Introduction to general relativity

Marc Mars

University of Salamanca

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99 years of General Relativity

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Outline

1. Lecture 1
   - Newtonian gravitation
   - Special Relativity
   - Equivalence principles and metric theories of gravitation
   - Reference frames in a spacetime
2. Lecture 2
   - Physics in curved spacetimes
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   - Einstein field equations
   - Stationary spacetimes
3. Lecture 3
   - Spherically symmetric spacetimes and Birkhoff theorem
   - Kerr spacetime
   - Uniqueness theorem of vacuum black holes
   - ADM energy-momentum and positivity of energy

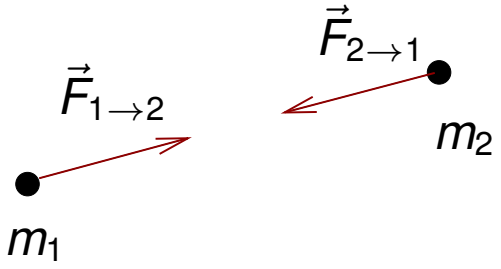
Basics of Newtonian gravitation

Newton's theory of Gravitation is based on four hypotheses:

- Newton's law:

$$F_{1\to 2} = -G\frac{m_1 m_2 \vec{r}}{|\vec{r}|^3},$$

$\vec{r}$ = position vector of 2 respect to 1.



- Linearity.
- Action at a distance between masses (equivalently, infinite velocity of propagation of the gravitational field).
- Gravitational mass m of a particle is the same as its inertial mass: $\vec{F} = m\vec{a}$.

A priori, very surprising equality. Why should gravitational charge agree with inertia, i.e. resistance to velocity change?

Newton realized that this requires experimental verification.

- The period of the pendulum is $T = 2\pi\sqrt{\frac{m_g g}{m_i l}}$
- He used pendula with massive bobs of different materials.
- Periods independent of materials: Concluded $\frac{m_g}{m_i} = 1 + O(10^{-3})$.

Basics of Newtonian Gravitation (II)

Newtonian gravity: very successful theory from an observational point of view:

- Explains tides, or why objects fall the way they fall on Earth.
- Explains the motion of planets in the Solar system to great accuracy.
- Deviations in the observations from the theory predictions led Le Verrier to postulate the existence of a new planet: Neptune discovered in 1846.

There was one anomaly in the Solar system motion not explained by Newtonian gravity.

- Mercury's perihelion predicted to rotate by 531 arc sec/century. Precession of Mercury.
- Observationally, precession of 574 arc sec/century: 43 arc sec/century not accounted for.

Newton's gravity had more serious problem from a conceptual point of view:

- Requires an absolute time (interaction propagates at infinite speed).
- Fine in XIX century physics, but inconsistent with special relativity, Einstein 1905.
- A relativistic version of Newton's gravity became necessary.

Basics of special relativity

- The set of events (i.e. all possible locations and instants of time) form a smooth four-dimensional manifold, called *Minkowski spacetime*, $(\mathbb{M}^{1,3}, \eta)$.
- As a manifold $\mathbb{M}^{1,3} \simeq \mathbb{R}^4$. Geometry determined by the Minkowski metric η.
- There exists a (non-empty) class $\mathcal{S}$ of diffeomorphisms

$$\begin{aligned} \Phi : \mathbb{M}^{1,3} &\longrightarrow \mathbb{R}^4 \\ p &\longrightarrow x^\alpha, \qquad \alpha, \beta = 0, 1, 2, 3 \end{aligned}$$

where the metric η takes the form

$$\eta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 := \eta_{\alpha\beta} dx^\alpha dx^\beta. \qquad (\eta) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- $\{x^\alpha\}$ are called Minkowskian coordinates.
- The ortochronus Poincaré group is defined

$$\mathcal{P} = \left\{ \begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \\ x & \longrightarrow & \Lambda x + a \end{array}, \quad \Lambda \in GL(\mathbb{R}^4), \ a \in \mathbb{R}^4; \quad \Lambda^T(\eta)\Lambda = (\eta), \quad \Lambda^0_0 > 0 \right\}.$$

- Fixed $\Phi_0 \in \mathcal{S}$, elements $\Phi_1 \in \mathcal{S}$ parametrized by $\mathcal{P}$ as $\Phi_1 = P \circ \Phi_0, \quad P \in \mathcal{P}$.
- $\mathcal{P}$ is 10-dim Lie group. $\{a = 0\} \subset \mathcal{P}$: 6-dim subgroup; ortochronus Lorentz group.

Basics of special relativity

Classification of tangent vectors. A vector $u \in T_p\mathbb{M}^{1,3}$, $p \in \mathbb{M}^{1,3}$ is:

- Timelike: $\eta(u,u) < 0$
- Null (or lightlike): $\eta(u,u) = 0$

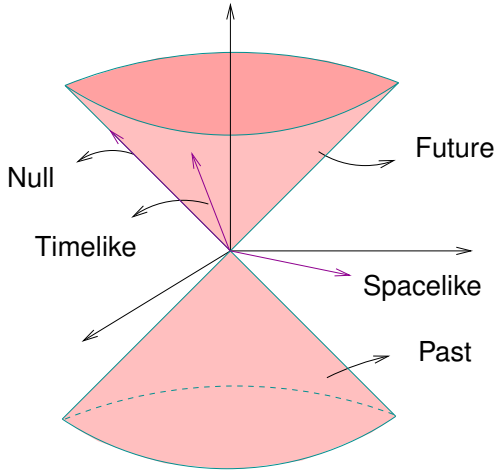
$\left.\right\}$ Causal

- Spacelike: $\eta(u,u) > 0$.

For causal vectors:

- Future: In Minkowskian coordinates $u^0 \geq 0$.
- Past: In Minkowskian coordinates $u^0 \leq 0$.

Definition independent of choice of Minkowskian coordinates.



Each diffeomorphism in $\mathcal{S}$ defines an inertial Cartesian reference frame S:

- For any event p, the time when it happens is $t(p) = \frac{x^0(p)}{c}$.
  - c universal constant with units of velocity.
- $\{x^1(p), x^2(p), x^3(p)\}$ are the Cartesian coordinates of the location of p.

Motion of particles

- Free pointlike particles of inertial mass m move in straight lines

$$x^\alpha(s) = x_0^\alpha + u^\alpha(s - s_0), \qquad u^\alpha = \text{const.} \quad \begin{array}{ll} \text{if } m > 0: & u \text{ future timelike} \\ \text{if } m = 0: & u \text{ future null } \neq 0. \end{array}$$

- Allowed paths of particles with $m > 0$ (called world-lines) are smooth maps

$$\begin{array}{rcl} \gamma : I \subset \mathbb{R} & \longrightarrow & \mathbb{M}^{1,3} \\ s & \longrightarrow & \gamma(s) \end{array} \qquad \text{with tangent vector } \dot\gamma \text{ timelike and future.}$$

Parameter s measures time elapsed to a clock moving along with the particle iff $\eta(\dot\gamma, \dot\gamma) = -c^2$. Called proper time and denoted by τ.

- Allowed paths of particles with $m = 0$: smooth curves $\gamma(s)$ with $\dot\gamma$ null, future and non-zero.

Future causal curve: Smooth curve $\gamma(s)$ with future, null $\dot\gamma(s)$.

Causality in Special Relativity

- An event $q \in \mathbb{M}^{1,3}$ lies to the causal future of $p \in \mathbb{M}^{1,3}$ iff in one inertial reference frame $\{x^\alpha\}$:

$$\eta_{\alpha\beta}(x^\alpha(q) - x^\alpha(p))(x^\beta(q) - x^\beta(p)) \leq 0, \qquad x^0(q) - x^0(p) \geq 0.$$

- q lies in the chronological future of p iff

$$\eta_{\alpha\beta}(x^\alpha(q) - x^\alpha(p))(x^\beta(q) - x^\beta(p)) < 0, \qquad x^0(q) - x^0(p) > 0.$$

- Definitions independent of choice of Minkowskian coordinate system.

$$J^+(p) = \{q \in \mathbb{M}^{1,3}, \quad q \text{ lies in the causal future of } p\}$$
$$I^+(p) = \{q \in \mathbb{M}^{1,3}, \quad q \text{ lies in the chronological future of } p\}$$

Similar definitions for causal or chronological past.

Definition adapted to the affine structure of the Minkowski spacetime. Alternatively:

- $q \in J^+(p) \iff$ there exist a future causal curve from p to q
- $q \in I^+(p) \iff$ there exists a world-line from p to q.

Principle of relativity

- Fundamental physical principle of special relativity: no physical experiment can distinguish between inertial reference frames.
- Equivalently, the physical laws are invariant under the (ortochronus) Poincaré group.

Electromagnetism is a good example:

- Electromagnetic field: Two-form $\boldsymbol{F}$ on the Minkowski spacetime $\mathbb{M}^{1,3}$.
- Field equations outside the sources: In an inertial reference frame:

$$d\boldsymbol{F} = 0, \quad \partial_{x^\alpha} F^{\alpha\beta} = 0 \quad \text{indices raised with } \eta^{\alpha\beta} = \text{inverse of } \eta_{\alpha\beta}.$$

- Can be written in any local coordinate system in $\mathbb{M}^{1,3}$ as

$$d\boldsymbol{F} = 0, \qquad \nabla^{\mathbb{M}}_{\alpha} F^{\alpha\beta} = 0, \qquad \nabla^{\mathbb{M}} : \text{ Levi-Civita connection of } \eta.$$

- Four-force acting on a particle with charge q with word-line $\gamma(\tau)$.

$$\mathcal{F}^{\mu} = qF^{\mu}{}_{\nu}\dot{\gamma}^{\nu}.$$

- Four-force enters the equations of motion of the particle (relativistic Newton's law):

$$m\nabla^{\mathbb{M}}_{\dot{\gamma}}\dot{\gamma} = \mathcal{F} + \text{rest of four-forces acting on the particle.}$$

Energy-momentum tensor of electromagnetic field

Electric and magnetic parts of $F_{\mu\nu}$ in an inertial frame $\{x^\alpha\}$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix} \qquad \begin{array}{l} \vec{E} \text{ electric field} \\ \vec{B} \text{ magnetic field} \end{array} \tag{2}$$

Electromagnetic field has associated an energy-momentum tensor:

$$T_{\mu\nu} = c\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right).$$

- T_{00} = field's energy-density: $T_{00} = \frac{1}{2c}\left(\vec{E}^2 + c^2\vec{B}^2\right)$.
- T_{i0} = field's energy flux along x^i. $T_{i0} = -(\vec{E}\times\vec{B})_i$. Poynting vector.
- T_{0i} = field's linear momentum density along the direction x^i. $T_{0i} = -(\vec{E}\times\vec{B})_i$
- T_{ij} = flux along the direction x^i of the field's linear momentum along x^j.

$$T_{ij} = -\frac{1}{c}\left(E_iE_j + c^2B_iB_j - \frac{1}{2}\delta_{ij}(\vec{E}^2 + c^2\vec{B}^2)\right) \qquad \text{Maxwell's stress tensor.}$$

Fundamental properties:

- $T_{\mu\nu}$ is symmetric and in Minkowskian coordinates:

$$\partial_\alpha T^{\alpha\beta} = 0 \qquad \text{energy-momentum conservation.}$$

Energy-momentum tensor conservation

Similarly: All fields satisfying the principle of relativity admit a symmetric tensor describing its energy-momentum contents.

