ we have

$$-\rho_q(q) = -qqq^{-1} = -q\,.$$

Next, suppose that p is orthogonal to q, from (B.1.37) one finds

$$p \perp q \iff p\overline{q} + q\overline{p} = 0 \iff p\overline{q} = -q\overline{p}\,.$$

We then obtain

$$-\rho_q(p) = -qp\underbrace{q^{-1}}_{=\overline{q}} = -q\underbrace{p\overline{q}}_{=-q\overline{p}} = q^2\overline{p} = -\overline{p} = p\,,$$

so that ρ_q is indeed the claimed reflection.

To finish the proof, suppose that $\rho_q p = p$ for each pure quaternion $p = \vec{y}$. Writing q as $w + \vec{x}$ we have

$$\begin{aligned} \rho_q p = p &\iff qp = pq \\ &\iff w\vec{y} + \vec{x}\times\vec{y} = (w+\vec{x})\vec{y} = \vec{y}(w+\vec{x}) = w\vec{y} + \vec{y}\times\vec{x} \\ &\iff \vec{x}\times\vec{y} = 0\,, \end{aligned}$$

so that $\vec{x}$ is proportional to $\vec{y}$, possibly with a zero proportionality factor. Since this is true for all $\vec{y}$ we obtain $\vec{x} = 0$, and as $q \in S^3$ we conclude that $w = \pm 1$. □

B.2 Robinson–Trautman spacetimes.

The Robinson–Trautman (RT) metrics are vacuum metrics which can be viewed as evolving from data prescribed on a single null hypersurface.

From a physical point of view, the RT metrics provide examples of isolated gravitationally radiating systems. In fact, these metrics were hailed to be the first exact nonlinear solutions describing such a situation. Their discovery [381] was a breakthrough in the conceptual understanding of gravitational radiation in Einstein's theory.

The RT metrics were the only example of vacuum dynamical black holes without any symmetries and with exhaustively described global structure until the construction, in 2013 [171], of a large class of such spacetimes using "scattering data" at the horizon and at future null infinity. Further dynamical black holes have been meanwhile constructed in 2018 in [172, 276], by evolution of small perturbations of Schwarzschild initial data. See also [148] for asymptotically many-black-hole dynamical vacuum spacetimes with "a piece of $\mathscr{I}$", and [262] for a class of vacuum multi-black-holes with a positive cosmological constant.

There are several interesting features exhibited by the RT metrics: First, and rather unexpectedly, in this class of metrics the Einstein equations reduce to a single parabolic fourth order equation. Next, the evolution is unique within the class, in spite of a "naked singularity" at $r = 0$. Last but not least, they possess remarkable extendibility properties.

By definition, the Robinson–Trautman spacetimes can be foliated by a *null, hypersurface-orthogonal, shear-free, expanding* geodesic congruence. It has been shown by Robinson and Trautman [363] that in such a spacetime there always exists a coordinate system in which the metric takes the form

$$\mathbf{g} = -\Phi\, du^2 - 2du\, dr + r^2 e^{2\lambda} \underbrace{\mathring{g}_{ab}(x^c)\, dx^a\, dx^b}_{=:\mathring{g}}, \quad \lambda = \lambda(u, x^a)\,, \quad \text{(B.2.1)}$$

$$\Phi = \frac{R}{2} + \frac{r}{12m}\Delta_g R - \frac{2m}{r}\,, \quad R = R(g_{ab}) \equiv R(e^{2\lambda}\mathring{g}_{ab})\,, \qquad \text{(B.2.2)}$$

where the x^a's are local coordinates on the two-dimensional smooth Riemannian manifold $({}^2M, \mathring{g})$, $m \neq 0$ is a constant which is related to the total Trautman-Bondi mass of the metric, and R is the Ricci scalar of the metric $g := e^{2\lambda}\mathring{g}$. In writing (B.2.1)-(B.2.2) we have ignored those spacetimes which admit a congruence as above and where the parameter m vanishes.

The Einstein equations for a metric of the form (B.2.1) reduce to a single equation

$$\partial_u g_{ab} = \frac{1}{12m}\Delta_g R\, g_{ab} \quad \Longleftrightarrow \quad \partial_u \lambda = \frac{1}{24m} e^{-2\lambda} \Delta_{\mathring{g}}(e^{-2\lambda}(\mathring{R} - 2\Delta_{\mathring{g}}\lambda))\,, \quad \text{(B.2.3)}$$

where Δ_g is the Laplace operator of the two-dimensional metric $g = g_{ab}dx^a dx^b$, and $\mathring{R}$ is the Ricci scalar of the metric $\mathring{g}$.

Equation (B.2.3) will be referred to as the RT equation. It is first-order in the "time" u, fourth-order in the space-variables x^a, and belongs to the family of parabolic equations. The Cauchy data for (B.2.3) consist of a function $\lambda_0(x^a) \equiv \lambda(u = u_0, x^a)$, which is equivalent to prescribing the metric $g_{\mu\nu}$ of the form (B.2.1) on a null hypersurface $\{u = u_0,\ r \in (0, \infty)\} \times {}^2M$. Without loss of generality, translating u if necessary, we can assume that $u_0 = 0$.

Note that the initial data hypersurface asymptotes to a curvature singularity at $r = 0$, with the scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverging as r^{-6} when $r = 0$ is approached. This is a "white hole singularity", familiar to all known stationary black hole spaces-times.

The RT equation (B.2.3) has been considered in a completely different context by Calabi [80].

The function $\lambda \equiv 0$ solves (B.2.3) when $\mathring{g}$ is the unit round metric on the sphere. The metric (B.2.1) is then the Schwarzschild metric in retarded Eddington-Finkelstein coordinates.

It follows from the theory of parabolic equations that for $m < 0$ the evolution problem for (B.2.3) is locally well posed backwards in u, while for $m > 0$ the RT equation can be locally solved forwards in u. Redefining u to $-u$ transforms

(B.2.3) with $m < 0$, $u \leq 0$ to the same equation with a new mass parameter $-m > 0$ and with $u \geq 0$. Thus, when discussing (B.2.3) it suffices to assume $m > 0$. On the other hand, the global properties of the associated spacetimes will be different, and will need separate discussion.

Note that solutions of typical parabolic equations, including (B.2.3), immediately become analytic. This implies that for smooth but *not analytic* initial functions λ_0, the equation will *not* be solvable backwards in u when $m > 0$, or forwards in u when $m < 0$.

In [121, 122, 156] the following has been proved:

1. When $m > 0$ solutions of (B.2.3) with, say smooth, initial data at $u = 0$ exist for all $u \geq 0$. The proof consists in showing that all Sobolev norms of the solution remain finite during the evolution. The first key to this is the monotonicity of the Trautman-Bondi mass, which for RT metrics equals [395]
$$m_{\rm TB} = \frac{m}{4\pi}\int_{S^2} e^{3\lambda} d\mu_{\mathring{g}}\,. \tag{B.2.4}$$
The second is the monotonicity property of
$$\int_{{}^2M} (R - R_0)^2 \tag{B.2.5}$$
discovered by Calabi [80] and, independently, by Lukács, Perjes, Porter and Sebestyén [304].

2. Let $m > 0$. There exists a strictly increasing sequence of real numbers $\lambda_i > 0$, integers n_i with $n_1 = 0$, and functions $\varphi_{i,j} \in C^\infty({}^2M)$, $0 \leq j \leq n_i$, such that, possibly after performing a conformal transformation of $\mathring{g}$, solutions of (B.2.3) have a full asymptotic expansion of the form
$$\lambda(u, x^a) = \sum_{i\geq 1,\ 0\leq j\leq n_i} \varphi_{i,j}(x^a) u^j e^{-\lambda_i u/m}\,, \tag{B.2.6}$$
when u tends to infinity. The result is obtained by a delicate asymptotic analysis of solutions of the RT equation.

The decay exponents λ_i and the n_i's are determined by the spectrum of $\Delta_{\mathring{g}}$. For example, if $({}^2M, \mathring{g})$ is a round two sphere, we have [122]
$$\lambda_i = 2i\,,\ i \in \mathbb{N}\,,\ \text{with } n_1 = \ldots = n_{14} = 0\,,\ n_{15} = 1\,. \tag{B.2.7}$$

REMARK B.2.1 The first global existence result for the RT equation has been obtained by Rendall [365] for a restricted class of near-Schwarzschildian initial data. Global existence and convergence to a round metric for all smooth initial data has been established in [121]. There the uniformization theorem for compact two-dimensional manifolds has been assumed. An alternative proof of global existence, which establishes the uniformization theorem as a by-product, has been given by Struwe [400]. □

The RT metrics all possess a smooth conformal boundary *à la Penrose* at "$r = \infty$". To see this, one can replace r by a new coordinate $x = 1/r$, which brings the metric (B.2.1) to the form

$$\mathbf{g} = x^{-2}\left(-\left(\frac{Rx^2}{2} + \frac{x\Delta_g R}{12m} - 2mx^3\right) du^2 + 2du\,dx + e^{2\lambda}\mathring{g}\right) , \quad \text{(B.2.8)}$$

so that the metric $\mathbf{g}$ multiplied by a conformal factor x^2 smoothly extends to $\{x = 0\}$.

In what follows we shall take $({}^2M, \mathring{g})$ to be a two dimensional sphere equipped with the unit round metric. See [122] for a discussion of other topologies.

B.2.1 $m > 0$

Let us assume that $m > 0$. Following an observation of Schmidt reported in [402], the hypersurface "$u = \infty$" can be attached to the manifold $\{r \in (0,\infty)\,,\, u \in [0,\infty)\} \times {}^2M$ as a null boundary by introducing Kruskal–Szekeres-type coordinates $(\hat{u}, \hat{v})$, defined in a way identical to the ones for the Schwarzschild metric:

$$\hat{u} = -\exp\left(-\frac{u}{4m}\right) , \quad \hat{v} = \exp\left(\frac{u+2r}{4m} + \ln\left(\frac{r}{2m} - 1\right)\right) . \quad \text{(B.2.9)}$$

This brings the metric to the form

$$\begin{aligned}\mathbf{g} \;=\; & -\frac{32m^3\exp\left(-\frac{r}{2m}\right)}{r} d\hat{u}\,d\hat{v} + r^2 e^{2\lambda}\mathring{g} \\ & -16m^2\exp\left(\frac{u}{2m}\right)\left(\frac{R}{2} - 1 + \frac{r\Delta_g R}{12m}\right) d\hat{u}^2 \,. \quad \text{(B.2.10)}\end{aligned}$$

Note that $\mathbf{g}_{\hat{u}\hat{u}}$ vanishes when $\lambda \equiv 0$, and one recovers the Schwarzschild metric in Kruskal-Szekeres coordinates. Equations (B.2.6)-(B.2.7) imply that $\mathbf{g}_{\hat{u}\hat{u}}$ decays as $e^{u/2m} \times e^{-2u/m} = \hat{u}^6$. Hence g approaches the Schwarzschild metric as $O(\hat{u}^6)$ when the null hypersurface

$$\mathcal{H}^+ := \{\hat{u} = 0\}$$

is approached. A projection diagram, as defined in [151], with the ${}^2M$ factor projected out, can be found in Figure B.2.1.

