Elements of differential geometry

R.Beig (Univ. Vienna)

ESI-EMS-IAMP School on Mathematical GR, 28.7. - 1.8. 2014

1. tensor algebra
2. manifolds, vector and covector fields
3. actions under diffeos and flows
4. connections
5. pseudo-Riemannian manifolds
6. geodesics
7. curvature

Tensor algebra

Let $\mathbb{T}$ be an n-dimensional vector space over $\mathbb{R}$ and $\mathbb{T}^*$ its dual. Elements u, v of $\mathbb{T}$ are called vectors, elements ω, μ of $\mathbb{T}^*$ are called covectors. In a basis $\{e_i\}$ the vector u has the form $u = \sum_1^n u^i e_i = u^i e_i$ (note summation convention!) and the covector ω reads $\omega = \omega_i e^i$ in the dual basis given by $e^i(e_j) = \delta^i{}_j$. The action of ω on the v reads $\omega(v) = \omega_i v^i$. Although v^i and ω_i depend on the choice of basis, $\omega_i v^i$ does not. Reading $\omega_i v^i$ 'from right to left' gives the identification $\mathbb{T} \cong \mathbb{T}^{**}$. The spaces $\mathbb{T}^r_s$ consist of multilinear forms on $\mathbb{T}^* \times ...\mathbb{T}^* \times ..\mathbb{T}$ (r copies of $\mathbb{T}^*$, s copies of $\mathbb{T}$). They have r upper and s lower indices: $U = U^{i_1...i_r}{}_{j_1..j_s}\, e_{i_1} \otimes ...e_{i_r} \otimes e^{j_1} \otimes ..e^{j_s}$. In particular $\mathbb{T} \cong \mathbb{T}^1$ and $\mathbb{T}^* \cong \mathbb{T}_1$ and $\mathbb{T}^1_1 \cong L(\mathbb{T}, \mathbb{T})$.

A scalar product γ on $\mathbb{T}$ is given by $\gamma(u,v) = \gamma_{ij}u^i v^j$ with $\gamma_{ij} = \gamma_{ji}$ non-degenerate. γ of signature $(++,..+)$ is positive definite, γ with $(-,+,+,...+)$ a Lorentzian metric. γ gives rise to a unique quadratic form on $\mathbb{V}^*$ given by γ^{ij} where $\gamma^{ij}\gamma_{jk} = \delta^i{}_k$. The quantities γ^{ij} and γ_{ij} yield an isomorphism between elements $v \in \mathbb{T}$ and $\omega \in \mathbb{T}^*$ by means of 'raising and lowering of indices', e.g. $v^i(\omega) = \gamma^{ij}\omega_j =: \omega^i$. γ Lorentzian: a non-zero vector $v \in \mathbb{T}$ is

- timelike: $\gamma(v,v) = \gamma_{ij}v^i v^j = -(v^0)^2 + \sum_1^{n-1}(v^i)^2 < 0$
- null: $\gamma(v,v) = \gamma_{ij}v^i v^j = -(v^0)^2 + \sum(v^i)^2 = 0$
- spacelike $\gamma(v,v) = \gamma_{ij}v^i v^j = -(v^0)^2 + \sum(v^i)^2 > 0$