- In general, individual energy-momentum tensors are nor conserved.
- E.g. inside the electromagnetic sources the electromagnetic energy-momentum tensor is not conserved.
- However, the *sum* of all energy-momentum tensors of an isolated system is conserved

$$\partial_\alpha \left(\sum_i T_i^{\alpha\beta} \right) = 0.$$

- Energy-momentum conservation can be written in any local coordinate system

$$\nabla_\alpha^{\mathbb{M}} T^{\alpha\beta} = 0.$$

Newton and Galileo equivalence principles

Challenge at the beginning of the XX century:

- Develop a new theory gravitation consistent with the framework of special relativity and reducing the Newton's theory at a suitable limit.
- Einstein devoted more than 10 years to such effort.
- The result was not a theory consistent with special relativity. Instead, a new framework which, at a suitable limit, approached special relativity and at another limit approached Newton's theory.

Basic physical principle: Einstein equivalence principle.

- Newton postulated that inertial mass and gravitational mass are the same.
- Combined with Newton's second law and Newton's Gravitation theory leads to:

Principle (Weak equivalence principle (Galileo))

The motion of test particles with negligible self-gravity in free fall is independent of their properties.

- Mechanical experiments cannot distinguish accelerated frames from gravitational fields (on sufficiently small regions). In particular:

Mechanical effects of gravitational field effectively disappear in freely-falling reference frames.

Einstein equivalence principle

- Einstein's key insight: Postulate that all non-gravitational physics experiments must have the same outcome when carried out in freely falling reference frames or in inertial frames with absence of gravitation.

Principle (Einstein equivalence principle)

In sufficiently small regions of space and time, all non-gravitational laws of physics take the same form in a freely falling frame and in an inertial reference frame in absence of gravitational field.

- Not fully precise because of the term "sufficiently small regions of space and time". However, it is the key guiding principle for General Relativity (and other theories of Gravitation).

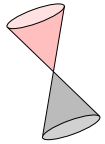
Lead Einstein to the following framework:

Metric theories of gravitation

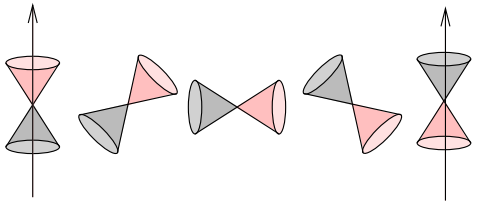
- The collection of all events: smooth 4-dim manifold $\mathcal{M}$ endowed with a smooth metric g of Lorentzian signature $\{-1, 1, 1, 1\}$: (M, g) is the spacetime.

A vector $v \in T_p\mathcal{M}$ is called: timelike iff $g(v, v) < 0$, null iff $g(v, v) = 0$, spacelike iff $g(v, v) > 0$.

- At any point $p \in M$, the set of timelike vectors split into two connected components.



- It may be impossible to make a continuous splitting across the spacetime $\longrightarrow$ no notion of future and past.



Definition

A spacetime is time-orientable if there exists a smooth timelike vector field.

- A time-orientable spacetime admits a continuous splitting of the null cones. The converse is also true.
- All spacetimes are assumed to be time-orientable with a choice of timelike vector field U declared to be future.
- A causal vector $v \in T_pM$ is future directed if $g(v, U) \leq 0$ and past directed if $g(v, U) \geq 0$.

Freely falling test particles move along the geodesics in (M, g): Timelike (massive particles) or null (massless particles).

- Geodesics are defined uniquely from an initial point and an initial vector $\longrightarrow$ weak equivalence principle incorporated.

Proper time measured by a clock moving along a future directed timelike trajectory $\gamma(s)$ (i.e. with $\dot{\gamma}$ future timelike) is

$$\tau(s_1) - \tau(s_0) = \frac{1}{c} \int_{s_0}^{s_1} \sqrt{-g(\dot{\gamma}, \dot{\gamma})}\, ds.$$

- Definition independent of the parametrization of $\gamma(s)$.

Spacetime metric g has a double role:

- g determines the motion of freely falling test particles. So, the gravitational field is encoded in g.
- g determines how time elapses, so metric properties of the spacetime are encoded in g.

All such theories are called metric theories of gravity. They differ in how the metric g is determined from the gravitational sources.

- General Relativity is an example.

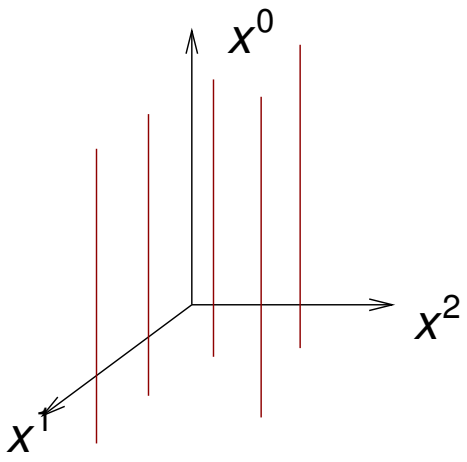
Freely falling frames

Einstein equivalence principle dictates how to incorporate non-gravitational physics laws into the spacetime.

- Take the physics laws in inertial systems to be compatible with special relativity.
- At sufficiently small scales, the laws must be identical in a freely falling reference system in (M, g) and in an inertial system in $(\mathbb{M}^{1,3}, \eta)$.

Necessary to define first what is a reference frame.

- How can an inertial frame be defined in the Minkowski spacetime in geometric terms?
- In a Minkowskian coordinate system $\{x^\alpha\}$, the world-lines of points at rest are $x^0(s) = s, x^i = x^i_0$.
- The vector field tangent to these trajectories is $\xi = \partial_{x^0}$: unit, future directed and $\nabla^{\mathbb{M}}\xi = 0$.



- The general solution of $\nabla^{\mathbb{M}}\xi = 0, \eta(\xi, \xi) = -1, \xi$ future directed is

$$\xi = \sqrt{1 - v^2/c^2}\partial_{x^0} + v^i/c\,\partial_{x^i}, \qquad v^2 := v^i v^j \delta_{ij} < c^2.$$

- Inertial reference frames in Minkowski are characterized by the integral curves of ξ.

Reference frames in a spacetime

- In a general spacetime, there are no timelike, covariantly constant vector fields.

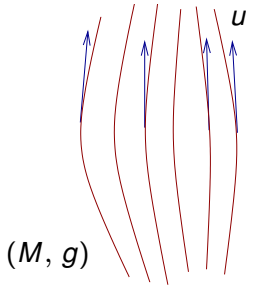
What are the basic properties of a reference frame?

- For any event, it needs to define a location in space and one instant of time.

Definition

A timelike congruence in a spacetime (M, g) is the set of integral curves of a future timelike unit vector field u.

- Describes a set of observers filling the spacetime.
- Any event happens at the location of one such observer and at the given time according to the observer's clock.



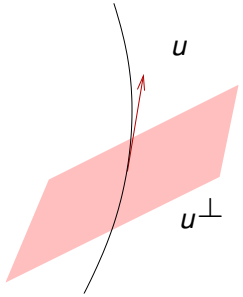
Defines a reference frame $\longrightarrow$ What is a freely falling reference frame?

Need a congruence of geodesics moving "parallely" to each other.

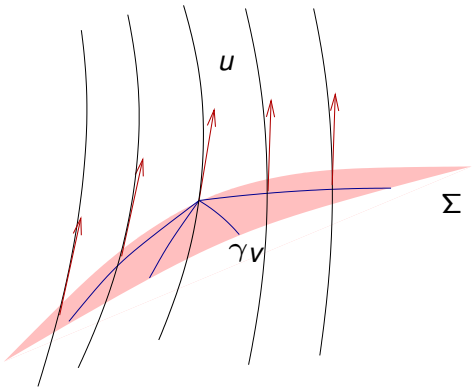
- Cannot be achieved in the large because:
  - (i) Parallelism is not absolute in a Riemannian manifold (it depends on the curve joining the two vectors).
  - (ii) World-lines initially parallel will accelerate relative to each other (because of gravitational field).
- Achieve this near one geodesic and for a short period of time.

Inertial reference frame

- Consider a timelike curve γ with future and unit tangent vector u.
- At an event $p \in \gamma$ the space relative to u is defined as $u^\perp = \{v \in T_pM;\ g(u,v) = 0\}$
- Denote γ_v: geodesic starting at p with tangent vector v.



There is $a > 0$ such that the set of points $\{\gamma_v(s),\ s \in [0,a),\ v \in u^\perp$ and unit $\}$ is a smooth spacelike surface Σ.



- At $q = \gamma_v(s) \in \Sigma$ define u_q by parallel transport of u_p along $\gamma_v \longrightarrow$ defines a timelike vector field along Σ.
- Define a timelike congruence (in a neighbourhood U_p of p) by solving the geodesic equations with initial vector u_q.
- A coordinate system on U_p can be constructed by:

- Choose an orthonormal basis $\{e_1, e_2, e_3\}$ of $u^\perp$.
- Any $q \in U_p$ has associated a unique triple $\{v = v^i e_i, s, \tau\}$. Define $\{x^0 = c\tau, x^i = sv^i\}$.
- In these coordinates: $g|_p = \eta_{\alpha\beta} dx^\alpha dx^\beta, \quad \Gamma^\alpha_{\beta\gamma}|_p = 0.$

Take this as definition of locally inertial reference frame.

