Lectures on Schramm–Loewner Evolution

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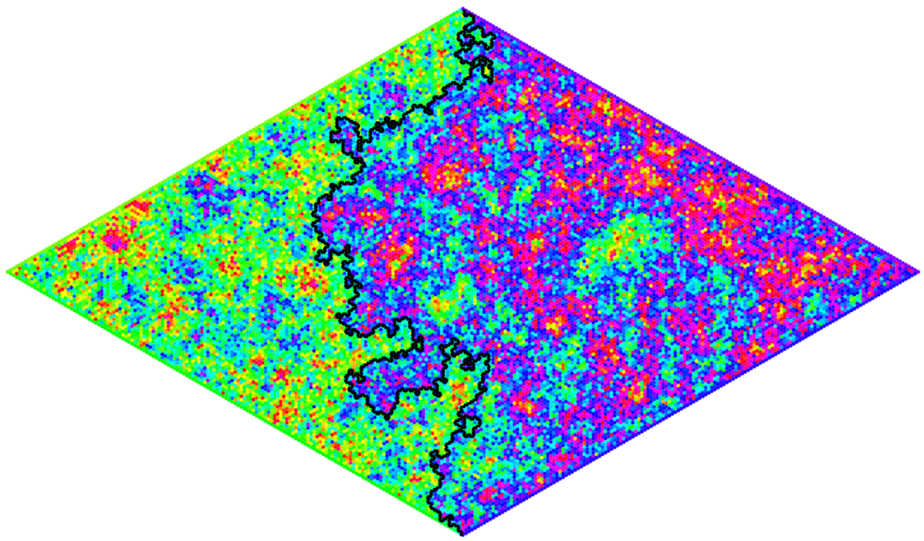

These notes are based on courses given to Masters students in Cambridge and Vienna. Their scope is the basic theory of Schramm–Loewner evolution, together with some underlying and related theory for conformal maps and complex Brownian motion. The structure of the notes is influenced by our attempt to make the material accessible to students having a working knowledge of basic martingale theory and Itô calculus, whilst keeping the prerequisities from complex analysis to a minimum.

Contents

1 Planar Brownian motion

1.1 Harmonic functions

A region $D \subset \mathbb{C}$ is called a *domain* if it is nonempty, open and connected. A real-valued function u defined on a domain $D \subseteq \mathbb{C}$ is *harmonic* if u is twice continuously differentiable on D with

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0$$

everywhere on D.

Harmonic functions are closely connected to Brownian motion. For instance, subject to continuity up to the boundary, harmonic functions are entirely determined by their boundary values; furthermore the unique harmonic function associated to given boundary values can be computed using a Brownian motion on $\mathbb{R}^2$, namely $B_t = (B_t^1, B_t^2)_{t \geqslant 0}$, where B^1, B^2 are independent real-valued Brownian motions.

Theorem 1.1 (Kakutani's formula)**.** *Let u be a bounded harmonic function defined on a domain D and having a continuous extension to the closure $\bar{D}$. Fix $z \in D$ and let $(B_t)_{t \geqslant 0}$ be a complex Brownian motion starting from z. Set $T(D) = \inf\{t \geqslant 0 : B_t \notin D\}$, and suppose that D is regular, i.e., for $z \in \partial D$, $\mathbb{P}_z(T(D) = 0) = 1$. Then*

$$u(z) = \mathbb{E}_z(u(B_{T(D)})).$$

Proof. Suppose for now that u is the restriction to D of a C^2 function on $\mathbb{C}$. Denote this function also by u. Define $(M_t)_{t \geqslant 0}$ by the Itô integral

$$M_t = u(z) + \int_0^t \nabla u(B_s) \cdot dB_s.$$

Then $(M_t)_{t \geqslant 0}$ is a continuous local martingale. By Itô's formula, $u(B_t) = M_t$ for all $t \leqslant T$. Hence the stopped process $M^T = (M_{T \wedge t})_{t \geqslant 0}$ is uniformly bounded and, by optional stopping,

$$u(z) = M_0 = \mathbb{E}_z(M_T) = \mathbb{E}_z(u(B_{T(D)})).$$

For each $n \in \mathbb{N}$, the restriction of u to $D_n = \{z \in D : \operatorname{dist}(z, \partial D) > 1/n\}$ has a C^2 extension to $\mathbb{C}$, regardless of whether u itself does. The preceding argument then shows that $u(z) = \mathbb{E}_z(u(B_{T(D_n)}))$ for all $z \in D_n$. Now $T(D_n) \uparrow T(D) < \infty$ as $n \to \infty$ almost surely (since D is bounded). Since B is continuous and u extends continuously to $\bar{D}$, we obtain the desired identity by bounded convergence on letting $n \to \infty$. $\square$

In fact, the validity of Kakutani's formula, even just in the special case where D is a disc centred at z, turns out to be a useful characterization of harmonic functions.

