BOUNDARY CONDITIONS AT SPATIAL INFINITY FROM A HAMILTONIAN POINT OF VIEW[1]

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INTRODUCTION

There are many, both conceptually and technically different ways to obtain the ADM expression for the energy of the gravitational field, some of the published methods containing inconsistencies, most of them raising doubts about uniqueness of the final result. The author wishes to present here a simple way of obtaining this expression in a geometrical Hamiltonian setting allowing for an exact analysis of all ambiguities present. One of the results of this study is a considerable weakening of the boundary conditions at spatial infinity, for which the energy-momentum of an initial data set is finite and well defined. The derivation of the ADM Hamiltonian presented here is the simplest one known to the author, as far as calculations are concerned.
The starting point of our derivation of the ADM Hamiltonian will be the so-called affine formulation of general relativity [1]. Many different formulations of general relativity may be used for this purpose, this is however in the general affine framework that the calculations are the simplest ones. It must be noted that vacuum general relativity is a somewhat pathological theory in this context (cf. the discussion following formula (1)), but this presents no difficulty in our approach: we shall start with a theory which contains a certain number of matter fields, find the Hamiltonian for this theory, and finally set the non-gravitational fields to zero, obtaining thus the Hamiltonian for vacuum general relativity. All the calculations required to obtain the final expression for the Hamiltonian are four-dimensional covariant, they do not necessitate a 3 + 1 decomposition of the fields. This requires some justification, because usually the phase space of general relativity is thought of as the space of functions (g_{ij}, P^{ij}) (the ADM data) on a three dimensional manifold, satisfying certain constraint equations and certain boundary conditions. However, by well known evolution theorems, every such set of functions gives rise to a four-dimensional Lorentzian manifold in which the four-dimensional field equations are satisfied - this shows that the phase space of general relativity (or, in fact, of any field theory) is isomorphic to the space of solutions of the equations of the theory, satisfying certain boundary conditions. In general relativity the ADM data give one possible parameterisation of this space (which is incomplete, since constraint equations are still present). Anybody acquainted with symplectic geometry knows that no coordinates are required to make sense of Hamilton's equations of motion (or, equivalently, any set of coordinates is fine) once the symplectic structure on the phase space is given. This leads one to expect that

[1]Published in *"Topological Properties and Global Structure of Space-Time"*, ed. by P. Bergmann, V. de Sabbata, pp. 49-59, Plenum Press, New York 1986

there should exist a framework in which four dimensional covariant quantities can be used in the calculations - such a framework has been recently constructed by J. Kijowski and W. Tulczyjew [2]. We shall not attempt here to review this construction and will just present how it works in general relativity.

THE AFFINE FORMULATION OF GENERAL RELATIVITY

As has been shown by M. Ferraris and J. Kijowski [1,3,4], every Lagrangian theory of gravitation and some matter fields ϕ^A satisfying field equations deriving from an action of the form

$$I[g_{\mu\nu},\phi^A]=\int(g^{\mu\nu}R_{\mu\nu}+L_m(\phi^A,\phi^A{}_{,\mu},g_{\mu\nu},g_{\mu\nu,\sigma}))(-det\ g)^{1/2}d^4x \qquad (1)$$

can be formulated as a "purely affine theory" in the following sense: the theory may be considered as a theory of a $GL(4,R)$ connection field $\Gamma^\lambda_{\mu\nu}$, the field equations deriving from an action of the form:

$$I[\Gamma^\lambda_{\mu\nu},\phi^A]=\int L(\Gamma^\lambda_{\mu\nu},\Gamma^\lambda_{\mu\nu,\sigma},\phi^A,\phi^A{}_{,\sigma})\ d^4x.$$

(Care must be taken when interpreting this result. The scalar density L appearing above is what the physicists call a Lagrangian only in the case of no constraints in the "infinitesimal configuration space", the reader is referred to ref. [2] for details. In the case of vacuum general relativity L above is defined only on the constraint hypersurface $R_{\mu\nu}=0$, its numerical value being zero. Let us also note, that Kijowski's theorem does not hold in presence of fermionic fields, because there is no natural formulation of such theories with an action of the form (1)).