Principle of General Covariance

- The equivalence principle dictates how the laws of physics should be written in locally inertial frames.
- But, locally inertial frames do not cover extended regions (unless (M, g) flat).

Prescription to write down the laws of physics in arbitrary reference frames is needed.

- Can one even dispense of using reference frames and use any coordinate system?

Principle (Principle of general covariance)

The laws of physics are written in terms of local geometric objects on manifolds.

- The primary local geometric objects on manifolds are tensor fields, but there are others (e.g. spinors fields, when the manifold admits them).
- It states that the natural language for physics is differential geometry.

This principle is logically independent of the Einstein equivalence principle (EEP).

- Free particle in Minkowski: $\nabla^{\mathbb{M}}_u u = 0$.
- Example of postulate consistent with general covariance but not EEP:

  Particles in free fall move along trajectories satisfying $\nabla_u u + (\mathrm{Scal}_g) u = 0$.

∇ : Levi-Civita connection of g $\qquad$ Scal_g: curvature scalar of g.

- Combination of both principles is a powerful tool. Example:
- Field equation of electromagnetism with sources in $(\mathbb{M}^{1,3}, \eta)$ (in flat coordinates):

$$\partial_\alpha F^{\beta\alpha} = \frac{1}{c} J^\beta \qquad F_{\alpha\beta} \text{ two-form}$$
$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.$$

- J^β is a timelike vector field, J^0 charge density, J^i change current. Necessarily $\partial_\alpha J^\alpha = 0$ (charge conservation).

Field equations of electromagnetism in the presence of gravitation

$$\nabla_\alpha F^{\beta\alpha} = \frac{1}{c} J^\alpha, \qquad d\boldsymbol{F} = 0.$$

- Fulfill both Einstein equivalence and general covariance principles.

Works for physical laws either first order or second order involving scalar fields.

- Second order equations have an ordering ambiguity: $[\nabla_X, \nabla_Y] \neq 0$ if $\mathrm{Riem}_g \neq 0$.
- Example: Special relativity field equations for electromagnetism in terms of electromagnetic potential $\boldsymbol{F} = dA$: $\partial_\alpha \partial^\mu A^\alpha - \partial_\alpha \partial^\alpha A^\mu = \frac{1}{c} J^\mu$.
- Admits inequivalent generalizations to (M, g)

$$\nabla_\alpha \nabla^\mu A^\alpha - \nabla_\alpha \nabla^\alpha A^\mu = \frac{1}{c} J^\mu, \qquad \nabla^\mu \nabla_\alpha A^\alpha - \nabla_\alpha \nabla^\alpha A^\mu = \frac{1}{c} J^\mu.$$

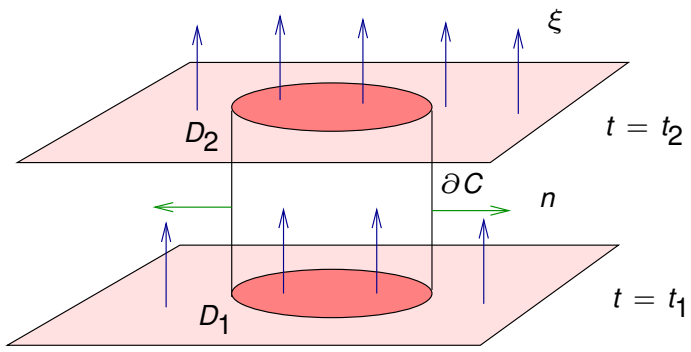
- However, only one implies $\nabla_\alpha J^\alpha = 0$.

How does the metric g encode the gravitational field?

In special relativity any field has an associated energy-momentum tensor $T^{\alpha\beta}$

$$T^{\alpha\beta} = T^{\beta\alpha}, \qquad \partial_\alpha T^{\alpha\beta} = 0.$$

- Last equation implies a conservation law.
- Consider a cylinder between two constant time hyperplanes, ξ future unit normal.



$$\int_{D_2} T(\xi,\xi)dV = \int_{D_1} T(\xi,\xi)dV + \int_{\partial C} T(\xi,n)dV$$

Energy at time t_2 = Energy at time t_1 + energy flux across the boundary.

- This conservation law follows because

$$\nabla^{\mathbb{M}}_\alpha (T^{\alpha\beta}\xi_\beta) = (\nabla^{\mathbb{M}}_\alpha T^{\alpha\beta})\xi_\beta = 0.$$

- Can be extended to any vector field ξ satisfying

$$\nabla^{\mathbb{M}}_\alpha \xi_\alpha + \nabla^{\mathbb{M}}_\beta \xi_\alpha = 0, \qquad \text{Killing vectors.}$$

- In Minkowski spacetime the general solution has 10 parameters (generators of the Poincaré group).

- A field in a spacetime (M, g) has associated a symmetric tensor satisfying

$$\nabla_\alpha T^{\alpha\beta} = 0.$$

- Implies no conservation law. Only special spacetimes admit non-trivial Killing vectors.
- In a local coordinate system

$$\partial_\alpha T^{\alpha\beta} + \Gamma^\alpha_{\alpha\mu} T^{\mu\beta} + \Gamma^\beta_{\alpha\mu} T^{\alpha\mu} = 0.$$

- Describes an interaction between $T^{\alpha\beta}$ and the gravitational field (through the Christoffel Γ symbols of g).

In Minkowski, the interaction of a field with external forces takes on a similar form.

- E.g. a charged fluid in an external electromagnetic field:

$$\nabla^{\mathbb{M}}_\alpha T^{\alpha\beta}_{\text{fluid}} - F^{\beta\mu} J_\mu = 0 \qquad \text{Second term is the Lorentz force.}$$

- Energy conservation is restored by taking into account the energy-momentum tensor of the charged sources.

Two main consequences:

- $\Gamma^\alpha_{\beta\mu}$ are physically the force of the gravitational field per unit energy.
- Since $\Gamma^\alpha_{\beta\mu}|_p = 0$ on a local inertial frame at p, this interaction between fields and gravity cannot be localized.

In particular, there is no energy-momentum tensor associated to the gravitational field.

Geodesic deviation and tidal forces

The analogy $\Gamma \leftrightarrow$ Gravitational field implies $g \leftrightarrow$ Gravitational potential.

- The weak equivalence principle means:

At sufficiently local scales, gravitational forces cannot be distinguished from inertial forces.

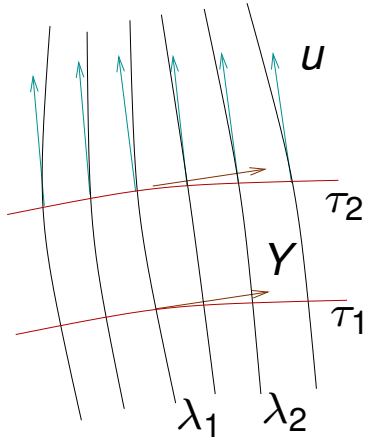
How can be truly gravitational effects be distinguished from inertial effects?

Consider a one-parameter family of geodesics, i.e. a map

$$\begin{aligned} \Phi : I \times (a,b) &\longrightarrow (M,g) \\ (\tau,\lambda) &\longrightarrow \Phi(\tau,s) \end{aligned} \qquad \text{with} \quad \gamma_s = \Phi(\cdot,s) \quad \text{a geodesic:} \quad \nabla_{\dot\gamma_s}\dot\gamma_s = 0.$$

- The vector field $Y = \Phi_\star(\partial_\lambda)$ satisfies, with $u = \dot\gamma_s$,

$$\nabla_u \nabla_u Y = -\mathrm{Riem}_g(Y,u)u.$$



Geometrically, Y is the displacement vector between close geodesics.

- Equation determines the acceleration of this displacement.

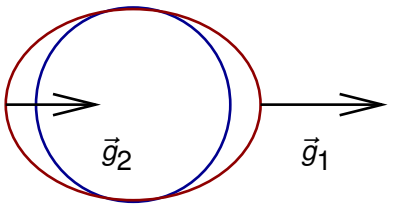
Measures how geodesics approach or separate each other.

- Geodesics accelerate relative to each other in spacetimes with non-zero curvature.

Newtonian gravity: relative acceleration of particles in free fall called "tidal acceleration" $\longrightarrow$ Responsible for the tides.

- Points at different point on the Earth fall in the gravitational field of Moon and Sun.

- Different distances to the sources $\longrightarrow$ Slightly different gravitational fields are felt $\longrightarrow$ Slightly different forces.



The Earth is bound object, gets deformed to compensate for this different forces.

- Less tightly bound $\longrightarrow$ Larger effects $\longrightarrow$ Tides in the oceans.

The effect is not due to gravitational field $\vec{g} = -\text{grad}(\Phi)$ but to its variation $\partial\vec{g} = -\text{Hess}(\Phi)$.

- Agrees with the interpretation that the metric $g \longleftrightarrow \Phi$.

Newtonian limit

- Any metric theory of gravitation compatible with experiments must reduce to the Newtonian theory in an appropriate limit.
- Requires slowly varying gravitational field.

Consider the simplest case of time-independent gravitational fields:

- A generator of a local isometry in (M, g) is a vector field $\xi \in \mathfrak{X}(M)$:

$$\mathcal{L}_\xi g = 0 \qquad \Longleftrightarrow \qquad \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \qquad \text{Killing vector}$$

- (M, g) time-independent: admits a Killing vector being timelike everywhere.
- (M, g) static: time-independent and distribution of spaces orthogonal to ξ integrable $\Longleftrightarrow$ (Fröbenius) $d\boldsymbol{\xi} \wedge \boldsymbol{\xi} = 0$ where $\boldsymbol{\xi} = g(\xi, \cdot)$.