In terms of $\hat{u}$ the expansion (B.2.6) becomes

$$\lambda(\hat{u}, x^a) = \sum_{i\geq 1\,,\; 0\leq j\leq n_i} \varphi_{i,j}(x^a)\,(-4m\log(|\hat{u}|))^j\,\hat{u}^{8i}\,, \quad \text{(B.2.11)}$$

which can be extended to $\hat{u} > 0$ as an even function of $\hat{u}$. This expansion carries over to similar expansions of R and $\Delta_g R$, and results in an asymptotic expansion of the form

$$\mathbf{g}_{\hat{u}\hat{u}}(\hat{u}, x^a) = \sum_{i\geq 1\,,\; 0\leq j\leq n_i} \psi_{i,j}(x^a)\,(\log|\hat{u}|)^j\,\hat{u}^{8i-2}\,, \quad \text{(B.2.12)}$$

for some functions ψ_{ij}. It follows from (B.2.2) that the even extension of $\mathbf{g}_{\hat{u}\hat{u}}$ will be of C^{117}–differentiability class.

In fact, any two such even functions $\mathbf{g}_{\hat{u}\hat{u}}$ can be continued into each other across $u = 0$ to a function of C^5–differentiability class. It follows that:

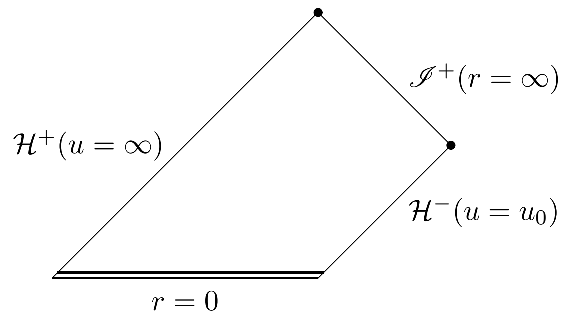

Figure B.2.1: A projection diagram for RT metrics with $m > 0$.

1. Any two RT metrics can be joined together as in Figure B.2.2 to obtain a spacetime with a metric of C^5–differentiability class. In particular g can be glued to a Schwarzschild metric beyond $\mathcal{H}$, resulting in a C^5 metric.

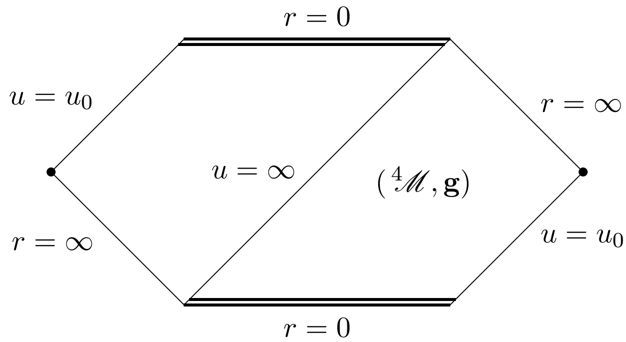

Figure B.2.2: Vacuum RT extensions beyond $\mathcal{H}^+ = \{u = \infty\}$. Any two RT metrics with the same mass parameter m can be glued across the null hypersurface $\mathcal{H}^+$, leading to a metric of C^5–differentiability class.

2. It follows from (B.2.2) that g can be glued to itself in the C^{117}–differentiability class.

The vanishing, or not, of the expansion functions $\varphi_{i,j}$ in (B.2.6) with $j \geq 1$ turns out to play a key role for the smoothness of the metric at $\mathcal{H}$. Indeed, the first non-vanishing function $\varphi_{i,j}$ with $j \geq 1$ will lead to a $\psi_{i,j} (\ln |\hat{u}|)^j \, \hat{u}^{8i-2}$ term in the asymptotic expansion of $\mathbf{g}_{\hat{u}\hat{u}}$. As a result, $\mathbf{g}_{\hat{u}\hat{u}}$ will be extendable to an even function of $\hat{u}$ of C^{8i-3}-differentiability class, but not better. It is shown in [156] that

1. Generic $\lambda(0, x^a)$ close to zero lead to a solution with $\psi_{15,1} \neq 0$, resulting in metrics which are extendible across $\mathcal{H}$ in the C^{117}-differentiability class, but not C^{118}, in the coordinate system above.

2. There exists an infinite-dimensional family of non-generic initial functions $\lambda(0, x^a)$ for which $\psi_{15,1} \equiv 0$. An even extension of $\mathbf{g}_{\hat{u}\hat{u}}$ across $\mathcal{H}$ results

in a metric of C^{557}-differentiability class, but not C^{558}, in the coordinate system above.

The question arises, whether the above differentiability issues are related to a poor choice of coordinates. By analysing the behaviour of the derivatives of the Riemann tensor on geodesics approaching $\mathcal{H}$, one can show [156] that the metrics of point 1 above cannot be extended across $\mathcal{H}$ in the class of spacetimes with metrics of C^{123}-differentiability class. Similarly the metrics of point 2 cannot be extended across $\mathcal{H}$ in the class of spacetimes with metrics of C^{564}-differentiability class. One expects that the differentiability mismatches are not a real effect, but result from a non-optimal inextendibility criterion used. It would be of some interest to settle this issue.

Summarising, we have the following:

THEOREM B.2.2 *Let $m > 0$. For any $\lambda_0 \in C^\infty(S^2)$ there exists a Robinson–Trautman spacetime $({}^4\mathscr{M}, \mathbf{g})$ with a "half-complete" $\mathscr{I}^+$, the global structure of which is shown in Figure B.2.1. Moreover:*

1. *$({}^4\mathscr{M}, \mathbf{g})$ is smoothly extendible to the past through $\mathscr{H}^-$. If, however, λ_0 is* not *analytic, then* no *vacuum Robinson–Trautman extensions through $\mathscr{H}^-$ exist.*

2. *There exist infinitely many non-isometric vacuum Robinson–Trautman C^5 extensions[3] of $({}^4\mathscr{M}, \mathbf{g})$ through $\mathscr{H}^+$, which are obtained by gluing to $({}^4\mathscr{M}, \mathbf{g})$ any other positive mass Robinson–Trautman spacetime, as shown in Figure B.2.2.*

3. *There exist infinitely many C^{117} vacuum RT extensions of $({}^4\mathscr{M}, \mathbf{g})$ through $\mathscr{H}^+$. One such extension is obtained by gluing a copy of $({}^4\mathscr{M}, \mathbf{g})$ to itself, as shown in Figure B.2.2.*

4. *For any $6 \le k \le \infty$ there exists an open set $\mathcal{O}_k$ of Robinson–Trautman spacetimes, in a $C^k(S^2)$ topology on the set of the initial data functions λ_0, for which no C^{123} extensions beyond $\mathscr{H}^+$ exist, vacuum or otherwise. For any u_0 there exists an open ball $\mathcal{B}_k \subset C^k(S^2)$ around the initial data for the Schwarzschild metric, $\lambda_0 \equiv 0$, such that $\mathcal{O}_k \cap \mathcal{B}_k$ is dense in $\mathcal{B}_k$.*

The picture that emerges from Theorem B.2.2 is the following: generic initial data lead to a spacetime which has no RT vacuum extension to the past of the initial surface, even though the metric can be smoothly extended (in the non–vacuum class); and generic data sufficiently close[4] to Schwarzschildian ones lead to a spacetime for which no smooth vacuum RT extensions exist beyond $\mathcal{H}^+$. This shows that considering smooth extensions across $\mathcal{H}^+$ leads to non–existence, while giving up the requirement of smoothness of extensions beyond

[3]By this we mean that the metric can be C^5 extended beyond $\mathscr{H}^+$; the extension can actually be chosen to be of $C^{5,\alpha}$–differentiability class, for any $\alpha < 1$.

[4]It is rather clear from the results of [156] that generic RT spacetimes will not be smoothly extendible across $\mathcal{H}^+$, without any restrictions on the "size" of the initial data; but no rigorous proof is available.

$\mathcal{H}^+$ leads to non–uniqueness. It follows that global well-posedness of the general relativistic initial value problem completely fails in the class of positive mass Robinson–Trautman metrics.

REMARK B.2.3 There are two striking differences between the global structure seen in Figure B.2.2 and the usual Penrose diagram for Schwarzschild spacetime. The first is the lack of past null infinity, which we have seen to be unavoidable in the RT case. The second is the lack of the past event horizon, sections of which can be technically described as a *marginally past outer trapped surfaces*. The existence of such surfaces in RT spacetimes is a non-trivial property which has been established in [402]. □

B.2.2 $m < 0$

Unsurprisingly, and as already mentioned, the global structure of RT spacetimes turns out to be different when $m < 0$, which we assume now. As already noted, in this case we should take $u \le 0$, in which case the expansion (B.2.6) again applies with $u \to -\infty$.

The existence of future null infinity as in (B.2.8) applies without further due, except that now the coordinate u belongs to $(-\infty, 0]$.