It has also recently been shown [5], that every purely affine theory with a Lagrangian of the form

$$L=L(R^\lambda{}_{\mu\nu\rho}) \qquad (2)$$

can be interpreted as an Einstein theory of gravitation, in which certain components of the connection can be considered as tensor matter fields. For example, in the simple case of a Lagrangian taking the form

$$L=L(K_{\mu\nu},F_{\mu\nu}),$$

where

$$K_{\mu\nu}=(R^\alpha{}_{\mu\alpha\nu}+R^\alpha{}_{\nu\alpha\mu})/2\ , \qquad F_{\mu\nu}=R^\alpha{}_{\alpha\mu\nu}\ .$$

the metric is obtained from the equation

$$((-det\ g)^{1/2}g^{\mu\nu})/16\pi=\pi^{\mu\nu}, \qquad (3)$$

$$\pi^{\mu\nu}=\partial L/\partial K_{\mu\nu}. \qquad (4)$$

When L is taken to be of the form

$$L=(-det\ K_{\alpha\beta})^{1/2}\ K^{\mu\alpha}\ K^{\nu\beta}\ F_{\mu\beta}\ F_{\nu\alpha}/16\pi,$$

where $K^{\alpha\beta}$ is the inverse tensor to $K_{\alpha\beta}$, the field equations are just Einstein-Maxwell equations [6,7] (in this case the quantity $A_\mu = \Gamma^\lambda_{\mu\lambda}$ has the interpretation of the electromagnetic potential). Kijowski and Tulczyjew [2] have derived the following formula for the Hamiltonian of the theory:

$$E(X, \Sigma) = \int_\Sigma (\pi_\alpha{}^{\gamma\mu\beta} \mathcal{L}_X \Gamma^\alpha_{\beta\gamma} - X^\mu L)\eta_\mu, \tag{5}$$

where Σ is any hypersurface of codimension 1 in the manifold M on which we study the dynamics of the gravitational field, X is any vector field on M, and

$$\pi_\lambda{}^{\mu\nu\rho} = \partial L / \partial \Gamma^\lambda_{\rho\mu,\nu} \tag{6}$$

(the "strange-looking" positioning of indices on $\pi_\alpha{}^{\beta\gamma\delta}$ in (5) and (6) comes from the conventions of ref. [8], which are used throughout this paper). We will show that E is indeed a Hamiltonian for translations generated by X, modulo some boundary terms which will be analysed later on. We will restrict ourselves to Lagrangians of the form (2). As has been pointed out above, this form of L is general enough to include the Einstein-Maxwell theory, and therefore sufficient to obtain what we finally aim to: the Hamiltonian for vacuum general relativity. To show that formula (4) provides a Hamiltonian on the phase space (it is, the space of fields satisfying the field equations, and some boundary conditions to be imposed later on) let us calculate the differential of E:

$$\begin{aligned} \delta E(X, \Sigma) &= \int_\Sigma (\ \delta\pi_\alpha{}^{\gamma\mu\beta} \mathcal{L}_X \Gamma^\alpha_{\beta\gamma} + \pi_\alpha{}^{\gamma\mu\beta} \delta \mathcal{L}_X \Gamma^\alpha_{\beta\gamma} - X^\mu \delta L\ )\eta_\mu \\ &= \int_\Sigma (\ \mathcal{L}_X \Gamma^\alpha_{\beta\gamma} \delta\pi_\alpha{}^{\gamma\mu\beta} - \mathcal{L}_X \pi_\alpha{}^{\gamma\mu\beta} \delta\Gamma^\alpha_{\beta\gamma}\ )\eta_\mu \\ &\quad + \int_\Sigma \Big(\ \mathcal{L}_X \pi_\alpha{}^{\gamma\mu\beta} \delta\Gamma^\alpha_{\beta\gamma} - X^\mu (\pi_\alpha{}^{\gamma\sigma\beta} \delta\Gamma^\alpha_{\beta\gamma})_{,\sigma} \Big)\, \eta_\mu, \end{aligned} \tag{7}$$

and we have used the formula

$$\delta L = (\pi_\alpha{}^{\gamma\sigma\beta} \delta\Gamma^\alpha_{\beta\gamma})_{,\sigma}\ ,$$

which holds in virtue of field equations. It is easily seen (using the definition of Lie derivatives) that the last integral in the right hand side of formula (7) is a total divergence, and one obtains

$$\delta E = \int_\Sigma (\ \mathcal{L}_X \Gamma^\alpha_{\beta\gamma} \delta\pi_\alpha{}^{\gamma\mu\beta} - \mathcal{L}_X \pi_\alpha{}^{\gamma\mu\beta} \delta\Gamma^\alpha_{\beta\gamma}\ )\ \eta_\mu + \int_{\partial\Sigma} \pi_\alpha{}^{\gamma\beta[\mu} X^{\nu]} \delta\Gamma^\alpha_{\beta\gamma} \eta_{\mu\nu}. \tag{8}$$