A static spacetime has the following local structure:

- For all $p \in M$, there is a (centered) neighbourhood $U_p = (-a, a) \times N$: (N, γ) Riemannian manifold, $\lambda : N \to \mathbb{R}^+$ such that

$$g = -c^2 \lambda \, dt^2 + \gamma, \qquad t: \text{ coordinate along } \mathbb{R} \text{ factor }, \qquad \xi = \partial_t$$

- For each p, there is a discrete isometry $\tau_p : U_p \longrightarrow U_p$, $\tau_p(t, x) = (-t, x)$

$$\tau_p(p) = p, \qquad d\tau_p(\xi) = -\xi, \qquad d\tau_p(X) = X, \qquad \forall X \in \xi^\perp \subset T_pM.$$

- Invariant under time inversion: future indistinguishable from past $\longrightarrow$ sources at rest.

Newtonian theory: trajectory of a freely falling particle satisfies, in Cartesian coord.,

$$\frac{d^2x(t)}{dt^2} = -\text{grad}\,(\Phi), \qquad \Phi \text{ gravitational potential.}$$

Assume (M, g) static spacetime. Trajectory $\{t(s), x(s)\}$, where $x : I \longrightarrow N$, geodesic iff

$$\frac{dt}{ds} = \frac{E}{c^2\lambda}, \qquad E \text{ const.}$$

$$u = \dot{x} \quad \text{satisfies} \qquad \nabla^N_u u = -\frac{E^2}{2c^2\lambda^2}\text{grad}_\gamma(\lambda)$$

- If the geodesic is causal, then $E > 0$ and $s(t)$. Vector $v = \frac{dx(t)}{dt}$ satisfies

$$\nabla^N_v v = -\frac{c^2}{2}\text{grad}_\gamma(\lambda) + v\frac{v(\lambda)}{\lambda}.$$

- Assume $\mathcal{M} = \mathbb{R} \times \mathbb{R}^3$. Absence of gravitation: $\Phi = 0 \quad \longleftrightarrow \quad \lambda = 1, \gamma = g_E$.
- Assume $\frac{v}{c} << 1$ (slow motion). The two equations agree iff

$$\lambda = 1 + \frac{2\Phi}{c^2} + o\left(\frac{\Phi}{c^2}\right), \qquad \gamma = g_E + O\left(\frac{\Phi}{c^2}\right).$$

- Newtonian limit is recovered in any metric theory of gravitation predicting this behaviour for slowly moving sources.

The first correction to the space metric is not determined.

Order of magnitude of the metric correction in the Newtonian approximation.

- Consider a spherically symmetric object of mass m. Newtonian potential

$$\Phi = -\frac{Gm}{r}, \qquad r \text{ distance to the center.}$$

$\frac{2\Phi}{c^2}$ for typical objects:

Surface of a ball 10^3 Kg, $R = 1$ m	$\approx -10^{-32}$
Surface of the Earth	$\approx -10^{-9}$
Surface of the Sun	$\approx -10^{-6}$
Surface of a White Dwarf	$\approx -10^{-3}$
Surface of a Neutron Star	$\approx -1/3$
Surface of a Black Hole	≈ -1

There are astrophysical objects for which the Newtonian approximation is not valid (even if at rest).

Einstein field equations

All results so far are valid for any metric theory of gravitation. Basically we addressed:

- How is the physics in the presence of a gravitational field, i.e. in a spacetime (M, g).
- In particular, how do particles move in free fall.

Fundamental missing ingredient: How is the gravitational field determined from its sources?

- What are the field equations for g?

Field equations in Newtonian gravity: Poisson equation

$$\Delta_{g_E}\Phi = 4\pi G\rho \qquad\qquad \rho : \text{mass density.}$$

- Special relativity forces mass and energy to be the same concept (mass is a measure of rest energy).
- Also different reference frames measure different energies $\longleftrightarrow$ Information encoded in the energy-momentum tensor $T_{\mu\nu}$.
- Field equations should reduce to Poisson equation in the Newtonian limit.
- By the covariance principle: Tensor equations.

Lovelock theorem

Equations are assumed to be of the form $S_{\alpha\beta}(g) = \chi T_{\alpha\beta}$ (χ universal constant).

Requirements:

- $S_{\alpha\beta}(g)$ is a tensor constructed from the metric, its first and second derivatives.
  - Required so that the equations can reduce to Poisson equation in some limit.
- $S_{\alpha\beta}(g)$ is symmetric.
  - Necessary, as $T_{\alpha\beta}$ is symmetric.
- $S_{\alpha\beta}(g)$ is identically divergence-free: $\nabla_\alpha S^{\alpha\beta}(g) = 0 \quad$ for any metric g.
  - Required so that the conservation law $\nabla_\alpha T^{\alpha\beta}$ does not impose additional equations.

Theorem (Lovelock, 1970)

The most general symmetric 2-cov tensor $S(g)$ on a 4-dimensional manifold constructed from a pseudo-riemannian metric g, its first and second derivatives and being identically divergence-free is

$$S(g) = a\, Ein_g + \Lambda g$$

where $a, \Lambda \in \mathbb{R}$ and Ein_g is the Einstein tensor of g.

- No a priori requirement that $S(g)$ is linear in second derivatives.

Lovelock theorem (II)

This theorem does not hold in higher dimensions:

Theorem (Lovelock theorem in arbitrary dimension, 1971)

The most general symmetric 2-cov tensor $S(g)$ on an n-dimensional manifold constructed from a pseudo-riemannian metric g, its first and second derivatives and being identically divergence-free is a linear combination of the $\left[\frac{n+1}{2}\right]$ tensors

$$S^{(m)}_{\alpha\beta}(g) = g_{\alpha\gamma}\left(\delta^{\gamma\mu_1\nu_1\cdots\mu_m\nu_m}_{\beta\rho_1\sigma_1\cdots\rho_m\sigma_m} Riem_g{}^{\rho_1}{}_{\mu_1}{}^{\sigma_1}{}_{\nu_1}\cdots Riem_g{}^{\rho_m}{}_{\mu_m}{}^{\sigma_m}{}_{\nu_m}\right), \qquad m = 0, \cdots, \left[\frac{n-1}{2}\right]$$

where

$$\delta^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k} = \begin{cases} +1\ (-1) & \text{if } \begin{pmatrix} \alpha_1 \cdots \alpha_k \\ \beta_1 \cdots \beta_k \end{pmatrix} \text{ is an even (odd) permutation} \\ 0 & \text{otherwise} \end{cases}$$

- Gives rise to an alternative to General Relativity in higher dimensions called Lovelock Gravity.
- Imposing the additional condition that $S(g)$ is linear in the second derivatives: only Ein_g and g survive.

Einstein equations of General Relativity

Back to four-dimensions:

- Field equations from Lovelock theorem:

$$a\,\mathrm{Ein}_g + \Lambda\, g = \chi T$$

- Necessarily $a \neq 0$ (Newtonian limit). Set $a = 1$ by redefining constants:

$$\mathrm{Ein}_g + \Lambda g = \chi T, \qquad \text{Einstein field equations, 1915.}$$

Λ : Constant with dimensions of $(\text{Length})^{-2}$: Cosmological constant

- Introduced originally by Einstein to admit static cosmological models.
- Discarded later by Einstein himself, when expansion of the Universe was discovered.
- Non-zero value according to present day observational evidence.

$$\Lambda \approx 10^{-52}\mathrm{m}^{-2} \qquad \text{relative uncertainty: 4\%}$$

- χ: coupling constant of matter and gravity. Necessarily related to Newton's constant. Units of χ: $(\text{time})^2(\text{mass})^{-1}(\text{length})^{-2}$. Same dimensions as $\frac{G}{c^4}$:

$$\chi = k\frac{8\pi G}{c^4}, \qquad k \text{ dimension-less number.}$$

Static field equations

- Equivalent form of Einstein field equations:

$$\mathrm{Ric}_g = \chi\left(T - \frac{1}{2}(\mathrm{tr}_g T)\, g\right) + \Lambda g.$$

The constant k is determined by the Newtonian limit: involves slowly moving sources.

- Consider the field equations for static spacetimes: (N, h) Riemannian manifold.

$$M = \mathbb{I} \times N, \qquad g = -c^2 \lambda\, dt^2 + \frac{1}{\lambda} h, \qquad \xi = \partial_t$$

- Ricci tensor of (M, g): $X, Y \in \mathfrak{X}(N)$

$$\mathrm{Ric}_g(\xi, \xi) = \frac{c^2 \lambda}{2}\left(\Delta_h \lambda - \frac{|\nabla^h \lambda|^2}{\lambda}\right),$$

$$\mathrm{Ric}_g(\xi, X) = 0,$$

$$\mathrm{Ric}_g(X, Y) = \left(\mathrm{Ric}_h - \frac{1}{2\lambda^2} d\lambda \otimes d\lambda + \frac{1}{2}\left(\frac{\Delta_h \lambda}{\lambda} - \frac{|\nabla^h \lambda|^2}{\lambda^2}\right) h\right)(X, Y).$$

Newtonian limit

- Poisson equation describes the gravitational field of a fluid at rest.

Energy-momentum tensor of a fluid:

$$T = \left(\rho + \frac{p}{c^2}\right) \boldsymbol{u} \otimes \boldsymbol{u} + p\, g \quad \Longleftrightarrow \quad T - \frac{1}{2}(\mathrm{tr}_g T) g = \left(\rho + \frac{p}{c^2}\right) \boldsymbol{u} \otimes \boldsymbol{u} + \frac{\rho c^2 - p}{2} g.$$

- ρ mass density (energy density $/c^2$), p pressure of the fluid.
- u fluid four-velocity ($g(u,u) = -c^2$). Integral lines: trajectories of the fluid.

For a static spacetime necessarily $u \propto \xi \longrightarrow$ Fluid at rest.

- ξ-ξ component of the static Einstein equations

$$\Delta_h \lambda - \frac{|\nabla^h \lambda|^2}{\lambda} = -2\Lambda + k \frac{8\pi G}{c^2} \left(\rho + \frac{3p}{c^2}\right).$$

Newtonian limit: Static metric with $\quad N = \mathbb{R}^3 \quad \lambda = 1 + \frac{2\Phi}{c^2} \quad h = g_E + O\left(\frac{\Phi}{c^2}\right)$.