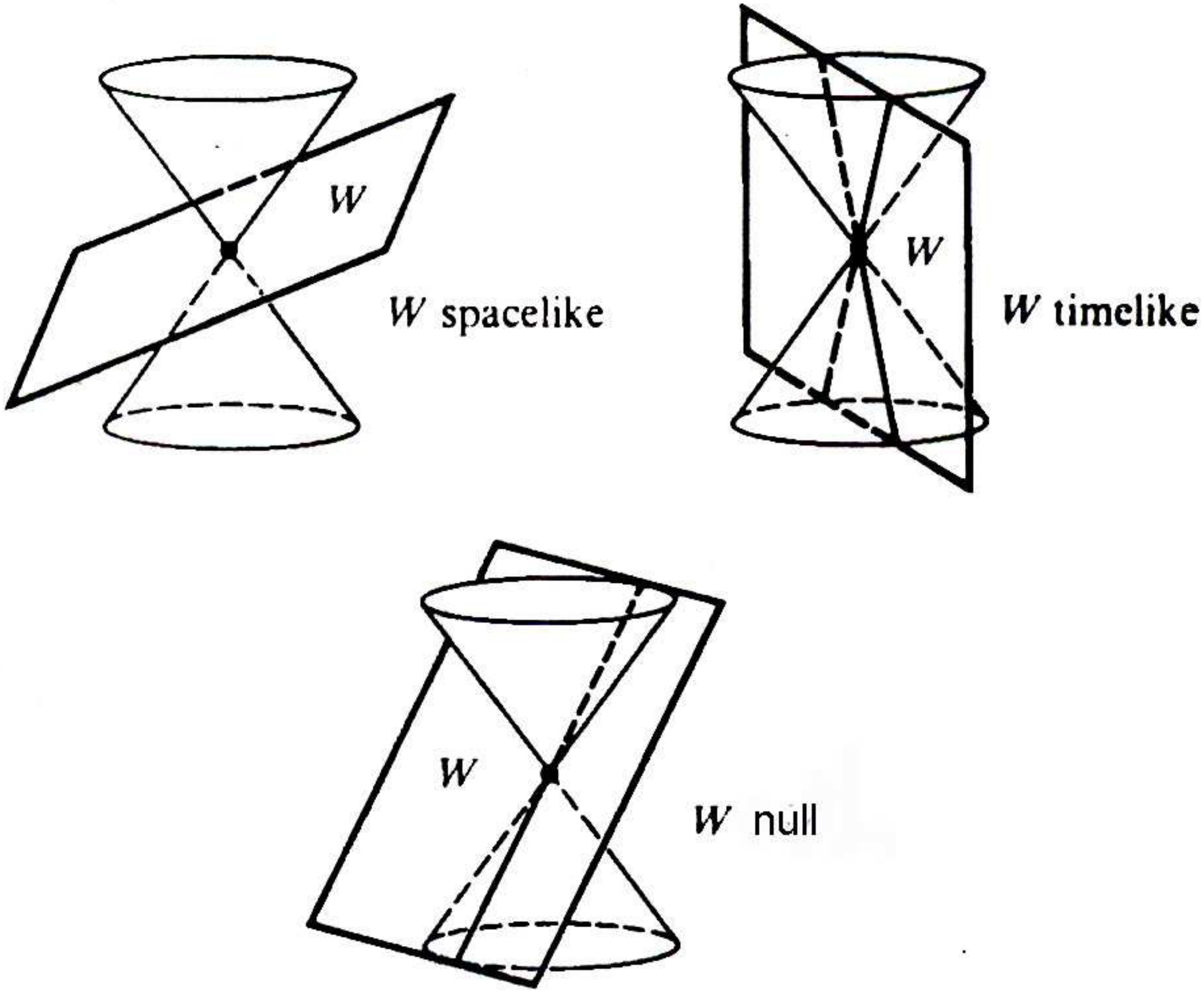
Null vectors form a double cone ('past and future light cone') $\mathcal{C}$, timelike inside.

Subspaces $\mathbb{W} \subset \mathbb{T}$ are

- spacelike: $\gamma(.,.)|_{\mathbb{W}}$ pos.def.
- null: $\gamma(.,.)|_{\mathbb{W}}$ degenerate, same as $\mathbb{T}$ tangent to $\mathcal{C}$
- timelike: $\gamma(.,.)|_{\mathbb{W}}$ Lorentzian

Fundamental inequalities:

- 'reverse C-S inequality': $(\gamma(u,v))^2 \geq \gamma(u,u)\gamma(v,v)$, provided that u or v (or both) are causal
- 'reverse Δ inequ.: $|u+v| \geq |u| + |v|$, where $|u| = \sqrt{-\gamma(u,u)}$ and u, v are causal, both future or both past directed. Is essence behind the 'twin paradox'


W
W spacelike
W
W timelike
W
W null

Manifolds

Stated somewhat informally a (C^∞, n-dimensional manifold) M is a topological space (Hausdorff, 2nd countable), equipped with a set of coordinate charts (U, x^i), i.e. U open and x^i map U bijectively into an open set in $\mathbb{R}^n$. These charts should cover M, s.th on overlapping charts $(U, x^i), (\bar{U}, \bar{x}^i)$, $U \cap \bar{U} \neq \{0\}$ they are smoothly related: $\bar{x}^i = F^i(x^j),\ F^i \in C^\infty(\mathbb{R}^n, \mathbb{R}), i = 1, ...n$. Smoothness of functions $f : M \to \mathbb{R}$ is defined 'chartwise', likewise smoothness of maps between more general manifolds.

$(\mathbb{T}, \gamma)$, viewed as an affine space, is a manifold, namely Minkowski spacetime, the realm of Special Relativity. Lorentzian manifolds, see later, are the realm of General Relativity.