This formula has a deep symplectic meaning, for details the reader is referred to ref. [2]. It can be used as a starting point of the canonical analysis of general relativity [9,10,11] and in fact it "looks like" Hamilton's equations of motion

$$dH = \dot{q}\ dp - \dot{p}\ dq\ ,$$

apart from the boundary term. The numerical value of E is given by equation

(5) - it takes a three lines calculation to show that E is a boundary integral, and to calculate its actual value [8]:

$$E = (\int_{\partial\Sigma} \pi_\delta{}^{\mu\alpha\beta}(X^\delta{}_{;\mu} + (\Gamma^\delta_{\sigma\mu} - \Gamma^\delta_{\mu\sigma})X^\sigma)\eta_{\beta\alpha})/2. \tag{9}$$

In what follows we shall confine our attention to pure gravity, in which case $\pi_\lambda{}^{\mu\nu\alpha}$ takes the form [8]

$$\pi_\lambda{}^{\mu\nu\alpha} = 2\ \pi^{\mu[\alpha}\delta^{\nu]}_\lambda, \tag{10}$$

$\pi^{\mu\nu}$ is related to the metric via (3) and, as a consequence of the field equations, $\Gamma^\lambda_{\mu\nu}$ is the symmetric Riemannian connection of $g_{\mu\nu}$. It is convenient to introduce the variable

$$A^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \delta^\lambda_\mu \Gamma^\sigma_\sigma.$$

Insertion of (10) into (8) leads to

$$\delta E = \theta_\Sigma + \int_{\partial\Sigma} \pi^{\alpha\beta} X^{[\mu}\delta A^{\nu]}_{\alpha\beta}\eta_{\mu\nu}\ ,$$

$$\theta_\Sigma = \int_\Sigma (\mathcal{L}_X A^\mu_{\alpha\beta}\delta\pi^{\alpha\beta} - \mathcal{L}_X \pi^{\alpha\beta}\delta A^\mu_{\alpha\beta})\eta_\mu\ . \tag{11}$$

In the pure vacuum case we are considering here, insertion of (10) into (9) leads to the Komar integral:

$$E = (\int_{\partial\Sigma} \nabla^\mu X^\nu \eta_{\mu\nu})/16\ \pi$$

(it is worthwhile noting, that this is the Komar expression with a "factor wrong by 1/2" - when evaluated for the Schwarzschild metric with $X = \partial/\partial t$, the Killing vector, it gives m/2).

ASYMPTOTICALLY FLAT SPACE-TIMES

In order to analyse the boundary terms in eq. (11) let us assume that Σ is a spacelike hypersurface extending up to infinity in an asymptotically flat space-time, where "asymptotic flatness" is to be understood as follows: outside a world tube there exists a coordinate system such that

$$g_{\mu\nu} = \mathring{\eta}_{\mu\nu} + h_{\mu\nu}, \tag{12}$$

where $\mathring{\eta}_{\mu\nu}$ is the Minkowski metric, and $h_{\mu\nu}$ satisfies

$$|h_{\mu\nu}| \le C/r^\alpha\ , \qquad |h_{\mu\nu,\sigma}| \le C/r^{\alpha+1}, \tag{13}$$

for some α to be specified later. It will be assumed, that X tends asymptotically to the vector $\partial/\partial t$. If we want the functional E to generate the time translations (it is, translations along X) we have to "kill" the boundary non-dynamical terms in formula (11) - and this can be done by imposing appropriate boundary conditions in the space of metrics we are working with. Formula (13) shows, that

$$\pi^{\alpha\beta}X^{[\mu}\delta A^{\nu]}_{\alpha\beta} \sim 1/r^{\alpha+1}.$$

If we required the boundary terms to vanish in the limit $r \to \infty$, we should have $\alpha > 1$ but then, due to the positive energy theorems, the metric $g_{\mu\nu}$ would have to be flat. This is related to the fact, that E is a Hamiltonian for time-translations in a space of functions, where certain leading order components of $\Gamma^{\lambda}_{\mu\nu}$ are kept fixed on the boundary — but this is not the way we have introduced asymptotically flat metrics. In (12) we are keeping fixed the leading order components of the metric, and not of the connection. A remedy to this problem is given by the following procedure: introduce, for large r, a fixed "background" metric $f_{\mu\nu}$ and let

$$\mathring{A}^{\alpha}_{\mu\nu} = \mathring{\Gamma}^{\alpha}_{\mu\nu} - \delta^{\alpha}_{\mu}\mathring{\Gamma}^{\sigma}_{\nu\sigma} ,$$

where $\mathring{\Gamma}^{\alpha}_{\beta\alpha}$ are the Christoffel symbols of the metric $f_{\mu\nu}$. Introduce

$$H = E - \int_{\partial\ \Sigma} \pi^{\alpha\beta} X^{[\mu} D^{\nu]}_{\alpha\beta}\ \eta_{\mu\nu}, \tag{14}$$

where

$$D^{\alpha}_{\beta\gamma} = A^{\alpha}_{\beta\gamma} - \mathring{A}^{\alpha}_{\beta\gamma}$$