The new aspect is the possibility of attaching a conformal boundary at past null infinity, $\mathscr{I}^-$, which is carried out by first replacing u with a new coordinate v defined as [385]

$$v = u + 2r + 4m \ln\left(\left|\frac{r}{2m} - 1\right|\right) . \tag{B.2.13}$$

In the coordinate system (v, r, x^a) the metric becomes

$$\begin{aligned}\mathbf{g} &= -\left(1 - \frac{2m}{r}\right) dv^2 + 2dv\,dr + r^2 e^{2\lambda} \mathring{g} \\ &\quad + \left(\frac{R}{2} - 1 + \frac{r}{12m}\Delta_g R\right)\left(dv - \frac{2dr}{1 - \frac{2m}{r}}\right)^2 .\end{aligned} \tag{B.2.14}$$

The last step is the usual replacement of r by $x = 1/r$:

$$\begin{aligned}\mathbf{g} &= x^{-2}\Bigg[- x^2(1 - 2mx)dv^2 - 2dv\,dx + e^{2\lambda}\mathring{g} \\ &\quad + \left(\frac{R-2}{2x^2} + \frac{12m\Delta_g R}{x^3}\right)\left(x^2 dv + \frac{2dx}{1-2mx}\right)^2\Bigg] .\end{aligned} \tag{B.2.15}$$

One notices that all terms in the conformally rescaled metric $x^2 \times \mathbf{g}$ extend smoothly to smooth functions of (v, x^a) at the conformal boundary $\{x = 0\}$ except possibly for

$$\left(\frac{R-2}{2x^2} + \frac{12m\Delta_g R}{x^3}\right) \times \left(\frac{4dx^2}{(1-2mx)^2} + \frac{4x^2 dv\,dx}{1-2mx}\right) . \tag{B.2.16}$$

Now, from the definition of v we have

$$\begin{aligned}\exp\left(-\frac{2u}{m}\right) &= \left(\frac{r}{2m} - 1\right)^8 \exp\left(\frac{2v - 4r}{|m|}\right) \\ &= \left(\frac{1-2mx}{2mx}\right)^8 \exp\left(\frac{2v}{|m|}\right) \exp\left(-\frac{4}{|m|x}\right) .\end{aligned}$$

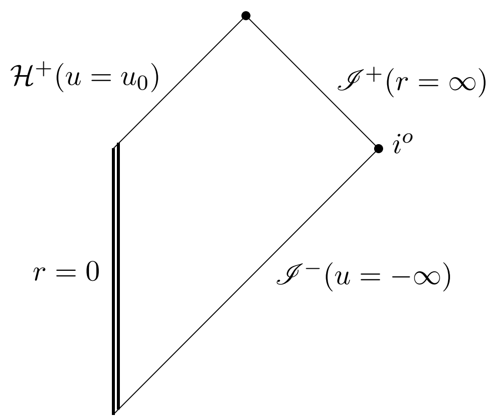

Figure B.2.3: A projection diagram for RT metrics with $m < 0$.

Using the fact that $\lambda = O(\exp(-2u/m))$, similarly for all angular derivatives of λ, we see that all three functions λ, $R - 2$ and $\Delta_g R$ decay to zero, as x approaches zero, faster than any negative power of x. In fact, the offending terms (B.2.16) extend smoothly by zero across $\{x = 0\}$. We conclude that the conformally rescaled metric $x^2 \times \mathbf{g}$ smoothly extends to $\mathscr{I}^- := \{x = 0\}$.

Summarising, we have:

THEOREM B.2.4 *Let $m < 0$. For any $\lambda_0 \in C^\infty({}^2M)$ there exists a unique RT spacetime $({}^4\mathscr{M}, \mathbf{g})$ with a complete i^0 in the sense of [13], a complete $\mathscr{I}^-$, and "a piece of $\mathscr{I}^+$", as shown in Figure B.2.3. Moreover*

1. *$({}^4\mathscr{M}, \mathbf{g})$ is smoothly extendible through $\mathscr{H}^+$, but*
2. *if λ_0 is* not *analytic, there exist* no *vacuum RT extensions through $\mathscr{H}^+$.*

The generic non-extendability of the metric through $\mathscr{H}^+$ in the vacuum RT class is rather surprising, and seems to be related to a similar non-extendability result for compact non–analytic Cauchy horizons in the polarized Gowdy class, *cf.* [142]. Since it may well be possible that there exist vacuum extensions which are not in the RT class, this result does not unambiguously demonstrate a failure of Einstein equations to propagate generic data forwards in u in such a situation; however, it certainly shows that the forward evolution of the metric via Einstein equations breaks down in the class of RT metrics with $m < 0$.

B.2.3 $\Lambda \neq 0$

So far we have assumed a vanishing cosmological constant. It turns out that there exists a straightforward generalisation of RT metrics to $\Lambda \neq 0$. The metric retains its form (B.2.1), with the function Φ of (B.2.2) taking instead the form

$$\Phi = \frac{R}{2} + \frac{r}{12m}\Delta_g R - \frac{2m}{r} - \frac{\Lambda}{3} r^2 \, . \tag{B.2.17}$$

We continue to assume that $m \neq 0$.

It turns out that the key equation (B.2.3) remains the same, thus λ tends to zero and Φ tends to the function

$$\mathring{\Phi} = \frac{\mathring{R}}{2} - \frac{2m}{r} - \frac{\Lambda}{3}r^2 \tag{B.2.18}$$

as u approaches infinity. It follows from the generalised Birkhoff theorem that these are the *Birmingham* metrics. The relevant projection diagrams can be found in Figures B.2.4-B.2.7.

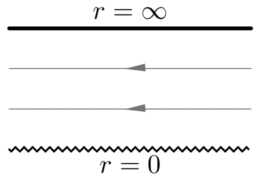

Figure B.2.4: The causal diagram when $m < 0$, $\Lambda > 0$ and $\mathring{\Phi}$ has no zeros.

The global structure of the spacetimes with $\Lambda \neq 0$ and $\lambda \not\equiv 0$ should be clear from the analysis of the case $\Lambda = 0$: One needs to cut one of the building blocs of Figures B.2.4-B.2.7 with a line with a ± 45-degrees slope, corresponding to the initial data hypersurface $u_0 = 0$. This hypersurface should *not* coincide with one of the Killing horizons there, where $\mathring{\Phi}$ vanishes. The Killing horizons with the opposite slope in the diagrams should be ignored. Depending upon the sign of m, one can evolve to the future or to the past of the associated spacetime hypersurface until a conformal boundary at infinity or a Killing horizon $\mathring{\Phi}(r_0) = 0$ with the same slope is reached.

The metric will always be smoothly conformally extendable through the conformal boundaries at infinity.

As discussed in [67], the extendibility properties across the horizons which are approached as $m \times u$ tends to infinity will depend upon the surface gravity of the horizon and the spectrum of $\mathring{g}$. For simplicity we assume that

$$^2M = S^2 \quad \Longleftrightarrow \quad \mathring{R} > 0\,,$$

a similar analysis can be carried out for other topologies.

Consider, first, a zero $r = r_0$ of $\mathring{\Phi}$ such that

$$c = \frac{\mathring{\Phi}'(r_0)}{2} > 0\,.$$

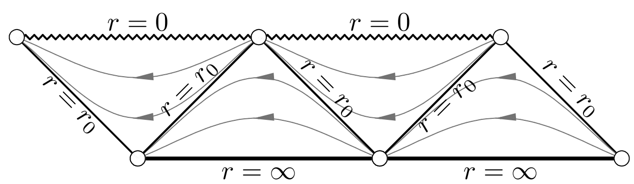

Figure B.2.5: The causal diagram for Kottler metrics with $\Lambda > 0$, and $\mathring{\Phi} \leq 0$, with $\mathring{\Phi}$ vanishing precisely at r_0.

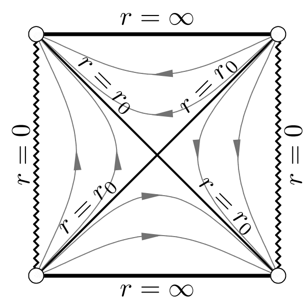

Figure B.2.6: The causal diagram for Kottler metrics with $m < 0$, $\Lambda > 0$ $\mathring{R} \in \mathbb{R}$, or $m = 0$ and $\mathring{R} = 1$, with r_0 defined by the condition $\mathring{\Phi}(r_0) = 0$. The set $\{r = 0\}$ is a singularity unless the metric is the de Sitter metric (${}^2M = S^2$ and $m = 0$), or a suitable quotient thereof so that $\{r = 0\}$ corresponds to a center of (possibly local) rotational symmetry.

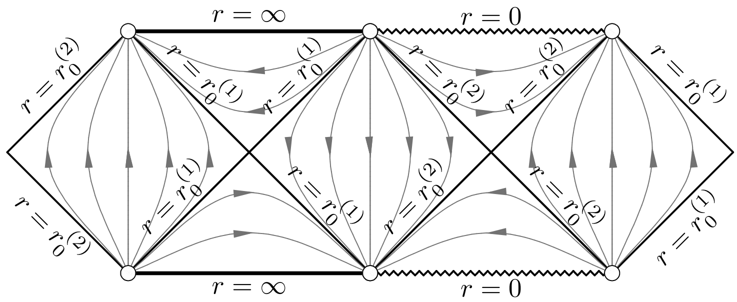

Figure B.2.7: The causal diagram for Kottler metrics with $\Lambda > 0$ and exactly two first-order zeros of $\mathring{\Phi}$.

Similarly to (B.2.9), introduce Kruskal–Szekeres-type coordinates $(\hat{u}, \hat{v})$ defined as

$$\hat{u} = -e^{-cu}, \quad \hat{v} = e^{c(u+2F(r))}, \tag{B.2.19}$$

where

$$F' = \frac{1}{\mathring{\Phi}}. \tag{B.2.20}$$

This brings the metric to the form

$$\mathbf{g} = -\frac{e^{-2cF(r)}\mathring{\Phi}}{c^2} d\hat{u}\, d\hat{v} + r^2 e^{2\lambda} \mathring{g} - \frac{e^{2cu}}{c^2} \underbrace{\left(\frac{R - \mathring{R}}{2} + \frac{r\Delta_g R}{12m} \right)}_{O(\exp(-2u/m))} d\hat{u}^2 . \tag{B.2.21}$$

It is elementary to show that $\mathbf{g}_{\hat{u}\hat{v}}$ extends smoothly across $\{r = r_0\}$. Next we have

$$\mathbf{g}_{\hat{u}\hat{u}} = O\left(e^{2(c-\frac{1}{m})u}\right) = O\left(\hat{u}^{2(\frac{1}{mc}-1)}\right) \tag{B.2.22}$$

which will extend continuously across a horizon $\{\hat{u} = 0\}$ provided that

$$\frac{1}{mc} > 1 \qquad \Longleftrightarrow \qquad mF'(r_0) < 2 . \tag{B.2.23}$$

In fact when (B.2.23) holds, then for any $\epsilon > 0$ the extension to any other RT solution will be of $C^{\lfloor 2(\frac{1}{mc}-1)\rfloor-\epsilon}$ differentiability class.