Let $p \in M$. The tangent space $T_p(M)$ at p can be defined as the vector space of derivations (see below) acting on smooth functions defined near p. Elements $v \in T_p(M)$ can be shown to be the same, in local coordinates (U, x^i) with $p \in M$, as directional derivatives, i.e. $v(f) = v^i \frac{\partial f}{\partial x^i}|_p$. Thus the 'coordinate vectors' $\frac{\partial}{\partial x^i}|_p = \partial_i|_p$ form a basis of $T_p(M)$. It follows that, under a change of chart:
$\bar{v}^i = (\partial_j \bar{x}^i) v^j$.

Example for tangent vector: a smooth curve $\gamma : I \to M$ with $\gamma(0) = p$ gives an element in $T_p(M)$ via its tangent vector defined by $\gamma'(0)(f) = \frac{d}{dt} f \circ \gamma(t)|_0$. By the chain rule $\gamma'(0) = \frac{dx^i}{dt}(0)\partial_i|_p$. All tangent vectors can be gotten in this way.

A vector field v on M is a smooth assignment to each $p \in M$ of a vector $v_p \in T_p(M)$. Or, v maps $C^\infty(M)$ into itself subject to

- $v(af + bg) = av(f) + bv(g) \qquad (a, b \in \mathbb{R}, f, g \in C^\infty(M))$
- $v(fg) = fv(g) + gv(f)$ 'Leibniz rule'

Here $v(f)(p) = v_p(f)$. The set of smooth vector fields is denoted by $\mathfrak{X}(M)$. It is a module over $C^\infty(M)$, addition and scalar multiplication being defined in the obvious way. In local coordinates $v \in \mathfrak{X}(M)$ can be written as $v = v^i(x)\partial_i$ or $v(f) = v^i(x)\partial_i f$.

Thus

$$\bar{v}^i(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j}(x(\bar{x}))v^j(x(\bar{x})) \qquad (*)$$

Given $v, w \in \mathfrak{X}(M)$, the map $f \in C^\infty(M) \mapsto v(w(f))$ does not define a vector field, but the Lie bracket

$$[v, w] = vw - wv = (v^j \partial_j w^i - w^j \partial_j v^i)\partial_i$$

does. Note $[\partial_i, \partial_j] = 0$.

Jacobi identity: $[v, [w, z]] + [z, [v, w]] + [w, [z, v]] = 0$

Covectors: a covector at p is an element ω_p of $T_p^*(M)$. A covector field or 1-form ω is defined in the obvious way. It is smooth if $\omega(v)(p) = \omega_p(v_p)$ is smooth for all $v \in \mathfrak{X}(M)$. Let $f \in C^\infty(M)$. The 1-form df is defined by $df(v) = v(f)$. In particular $dx^i(\partial_j) = \delta^i{}_j$...dual basis. $\omega = \omega_i(x)dx^i$, where $\omega_i = \omega(\partial_i)$.
E.g. $df = \partial_i f\, dx^i$.
Under change of chart: $\omega_i(x) = \frac{\partial \bar{x}^j}{\partial x^i}(x)\bar{\omega}_j(\bar{x}(x))$.
Higher order tensors (tensor fields) are defined in the obvious way, e.g. the $(1,1)$-tensor $t = t^i{}_j\, \partial_i \otimes dx^j$. Contraction, in a basis, is by summation over a pair of up-and downstairs indices.

The operation d sending functions to 1-forms is a special case of an operation d, sending p-forms, i.e. covariant, totally antisymmetric tensors $\omega_{i_1 \ldots i_p}, p < n$ into $p+1$- forms. E.g. when $p = 1$ we define (check this is a 2-tensor!)

$$d\omega(u, v) = u(\omega(v)) - v(\omega(u)) - \omega([u, v])$$

i.e. $d\omega_{ij} = \partial_i \omega_j - \partial_j \omega_i$. We have that $ddf = 0$, and $d\omega = 0$ implies $\omega = df$ when M is simply connected.