Proposition 1.2. *Let D be a domain, and let $u : D \to \mathbb{R}$ be a harmonic function on D. Then u satisfies the following* circle average property*: for all $z \in D$ and any $r \in (0, d(z, \partial D))$, we have*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta. \tag{1}$$

Conversely, suppose that $u : D \to [0, \infty]$ is a measurable function satisfying the circle average property. Then, either $u(z) = \infty$ for all $z \in D$, or u is harmonic.

Proof. Let $z \in D$ and let $0 < r < d(z, \partial D)$. Set $U = B(z, r)$. Then u is harmonic on U with a continuous extension to $\bar{U}$. By Kakutani's formula,

$$u(z) = \mathbb{E}_z(u(B_{T(U)})),$$

where $T(U) = \inf\{t > 0 : B_t \notin U\}$ is the first time B leaves U. However, rotational invariance of Brownian motion implies that the law of $B_{T(U)}$ is uniformly distributed on ∂U, from which (1) follows.

Conversely, suppose that u is not identically infinite. Let $z \in D$ such that $u(z) < \infty$. Then for any $0 < r < d(z, \partial D)$, by (1), we see that $u(w) < \infty$ almost everywhere on $\partial B(z, r)$ and thus (by Fubini's theorem), almost everywhere on $B(z, r)$. Let us show that u coincides with a harmonic function on $B(z, r/2)$. (The result then follows by applying the same argument inductively for all points in $B(z, r/2)$, eventually covering all of D by connectedness: more precisely, if x is any other point in D, we can find by path connectedness of D a path connecting x to z, and can cover this path with a finite number of balls of some fixed radius $r/2$ such that the corresponding ball of radius r is contained in D, showing that the function u coincides with a harmonic function at x.) Fix a radially symmetric smooth function $\phi : \mathbb{R}^2 \to \mathbb{R}$ (where smooth means infinitely differentiable in the sense of real-valued functions), such that $\operatorname{supp} \phi \subset B(0, r/2)$, $\phi \geqslant 0$ and $\int_{\mathbb{R}^2} \phi = 1$. Then, since ϕ is radially symmetric and u satisfies the circle average property, $u(x) = u \star \phi(x) = \int_{\mathbb{R}^2} u(y)\phi(x - y)dy$ for all $x \in B(z, r/2)$. As ϕ is smooth and u is measurable, the function $u \star \phi$ is therefore smooth (infinitely differentiable) in $B(z, r/2)$. Thus u is infinitely differentiable in $B(z, r/2)$. We may therefore do a Taylor expansion of u at order 2 near z, and find

$$u(y) = u(z) + \nabla u(z) \cdot (y - z) + \frac{1}{2}\Delta u(z)(z - y)^2 + \sum_{1 \leqslant i \neq j \leqslant 2} \frac{\partial^2 u}{\partial x_i \partial x_j}(y_i - z_i)(y_j - z_j) + o(z - y)^2,$$

where the term $o(z - y)^2$ is uniform in $B(z, r/2)$, say. Let us integrate this identity over $\partial B(z, \varepsilon)$ and use the circle average property. The term of order 1

$$\frac{1}{2\pi\varepsilon} \int_{\partial B(z,\varepsilon)} \nabla u(z) \cdot (y - z) dy = \frac{1}{2\pi\varepsilon} \nabla u(z) \cdot \int_{\partial B(z,\varepsilon)} (y - z) dy$$

vanishes by symmetry. Likewise, for $1 \leqslant i \neq j \leqslant 2$, the integral of the term containing the cross derivatives vanishes, because

$$\int_{\partial B(z,\varepsilon)} (y_i - z_i)(y_j - z_j) dy = 0$$

(consider the reflection of one coordinate, which preserves the uniform distribution on the circle). We deduce that

$$u(z) = u(z) + \frac{1}{2}\varepsilon^2 \Delta u(z) + o(\varepsilon^2).$$

The only possibility is therefore that $\Delta u(z) = 0$. This concludes the proof. $\square$

Lecture 2: Friday 10 March

Kakutani's formula implies immediately that a harmonic function u on a bounded domain D, which extends continuously to $\bar{D}$, cannot exceed the supremum of its values on the boundary ∂D. Moreover, as we now show, a harmonic function cannot achieve a maximum value on its domain, unless it is constant.