($D^{\alpha}_{\beta\gamma}$ is a tensor). From (11) and (14) one finds

$$\delta H = \delta E - \int_{\partial\Sigma} \delta\pi^{\alpha\rho} X^{\mu} D^{\nu}_{\alpha\beta}\ \eta_{\mu\nu} - \int_{\partial\Sigma} \pi^{\alpha\beta} X^{\mu} \delta D^{\nu}_{\alpha\beta}\ \eta_{\mu\nu} .$$

Since the background is fixed, $\delta D^{\alpha}_{\beta\gamma} = \delta A^{\alpha}_{\beta\gamma}$, therefore

$$\delta H = \theta_{\Sigma} + \int_{\partial\Sigma} X^{\mu} D^{\nu}_{\alpha\beta} \delta\pi^{\alpha\beta}\ \eta_{\mu\nu}. \tag{15}$$

If we consider metrics satisfying (13) we have

$$|\delta\pi^{\alpha\beta}| \le C r^{-\alpha} , \quad |D^{\alpha}_{\beta\gamma}| \le C r^{-\alpha-1} ,$$

$$|X^{\mu} D^{\nu}_{\alpha\beta} \delta\pi^{\alpha\beta}| \le C r^{-\alpha-1} .$$

The non-dynamical terms in (15) will give no contribution if we require $2\alpha+1 > 2$, it is $\alpha = 1/2+\varepsilon, \varepsilon$ being any strictly positive number. In this space of metrics we will simply have

$$\delta H = \theta_{\Sigma} .$$

The final formula for the Hamiltonian can be written in the form [12]:

$$H = (\int_{\partial\Sigma} E^{\alpha\beta} \eta_{\alpha\beta}) /\ \ 16\pi, \tag{16}$$

where

$$\begin{aligned} E^{\alpha\beta} &= (U_{\mu}{}^{\alpha\beta} X^{\mu} + g^{\lambda[\alpha}\delta^{\beta]}_{\mu} X^{\mu}{}_{|\lambda})(-\det g)^{1/2} \\ U_{\nu}{}^{\alpha\beta} &= g_{\nu\mu}(e^2 g^{\mu[\alpha} g^{\beta]\sigma})_{|\sigma} e^{-2} , \qquad e^2 = \det g_{\mu\nu} / \det f_{\mu\nu} \end{aligned} \tag{17}$$

and a bar refers to covariant differentiation with respect to the background metric[2]. It may be of some interest to mention, that the transition from H' to H is accomplished via a sort of Legendre transformation. A good analogy is provided by thermodynamics, where one defines the internal energy

$$dU = TdS + pdV$$

and one interprets the increments of U as the amount of energy required to change the state of, say, a gas by heating it while keeping its volume fixed. Another type of energy (the enthalpy) is obtained if one heats the gas while keeping its pressure fixed:

$$dH = d(U - pV) = TdS - Vdp\ .$$

Let us briefly analyse the expression (17). It must be emphasised, that it provides a Hamiltonian for the dynamics of the gravitational field for any background, provided the boundary integral in (15) vanishes - in particular, one can use it for asymptotically anti-de-Sitter space-times. It is also worthwhile noting, that the vector field X is still arbitrary in this formula. It seems that the discussion of the dependence of H upon X and the background metric has to be done separately for each class of space-times considered — from now on we will restrict the discussion to the usual dynamical description of asymptotically flat space-times, in the sense of (13), with $\alpha > 1/2$. Let us therefore specify Σ to be a $t =$ const hypersurface, $f_{\mu\nu}$ to be the flat metric $\mathring{\eta}_{\mu\nu}$, and X to be any translational Killing vector of the metric $\mathring{\eta}_{\mu\nu}$. Since X is now "background covariantly constant" the second term in (17) vanishes, therefore

$$H(X) = (\int_{\partial\Sigma} (-\det\ g)^{1/2}\ U_\mu{}^{\alpha\beta}\ X^\mu\ \eta_{\alpha\beta})/16\ \pi\ .$$

This formula is known as the "Freud superpotential" for the "Einstein energy-momentum pseudo-tensor". In the ADM notation it takes the following form (in the coordinates x^μ satisfying (13), with $\alpha > 1/2$):

$$P_0 = H(X = \partial/\partial t) = (\int_{\partial\Sigma} (g_{ik,k} - g_{kk,i})\ dS_i\ )/16\ \pi, \tag{18}$$

$$P_i = H(X = \partial/\partial x^i) = (\int_{\partial\Sigma} P^{ij}\ dS_j\ )/8\ \pi. \tag{19}$$

The above formulae are the well known ADM expressions for the energy-momentum of an initial data set.