When $\Lambda > 0$, the parameter $c = c(m, \Lambda)$ can be made as small as desired by making m approach from below the critical value

$$m_c = \frac{1}{3\sqrt{\Lambda}},$$

for which c vanishes. For $m > m_c$ the function $\mathring{\Phi}$ has no (real) zeros, and for $0 < m < m_c$ all zeros are simple.

It follows from Figure B.2.8 that the extension through the black hole event horizon is at least of C^6-differentiability class, and becomes as differentiable as desired when the critical mass is approached.

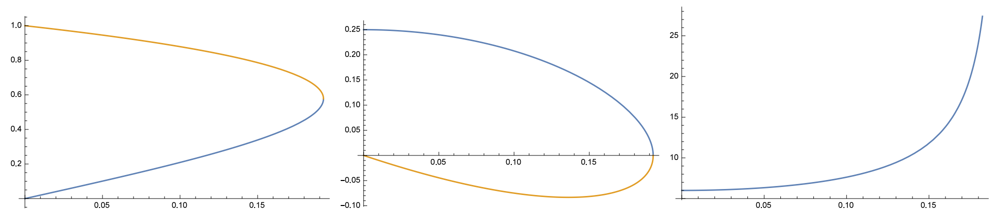

Figure B.2.8: The value of the real positive zero of $\mathring{\Phi}$ (left plot), the product $m \times c(m, \Lambda)$ (middle plot), and the function $2/(m \times c(m, \Lambda)) - 2$ (which determines the differentiability class of the extension through the black hole event horizon; right plot) as functions of m, with $\mathring{R} = 2$ and $\Lambda = 3$.

The calculation above breaks down for degenerate horizons, where $m = m_c$, for which $c = 0$. In this case an extension across a degenerate horizon can be obtained by replacing u by a coordinate v defined as

$$v = u + 2F(r)\,, \text{ with again } \frac{dF}{dr} = \frac{1}{\mathring{\Phi}}\,. \tag{B.2.24}$$

An explicit formula for F can be found, which is not very enlightening. Since $\mathring{\Phi}$ has a quadratic zero, we find that for r approaching r_0 we have, after choosing an integration constant appropriately,

$$u \approx v + \frac{1}{3(r-r_0)} \quad \Longrightarrow \quad \mathring{\Phi} \sim u^{-2} \text{ and } e^{-\frac{2u}{m}} \sim e^{-\frac{2}{3m(r-r_0)}}\,, \tag{B.2.25}$$

where $f \sim g$ is used to indicate that $|f/g|$ is bounded by a positive constant both from above and below over compact intervals of v.

Using $du = dv - 2dr/\mathring{\Phi}$ we find

$$\mathbf{g} = -\Phi dv^2 + 2\underbrace{\frac{2\Phi - \mathring{\Phi}}{\mathring{\Phi}}}_{1+O(\exp(-2u/m))} dv\,dr - 4\underbrace{\frac{\Phi - \mathring{\Phi}}{\mathring{\Phi}^2}}_{O(u^4\exp(-2u/m))} dr^2 + r^2 e^{2\lambda}\mathring{g}\,. \tag{B.2.26}$$

It easily follows that $\mathbf{g}_{vr}$ can be *smoothly* extended by a constant function across $r = r_0$, and that $\mathbf{g}_{rr}$ can be *again smoothly* extended by the constant function

0. We conclude that any RT metric with a degenerate horizon can be smoothly continued across the horizon to a Schwarzschild-de Sitter or Schwarzschild-anti de Sitter metric with the same mass parameter m, as first observed in [67].

INCIDENTALLY: Some results on higher-dimensional generalisations of RT metrics can be found in [351].

B.3 Birmingham (Kottler–Schwarzschild-(A)(de)Sitter) metrics

The Schwarzschild-de Sitter ($\Lambda > 0$), and Schwarzschild-anti de Sitter ($\Lambda < 0$) metrics, first written down by Kottler [280], are special cases of the family of vacuum metrics discovered by Birmingham [66]. The metrics take the form

$$g = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \underbrace{\breve{h}_{AB}(x^C)dx^A dx^B}_{=:\breve{h}}, \tag{B.3.1}$$

where $\breve{h}$ is a *Riemannian Einstein metric* on a compact $(n-1)$-dimensional manifold N, and where we denote by x^A the local coordinates on N. As first pointed out by Birmingham in [66], for any $m \in \mathbb{R}$ and

$$\ell \in \mathbb{R}^* \cup \sqrt{-1}\mathbb{R}^*$$

the function

$$f = \frac{\breve{R}}{(n-1)(n-2)} - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2}, \tag{B.3.2}$$

where $\breve{R}$ is the (constant) scalar curvature of $\breve{h}$, leads to a vacuum metric,

$$R_{\mu\nu} = \frac{n}{\ell^2} g_{\mu\nu}, \tag{B.3.3}$$

where ℓ is a constant related to the cosmological constant $\Lambda \in \mathbb{R}$ as

$$\frac{1}{\ell^2} = \frac{2\Lambda}{n(n-1)}. \tag{B.3.4}$$

A comment about negative Λ, and thus purely complex ℓ's, is in order. When considering a negative cosmological constant (B.3.4) requires $\ell \in \sqrt{-1}\mathbb{R}$, which is awkward to work with. So when $\Lambda < 0$ it is convenient to change r^2/ℓ^2 in f to $-r^2/\ell^2$, change the sign in (B.3.4), and use a real ℓ. We will often do this without further ado.

Clearly, n cannot be equal to two in (B.3.2), and we therefore exclude this dimension in what follows.

The multiplicative factor two in front of m is convenient in dimension three when $\breve{h}$ is a unit round metric on S^2, and we will keep this form regardless of topology and dimension of N.

There is a rescaling of the coordinate $r = b\bar{r}$, with $b \in \mathbb{R}^*$, which leaves (B.3.1)-(B.3.2) unchanged if moreover

$$\bar{\breve{h}} = b^2\breve{h}, \quad \bar{m} = b^{-n}m, \quad \bar{t} = bt. \tag{B.3.5}$$

We can use this to achieve

$$\beta := \frac{\breve{R}}{(n-1)(n-2)} \in \{0, \pm 1\}\,. \tag{B.3.6}$$

The set $\{r = 0\}$ corresponds to a singularity when $m \neq 0$. Except in the case $m = 0$ and $\beta = -1$, by an appropriate choice of the sign of b we can always achieve $r > 0$ in the regions of interest.

Appendix C

Conformal rescalings

Consider a metric $\widetilde{g}$ related to g by a conformal rescaling:

$$\widetilde{g}_{ij} = \varphi^{\ell} g_{ij} \quad \Longleftrightarrow \quad \widetilde{g}^{ij} = \varphi^{-\ell} g^{ij} \,, \tag{C.0.1}$$

where φ is a function and ℓ is a real number.

C.1 Christoffel symbols

Under (C.0.1) the the Christoffel symbols transform as follows:

$$\begin{aligned} \widetilde{\Gamma}^i{}_{jk} &= \frac{1}{2}\widetilde{g}^{im}(\partial_j \widetilde{g}_{km} + \partial_k \widetilde{g}_{jm} - \partial_m \widetilde{g}_{jk}) \\ &= \frac{1}{2}\varphi^{-\ell} g^{im}(\partial_j(\varphi^{\ell} g_{km}) + \partial_k(\varphi^{\ell} g_{jm}) - \partial_m(\varphi^{\ell} g_{jk})) \\ &= \Gamma^i{}_{jk} + \frac{\ell}{2\varphi}(\delta^i_k \partial_j \varphi + \delta^i_j \partial_k \varphi - g_{jk} D^i \varphi) \,, \end{aligned} \tag{C.1.1}$$

where D denotes the covariant derivative of g. Equation (C.1.1) can be rewritten as

$$\tilde{D}_X Y = D_X Y + C(X,Y) \,, \tag{C.1.2}$$

with

$$C(X,Y) = \frac{\ell}{2\varphi}\Big(Y(\varphi)X + X(\varphi)Y - g(X,Y)D\varphi\Big) \tag{C.1.3a}$$

$$= \frac{\ell}{2\varphi}\Big(Y(\varphi)X + X(\varphi)Y - \tilde{g}(X,Y)\tilde{D}\varphi\Big) \,. \tag{C.1.3b}$$

C.2 The curvature

Let $\widetilde{\mathrm{Riem}}$ denote the curvature tensor of a connection of the form (C.1.2); from (C.1.3) we obtain

$$\begin{aligned} \tilde{R}(X,Y)Z &= \Big(\tilde{D}_X \tilde{D}_Y Z - X \leftrightarrow Y\Big) - \tilde{D}_{[X,Y]} Z \\ &= \Big(D_X(D_Y Z + C(Y,Z)) + C(X,(D_Y Z + C(Y,Z)) - X \leftrightarrow Y\Big) \end{aligned}$$

$$
\begin{aligned}
& -D_{[X,Y]}Z - C(\underbrace{[X,Y]}_{=D_XY-D_YX},Z) \\
= \quad & R(X,Y)Z + \Big((D_XC)(Y,Z) + C(D_XY,Z) + C(Y,D_XZ) \\
& +C(X,D_YZ) + C(X,C(Y,Z)) - X \leftrightarrow Y\Big) - C(D_XY,Z) + C(D_YX,Z) \\
= \quad & R(X,Y)Z + \Big((D_XC)(Y,Z) + C(X,C(Y,Z)) - X \leftrightarrow Y\Big) .
\end{aligned}
$$

In index notation this can be rewritten as

$$
\tilde{R}^i{}_{jk\ell} = R^i{}_{jk\ell} + C^i{}_{\ell j;k} - C^i{}_{kj;\ell} + C^i{}_{km}C^m{}_{j\ell} - C^i{}_{\ell m}C^m{}_{jk} \ . \tag{C.2.1}
$$

C.2.1 The Weyl conformal connection

There is a natural generalisation of (C.1.2)-(C.1.3) to *Weyl conformal connections*, obtained by the replacement

$$
\frac{\ell\partial_a\varphi}{2\varphi} \ \longrightarrow \ f_a \tag{C.2.2}
$$

there, where $f_a dx^a$ is an arbitrary one-form, not necessarily exact (compare [213]). In other words, one sets

$$
C^i{}_{jk} = \delta^i_j f_k + \delta^i_k f_j - g^{i\ell} f_\ell \, g_{jk} \, . \tag{C.2.3}
$$

Since $C^i{}_{jk}$ is symmetric in j and k, the connection $\tilde{D}$ is always torsion-free.