- ξ-ξ component of Einstein equations:

$$\Delta_{g_E} \Phi = 4\pi k G \rho - c^2 \Lambda + O\left(\frac{\Phi^2}{c^4}\right).$$

- $k = 1$ and $\Lambda << \frac{4\pi G \rho}{c^2}$.
- At the center of the Sun $\frac{4\pi G \rho}{c^2} \approx 10^{-21} \mathrm{m}^{-2}$: Compatible with cosmological observations of Λ.

Newtonian limit of space metric in General Relativity

- The Newtonian limit by itself does not determine the space part of the metric.
- Different metric theories of gravitation make different predictions for the space metric in the Newtonian limit.

In General Relativity:

- For static fluids: $g = -c^2 \lambda\, dt^2 + \lambda^{-1} h, \qquad T = \left(\rho + \frac{p}{c^2}\right) \boldsymbol{u} \otimes \boldsymbol{u} + p g_{\alpha\beta}.$
- The space-space component of the Einstein equations is (with $\Lambda = 0, p = 0$)

$$Ric_h = \frac{1}{2\lambda^2} d\lambda \otimes d\lambda.$$

- Newtonian limit: $\qquad N = \mathbb{R}^3, \qquad \lambda = 1 + \frac{2\Phi}{c^2}.$

$$\mathrm{Ric}_h = O\left(\Phi^2/c^4\right) \qquad \Longrightarrow \qquad h = g_E + O\left(\frac{\Phi^2}{c^4}\right) \qquad \Longrightarrow$$

$$g = -c^2\left(1 + \frac{2\Phi}{c^2}\right) dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) g_E + O(\Phi^2/c^4).$$

Has observational consequences.

- E.g. General Relativity makes different predictions for the advance of Mercury's periastron than other metric theories of gravity.

Einstein-Hilbert action

Einstein field equations:

$$\mathrm{Ein}_g + \Lambda\, g = \frac{8\pi G}{c^4} T$$

- The case $T = 0$ is called Einstein vacuum field equations.

For many applications Λ can be neglected. Not when working at very large scales.

- From now on, vacuum here means $T = 0$ and $\Lambda = 0$.

Is General Relativity a field theory, i.e. admits a principle of least action?

- Assume that the gravitating fields are field-theoretical,i.e. there is an action

$$S_{\mathrm{mat}}[\Psi] = \int \mathcal{L}_{\mathrm{mat}}(\Psi, \nabla\Psi, \cdots)\boldsymbol{\eta_g} \qquad \boldsymbol{\eta_g}: \quad \text{Volume form.}$$

  Ψ_0 satisfies the field equations $\iff$ $S_{\mathrm{mat}}(\Psi_0)$ is a stationary point with respect to compactly supported variations.

- Energy-momentum tensor:

$$T^{\alpha\beta}\boldsymbol{\eta_g} = \frac{\partial(\mathcal{L}_{\mathrm{mat}}\boldsymbol{\eta_g})}{\partial g_{\alpha\beta}}.$$

- Example: electromagnetism. $\mathcal{L}_{\mathrm{EM}}(\boldsymbol{F}) = -\frac{c}{4} F_{\alpha\beta} F_{\mu\nu} g^{\alpha\mu} g^{\beta\nu}$.

The Einstein equations are the Euler-Lagrange equation of the action

$$S = S_{\text{grav}} + S_{\text{mat}} = \frac{1}{2\chi}\int_M (\text{Scal}_g - 2\Lambda)\boldsymbol{\eta_g} + \int_M \mathcal{L}_{\text{mat}}\,\boldsymbol{\eta_g}$$

for variations of compact support on M.

- The gravitational term $S_{\text{grav}} = \frac{1}{2\chi}\int_M \text{Scal}_g \boldsymbol{\eta_g}$ is the Einstein-Hilbert action.
- Lagrangian density depends on g, ∂g, $\partial\partial g$, but field equations still second order.

All second derivatives in the variation arise in divergence form.

- The difference between two connections on a manifold is a tensor field.
- h an arbitrary symmetric two-tensor with compact support. Variation of g: $g_\epsilon = g + \epsilon h$ (Lorentzian metric for ϵ small). Define $W = \frac{d\Gamma_\epsilon}{d\epsilon}\Big|_\epsilon$.

$$\frac{d(\text{Scal}_{g_\epsilon})\boldsymbol{\eta_{g_\epsilon}}}{d\epsilon}\Bigg|_{\epsilon=0} = \left(-\text{Ein}_g^{\alpha\beta} h_{\alpha\beta} + \text{div}_g V\right)\boldsymbol{\eta_g} \qquad V^\alpha := W^\alpha_{\beta\gamma} g^{\beta\gamma} - W^\beta_{\beta\gamma} g^{\alpha\gamma}$$

- Variation of the action

$$\frac{dS_{\text{grav}}[g+\epsilon h]}{d\epsilon}\Bigg|_{\epsilon=0} = -\frac{1}{2\chi}\int_M (\text{Ein}_g^{\alpha\beta} + \Lambda g^{\alpha\beta}) h_{\alpha\beta}\,\boldsymbol{\eta_g}.$$

- Euler-Lagrange equations $\frac{dS[g+\epsilon h]}{d\epsilon} = 0\Big|_{\epsilon=0}$ yields $\qquad \text{Ein}_g + \Lambda g = \chi T.$

Stationary spacetimes

Equilibrium configurations play a fundamental role to understand any physical theory:

- (M, g) stationary: The spacetime admits a Killing vector with complete integral curves which are timelike on a non-empty open subset $U \subset M$.
  - An integral curve is complete if defined as a map $\gamma : (-\infty, \infty) \longrightarrow M$.
- (M, g) strictly stationary: stationary and $U = M$.
- (M, g) time-independent: admits a Killing vector which is timelike somewhere.

"Stationary" is stronger than "time-independent".

Theorem (Global structure theorem for strictly stationary (M, g) [Harris, 1992])

A strictly stationary chronological spacetime is of the form $M = \mathbb{R} \times S$ with ξ tangent to the $\mathbb{R}$ factor.

- Chronological: there are no closed world-lines.

Uniqueness of strictly stationary spacetimes

- Consider a gravitational source in equilibrium and $\Lambda = 0$.
- We can ask how should behave the gravitational field far away from the source.
- The natural answer is that it should approach a state of absence of gravitation, i.e. the Minkowski spacetime.
- However, a priori it could approach any state of "pure" gravitational field in equilibrium, i.e. a strictly spacetime with no sources of any kind.

Fundamental question: Is the Minkowski spacetime the only one with these properties?

Theorem (Lichnerowicz, 1955)

If a spacetime (M, g) is strictly stationary, chronological, complete, vacuum and approaches (in a suitable sense) the Minkowski spacetime "at infinity", then (M, g) is the Minkowski spacetime.

- "Vacuum and complete" encode the property that there are no sources.
- "Chronological" needed to apply the global structure theorem.
- The notion of "approaching Minkowski at infinity" requires an appropriate definition.

- The assumption of asymptotic flatness is strong.
- Implicitly assumes that Minkowski is the only state of "pure" gravitational field.

Theorem (Anderson, 2000)

Let (M, g) be a geodesically complete, chronological, strictly stationary, vacuum spacetime. Then (M, g) is either the Minkowski spacetime $(\mathbb{M}^{1,3}, \eta)$ or a quotient of this spacetime by a discrete group of isometries commuting with a time translation.

- In physical terms one can state this theorem as

  There are no gravitational solitons, i.e. non-trivial gravitational field with no sources and in equilibrium.

- Other non-linear physical theories do admit such states, so this is a highly non-trivial statement about General Relativity.
- Justifies the condition that far away from the sources, the field approaches Minkowski.

Spherical symmetry

- A Lie group G acts a group of transformations on a manifold $\mathcal{M}$ is there exists a smooth map:
$$\Phi : G \times \mathcal{M} \longrightarrow \mathcal{M}$$
$$(\boldsymbol{a}, p) \longrightarrow \Phi(\boldsymbol{a}, p) := \Phi_{\boldsymbol{a}}(p)$$
such that $\Phi_{\boldsymbol{a}} : \mathcal{M} \to \mathcal{M}$ is a diffemorphism and $\Phi_{\boldsymbol{a_1}} \circ \Phi_{\boldsymbol{a_2}}(p) = \Phi_{\boldsymbol{a_1} \cdot \boldsymbol{a}_2}(p)$.
- At any point $p \in \mathcal{M}$, the orbit $\mathcal{O}_p$ is the set $\{\Phi_{\boldsymbol{a}}(p) \, : \boldsymbol{a} \in G\} \subset \mathcal{M}$.
- A group of transformations Φ on a Riemannian manifold (M, g) is a group of isometries $\iff$ $\Phi^{\star}_{\boldsymbol{a}}(g) = g$.

Definition (Spherical symmetry)

A spacetime (M, g) is spherically symmetric if the the group $SO(3)$ acts a group of isometries with spacelike codimension-two orbits (or points).

- Analogous definition works in arbitrary dimension ($SO(3)$ replaced by $SO(n)$).
- Spherically symmetric spacetime: natural smooth function (area radius function)
$$r : \mathcal{M} \to \mathbb{R}$$
$$p \to \sqrt{|\mathcal{O}_p|/4\pi}.$$
- r is smooth even at points where $r = 0$ (i.e. $\mathcal{O}_p$ is a point).

Penrose diagrams

Spherically symmetric spacetimes admit a pictorial representation: Penrose diagrams.

- (M, g) spherically symmetric and $r : M \longrightarrow \mathbb{R}$ area radius function.
- Quotient $M^{(2)} = (M \setminus \{r = 0\}) / SO(3)$: smooth manifold.
  - Two points are in the same equivalence class if they are on the same orbit of the isometry group $SO(3)$.
- r defines a smooth function on $M^{(2)}$ (still denoted by r).
- There is a Lorentzian metric h in $M^{(2)}$ such that (with π projection: $M \longrightarrow M^{(2)}$).

$$g = \pi^\star(h) + r^2 g_{\mathbb{S}^2}, \qquad g_{\mathbb{S}^2} \text{ standard metric on } \mathbb{S}^2.$$

- Any two-dimensional metric is locally conformally flat, i.e. exists local coordinates $\{t, x\}$

$$h = \Omega^2(t, x)(-dt^2 + dx^2).$$

The Penrose diagram assumes this to be true globally:

- There is a diffeomorphism $\Phi : M^{(2)} \to \mathcal{U} \subset \mathbb{M}^{1,1}$, U domain with compact closure.
- There is a smooth map $\Omega : \mathcal{U} \to \mathbb{R}^+$ such that $h = \Phi^\star(\Omega^{-2}\eta)$.
- Ω is allowed to approach zero at $\partial\mathcal{U} \longrightarrow$ points at infinity represented on ∂U.