[2]The introduction of the background metric is motivated by the way we defined asymptotic flatness, and also by the fact that we want the integrand of H to have correct transformation properties. Instead of considering (14) we could consider

$$H' = E - \int_{\partial\Sigma} \pi^{\alpha\beta} X^\mu A^\nu_{\alpha\beta}\eta_{\mu\nu}$$

Although the integrand of H' ceases to be a two-form density from a four-dimensional point of view, it can be shown that it is intrinsically defined by X and the geometry of $\partial\Sigma$. The "background metric" approach seems however more convenient for further purposes.

While inspecting formulae (18) and (19) three questions arise immediately:

1) what does the symbol $\int_{\partial\Sigma}$ mean ? Such an integral is usually understood as the limit of integrals over spheres, while the radii of spheres tend to infinity. Does such a limit exist and, if so, does it depend upon the family of spheres used to perform this calculation?

2) If these limits exist in some sense, are they finite ?

3) Do these limits depend upon the particular system of coordinates (satisfying (13)) used to perform the calculations ? Neither (18) nor (19) are defined in an intrinsic way on Σ — (19) contains a free vector index, and (18) contains partial derivatives of a tensor.

We will analyse these problems in the case of a fixed Cauchy hypersurface Σ (it can be shown, that P_μ transforms as a Lorentz convector under boosts of Σ, this will however be discussed elsewhere). Before giving the precise statement of the theorems, it is useful to introduce first some terminology. Suppose one is given a pair (g,ϕ), where

1) g is a Riemannian metric on a three dimensional manifold N, N diffeomorphic to $\mathbb{R}^3\setminus B(R)$, where $B(R)$ is a closed ball (N can be thought of as one of (possible many) "ends" of Σ).

2) ϕ is a coordinate system in the complement of a compact set K of N such that, in local coordinates $\phi^i(p)=x^i$ the metric takes the following form:

$$g_{ij}=\delta_{ij}+k_{ij}, \tag{20}$$

and k_{ij} satisfies

$$\forall_{i,j,k,x}\,|k_{ij}(x)|\le\; C/(r{+}1)^\alpha \quad |\partial k_{ij}/\partial x^k(x)|\le\; C/(r{+}1)^{\alpha+1} \quad (\;r(x)=(\Sigma(x)^2\;)^{1/2}\;), \tag{21}$$

for some constant $C\in R$. Such a pair (g,ϕ) will be called α–admissible. Let us restate the remaining boundary conditions (13) in the ADM language:

$$\forall i,j,x \quad |(N-1)(x)|\le C/(r+1)^\alpha\;, \qquad |N^i(x)|\le C/(r+1)^\alpha\;,$$
$$|N_{,i}(x)|\le C/(r{+}1)^{\alpha+1}\;,\quad |P_{ij}(x)|\le C/(r{+}1)^{\alpha+1}\;,\quad |N^i{}_{,j}(x)|\le C/(r{+}1)^{\alpha+1}\;. \tag{22}$$

<u>Theorem 1</u>: Suppose that

1) (g,ϕ) is α–admissible, with $\alpha>1/2$,

2) the conditions (22) are satisfied,

3) (g_{ij},P_{ij}) satisfy the constraint equations, with integrable sources.

Let $S(R)$ be any one-parameter family of differentiable spheres, such that $r(S(R))=\min_{x\in S(R)}r(x)$ tends to infinity, as R does. Define

$$\begin{aligned} m(g,\phi) &= \lim_{R\to\infty}\frac{1}{16\pi}\int_{S(R)}(g_{ik,i}-g_{ii,k})dS_k\;,\\ P_i(g,\phi) &= \lim_{R\to\infty}\frac{1}{8\pi}\int_{S(R)}P^{ij}dS_j \end{aligned}$$

(these integrals have to be calculated in the local α–admissible coordinates $\phi^i(p)=x^i$). m and P_i are finite, independent upon the particular family of spheres $S(R)$ chosen, provided $r(S(R))$ tends to infinity as R does.