Inserting into (C.2.1) one finds the following formula for the curvature tensor of a Weyl connection

$$
\tilde{R}^i{}_{jk\ell} = R^i{}_{jk\ell} + 2\Big(f_{j;[k}\delta^i_{\ell]} + \delta^i_j f_{[\ell;k]} - f^i{}_{;[k}g_{\ell]j} + \delta^i_{[k}f_{\ell]}f_j - g_{j[k}f_{\ell]}f^i - \delta^i_{[k}g_{\ell]j}f_mf^m\Big) \, . \tag{C.2.4}
$$

Contracting over i and k one obtains the Ricci tensor of $\tilde{D}$

$$
\begin{aligned}
\tilde{R}_{j\ell} \ &:= \ \mathrm{Ric}(\tilde{g})_{ij} \\
&= \ R_{j\ell} + (1-n)f_{j;\ell} + f_{\ell;j} - f^i{}_{;i}g_{j\ell} + (n-2)(f_jf_\ell - g_{j\ell}f_mf^m) \, .
\end{aligned} \tag{C.2.5}
$$

(Note that $\tilde{R}_{j\ell}$ is not symmetric in general.) We calculate the Ricci scalar of the Weyl connection by taking the trace of $\tilde{R}_{j\ell}$ using the metric g:

$$
g^{j\ell}\tilde{R}_{j\ell} \ = \ R - (n-1)(2f^i{}_{;i} + (n-2)f_mf^m) \tag{C.2.6}
$$

(the reader is warned that this is *not* the curvature scalar $\tilde{g}^{j\ell}\tilde{R}_{j\ell}$ of the metric $\tilde{g}$ when f_a is expressed in terms of φ using (C.2.2), see (C.2.14) below).

For $n \neq 2$ it is convenient to introduce the tensor

$$
\tilde{L}_{ij} = \frac{1}{n-2}\left(\tilde{R}_{(ij)} - \frac{n-2}{n}\,\tilde{R}_{[ij]} - \frac{1}{2\,(n-1)}\,g_{ij}\,g^{kl}\tilde{R}_{kl}\right) . \tag{C.2.7}
$$

which is a natural generalisation of the *Schouten tensor* A_{ij} associated to a metric g:

$$A_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{1}{2(n-1)} g_{ij} g^{k\ell} R_{k\ell}\right) . \tag{C.2.8}$$

From (C.2.5)-(C.2.8) one finds

$$D_i f_j - f_i f_j + \frac{1}{2} g_{ij} f_k f^k = A_{ij} - \tilde{L}_{ij} . \tag{C.2.9}$$

C.2.2 The Weyl tensor

Using (C.2.9) to eliminate the derivatives of f_i from (C.2.4) one obtains

$$\tilde{R}^i{}_{jk\ell} = 2\{\delta^i_{[k} \tilde{L}_{\ell]j} - \delta^i_j \tilde{L}_{[k\ell]} - g_{j[k} \tilde{L}_{\ell]}{}^i\} + C^i{}_{jk\ell} , \tag{C.2.10}$$

where the *Weyl tensor* $C^i{}_{jk\ell}$, often also denoted as $W^i{}_{jk\ell}$, is defined as

$$\begin{aligned} C_{ijkl} &:= R_{ijkl} - \frac{1}{n-2} (g_{ik} R_{jl} - g_{il} R_{jk} - g_{jk} R_{il} + g_{jl} R_{ik}) \\ &\quad + \frac{1}{(n-1)(n-2)} R(g_{ik} g_{jl} - g_{il} g_{jk}) \\ &= R_{ijkl} - A_{ik} g_{jl} + A_{il} g_{jk} + A_{jk} g_{il} - A_{jl} g_{ik} . \end{aligned} \tag{C.2.11}$$

The Weyl tensor has the important property that all its traces vanish, in particular

$$C^i{}_{jik} = 0 .$$

C.2.3 The Ricci tensor and the curvature scalar

We now return to (C.1.3); in this case $\tilde{R}_{ij}$ is the Ricci tensor of the metric $\tilde{g}_{ij}$, hence $\tilde{L}_{ij} = \tilde{A}_{ij}$, the Schouten tensor of $\tilde{g}_{ij}$. Equation (C.2.10) implies that $C^i{}_{jk\ell}$ is invariant under conformal changes of the metric:

$$\tilde{C}^i{}_{jk\ell} = C^i{}_{jk\ell} .$$

Next, (C.2.9) can be viewed as a transformation law of the Schouten tensor under conformal changes. Indeed, expressing f_a in terms of φ by inverting (C.2.2), Equation (C.2.9) can be rewritten as

$$\tilde{A}_{ij} = A_{ij} - \frac{\ell}{2\varphi} D_i D_j \varphi + \frac{\ell}{4\varphi^2}\Big((2+\ell) D_i\varphi D_j\varphi - \frac{\ell}{2} g_{ij} D_k\varphi D^k\varphi\Big) , \tag{C.2.12}$$

which does not have any dimension-dependent coefficients, and which simplifies somewhat when $\ell = -2$. Similarly, (C.2.5) gives

$$\begin{aligned} \tilde{R}_{ij} &= R_{ij} - \frac{(n-2)\ell}{2\varphi} D_i D_j \varphi + \frac{(n-2)\ell(\ell+2)}{4\varphi^2} D_i\varphi D_j\varphi - \frac{\ell}{2\varphi} \Delta_g \varphi g_{ij} \\ &\quad - \frac{(n-2)\ell^2 - 2\ell}{4\varphi^2} D^k\varphi D_k\varphi g_{ij} . \end{aligned} \tag{C.2.13}$$

Taking a $\tilde{g}$–trace one obtains

$$\begin{aligned}\tilde{R} &:= \tilde{g}^{ij}\tilde{R}_{ij} = \varphi^{-\ell}g^{ij}\tilde{R}_{ij} \\ &= \varphi^{-\ell}\left(R - \frac{(n-1)\ell}{\varphi}\Delta_g\varphi - \frac{(n-1)\ell\,\{(n-2)\ell-4\}}{4\varphi^2}D^i\varphi D_i\varphi\right) .\end{aligned} \tag{C.2.14}$$

For $n \neq 2$ a very convenient choice is

$$(n-2)\ell = 4\,, \tag{C.2.15}$$

leading to

$$\tilde{g}_{ij} = \varphi^{\frac{4}{n-2}}g_{ij}\,, \qquad \tilde{R} = \varphi^{-\frac{4}{n-2}}\left(R - \frac{4(n-1)}{(n-2)\varphi}\Delta_g\varphi\right) . \tag{C.2.16}$$

An immediate useful consequence of (C.2.16) is the following: if $R = 0$ and if φ is g–harmonic (*i.e.*, $\Delta_g\varphi = 0$), then $\tilde{g}$ also has vanishing scalar curvature, and φ is $\tilde{g}$–harmonic.

It is sometimes useful to take $\varphi = e^u$ and $\ell = 1$, which gives

$$\tilde{g}_{ij} = e^u g_{ij}\,, \qquad \tilde{R} = e^{-u}\left(R - (n-1)\Delta_g u - \frac{(n-1)(n-2)}{4}|du|^2_g\right) \tag{C.2.17}$$

When $n = 2$ one obtains

$$\tilde{g}_{ij} = e^u g_{ij}\,, \qquad \tilde{R} = e^{-u}\left(R - \Delta_g u\right) . \tag{C.2.18}$$

For the record we note the metric version of (C.2.10),

$$R^i{}_{jk\ell} = 2\{\delta^i_{[k}A_{\ell]j} - g_{j[k}A_{\ell]}{}^i\} + C^i{}_{jk\ell}\,. \tag{C.2.19}$$

C.3 The Beltrami-Laplace operator

Under a conformal transformation as in (C.2.16) we have the following transformation law for the Laplacian acting on functions:

$$\begin{aligned}\Delta_{\tilde{g}}f &= \frac{1}{\sqrt{\det\tilde{g}_{ij}}}\partial_k(\sqrt{\det\tilde{g}_{ij}}\tilde{g}^{k\ell}\partial_\ell f) \\ &= \frac{\varphi^{-\frac{2n}{n-2}}}{\sqrt{\det g_{ij}}}\partial_k\left(\underbrace{\varphi^{\frac{2n}{n-2}-\frac{4}{n-2}}}_{\varphi^2}\sqrt{\det g_{ij}}g^{k\ell}\partial_\ell f\right) \\ &= \varphi^{-\frac{4}{n-2}}(\Delta_g f + 2\varphi^{-1}g^{k\ell}\partial_k\varphi\partial_\ell f)\,.\end{aligned}$$

This implies

$$\begin{aligned}&\left(\Delta_{\tilde{g}} - \frac{(n-2)}{4(n-1)}\tilde{R}\right)f \\ &= \varphi^{-\frac{4}{n-2}}\left(\Delta_g f + 2\varphi^{-1}g^{k\ell}\partial_k\varphi\partial_\ell f - \frac{(n-2)}{4(n-1)}Rf + \varphi^{-1}f\Delta_g\varphi\right) \\ &= \varphi^{-\frac{4}{n-2}-1}\left(\Delta_g(f\varphi) - \frac{(n-2)}{4(n-1)}Rf\varphi\right) .\end{aligned}$$

Hence the operator

$$\Delta_g - \frac{(n-2)}{4(n-1)}R$$

is conformally-covariant: if $\tilde{g}_{ij} = \varphi^{\frac{4}{n-2}} g_{ij}$, then

$$\left(\Delta_{\tilde{g}} - \frac{(n-2)}{4(n-1)}\tilde{R}\right) f = \varphi^{-\frac{n+2}{n-2}} \left(\Delta_g - \frac{(n-2)}{4(n-1)}R\right) f\varphi\,; \tag{C.3.1}$$

equivalently

$$\left(\Delta_g - \frac{(n-2)}{4(n-1)}R\right) h = \varphi^{\frac{n+2}{n-2}} \left(\Delta_{\tilde{g}} - \frac{(n-2)}{4(n-1)}\tilde{R}\right) \left(\frac{h}{\varphi}\right). \tag{C.3.2}$$

C.4 The Cotton tensor

Given any pseudo-Riemannian metric g_{ij}, the Cotton tensor B_{ijk} is defined as

$$B_{ijk} = A_{i[j;k]}\,, \tag{C.4.1}$$

where A_{ij} is the Schouten tensor (C.2.8). The tensor B_{ijk} has the following properties

$$\underbrace{B_{ijk} = B_{i[jk]}}_{(a)}\,, \quad \underbrace{B^i{}_{ik}}_{(b)} = 0\,, \quad \underbrace{B_{[ijk]}}_{(c)} = 0\,, \tag{C.4.2}$$

which, from a purely algebraic point of view, allows a five-dimensional vector space of such tensors at each space point. (The first property in (C.4.2) follows immediately from the definition; similarly the last one is obvious in view of the symmetry of A_{ij} in its indices. The middle-one coincides with the contracted Bianchi identity, $R_i{}^j{}_{;j} = \frac{1}{2}R_{;j}$.)