- Conformal rescaling of a metric preserves the causal character of any vector.

Null (timelike, spacelike) direction on $(\mathcal{U}, \eta)$ is also null (timelike, spacelike) in $(M^{(2)}, h)$.

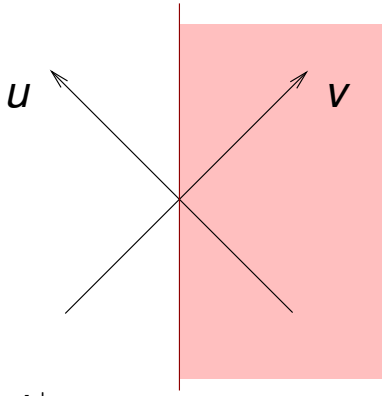
- The shape of the domain $\mathcal{U} \subset (\mathbb{M}^{1,1}, \eta)$ gives useful global information.

Example: The Minkowski spacetime $(\mathbb{M}^{1,3}, \eta)$

- Use double null, spherical coordinates

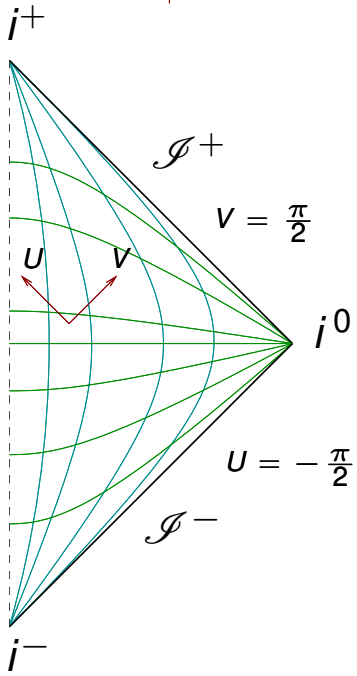
$$\eta = -du\,dv + \frac{(v-u)^2}{4} g_{\mathbb{S}^2}.$$

$u, v \in$



- Diffeomorphism: $u = \tan(U)$, $v = \tan(V)$.

- $U \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $V \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $V > U$.
- The projection of null curves on $M^{(2)}$ have a past endpoint at $\mathscr{I}^-$ and a future endpoint at $\mathscr{I}^+$.
- Timelike geodesics start at i^- and end at i^+.
- Spacelike geodesics go from i^0 back to i^0.



There are three types of infinity: past, null and spacelike.

- Very different properties.

Birkhoff theorem

- Fundamental result in General Relativity: spherically symmetric vacuum spacetimes are classified by a real parameter.

Theorem (Birkhoff, 1923)

Let (M, g) be spherically symmetric and vacuum. Then $\mathcal{X} := \{p \in \mathcal{M}; |\nabla r|^2 = 0\}$ has empty interior and $\exists m \in \mathbb{R}$ and local coordinates on $\mathcal{M} \setminus \mathcal{X}$ such that

$$g = -c^2 \left(1 - \frac{2Gm}{c^2 r}\right)^2 dt^2 + \frac{dr^2}{1 - \frac{2Gm}{c^2 r}} + r^2 g_{\mathbb{S}^2}, \qquad \textit{Schwarzschild metric, 1916.}$$

- Analog Newtonian result: the only spherical gravitational potential is $\Phi = -\frac{Gm}{r}$.

Besides spherically symmetric, the metric is time independent.

- Spherical sources with radial motions do not emit gravitational waves (independently of how these are defined, time-independent metrics cannot qualify as "waves").
- Birkhoff's theorem extends to arbitrary $(n+1)$-dimension replacing the metric by

$$g = -c^2 \left(1 - \frac{2a}{r^{n-2}}\right)^2 dt^2 + \frac{dr^2}{1 - \frac{2a}{r^{n-2}}} + r^2 g_{\mathbb{S}^{n-1}}, \qquad a \in \mathbb{R}$$

- Also extends to $\Lambda \neq 0$.

Physical properties

Interpretation of m:

- The spacetime is static.
- At large values of r the metric approaches the Minkowski metric $\longrightarrow$ admits a Newtonian limit

$$-c^2\left(1-\frac{2Gm}{c^2 r}\right)dt^2 = -c^2\left(1+\frac{2\Phi}{c^2}\right)dt^2, \qquad \text{for large } r.$$

- $\Phi = -\frac{Gm}{r}$: Gravitational potential of a particle of mass m.
- m is the analogous to the Newtonian mass. Simply called mass of the spacetime.
- Physical objects have positive gravitational masses $\longrightarrow$ $m \geq 0$ for physical systems.

The Schwarzschild metric provides two classical tests of General Relativity.

- The spacetime geometry at the Solar System is very approximately a Schwarzschild spacetime of mass $m = M_\odot$.
- Planets move along timelike geodesics, light along null geodesics.

Solving the geodesic equations of the Schwarzschild metric:

- The perihelion of Mercury advances 43 arc sec/century, exactly as observed.
- Light from distant stars passing by the Sun is deflected an angle 1.75 arc sec. Measured by Eddington, 1919.

Kruskal spacetime

The Birkhoff theorem is a local result. Is there a global analog?

- Choose units $G = c = 1$.

Theorem (Kruskal '1960)

Given $m \in \mathbb{R}$ there is a unique maximal spherically symmetric smooth vacuum spacetime of mass m.

- Maximal: not a proper subset of a spacetime with the same properties.

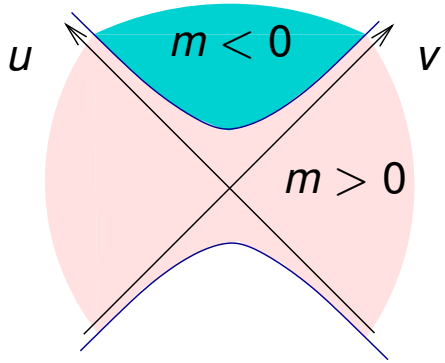
This spacetime is called Kruskal spacetime, denoted by $(M_{\text{Kr}}, g_{\text{Kr}})$.

- For $m \neq 0$: $\quad M_{\text{Kr}} = \mathbb{R}^2 \times \mathbb{S}^2$

$$g_{\text{Kr}} = -\frac{32m^3}{r} e^{-\frac{r}{2m}} du\, dv + r^2 g_{\mathbb{S}^2},$$

$$r : U \longrightarrow \mathbb{R}^+, \qquad uv = e^{\frac{r}{2m}}(1 - \frac{r}{2m}).$$

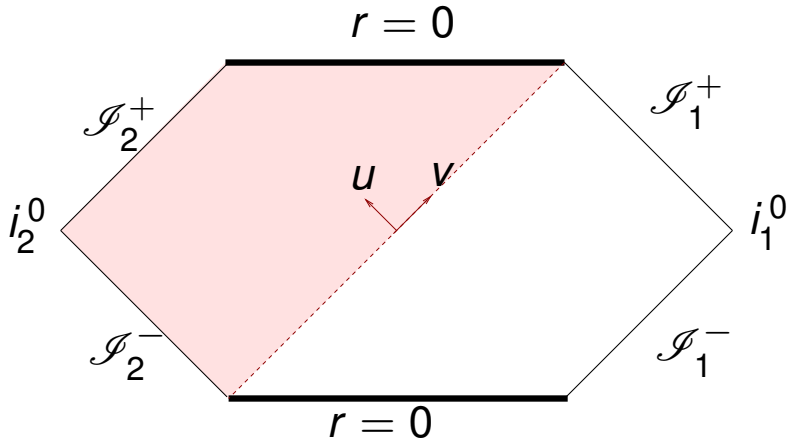
$u, v \in$



- Singularity at $uv = 1$ (a curvature singularity).
- Approaches Minkowski for large r.

Penrose diagram for $m > 0$.

- There are two asymptotic regions.
- There is no i^+ or i^-.
- Causal curves starting at $u > 0$ cannot reach $\mathscr{I}_1^+$



There are points causally disconnected from infinity. Cannot be seen by an observer far away.

- Event horizon $\mathcal{H}$: Boundary of the region causally disconnected from infinity.
  - In Kruskal: $u = 0$ or $v = 0$ (depending on selected infinity).
  - $\{u = 0\} \cup \{v = 0\} = \{\nabla r|^2 = 0\} = \{r = 2m\}$.
- Hypersurface ruled by null geodesics.

Besides being spherically symmetric, Kruskal admits the Killing vector

$$\xi = (4m)^{-1} \left(-u\partial_u + v\partial_v\right).$$

- Timelike on $\{uv < 0\}$, spacelike on $\{uv > 0\}$ and null on $\{uv = 0\}$.
- Vanishes on $\{u = v = 0\}$: Spacelike surface called bifurcation surface
- $\{u = 0, v > 0\}$ is a null hypersurface where $\xi \neq 0$, null and tangent. Similarly for $\{u = 0, v < 0\}$, $\{u > 0, v = 0\}$, $\{u < 0, v = 0\}$.

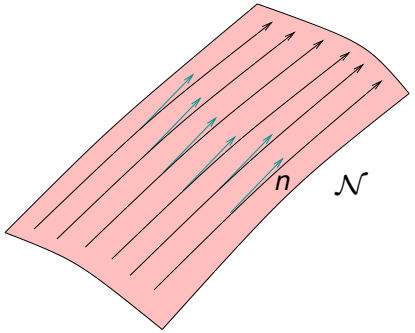
They are examples of Killing horizons.

A null hypersurface of an n-dimensional spacetime (M, g) is an embedded hypersurface $\Psi : \mathcal{N} \longrightarrow M$ such that $g_{\mathcal{N}} = \Psi^\star(g)$ is degenerate everywhere.

- For all $p \in \mathcal{N}$, exists $v \neq 0 \in T_p\mathcal{N}$ such that $g_{\mathcal{N}}(v, \cdot) = 0$ (degeneration vector).