Proof: Let $f_{\mu\nu}$ be the flat metric $ds^2 = -dt^2 + \Sigma(dx^i)^2$ (where the dx^i refer to the local coordinate system ϕ^i on Σ). Define

$$A(X,R,f) = \int_{S(R)} U_\mu{}^{\alpha\beta} X^\mu \quad (-\det g)^{1/2} \eta_{\alpha\beta} \ .$$

The Einstein-von Freud identity takes the following form:

$$\begin{aligned} A(X,R_2,f) - A(X,R_1,f) \quad = \quad & \int_{\Gamma(R_1,R_2)} (\text{“expression quadratic in } (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu})\text{”} \\ & + \text{“expression linear in } T_{\mu\nu}\text{”})\ d^3x \ . \end{aligned}$$

(see, for example, ref. [12] for the explicit form of the volume integrand), where $\Gamma(R_1,R_2)$ is the “annulus” lying between $S(R_1)$ and $S(R_2)$. For $X = \partial/\partial t$ or $X = \partial/\partial x^i$, and $r(S(R_2)) > r(S(R_1))$ the volume integral above is bounded by a constant independent of R_2 in virtue of our hypotheses, tending to zero as R_1 tends to infinity. The reader may easily establish all the claimed properties of $m(g,\phi)$ and $P_i(g,\phi)$ using this observation.

In the proof of the theorem to follow we will need the following simple lemma:

Lemma 1: Let (g,ϕ_1) and (g,ϕ_2) be α_1 and α_2–admissible, respectively, with any $\alpha_a > 0$. Let $\phi_1 \circ \phi_2^{-1} : \mathbb{R}^3 \backslash K_2 \to \mathbb{R}^3 \backslash K_1$ be a twice differentiable diffeomorphism, for some compact sets K_1 and $K_2 \subset \mathbb{R}^3$. Then, in local coordinates

$$\phi_1^i(p) = x^i \qquad \phi_2^i(p) = y^i \ ,$$

the diffeomorphisms $\phi_1 \circ \phi_2^{-1}$ and $\phi_2 \circ \phi_1^{-1}$ take the form

$$x^i(y) = \omega^i{}_j\ y^i + \eta^i(y) \ , \qquad y^i(x) = (\omega^{-1})^i{}_j\ x^i \ + \zeta^i(x) \ ,$$

ζ^i and η^i satisfy, for some constant $C \in \mathbb{R}$,

$$\begin{aligned} |\zeta^i(x)| \le C(r(x)+1)^{1-\alpha} \ , \qquad & |\zeta^i{}_{,j}(x)| \le C(r(x)+1)^{-\alpha} \ , \\ |\eta^i(y)| \le C(r(y)+1)^{1-\alpha} \ , \qquad & |\eta^i{}_{,j}(y)| \le C(r(y)+1)^{-\alpha} \ , \\ r(x) = (\textstyle\sum (x^i)^2)^{1/2} \ , \qquad & r(y) = (\textstyle\sum (y^i)^2)^{1/2} \ , \end{aligned}$$

with $\alpha = \min(\alpha_1,\alpha_2), \omega^i{}_j$ is an $O(3)$ matrix, and r^0 is to be understood as $\ln r$.

Proof: This lemma is intuitively obvious, there are however a few technicalities needed to make the proof mathematically rigorous. Let us first note, that both (g,ϕ_1) and (g,ϕ_2) are α–admissible, so that we do not have to worry about two constants α_1 and α_2. In (21) we can also take a common constant $C = \max(C_1,C_2)$. Let g^1_{ij} and g^2_{ij} be the representatives of g in local coordinates ϕ_1 and ϕ_2. (21) implies, that g^1_{ij} and g^2_{ij} are “uniformly elliptic”, it is there exist positive constants C'_1 and C'_2 such that

$$\forall X^i \in \mathbb{R}^3 \quad \forall x \in \mathbb{R}^3 \setminus K_a \qquad C_a'^{-1} \Sigma (X^i)^2 \le g^a_{ij}\ X^i X^j \le C'_a \Sigma(X^i)^2 \qquad (24)$$

$a = 1, 2$. From now on C, C', etc. will denote constants which may vary from line to line, their exact values can be estimated at each step but are irrelevant for further purposes. Let us write down the equations following from the transformation properties of the metric

$$g^2_{ij}(y) = g^1_{k\ell}(x(y))\frac{\partial x^k}{\partial y^i}\frac{\partial x^\ell}{\partial y^j} \tag{25}$$

$$g^1_{ij}(x) = g^2_{k\ell}(y(x))\frac{\partial y^k}{\partial x^i}\frac{\partial y^\ell}{\partial x^j} \ . \tag{26}$$