The Cotton tensor further satisfies the differential identity

$$B_{i[jk;l]} = 0\,. \tag{C.4.3}$$

One can think of the Cotton tensor as the three-dimensional counterpart of the Weyl tensor. Indeed, the Weyl tensor vanishes identically in dimension three so it is not of much interest there. The key property of B is its invariance under conformal transformation when $n = 3$.

In dimension three, an object equivalent to the Cotton tensor is the tensor

$$H_{ij} = \frac{1}{2}\epsilon^{kl}{}_i B_{jkl}\,. \tag{C.4.4}$$

The tensor H_{ij} is symmetric, tracefree and divergence-free. Indeed, the vanishing of its trace is precisely (C.4.2)(c). The vanishing of the divergence is (C.4.3). To see the symmetry, we calculate as follows, where underbracing an equality sign with "def." means "equal by definition":

$$\begin{aligned} H_{12} \underbrace{=}_{\text{def.}} B_{223} &= \underbrace{B_{113} + B_{223} + B_{333}}_{=0 \text{ by (C.4.2)(b)}} - B_{113} - \underbrace{B_{333}}_{=0 \text{ by (C.4.2)(a)}} = - \underbrace{B_{113}}_{=-B_{131} \text{ by (C.4.2)(a)}} \\ &\underbrace{=}_{\text{def.}} H_{21}\,. \end{aligned}$$

Finally, one readily verifies the inversion formula

$$B_{ijk} = \epsilon_{jk}{}^{\ell} H_{i\ell}\,. \tag{C.4.5}$$

C.5 The Bach tensor

The Bach tensor is defined by the formula

$$B_{ab} = D^c D^d C_{abcd} + \frac{1}{2} R^{cd} C_{acbd} \, . \tag{C.5.1}$$

Its interest arises from the fact that it is conformally covariant in four dimensions,

$$g_{ij} \to \omega^2 g_{ij} \quad \Longrightarrow \quad B_{ij} \to \omega^{-2} B_{ij} \, .$$

Whatever the dimension, B_{ij} vanishes if g is Einstein. This follows from the fact that $D^d C_{abcd}$ vanishes for Einstein metrics by the Bianchi identity, while the second term in (C.5.1) becomes a trace in the second and third index, which is zero for the Weyl tensor.

C.6 The Graham-Hirachi theorem, and the Fefferman-Graham obstruction tensor

C.6.1 The Fefferman-Graham tensor

Let, as elsewhere, $n+1$ denote spacetime dimension. In what follows we assume that n is odd. The Fefferman-Graham tensor $\mathcal{H}$ is a conformally covariant tensor, built out of the metric g and its derivatives up to order $n+1$, of the form

$$\mathcal{H} = (\nabla^* \nabla)^{\frac{n+1}{2}-2} [\nabla^* \nabla (A) + \nabla^2 (trA)] + \mathcal{F}^n \, , \tag{C.6.1}$$

where A is the Schouten tensor (C.2.8), and where $\mathcal{F}^n$ is a tensor built out of lower order derivatives of the metric (see, e.g., [223], where the notation $\mathcal{O}$ is used in place of $\mathcal{H}$). It turns out that $\mathcal{F}^n$ involves only derivatives of the metric up to order $n-1$: this is an easy consequence of Equation (2.4) in [223], using the fact that odd-power coefficients of the expansion of the metric g_x in [223, Equation (2.3)] vanish. (For $n = 3, 5$ this can also be verified by inspection of the explicit formulae for $\mathcal{F}^3$ and $\mathcal{F}^5$ given in [223].)

The system of equations

$$\mathcal{H} = 0 \tag{C.6.2}$$

will be called the Anderson-Fefferman-Graham (AFG) equations. It has the following properties [223]:

1. The system (C.6.2) is conformally invariant: if g is a solution, so is $\varphi^2 g$, for any positive function φ.

2. If g is conformal to an Einstein metric, then (C.6.2) holds.

3. $\mathcal{H}$ is trace-free.

4. $\mathcal{H}$ is divergence-free.

The tensor $\mathcal{H}$ was originally discovered by Fefferman and Graham [199] as an obstruction to the existence of a formal power series expansion for conformally compactifiable Einstein metrics, with conformal boundary equipped with the conformal equivalence class $[g]$ of g. Indeed. $\mathcal{H}$ is the tensor $\tilde g_{\log}$ arising as the coefficient of the $x^n \log x$ term in the expansion (**??**), p. **??**. This geometric interpretation is irrelevant from our point of view, as here we are interested in (C.6.2) as an equation on its own.

C.6.2 The Graham-Hirachi theorem

It is of interest to classify all conformally-covariant tensors which are polynomial in the metric, its inverse, and in the derivatives of the metric. Such tensors will be called natural. Now, one may construct further covariants from known ones by taking tensor products and contracting. A tensor will be called irreducible if it cannot be constructed in that fashion in a non-trivial way.

The following theorem of Hirachi-Graham shows that up to quadratic and higher terms in curvature, the Weyl tensor, or the Cotton tensor in dimension 3, and the obstruction tensor are the only irreducible conformally invariant tensors:

THEOREM C.6.1 (Graham-Hirachi [223]) *A conformally covariant irreducible natural tensor of n-dimensional oriented Riemannian manifolds is equivalent modulo a conformally covariant natural tensor of degree at least 2 in curvature with a multiple of one of the following:*

1. $n = 3$*: the Cotton tensor* $C_{ijk} = A_{ij;k} - A_{ik;j}$
2. $n = 4$*: the self-dual or anti-self dual Weyl tensor*

$$C^{\pm}_{ijkl} := C_{ijkl} \pm \frac{1}{2}\epsilon_{ij}{}^{mn} C_{mnkl}$$

   or the Bach tensor $B_{ij} = \mathcal{O}_{ij}$

3. $n \geq 5$ *odd: the Weyl tensor* C_{ijkl}
4. $n \geq 6$ *even: the Weyl tensor* C_{ijkl} *or the obstruction tensor* $\mathcal{O}_{ij}$

C.7 Frame coefficients, Dirac operators

In order to calculate the transformation law of the connection coefficients, we will consider a conformal rescaling of the form $\bar g_{ij} = e^{2u} g_{ij}$. Let $\bar\theta^i$ be an orthonormal coframe for $\bar g$, then

$$\theta^i := e^{-u}\bar\theta^i$$

is an orthonormal coframe for $\bar g$. We claim that:

$$\bar\omega_{ij}(e_k) = \omega_{ij}(e_k) - e_i(u) g_{jk} + e_j(u) g_{ik}\,, \qquad \text{(C.7.1)}$$

equivalently

$$\bar{\omega}_{ij} = \bar{\omega}_{ij}(e_k)\theta^k = \omega_{ij} - e_i(u)\theta_j + e_j(u)\theta_i \,. \tag{C.7.2}$$

To verify this equation, notice that $\bar{\omega}_{ij}$ as given by this equation is anti-symmetric in i and j; further, it is straightforward to check that

$$d\bar{\theta}^i + \bar{\omega}^i{}_j \wedge \bar{\theta}^j = 0 \,,$$

and (C.7.1) follows from uniqueness of $\bar{\omega}_{ij}$.

Let e_i be an orthonormal frame for g. Recall that the Dirac operator $\nabla\!\!\!\!/$ is defined by the formula

$$\nabla\!\!\!\!/\,\psi := \gamma^k \nabla_{e^k}\psi = \gamma^k(e_k(\psi) - \frac{1}{4}\omega_{ijk}\gamma^i\gamma^j)\psi \,.$$

The corresponding Dirac operator $\bar{\nabla\!\!\!\!/}$ associated to the metric $\bar{g}$ reads

$$\bar{\nabla\!\!\!\!/}\,\psi := \gamma^k \bar{\nabla}_{\bar{e}^k}\psi = \gamma^k(\bar{e}_k(\psi) - \frac{1}{4}\bar{\omega}_{ijk}\gamma^i\gamma^j)\psi \,.$$

Using (C.7.1) one finds

$$\bar{\nabla\!\!\!\!/}\,\psi = e^{-\frac{(n+1)u}{2}}\nabla\!\!\!\!/\,(e^{\frac{(n-1)u}{2}}\psi) \,. \tag{C.7.3}$$

C.8 Elements of bifurcation theory

Bifurcation theory provides one of the tools for constructing solutions of elliptic PDEs. For the reader's convenience, we state here some results that are used in this work. Detailed accounts of this theory can be found in [167, 354, 356–358], [337, Sections 3.2 and 3.3] and references therein.

Consider a continuous mapping $F : I \times \Omega \to B$, where I is a non-empty interval of $\mathbb{R}$, Ω is a subset of a Banach space A and B is another Banach space. We want to study the zero-level set of F, i.e. the set of pairs $(\lambda, x) \in I \times A$ for which $F(\lambda, x) = 0$.

A pair $(\lambda_0, x_0) \in I \times \Omega$ is called a bifurcation point for F if there exists a sequence $(\lambda_i)_{i>0}$ converging to λ_0 with $\lambda_i \in I$ and two sequences $(x_i^1)_{i>0}$, $(x_i^2)_{i>0}$ in Ω such that

- $\forall i, x_i^1 \neq x_i^2$,
- $x_i^1, x_i^2 \to x_0$,
- $F(\lambda_i, x_i^1) = F(\lambda_i, x_i^2) = 0$.