General properties (identify $\mathcal{N}$ and $\Phi(\mathcal{N})$):

- Let $\boldsymbol{n}$ be a normal one-form (unique up to non-zero rescaling). Corresponding normal vector n null and tangent to $\mathcal{N}$.



- Any degeneration vector $v \in T_p\mathcal{N}$ is proportional to $n|_p$.
- For any $p \in \mathcal{N}$, the spacetime geodesic at p with tangent vector $n|_p$ lies in $\mathcal{N}$.

Definition (Killing horizon)

Let (M, g) with a Killing vector ξ. A Killing horizon of ξ is a connected null hypersurface $\mathcal{H}_\xi$ such that $\forall p \in \mathcal{H}_\xi$, $\xi|_p \neq 0$, null, and tangent to $\mathcal{H}_\xi$.

- The surface gravity κ_ξ of a Killing horizon $\mathcal{H}_\xi$ is defined by $\nabla_\xi \xi \overset{\mathcal{H}_\xi}{=} \kappa_\xi \xi$

In the Kruskal spacetime there are four Killing horizons. Surface gravity $\kappa_\xi = \frac{1}{4m}$.

- A Killing horizon is degenerate of $\kappa_\xi = 0$ and non-degenerate if $\kappa_\xi \neq 0$.

Kerr spacetime

The Schwarzschild (Kruskal) spacetime is a particular case of a fundamental class of vacuum spacetimes. The Kerr family.

- Each element is identified by two real numbers m and a. Global properties depend on their values.
- When $a \neq 0$, no global chart exists. Useful to restrict to a suitable open subset.

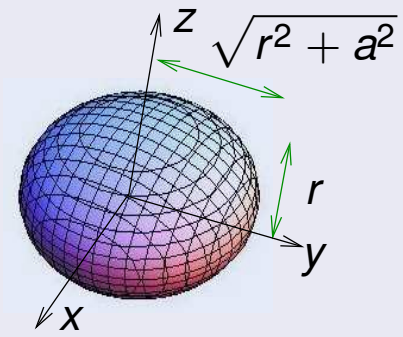
Definition (Kerr spacetime)

For $m, a \in \mathbb{R}$ let $M_a = \mathbb{R} \times \left(\mathbb{R}^3 \setminus \{x^2 + y^2 \leq a^2, z = 0\}\right)$, with (x, y, z) Cartesian coordinates in $\mathbb{R}^3$. The Kerr spacetime of mass m and specific angular momentum a is the spacetime $(M_a, g_{m,a})$ where

$$g_{m,a} = \underbrace{-dt^2 + dx^2 + dy^2 + dz^2}_{\eta} + \frac{2mr^3}{r^4 + a^2 z^2}\, \ell \otimes \ell,$$

where $r : M_a \longrightarrow \mathbb{R}^+$ defined by $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$ and

$$\ell = dt + \frac{r}{r^2 + a^2}(xdx + ydy) + \frac{a}{r^2 + a^2}(ydx - xdy) + \frac{zdz}{r}.$$



Some properties:

- $(M_a, g_{m,a})$ is a solution of the Einstein vacuum field equations ($T_{\alpha\beta} = 0$).
- The spacetime admits two Killing vectors

$$\xi = \partial_t, \qquad \eta = x\partial_y - y\partial_x.$$

In fact, the spacetime is axially symmetric:

- The group $SO(2)$ acts an as isometry group of M. $\Psi : SO(2) \times M \longrightarrow M$.
- The set of fixed points $\Psi(\alpha, p) = p, \quad \forall \alpha \in SO(2)$ is a codimension-two timelike surface $\longrightarrow$ axis of symmetry.

The generator of the axial isometry is η. Axis of symmetry $\{x = y = 0\}$.

- The spacetime admits Killing horizons iff $|a| \leq m \neq 0$.
- There are two Killing horizons located at the hypersurfaces $\mathcal{H}_{r_+} = \{r = r_+\}$ and $\mathcal{H}_{r_-} = \{r = r_-\}$ where

$$r_\pm := m \pm \sqrt{m^2 - a^2}.$$

- Topology $\mathbb{R} \times \mathbb{S}^2$, associated Killing vectors $\xi_\pm = \xi + \frac{a}{2mr_\pm}\eta$, surface gravity

$$\kappa_\pm = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}. \qquad \text{Degenerate when } |a| = m: \qquad \text{Extreme Kerr.}$$

- $(M_a, g_{m,a})$ with $a = 0$ corresponds to the region $\{v > 0\}$ of Kruskal.

Asymptotic flatness and definition of black hole

- The Kerr metric satisfies $g_{m,a} = \eta + O(1/\sqrt{x^2+y^2+z^2})$.
- Yields a possible definition of asymptotic flatness.
  - Different contexts have different suitable definitions. Not equivalent to each other.

Definition

A spacetime (M, g) is asymptotically flat if it admits an asymptotically flat 4-end, i.e.

- An open submanifold $M^\infty \simeq \mathbb{R} \times (\mathbb{R}^3 \setminus \overline{B}(r_0))$ such that
- $\exists C > 0$ such that $g_{\mu\nu}$ at M^∞ in Cartesian coordinates $(t, \vec{x})$ satisfies

$$|g_{\mu\nu}| + |g^{\mu\nu}| + |\vec{x}|\,|g_{\mu\nu} - \eta_{\mu\nu}| + |\vec{x}|^2\,|\partial_\sigma g_{\mu\nu}| + |\vec{x}|^3\,|\partial_\sigma\partial_\rho g_{\mu\nu}| \leq C.$$

A black hole is a region of a spacetime which cannot send signals to infinity:

- For any r define $M_r = \{p \in M^\infty : |\vec{x}(p)| > r\}$.
- The black hole region (for the asymptotically flat 4-end) is

$$\mathcal{B}^+ := \{p \in M;\ \exists r_p \text{ such that all future directed causal curves starting at } p \text{ lie in } M \setminus M_{r_p}\}.$$

- If $\mathcal{B}^+$ is non-empty, (M, g) is a black hole spacetime.
- Define also $\mathcal{B}^-$ (white hole region) replacing future by *past*.

Event horizon and general properties

- The topological boundary of the black hole region is the event horizon: $\mathcal{H} = \partial\mathcal{B}^+$
- Necessarily a Lipschitz null hypersurface ruled by null geodesics.
- A future null geodesic γ which is tangent to $\mathcal{H}$ at one point $p = \gamma(s_0)$ remains fully contained in $\mathcal{H}$ for $s \geq s_0$ (not true in general for $s < s_0$).

A spacetime $(\mathcal{M}, g)$ satisfies the null energy condition (NEC) iff

$$\mathrm{Ein}_g(k,k) \geq 0, \qquad \forall p \in \mathcal{M} \quad \text{and} \quad \forall k \in T_p\mathcal{M} \quad \text{null.}$$

A hypersurface Σ is achronal if no distinct points in Σ can be joined by a timelike curve.

General properties of $\mathcal{H}$ (see [Chuściel, Delay, Galloway, 2001] for precise statements)

- If the spacetime is stationary and satisfies NEC then $\mathcal{H}$ is smooth.
- Area theorem [Hawking 1972]: Assume (M, g) satisfies NEC. Let Σ_a $(a = 1, 2)$ be achronal, spacelike hypersurfaces and $\mathcal{H}_{\Sigma_a} := \mathcal{H} \cap \Sigma_a$ $(a = 1, 2)$. Assume every point $p \in \mathcal{H}_{\Sigma_1}$ can be joined to $\mathcal{H}_{\Sigma_2}$ by a future causal curve. Then

$$|\mathcal{H}_{\Sigma_1}| \leq |\mathcal{H}_{\Sigma_2}|.$$

The area of cross sections of the event horizon cannot decrease to the future.

Uniqueness theorem of vacuum black holes

- A Kerr spacetime with $|a| \leq m \neq 0$ is a black hole spacetime.
- The event horizon is the hypersurface $\mathcal{H}_{r_+}$.

A fundamental field of research of General Relativity is the classification of stationary black hole spacetimes.

Uniqueness of stationary vacuum back holes – rough statement –

A stationary, asymptotically flat vacuum black hole is a member of the Kerr family of spacetimes with $|a| \leq m$.

Not yet known to hold in such generality (see details in [Chruściel, Costa, Heusler, 2012]).

- There are versions for electrovacuum spacetimes ($T_{\mu\nu}$ of an electromagnetic field without sources).

Assumptions:

- Stationary: $\mathbb{R}$ acts as an isometry group with timelike orbits near infinity. Associated Killing vector ξ.

Simple consequences:

- $\mathcal{H}$ is invariant under the stationary isometry (i.e. ξ is tangent to $\mathcal{H}$).
- Let ℓ tangent to null geodesic generators of $\mathcal{H}$. $\xi|_{\mathcal{H}}$ may or may not be $\propto \ell$.

Fundamental first step:

- Staticity theorem [Sudarsky & Wald, 1992]: If $\xi|_{\mathcal{H}} \propto \ell$ then ξ is static: $\boldsymbol{\xi} \wedge d\boldsymbol{\xi} = 0$.
- Rigidity theorem [Hawking, 1972]: If $\xi|_{\mathcal{H}} \not\propto \ell$ and (M, g) is analytic then:
  - There is a second, axial Killing vector η.
  - $\mathcal{H}$ is a Killing horizon with Killing vector $\xi + \Omega_{\mathcal{H}}\eta$ ($\Omega_{\mathcal{H}}$ constant).

The problem splits into uniqueness of static black holes and uniqueness of stationary and axially symmetric black holes.

- Uniqueness of static vacuum black holes: ✓

Theorem ([Israel '67, Masood-ul-Alam '92, Chruściel '99, Chruściel, Reall, Tod '06])

Let (M, g) be a vacuum, stationary black hole spacetime with integrable Killing generator ξ. Then, there exists $m > 0$ such that the exterior region $M \setminus (\mathcal{B}^+ \cup \mathcal{B}^-)$ is isometric to the region $\{u < 0, v > 0\}$ of the Kruskal spacetime of mass m.

- Stationary, axially symmetric black hole spacetimes relatively well understood ✓
  - Yields Kerr spacetime assuming event horizon has one or two connected components.
- Analyticity assumption is unjustified: Strong a priori restriction.