Contracting (25) with g_1^{ij} ((26) with g_2^{ij}) and using the uniform ellipticity of g^1_{ij} (g^2_{ij}) one obtains

$$\sum_{k,i}|\frac{\partial x^k}{\partial y^i}| \le C \ , \qquad \sum_{k,i}|\frac{\partial y^k}{\partial x^i}| \le C \ . \tag{27}$$

Inequalities (27) show that all the derivatives of $x(y)$ and $y(x)$ are uniformly bounded. Let Γ_x be the ray joining x and K_1, and let $y^i_0(x)$ be the image by $\phi_2 \circ \phi_1^{-1}$ of the intersection point of K_1 with Γ_x (if there is more than one, choose the one which is closest to x). We have, in virtue of (27)

$$|y^i(x) - y^i_0(x)| = |\int_{\Gamma_x}(\partial y^i/\partial x^k)dx^k| \le C\ r(x) \ ,$$

so that

$$r(y(x)) \le C\ r(x) + C^- \ . \tag{28}$$

A similar reasoning shows

$$r(x(y)) \le C\ r(y) + C^- \ . \tag{29}$$

(28) and (29) can be combined into a single inequality

$$r(y(x))/C - C^- \le r(x) \le C\ r(y(x)) + C^- \ . \tag{30}$$

(30) shows, that any quantity which is $O(r(x)^{-\beta})$ $(O(r(y)^{-\beta}))$[3] is also $O(r(y)^{-\beta})$ $(O(r(x)^{-\beta}))$, when composed with $\phi_2 \circ \phi_1^{-1}$ $(\phi_1 \circ \phi_2^{-1})$. Moreover, due to (21), (27) and (30)

$$\partial O(r(y(x))^{-\alpha})/\partial x^i = O(r(y(x))^{-\alpha-1}) \ , \qquad \partial O(r(x(y))^{-\alpha})/\partial y^i = O(r(x(y))^{-\alpha-1}) \tag{31}$$

((31) holding for the functions appearing in the metric). (27) and (31) allow us to write (25) and (26) in the following form

$$\sum_k \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j} = \delta_{ij} + O(r^{-\alpha}) \tag{32}$$

$$\sum_k \frac{\partial x^k}{\partial y^i}\frac{\partial x^k}{\partial y^j} = \delta_{ij} + O(r^{-\alpha}) \ . \tag{33}$$

[3] $f(s) = O(s^\gamma)$ is used here to denote a function satisfying $|f(s)| \le C(s+1)^\gamma$ for some positive constant C.

In (32) and (33) it is irrelevant whether $O(r^{-\alpha})$ is $O(r(x)^{-\alpha})$ or $O(r(y)^{-\alpha})$, in virtue of (30). Let us introduce

$$
\begin{aligned}
&A^i{}_j = \partial y^i/\partial x^j \ , \qquad B^i{}_j = \partial x^i/\partial y^j \ , \\
&C_{ijk} = A^m{}_i g^2_{m\ell} \partial A^\ell{}_j/\partial x^k = g^2_{m\ell}(\partial y^m/\partial x^i)\partial^2 y^\ell/\partial x^j \partial x^k \ , \\
&D_{ijk} = B^m{}_i g^1_{m\ell} \partial B^\ell{}_j/\partial y^k \ .
\end{aligned}
$$

Differentiating (26) with respect to x, taking into account (27), (30) and (31) leads to

$$C_{ijk} + C_{jik} = O(r^{-\alpha-1}) \ .$$

A standard cyclic permutation calculation, using the symmetry of C_{ijk} in the last two indices yields

$$C_{ijk} = O(r^{-\alpha-1}) \ .$$

This equality, (24), (27) and the definition of C_{ijk} imply

$$\partial^2 y^i/\partial x^j \partial x^k = O(r^{-\alpha-1}) \ . \tag{34}$$

In a similar way one establishes

$$\partial^2 x^i/\partial y^j \partial y^k = O(r^{-\alpha-1}) \ . \tag{35}$$

It is elementary to show, using (32), (33), (34) and (35) that the following quantities

$$
\begin{aligned}
\mathring{A}^i{}_j &= \lim_{r\to\infty} A^i{}_j(r\vec{n}) \ , \\
\mathring{B}^i{}_j &= \lim_{r\to\infty} B^i{}_j(r\vec{n}) \ ,
\end{aligned}
$$