Theorem 1.3 (Maximum principle)**.** *Let u be a harmonic function defined on a domain D. Suppose there exists a point $z \in D$ such that $u(w) \leqslant u(z)$ for all $w \in D$. Then u is constant.*

Proof. It will suffice to consider the case where u has a finite supremum value m, say, on D. Consider the set $D_0 = \{z \in D : u(z) = m\}$. Then D_0 is relatively closed in D, since u is continuous. On the other hand, if $z \in D_0$, then for $\varepsilon > 0$ sufficiently small, the disc $B(z, \varepsilon)$ of radius ε and centre z is contained in D. So, by Kakutani's formula

$$m = u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + \varepsilon e^{i\theta}) d\theta.$$

Since u is continuous and bounded above by m, this implies that $w \in D_0$ whenever $|w - z| = \varepsilon$. Hence D_0 is open. Since D is connected, D_0 can only be non-empty if it is the whole of D. $\square$

As a corollary we obtain:

Corollary 1.4. *Suppose u is a harmonic function on a domain D. Then*

$$\sup_{z \in D} u(z) = \lim_{r \to 0} \sup_{z : \mathrm{dist}(z, \partial D) = r} u(z).$$

Proof. Suppose without loss of generality that u is nonconstant, otherwise the result is trivial. Let $D_r = \{z : \mathrm{dist}(z, \partial D) > r\}$. Then u is harmonic on D_r with a continuous extension to $\bar{D}_r$. By the maximum principle (Theorem 1.3), or directly from Kakutani's formula,

$$f(r) := \sup_{z \in \bar{D}_r} u(z) = \max_{z \in \partial D_r} u(z).$$

The left hand side is clearly increasing as r decreases to zero, so the right hand side has a limit as $r \to 0$, let us call it ℓ. We need to show that $\ell = \sup u$. Clearly, $\ell \leqslant \sup u$ from the definition of ℓ. Let $\varepsilon > 0$ and find a point z such that $u(z) \geqslant m - \varepsilon$, where $m = \sup u$. As $z \in D$, $\mathrm{dist}(z, \partial D) > 0$ and we can find $r > 0$ such that $z \in D_r$. Thus $m - \varepsilon \leqslant u(z) \leqslant f(r) \leqslant \ell$. As $\varepsilon > 0$ is arbitrary, we deduce $\ell = m$, as desired. $\square$

1.2 Conformal Invariance of Brownian motion

Let D be a domain and let $f : D \to \mathbb{C}$. We say that f is **holomorphic** (or **analytic**) if, for all $z_0 \in D$, f is differentiable (in the complex sense) at the point z_0:

$$\lim_{|z|\to 0} \frac{f(z_0+z)-f(z_0)}{z}$$

exists, in which case we call it $f'(z_0)$. The notion of holomorphicity has a geometric flavour which should be kept in mind. Namely, using a Taylor expansion, for $z \in \mathbb{C}$ with $|z|$ small,

$$f(z_0+z) \approx f(z_0) + zf'(z_0). \tag{2}$$

Suppose $f'(z_0) \neq 0$. If we write the complex number $f'(z_0)$ as $f'(z_0) = re^{i\theta}$, then (2) says that f behaves, locally around the point z_0, approximately as a dilation (by a factor r) and a rotation (with angle θ). In particular, the image of a circle of radius ε around z_0 is approximately a circle of radius $r\varepsilon$ around $f(z_0)$. For the same reason the image by f of two curves which meet at z_0 with an angle θ will be two curves meeting at $f(z_0)$, with the *same* angle. We say that f is **conformal** (hence a **conformal map** over a domain D means a holomorophic function f defined on D such that $f'(z) \neq 0$ for all $z \in D$).

If f is a holomorphic function on D, and $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$, then u, v satisfy the Cauchy–Riemann equations:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases}$$

Differentiating a second time (which turns out to be possible under the assumption that f is holomorphic), we see that $\Delta u = \Delta v = 0$, so u and v are both harmonic on D.

Hence, if f is a bounded holomorphic on a domain D and extends continuously to $\bar{D}$, then f may be recovered from its boundary values, just as in Kakutani's formula: for all $z \in D$

$$f(z) = \mathbb{E}_z(f(B_{T(D)})) \tag{3}$$

and we have the estimate

$$|f(z)| \leqslant \sup_{w\in\partial D} |f(w)|.$$

Then a small variation on the argument for the maximum principle leads to the following result.

Theorem 1.5 (Maximum modulus principle)**.** *Let f be a holomorphic function defined on a domain D. Suppose there exists a point $z \in D$ such that $|f(w)| \leqslant |f(z)|$ for all $w \in D$. Then f is constant.*

A fundamental property of planar Brownian motion is that it is invariant under **conformal isomorphisms** (i.e., one-to-one holomorphic transformations), up to a change of time. This property is called "conformal invariance of Brownian motion".