Action under flows

Let $\Phi : M \to N$ be a diffeomorphism, i.e. a (smooth) mapping with smooth inverse. The push-forward $\Phi_* v \in \mathfrak{X}(N)$ of $v \in \mathfrak{X}(M)$ is defined by ($f \in C^\infty(N)$)

$$(\Phi_* v)(f)(p) = v(f \circ \Phi)(\Phi^{-1}(p))$$

Locally $y^A = \phi^A(x^i)$ and

$$(\Phi_* v)^A(y) = \frac{\partial \phi^A}{\partial x^j}(\phi^{-1}(y))v^j(\phi^{-1}(y))$$

Next let $\Phi : M \to N$ be smooth and ω a 1-form on N. Then the pull-back $\phi^* \omega$ on M is defined as $(\Phi^* \omega)(v)(p) = \omega(\Phi_* v(\Phi(p))$. (Note: does not require Φ invertible.)

In coordinates

$$(\Phi^*\omega)_i(x) = \frac{\partial\phi^A}{\partial x^i}(x)\,\omega_A(\phi(x))$$

Pull-back on higher covariant tensor fields analogous. Pull-back $\Phi^* f \in C^\infty(N)$ of functions $f \in C^\infty(N)$ is simply $\Phi^* f = f \circ \Phi$. For mixed tensors, say on N, their pull-back to M is defined by 'pull-back under Φ w.r. to the downstairs indices' and 'pull-back under Φ^{-1} w.r. to the upstairs indices'.

Vector fields define a local(-in-t) 1-parameter family Ψ_t of maps $M \to M$ via their flow, i.e.

$$\frac{d\psi_t^i}{dt} = v^i(\psi_t), \qquad \Psi_0(p) = \mathrm{id}$$

The Ψ_t's are local diffeomorphisms of M into itself in that they map small neighbourhoods of each $p \in M$ diffeomorphically onto their image. This is enough in order for the *Lie derivative* of w w.r. to v, i.e. $L_v w = \frac{d}{dt}|_{t=0}(\Psi_{-t})_* w$ to be defined. It turns out that $L_v w = [v, w]$ or

$$(L_v w)^i = v^j \partial_j w^i - w^j \partial_j v^i$$

Next $L_v \omega$ is defined by $L_v\, \omega = \frac{d}{dt}|_{t=0}(\Psi_t)^* \omega$. It turns out that, locally,

$$(L_v \omega)_i = v^j \partial_j \omega_i + \omega_j \partial_i v^j, \qquad L_v f = v(f) = v^i \partial_i f$$

Similarly, for a 2-tensor g_{ij},

$$(L_v g)_{ij} = v^k \partial_k g_{ij} + g_{ik}\, \partial_j v^k + g_{kj}\, \partial_i v^k$$

Geometrically, the equation $L_v t = 0$ means that the structure defined by the object t is invariant under the flow generated by v. E.g. for g_{ij} a symmetric tensor of Riemannian or Lorentz signature the *Killing vector field* v satisfying $L_v g = 0$ generates a flow leaving the Riemannian (Lorentzian) structure invariant.

The operations d and L_v are 'natural' in that they, appropriately, commute with general diffeomorphisms. This means they require no structure an M. In contrast, ∇_v, defined presently, is not natural.

Pseudo-Riemannian manifolds

A manifold is called pseudo-Riemannian if it is provided with a symmetric $(0,2)$- tensor field $g = g_{ij} dx^i \otimes dx^j$ with $g_{ij} = g(\partial_i, \partial_j) = g_{ji}$ non-degenerate. It is called Riemannian if g is furthermore positive definite and Lorentzian if it has Lorentzian signature at each $p \in M$. Note that, e.g. in the Lorentzian case, there is in general no chart near $p \in M$, for which $g_{ij}(x) = \eta_{ij} = \mathrm{const}$. This phenomenon is related to the presence of curvature.

Linear connections

A *linear connection* ∇ on M is an $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$(u, v) \mapsto \nabla_u v$ with ($f \in C^\infty(M)$)

- $\nabla_{fu} v = f \nabla_u v$
- $\nabla_u fv = u(f)v + f \nabla_u v$

So $\nabla_u v$ is tensorial w.r. to u, i.e. defines a $(1, 1)$ tensor. In local coordinates (x^i), $\nabla_u v = u^j (\nabla_j v^i) \partial_i$, where

$$\nabla_i v^j = v^j{}_{;i} = \partial_i v^j + \Gamma^j_{ik} v^k \,, \qquad \nabla_{\partial_j} \partial_k = \Gamma^i_{jk} \partial_i$$

Note $\nabla_{\Phi_* u} \Phi_* v \neq \Phi_* \nabla_u v$ except if Φ leaves connection invariant.