($\vec{n}$ -any vector satisfying $\sum(n^i)^2 = 1$) exist and are constant matrices (n^i independent), with $A = B^{-1}$. Define

$$
\begin{aligned}
\zeta^i(x) &= y^i(x) - \mathring{A}^i{}_j \ x^j \\
\eta^i(y) &= x^i(y) - \mathring{B}^i{}_j \ y^j \ .
\end{aligned} \tag{37}
$$

(35) leads to

$$A^i{}_j(r_2\vec{n}) - A^i{}_j(r_1\vec{n}) = \int_{r_1}^{r_2} (\partial^2 x^i(r\vec{n})/\partial x^j \partial x^k) \ n^k \ dr = O(r_1^{-\alpha})$$

for $r_2 > r_1$. Going with r_2 to infinity, making use of (36) and (37) one obtains

$$\zeta^i{}_{,j}(x) = O(r^{-\alpha}) \ ,$$

which implies

$$\zeta^i(x) = O(r^{1-\alpha}) \ ,$$

(where $O(r^0)$ is understood as $O(\ln r)$) — this establishes lemma 1.

Theorem 2: Let $(g, \phi_a), a = 1, 2$, satisfy the hypotheses of theorem 1 and lemma 1. Then

1) $m(g,\phi_1) = m(g,\phi_2)$
2) $P_i(g,\phi_1) = \omega_i{}^j P_j(g,\phi_2)$
($\omega \in O(3)$, given by lemma 1).

Proof: Point 2) above is trivial, point 1) follows by a well known argument from the result of lemma 1, we will repeat it here for completeness. From lemma 1 we have

$$k^2_{ij} = g^2_{ij} - \delta_{ij} = \mathring{B}^k{}_i B^\ell{}_j k^1_{k\ell}(x(y)) + \mathring{B}^\ell{}_j \eta^\ell{}_{,i}(y) + \mathring{B}^\ell{}_i \eta^\ell{}_{,j}(y) + O(r^{-2\alpha})$$

$$(k^1_{ij} = g^1_{ij} - \delta_{ij}) \ .$$

Therefore

$$\begin{aligned} \partial g^2_{ij}(y)/\partial y^j - \partial g^2_{jj}(y)/\partial y^i &= \mathring{B}^k{}_i(\partial k^1_{kj}(x(y))/\partial x^j - \partial k^1_{jj}(x(y))/\partial x^k) \\ &+ (\mathring{B}^\ell{}_i \partial\eta^\ell/\partial y^j - \mathring{B}^\ell{}_j \partial\eta^\ell/\partial y^i)_{,j} + O(r^{-2\alpha-1}) \ . \end{aligned} \tag{38}$$

While integrated over the sphere $r(y) = \text{const}$, the last term in (38) will give no contribution in the limit $r(y) \to \infty$ if $2\alpha + 1 > 2$, the next to last term in (38) will give no contribution being the divergence of an antisymmetric quantity, the first gives the ADM mass of the metric g^1_{ij} (the $B^i{}_j$ factor cancels with a similar factor coming from the surface forms dS_k).

The condition $\alpha > 1/2$ is the best possible, in the following sense[4]:

Proposition 1: The ADM mass of 1/2–asymptotically flat metrics is either infinite, or can take any value greater than some number in the class of 1/2–admissible coordinate systems.

Proof: We shall establish proposition 1 for the flat metric $ds^2 = \sum(dx^i)^2$, the general result can be obtained by the same method. The new coordinates y^i implicitly defined by

$$x^i = (1 + a\ r(y)^{-1/2})y^i \ , \qquad a \in \mathbb{R} \ ,$$

are easily seen to be 1/2–admissible. The "ADM mass" of the flat metric in the coordinates y^i can be calculated to be

$$m = a^2/8,$$

which establishes proposition 1.

Let us finally remark, that all the above results can be stated in terms of the $H_{s,\delta}$ spaces of Y. Choquet-Bruhat and D. Christodoulou. The theorems of D. Christodoulou and N.O Murchadha [14] show that non- trivial α–asymptotically flat space-times satisfying Einstein equations exist with any $\alpha > 0$, and that

[4] This proposition is essentially due to V.I. Denisov and V.O. Solobev [13]. Theorems 1 and 2 above show in what sense the remaining claims of these authors are erroneous.

the boost problem is solvable in this class of space-times (all this holding under some supplementary conditions on the weak derivatives of the metric). The positivity of m for α–asymptotically flat space-times, for $\alpha > 1/2$, can probably be established along Witten's argument lines using the results of O. Reula [15], whose proof of existence of solutions of Witten's equation holds in this class of metrics.

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