If the mapping F is C^1, this imposes that the differential $\partial_x F$ is not invertible at (λ_0, x_0) since, otherwise, this would contradict the conclusion of the implicit function theorem. Two types of bifurcation points arise in our work, described in the next propositions, see Figure C.8.1.

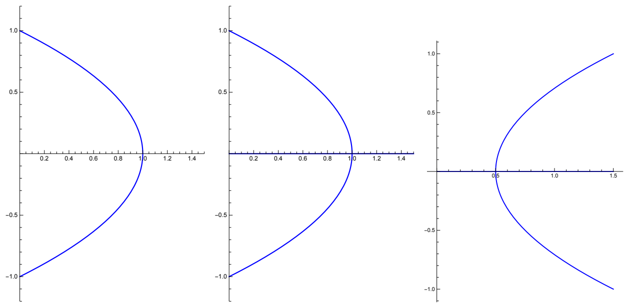

Figure C.8.1: Examples of Fold Bifurcation (left figure), and Pitchfork Bifurcations (middle and right figure).

PROPOSITION C.8.1 (Fold bifurcation) *Let* (λ_0, x_0) *be a point in the zero set of* F*. Assume that* $\partial_x F(\lambda_0, x_0)$ *is Fredholm with index* 0 *and that the kernel of* $\partial_x F(\lambda_0, x_0)$ *has dimension* 1 *and is generated by* y_1*. Assume further that* $\partial_\lambda F(\lambda_0, x_0)$ *is not in the range of* $\partial_x F(\lambda_0, x_0)$*. Then there exists a neighborhood* $J \times \Omega' \subset I \times \Omega$ *of* (λ_0, x_0) *and a* C^1*-curve* $\gamma : U \to J \times \Omega'$*, where* $U \subset \mathbb{R}$ *is a neighborhood of* 0*, such that*

- $\gamma(0) = (\lambda_0, x_0)$,
- $\dot{\gamma}(0) = (0, y_1)$,
- $\forall (\lambda, x) \in J \times \Omega',\ F(\lambda, x) = 0 \quad \Longleftrightarrow \quad \exists t \in U, (\lambda, x) = \gamma(t)$.

For a proof of this proposition, we refer the reader to [354, Theorem 2.3.1]. More information can be gained if we assume that F is C^2:

PROPOSITION C.8.2 (Fold bifurcation 2) *Under the assumptions of Proposition C.8.1, there exists a linear form* $\mu \in B^*$*,* $\mu \neq 0$*, whose kernel is the range of* $\partial_x F(\lambda_0, x_0)$*. Assuming further that* F *is* C^2 *and*

$$\mu(\partial_x^2 F(\lambda_0, x_0)(y_1, y_1)) \neq 0 \,,$$

then the curve $\gamma = (\gamma_\lambda, \gamma_x)$ *is* C^2 *and*

$$\ddot{\gamma}_\lambda(0) = \frac{\mu(\partial_\lambda F(\lambda_0, x_0))}{\mu(\partial_x^2 F(\lambda_0, x_0)(y_1, y_1))} \,.$$

In particular, upon shrinking the neighborhood $J \times \Omega'$ *of* (λ_0, x_0)*, we have:*

- *If* $\ddot{\gamma}_\lambda(0) > 0$*, then, for any* $\lambda \in J$*,*

$$\#\{x \in \Omega', F(\lambda, x) = 0\} = \begin{cases} 0 & \text{if } \lambda < \lambda_0 \,, \\ 1 & \text{if } \lambda = \lambda_0 \,, \\ 2 & \text{if } \lambda > \lambda_0 \,. \end{cases}$$

- *If* $\ddot{\gamma}_\lambda(0) < 0$*, then, for any* $\lambda \in J$*,*

$$\#\{x \in \Omega', F(\lambda, x) = 0\} = \begin{cases} 2 & \text{if } \lambda < \lambda_0 \,, \\ 1 & \text{if } \lambda = \lambda_0 \,, \\ 0 & \text{if } \lambda > \lambda_0 \,. \end{cases}$$

The second type of bifurcation was discovered in [167]:

PROPOSITION C.8.3 (Pitchfork bifurcation) *Assume that F is C^2 and that*

$$U \ni t \mapsto \gamma(t) \equiv (\gamma^\lambda(t), \gamma^x(t)) \in I \times \Omega$$

is a C^1 curve of solutions:

$$\forall t \in U,\ F(\gamma(t)) = 0\,,$$

with U a neighborhood of 0 *in* $\mathbb{R}$ *such that*

$$\gamma(0) = (\lambda_0, x_0)\,, \quad \dot\gamma^\lambda(0) \neq 0\,.$$

Assume further that $\partial_x F(\lambda_0, x_0)$ has a 1-dimensional kernel spanned by $v \in A$ and that $D^2F(\lambda_0, x_0)(\dot\gamma(0), v) \neq 0$. Then (λ_0, x_0) is a bifurcation point for F and there exists a neighborhood $J \times \Omega'$ of (λ_0, x_0) such that the set of solutions of $F(\lambda, x) = 0$ consists of the union of two C^2 curves (γ and another one) intersecting (transversally) only at (λ_0, x_0).

Appendix D

A collection of identities

We include here a collection of useful identities, mostly compiled by Erwann Delay. I am grateful to Erwann for allowing me to include his list here.

D.1 ADM notation

Letting $\tilde{g}^{ij}$ denote the inverse matrix to g_{ij}, using the Arnowitt-Deser-Misner notation we have

$$g^{kl} = \tilde{g}^{kl} - \tfrac{N^l N^k}{N^2},\ g_{0k} = N_k,\ g^{0k} = \tfrac{N^k}{N^2},\ N^2 = -\tfrac{1}{g^{00}},\ g_{00} = N^k N_k - N^2. \tag{D.1.1}$$

where $N^k := \tilde{g}^{kl} N_l$. The associated decomposition of the Christoffel symbols reads

$$\Gamma^0_{k0} = \partial_k \log N - \frac{N^l}{N} K_{lk}\,, \quad \Gamma^0_{00} = \partial_0 \log N + N^k \partial_k \log N - \frac{N^l N^k}{N} K_{lk}$$

(recall that $K_{kl} = -N\Gamma^0_{kl} = \frac{1}{2N}(D_l N_k + D_k N_l - \partial_0 g_{kl})$). Furthermore,

$$\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij} + \frac{N^k}{N} K_{ij}\,, \quad \Gamma^k_{0j} = D_j N^k - N K^k{}_j + \frac{N^k}{N}\Big(N^l K_{lj} - D_j N\Big)\,.$$

D.2 Some commutators

Here are some formulae for the commutation of derivatives:

$$\nabla_m \nabla_l t_{ik} - \nabla_l \nabla_m t_{ik} = R^p{}_{klm} t_{ip} + R^p{}_{ilm} t_{kp}\,,$$

$$\nabla_i \nabla_j V^l - \nabla_j \nabla_i V^l = R^l{}_{kij} V^k\,,$$

$$\nabla^k \nabla_k |df|^2 = 2(\nabla^l f \nabla_l \nabla^k \nabla_k f + Ric(\nabla f, \nabla f) + |\nabla\nabla f|^2)\,,$$

$$\nabla^k \nabla_k \nabla_i \nabla_j f - \nabla_i \nabla_j \nabla^k \nabla_k f - R_{kj} \nabla^k \nabla_i f - R_{ki} \nabla^k \nabla_j f + 2R_{qjli} \nabla^q \nabla^l f$$

$$= (\nabla_i R_{kj} + \nabla_j R_{ki} - \nabla_k R_{ij}) \nabla^k f\,.$$

D.3 Bianchi identities

The Bianchi identities for a Levi-Civita connection:

$$R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj} = 0\,,$$
$$\nabla_l R^t{}_{ijk} + \nabla_k R^t{}_{ilj} + \nabla_j R^t{}_{ikl} = 0\,,$$
$$\nabla_t R^t{}_{ijk} + \nabla_k R_{ij} - \nabla_j R_{ik} = 0\,,$$
$$\nabla^k R_{ik} - \frac{1}{2}\nabla_k R = 0\,.$$

D.4 Linearisations

Linearisations for various objects of interest:

$$D_g\Gamma^k_{ij}(g)h = \frac{1}{2}(\nabla_i h^k_j + \nabla_j h^k_i - \nabla^k h_{ij})\,,$$
$$2[D_g\mathrm{Riem}(g)h]_{sklm} = \nabla_l\nabla_k h_{sm} - \nabla_l\nabla_s h_{km} + \nabla_m\nabla_s h_{kl} - \nabla_m\nabla_k h_{sl} + R^p{}_{klm}h_{ps} + R^p{}_{sml}h_{pk}\,,$$
$$2[D_g\mathrm{Riem}(g)h]^i_{klm} = \nabla_l\nabla_k h^i_m - \nabla_l\nabla^i h_{km} + \nabla_m\nabla^i h_{kl} - \nabla_m\nabla_k h^i_l + g^{is}R^p{}_{sml}h_{pk} - R^p{}_{klm}h^i_p\,,$$
$$D_g Ric(g)h = \frac{1}{2}\Delta_L h - \mathrm{div}^*\mathrm{div}(Gh)\,,$$
$$\Delta_L h_{ij} = -\nabla^k\nabla_k h_{ij} + R_{ik}h^k{}_j + R_{jk}h^k{}_j - 2R_{ikjl}h^{kl}\,,$$
$$Gh = h - \frac{1}{2}tr\ hg,\quad (\mathrm{div}h)_i = -\nabla^k h_{ik},\quad \mathrm{div}^* w = \frac{1}{2}(\nabla_i w_j + \nabla_j w_i)\,,$$
$$D_g R(g)h = -\nabla^k\nabla_k(tr\ h) + \nabla^k\nabla^l h_{kl} - R^{kl}h_{kl}\,,$$
$$[D_g R(g)]^* f = -\nabla^k\nabla_k f g + \nabla\nabla f - f\ Ric(g)\,.$$

D.5 Warped products

Let (M,g), $\nabla := \nabla_g$, $f : M \to \mathbb{R}$ and

$$(\mathcal{M} = M \times_f I, \widetilde{g} = -f^2dt^2 + g)\,,$$

then for X, Y tangent to M and V, W tangent to I, we have

$$Ric(\widetilde{g})(X,Y) = Ric(g)(X,Y) - f^{-1}\nabla\nabla f(X,Y)\,,$$
$$Ric(\widetilde{g})(X,V) = 0 = \widetilde{g}(X,V)\,,$$
$$Ric(\widetilde{g})(V,W) = -f^{-1}\nabla^k\nabla_k f\widetilde{g}(V,W)\,.$$