∇ can be naturally extended to act on tensor fields as follows: $\nabla_u f := u(f)$ for $f \in C^\infty(M)$. Then, for 1-forms ω, require that $\nabla_u(v(\omega)) = (\nabla_u v)(\omega) + u(\nabla_u \omega)$, finally that ∇ satisfy the Leibniz rule w.r. to $\otimes$ and be linear under addition. E.g.

$$(\nabla_u t)^j{}_i \partial_j \otimes dx^i = u^k \underbrace{(\partial_k t^j{}_i + \Gamma^j_{kl} t^l{}_i - \Gamma^l_{ki} t^j{}_l)}_{\nabla_k t^j{}_i} \partial_j \otimes dx^i$$

∇ is *symmetric* (torsion-free) if $[u, v] = \nabla_u v - \nabla_v u$ for all $u, v \in C^\infty(M)$. This in a local chart means that $\Gamma^i_{jk} = \Gamma^i_{kj}$.
Let $C^i_{jk} = C^i_{kj}$ be a globally defined (1,2)-tensor field. Then, given ∇, $\nabla'_u v = \nabla_u v + C^i_{jk} u^j v^k \partial_i$ is also a symmetric connection.

∇ on pseudo-Riemannian manifold (M, g) is called *metric* when

$$\nabla_u \, g(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$$

Given g, there $\exists$ unique symmetric linear ('Levy-Civita')connection which is metric. It is given by

$$\Gamma^i_{jk} = \frac{1}{2} \, g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk})$$

Henceforth ∇ will be Levy-Civita.

Geodesics

Let $\gamma : I \to M$ be a smooth curve on M and v a vector field along γ. The covariant derivative of v along γ is defined as

$$\frac{Dv}{Dt} = \nabla_{\gamma'} v = \left(\frac{dv^i}{dt} + \Gamma^i_{jk} \frac{dx^j}{dt} v^k \right) \partial_i$$

The curve $t \mapsto \gamma(t)$ is *geodesic* when $\frac{D\gamma'}{Dt}$ is zero, i.e.

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

Prop.: Given $p \in M$ and $v \in T_p(M)$, there $\exists$ interval I about $t = 0$ and a unique geodesic $\gamma : I \to M$, s.th. $\gamma(0) = p$ and $\gamma'(0) = v$.

Because of

$$\frac{d\,g(u,v)}{dt} = g(\frac{Du}{Dt}, v) + g(u, \frac{Dv}{Dt})$$

the causal character of the geodesic, in the Lorentzian case, is preserved. Timelike geodesics model freely falling pointlike bodies, null geodesics play the role of light rays.

Curvature

Curvature, i.e. deviation from flatness, can be measured by the degree of non-commutativity of ∇ acting on tensor fields, which in turn is measured by the Riemann tensor. The basic observation is the

Prop.: The vector field given by ($u, v, w \in \mathfrak{X}(M)$)

$$R(u,v)w := \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

is tensorial in (u, v, w), i.e. defines a $(1,3)$- tensor field ${R_{ij}}^k{}_l$:

$${R_{ij}}^k{}_l = 2\,\partial_{[i}\Gamma^k_{j]l} + 2\,\Gamma^k_{m[i}\Gamma^m_{j]l}$$

$R_{ijkl} = g_{km} R_{ij}{}^{m}{}_{l}$ can be shown to have the following symmetries

- $R_{ijkl} = -R_{jikl} = -R_{ijlk}$
- $R_{ijkl} + R_{kijl} + R_{jkil} = 0$ '1st Bianchi identity'
- $R_{ijkl} = R_{klij}$

Here the last property follows from the other ones. For $n = 4$ the number of algebraically independent components of R_{ijkl} is 20. Furthermore there is the following ('2nd Bianchi') differential identity

$$\nabla_i R_{jklm} + \nabla_k R_{ijlm} + \nabla_j R_{kilm} = 0$$

One can infer the Bianchi identities from the equivariance (see Kazdan, 1981)

$$R_{ijkl}[\Phi^* g] = (\Phi^* R)_{ijkl}[g]$$

where Φ is a diffeomorphism $M \to M$.

The identities fulfilled by R_{ijkl} imply that the *Ricci tensor* $R_{ij} = R_{ki}{}^{k}{}_{j}$ satisfies $R_{ij} = R_{ji}$. Furthermore the *Einstein tensor* $G_{ij} = R_{ij} - \frac{1}{2} g_{ij}\, g^{kl} R_{kl}$ is divergence-free, i.e.

$$g^{ij} \nabla_i G_{jk} = 0$$