Open problem: Remove the analyticity assumption and allow any number of components in the stationary, axially symmetric case.

ADM energy-momentum and positivity of energy

From equivalence principle: Energy and momentum of gravitational field cannot be localized.
Two possibilities:

- Use quasi-local definitions: Define energy and momentum inside a spacelike, two-dimensional surface enclosing a spacelike domain.
  - Several possible definitions, active area of research.
- Define total energy and momentum "at infinity".

Needs a concept of "infinity": asymptotically flat spacetimes.

- Σ hypersurface embedded in $(M, g) \supset (M^\infty, g|_{M^\infty})$.
- Σ is asymptotically flat at M^∞ if $\Sigma^\infty = \Sigma \cap M^\infty$ can be written as a graph $(x^i, x^0 = f(x^i))$

$$\begin{aligned} f : \mathbb{R}^3 \setminus \overline{B}(r_0) &\longrightarrow \mathbb{R} \\ x^i &\longrightarrow f(x^i) \end{aligned}$$

  with $f \longrightarrow a_i x^i$ at a suitable rate when $|\vec{x}| \longrightarrow \infty$ and $a_i \in \mathbb{R}$ satisfy $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2} < 1$.
- Asymptotic flatness for Σ can be defined directly in terms of its intrinsic and extrinsic geometry.

- For each such Σ a notion of total energy E_{ADM} and total linear-momentum $\vec{P}_{\text{ADM}}$ can be defined.

ADM: Arnowitt-Deser-Misner, 1959.

- The combination $P^{\mu}_{\text{ADM}} = (E_{\text{ADM}}, \vec{P}_{\text{ADM}})$ is called ADM energy-momentum vector or ADM four-momentum.

Fundamental result in General Relativity: ADM energy-momentum has similar properties as the energy-momentum vectors of particles or fields in special relativity.

A spacetime $(\mathcal{M}, g)$ satisfies the dominant energy condition (DEC) iff

$$\text{Ein}_g(u, v) \geq 0, \qquad \forall p \in \mathcal{M} \quad \text{and} \quad \forall u, v \in T_p\mathcal{M} \quad \text{causal and future directed.}$$

- Via the Einstein field equations, it is a condition on the energy-momentum tensor of the matter fields.
  - Means that the energy-momentum vector is causal future directed when measured by any spacetime reference frame.
- Automatically satisfied e.g. for electromagnetic fields. Physically reasonable for all classical fields.

Theorem (Positive mass theorem, Schoen & Yau 1979, Witten 1981)

Let (M, g) be spacetime such that:

(i) *The dominant energy condition holds.*

(ii) *(M, g) admits an asymptotically flat four-end.*

(iii) *(M, g) admits an asymptotically flat spacelike hypersurface Σ such that $\Sigma \setminus \Sigma^\infty$ is compact.*

Then the ADM four-momentum P^μ_{ADM} of Σ is timelike and future directed or else $P^\mu_{ADM} = 0$. Moreover, $P^\mu_{ADM} = 0$ if and only if (M, g) is an open subset of the Minkowski spacetime $(\mathbb{M}^{1,3}, \eta)$.

- In particular $E_{\text{ADM}} > 0$ and zero iff (M, g) is Minkowski.
- A total mass M_{ADM} can be defined for spacetimes satisfying the conditions of the theorem.

$$M_{\text{ADM}} = \sqrt{E^2_{\text{ADM}} - \delta_{ij} P^i_{\text{ADM}} P^j_{\text{ADM}}}.$$

Diffeomorphism invariance

The Einstein field equations $\mathrm{Ein}_g = \chi T$ are a non-linear set of second order field equations for the spacetime metric g.

- Have the particularity that M needs to be constructed simultaneously with the metric g.
- Also, g may look very differently in different coordinate systems, while still representing the same spacetime.
    - Lack of any uniqueness of the Einstein field equations as P.D.E:
- (M, g) be a spacetime satisfying $\mathrm{Ein}_g = \chi T$ and Φ any diffeomorphism of M $\Phi : M \longrightarrow M$. Then $(M, \Phi^\star(g))$ solves

$$\mathrm{Ein}_{\Phi^\star(g)} = \chi \Psi^\star(T).$$

In particular: (M, g) vacuum $\Longleftrightarrow$ $(M, \Phi^\star(g))$ vacuum.

- Consequence of the following identities, valid for any diffeomorfism g:

$$\mathrm{Riem}_{\Phi^\star(g)} = \Phi^\star(\mathrm{Riem}_g), \qquad \mathrm{Ric}_{\Phi^\star(g)} = \Phi^\star(\mathrm{Ric}_g), \qquad \mathrm{Ein}_{\Phi^\star(g)} = \Phi^\star(\mathrm{Ein}_g).$$

Linearized gravity

Linearized gravity as an example of some of the issues.

- Fix a spacetime (M, g) and consider a family of Lorentzian metrics g_ϵ, $\epsilon \in (-a, a) \subset \mathbb{R}$ satisfying:

  (i) $g_{\epsilon=0} = g$

  (ii) g_ϵ depends smoothly on ϵ

- Define $h = \frac{dg_\epsilon}{d\epsilon}\Big|_{\epsilon=0}$. Symmetric 2-cov tensor. We can compute the variation of the Ricci tensor

$$\frac{d(\mathrm{Ric}_{g_\epsilon})_{\alpha\beta}}{d\epsilon}\Bigg|_{\epsilon=0} := D\mathrm{Ric}_g(h)_{\alpha\beta} =$$

$$= -\frac{1}{2}\left(\nabla_\mu\nabla^\mu h_{\alpha\beta} + \nabla_\alpha\nabla_\beta\left(\mathrm{tr}_g h\right) - \nabla_\mu\nabla_\alpha h^\mu{}_\beta - \nabla_\mu\nabla_\beta h^\mu{}_\alpha\right)$$

- Consider $\xi \in \mathfrak{X}(M)$ and the local group of diffeomorphisms Ψ_ϵ it generates:

$$\Psi_{\epsilon=0} = \mathrm{Id}, \qquad \Psi_{\epsilon_1+\epsilon_2} = \Psi_{\epsilon_1} \circ \Psi_{\epsilon_2} \qquad \frac{d\Psi_\epsilon(p)}{d\epsilon}\Bigg|_{\epsilon=0} = \xi(p), \qquad \forall p \in M$$

- The family of metrics $g_\epsilon^\xi := \Psi_\epsilon(g_\epsilon)$ is geometrically equivalent. Satisfies

$$h^\xi = h + \mathcal{L}_\xi g.$$

The following identity holds: $D\mathrm{Ric}_g(\mathcal{L}_\xi g) \equiv 0$.

- Assume for definiteness that g_ϵ solve the Einstein field equations. Then

$$D\mathrm{Ric}_g(h) = 0$$

- Second order linear partial differential equation for h: linearized vacuum field equations. Describe small perturbations around the background (M, g).
- For any solution h and $\xi \in \mathfrak{X}(M)$, $h + \mathcal{L}_\xi g$ also solves the equations. Physically represent the same solution. $\longrightarrow$ Linearized gravity is a gauge theory.

Example: Linearized gravity around Minkowski spacetime $(\mathbb{M}^{1,3}, \eta)$.

- Lorentz invariant field theory in flat spacetime.
- There exists a choice of gauge transverse traceless (TT) in which

$$\nabla^{\mathbb{M}}_\beta h^{\alpha\beta} = 0, \qquad \mathrm{tr}_\eta h = 0, \qquad h(u, \cdot) = 0$$

  where u unit timelike constant vector field.
- Field equations in this gauge

$$\Box_{\mathbb{M}} h := \nabla^{\mathbb{M}}_\mu \nabla^{\mathbb{M}\mu} h = 0, \qquad \text{Wave equation in Minkowski}$$

- (Linearized) gravitational waves propagate at the speed of light in Minkowski.

Plane wave solutions:

- Choose Minkowskian coordinates $\{x^\alpha\}$. A plane wave $h(x) = A_k \cos(\eta(k,x))$,

  k constant vector, A_k constant symmetric, trace-free 2-cov tensor

  is a solution of the linearized field equations iff

$$\eta(k,k) = 0, \quad \text{propagation along a null direction.}$$
$$A_k(k,\cdot) = 0 \qquad A_k(u,\cdot) = 0$$

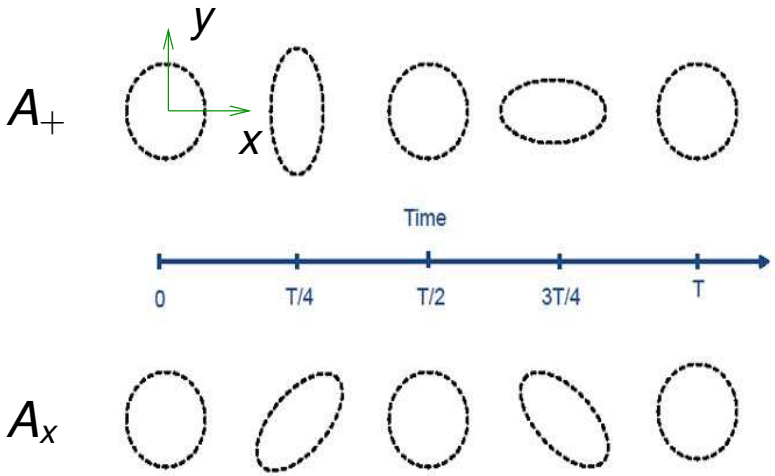
- For given k, the vector space $\{A_k\}$ is two-dimensional.

Linear plane gravitational waves propagating in Minkowski have two polarization states.

- Example: if $k = k\,(\partial_{x^0} + \partial_{x^3})$ and $u = \partial_{x^0}$:

$$A_k = A_+\,(dx \otimes dx - dy \otimes dy) + 2A_\times\, dx \otimes dy$$

- Effect on freely falling particles (with zero average velocity with respect to u):
- Displacement of particles are transversal to the direction of propagation.



(Linear) gravitational waves are transverse waves.

- Direct detection of gravitational waves is a major ongoing project: Ligo, Virgo, Geo600, Tama and future space detectors e-Lisa.

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Ligo Hanford. Source: caltech.edu

Ligo Livingston. Source: caltech.edu