Let (M,g), $\nabla := \nabla_g$, $f : M \to \mathbb{R}$ and let $(\mathcal{M} = M\times_f I, \widetilde{g} = \epsilon f^2dt^2 + g)$, $\epsilon = \pm 1$.
$x^a = (x^0 = t, x^i = (x^1, \ldots, x^n))$.

$$\widetilde{\Gamma}^0_{00} = \widetilde{\Gamma}^0_{ij} = \widetilde{\Gamma}^k_{i0} = 0,\ \ \widetilde{\Gamma}^0_{i0} = f^{-1}\partial_i f,\ \ \widetilde{\Gamma}^k_{00} = -\epsilon f\nabla^k f,\ \ \widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij}\,,$$
$$\widetilde{R}^l{}_{ijk} = R^l{}_{ijk},\ \ \widetilde{R}^l{}_{0j0} = -\epsilon f\nabla_j\nabla^l f,\ \ \widetilde{R}^0{}_{ij0} = f^{-1}\nabla_j\nabla_i f\,,$$
$$\widetilde{R}^0{}_{ijk} = \widetilde{R}^l{}_{ij0} = \widetilde{R}^l{}_{0jk} = \widetilde{R}^0{}_{0jk} = \widetilde{R}^0{}_{0j0} = 0\,,$$
$$\widetilde{R}_{mijk} = R_{mijk},\ \ \widetilde{R}_{0ijk} = 0,\ \ \widetilde{R}_{0ij0} = \epsilon f\nabla_j\nabla_i f\,,$$
$$\widetilde{R}_{ik} = R_{ik} - f^{-1}\nabla_k\nabla_i f,\ \ \widetilde{R}_{0k} = 0,\ \ \widetilde{R}_{00} = -\epsilon f\nabla^i\nabla_i f\,,$$
$$\widetilde{R} = R - 2f^{-1}\nabla^i\nabla_i f\,.$$

D.6 Hypersurfaces

Let M be a non-isotropic hypersurface in $\widetilde{M}$, with ν normal, and u, v tangent to M at m, we have

$$II(u,v) = (\widetilde{\nabla}_U V - \nabla_U V)_m = (\widetilde{\nabla}_U V)^{\perp}_m = II(v,u) = -l(u,v)\nu_m\,.$$

Setting $S(u) = \widetilde{\nabla}_u \nu \in T_m M$, one has

$$< S(u), v >=< \widetilde{\nabla}_u \nu, v >=< -\nu, \widetilde{\nabla}_u V >= l(u,v)\,.$$

If x, y, u, v are tangent to M, then

$$R(x,y,u,v) = \widetilde{R}(x,y,u,v) + l(x,u)l(y,v) - l(x,v)l(y,u)\,.$$

The *Gauss-Codazzi* equations read

$$\widetilde{R}(x,y,u,\nu) = \nabla_y l(x,u) - \nabla_x l(y,u)\,.$$

The Ricci tensor can be decomposed as:

$$\widetilde{R}(y,v) = R(y,v) + II \circ II(y,v) - trII\; II(y,v) + \widetilde{R}(\nu,y,\nu,v)\,,$$

$$\widetilde{R}(y,\nu) = -\nabla_y trII + y^j \nabla^i II_{ij}\,,$$

$$\widetilde{R} = R + |II|^2 - (trII)^2 + 2\widetilde{R}(\nu,\nu)\,.$$

D.7 Conformal transformations

The Weyl tensor:

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) + \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il})\,.$$

We have

$$W_i{}^j{}_{kl}(e^f g) = W_i{}^j{}_{kl}(g)\,.$$

The Schouten tensor

$$S_{ij} = \frac{1}{n-2}[2R_{ij} - \frac{R}{n-1}g_{ij}]\,.$$

Under a conformal transformation $g' = e^f g$, we have

$$\Gamma'^k_{ij} - \Gamma^k_{ij} = \frac{1}{2}(\delta^k_j \partial_i f + \delta^k_i \partial_j f - g_{ij}\nabla^k f)\,.$$

$$R'_{ij} = R_{ij} - \frac{n-2}{2}\nabla_i\nabla_j f + \frac{n-2}{4}\nabla_i f \nabla_j f - \frac{1}{2}(\nabla^k\nabla_k f + \frac{n-2}{2}|df|^2)g_{ij}$$

$$R' = e^{-f}[R - (n-1)\nabla^i\nabla_i f - \frac{(n-1)(n-2)}{4}\nabla^i f \nabla_i f]\,.$$

Specialising to $g' = e^{\frac{2}{n-2}u}g$,

$$R'_{ij} = R_{ij} - \nabla_i\nabla_j u + \frac{1}{n-2}\nabla_i u\nabla_j u - \frac{1}{n-2}(\nabla^k\nabla_k u + |du|^2)g_{ij}\,.$$

In the notation $g' = v^{\frac{2}{n-2}}g$,

$$R'_{ij} = R_{ij} - v^{-1}\nabla_i\nabla_j v + \frac{n-1}{n-2}v^{-2}\nabla_i v\nabla_j v - \frac{1}{n-2}v^{-1}(\nabla^k\nabla_k v)g_{ij}\,.$$

If we write instead $g' = \phi^{4/(n-2)}g$, then

$$R'_{ij} = R_{ij} - 2\phi^{-1}\nabla_i\nabla_j\phi + \frac{2n}{n-2}\phi^{-2}\nabla_i\phi\nabla_j\phi - \frac{2}{n-2}\phi^{-1}(\nabla^k\nabla_k\phi + \phi^{-1}|d\phi|^2)g_{ij}\,,$$

$$R'\phi^{(n+2)/(n-2)} = -\frac{4(n-1)}{n-2}\nabla^k\nabla_k\phi + R\phi\,.$$

When we have two metrics g and g' at our disposal, then

$$T^k_{ij} := \Gamma'^k_{ij} - \Gamma^k_{ij} = \frac{1}{2}g'^{kl}(\nabla_i g'_{lj} + \nabla_j g'_{li} - \nabla_l g'_{ij})\,.$$

$$\mathrm{Riem}'^i{}_{klm} - \mathrm{Riem}^i{}_{klm} = \nabla_l T^i_{km} - \nabla_m T^i_{kl} + T^i_{jl}T^j_{km} - T^i_{jm}T^j_{kl}\,.$$

Under $g' = e^f g$, the Laplacian acting on functions transforms as

$$\nabla'^k\nabla'_k v = e^{-f}(\nabla^k\nabla_k v + \frac{n-2}{2}\nabla^k f\nabla_k v)\,.$$

For symmetric tensors we have instead

$$\begin{aligned}\nabla'^k\nabla'_k u_{ij} \;=\; & e^{-f}\Big[\nabla^k\nabla_k u_{ij} + \frac{n-6}{2}\nabla^k f\nabla_k u_{ij} - (\nabla_i f\nabla^k u_{kj} + \nabla_j f\nabla^k u_{ki}) \\ & +(\nabla^k f\nabla_i u_{kj} + \nabla^k f\nabla_j u_{ki}) + (\frac{3-n}{2}\nabla^k f\nabla_k f - \nabla^k\nabla_k f)u_{ij} \\ & -\frac{n}{4}(\nabla_i f\nabla^k f u_{kj} + \nabla_j f\nabla^k f u_{ki}) + \frac{1}{2}\nabla_i f\nabla_j f u^k_k + \frac{1}{2}g_{ij}u_{kl}\nabla^k f\nabla^l f\Big]\,.\end{aligned}$$

D.8 Laplacians on tensors

For symmetric u's and arbitrary T's let

$$(Du)_{kij} := \frac{1}{\sqrt{2}}(\nabla_k u_{ij} - \nabla_j u_{ik}),$$

then

$$(D^*T)_{ij} = \frac{1}{2\sqrt{2}}(-\nabla^k T_{kij} - \nabla^k T_{kji} + \nabla^k T_{ijk} + \nabla^k T_{jik})\,.$$

Further

$$D^*Du_{ij} = -\nabla^k\nabla_k u_{ij} + \frac{1}{2}(\nabla^k\nabla_i u_{jk} + \nabla^k\nabla_j u_{ik})\,,$$

and

$$\mathrm{div}^*\mathrm{div}u = -\frac{1}{2}(\nabla_i\nabla^k u_{jk} + \nabla_j\nabla^k u_{ik})\,,$$

thus

$$(D^*D + \mathrm{div}^*\mathrm{div})u_{ij} = -\nabla^k\nabla_k u_{ij} + \frac{1}{2}(R_{kj}u^k_i + R_{ki}u^k_j - 2R_{qjli}u^{ql})\,.$$

D.9 Stationary metrics

Let (M, γ) be a Riemannian or pseudo-Riemannian three dimensional manifold, define $\lambda : M \to \mathbb{R}$, $\xi : M \to T^*M$, $(N = I \times M, g)$ by the formulae

$$g(t, x) = \begin{pmatrix} \lambda & {}^t\xi \\ \xi & \lambda^{-1}(\xi\ {}^t\xi - \gamma) \end{pmatrix} = \lambda(dt + \lambda^{-1}\xi_i dx^i)^2 - \lambda^{-1}\gamma_{ij}dx^i dx^j \,.$$

Let $w = -\lambda^2 *_\gamma d(\lambda^{-1}\xi)$. $\nabla = \nabla_g$, $E^i = \gamma^{is}E_s$. Then

$$\mathrm{Ric}(\gamma)_{ij} = \frac{1}{2}\lambda^{-1}(\nabla_i\lambda\nabla_j\lambda + w_i w_j) + \lambda^{-2}(Ric(g)_{ij} - Ric(g)_{cd}\xi^c\xi^d\gamma_{ij})\,,$$

$$\nabla^i\nabla_i\lambda = \lambda^{-1}(|d\lambda|^2 - |w|^2) - 2\lambda^{-1}Ric(g)_{ab}\xi^a\xi^b\,,$$

$$\nabla^i(\lambda^{-2}w_i) = 0\,,$$

$$\lambda(*_\gamma dw)^i = -2\lambda^{-1}T(g)^i_c\xi^c, \quad Ric(g) = G(T(g))\,.$$

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