Theorem 1.6. *Let D and D' be domains and let $\phi : D \to D' = \phi(D)$ be a conformal isomorphism. Fix $z \in D$ and set $z' = \phi(z)$. Let $(B_t)_{t\geqslant 0}$ and $(B'_t)_{t\geqslant 0}$ be complex Brownian motions starting from z and z' respectively. Set*

$$T = \inf\{t \geqslant 0 : B_t \notin D\}, \quad T' = \inf\{t \geqslant 0 : B'_t \notin D'\}.$$

Set $\tilde{T} = \int_0^T |\phi'(B_t)|^2 dt$ and define for $t < \tilde{T}$

$$\tau(t) = \inf\left\{ s \geqslant 0 : \int_0^s |\phi'(B_r)|^2 dr = t \right\}, \quad \tilde{B}_t = \phi(B_{\tau(t)}).$$

Then $(\tilde{T}, (\tilde{B}_t)_{t<\tilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution.

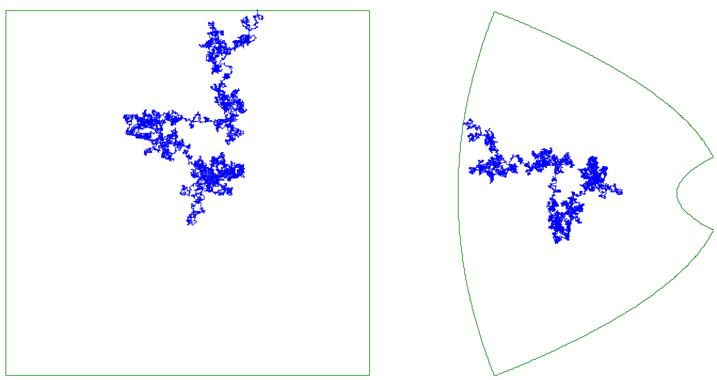

Figure 1: A Brownian motion stopped upon leaving the unit square, and its image under a conformal isomorphism.

Proof. Assume for now that D is bounded and ϕ has a C^1 extension to $\bar{D}$. Then $T < \infty$ almost surely and we may define a continuous semimartingale[1] Z and a continuous adapted process A by setting

$$Z_t = \phi(B_{T\wedge t}) + (B_t - B_{T\wedge t}), \quad A_t = \int_0^{T\wedge t} |\phi'(B_s)|^2 ds + (t - (T \wedge t)).$$

Moreover, almost surely, A is a homeomorphism of $[0, \infty)$, whose inverse is an extension of τ. Denote the inverse homeomorphism also by τ. Write $\phi = u + iv$, $B_t = X_t + iY_t$ and $Z_t = M_t + iN_t$. By Itô's formula, for $t < T$,

$$dM_t = \frac{\partial u}{\partial x}(B_t)dX_t + \frac{\partial u}{\partial y}(B_t)dY_t, \quad dN_t = \frac{\partial v}{\partial x}(B_t)dX_t + \frac{\partial v}{\partial y}(B_t)dY_t$$

[1]Here and below, where we use notions depending on a choice of filtration, such as martingale or stopping time, unless otherwise stated, these are to be understood with respect to the natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$ of $(B_t)_{t\geqslant 0}$.

and so, using the Cauchy–Riemann equations,

$$[dM_t, dM_t] = |\phi'(B_t)|^2 dt = dA_t = [dN_t, dN_t], \quad [dM_t, dN_t] = 0.$$

On the other hand, for $t \geqslant T$,

$$dM_t = dX_t, \quad dN_t = dY_t, \quad [dM_t, dM_t] = dt = dA_t = [dN_t, dN_t], \quad [dM_t, dN_t] = 0.$$

Hence $(M_t)_{t\geqslant 0}$, $(N_t)_{t\geqslant 0}$, $(M_t^2 - A_t)_{t\geqslant 0}$, $(N_t^2 - A_t)_{t\geqslant 0}$ and $(M_t N_t)_{t\geqslant 0}$ are all continuous local martingales. Set $\tilde{M}_s = M_{\tau(s)}$ and $\tilde{N}_s = N_{\tau(s)}$. Then, by optional stopping, $(\tilde{M}_s)_{s\geqslant 0}$, $(\tilde{N}_s)_{s\geqslant 0}$, $(\tilde{M}_s^2 - s)_{s\geqslant 0}$, $(\tilde{N}_s^2 - s)_{s\geqslant 0}$ and $(\tilde{M}_s\tilde{N}_s)_{s\geqslant 0}$ are continuous local martingales for the filtration $(\tilde{\mathcal{F}}_s)_{s\geqslant 0}$, where $\tilde{\mathcal{F}}_s = \mathcal{F}_{\tau(s)}$. Define $(\tilde{Z}_s)_{s\geqslant 0}$ by $\tilde{Z}_s = \tilde{M}_s + i\tilde{N}_s$. Then, by Lévy's characterization of Brownian motion, $(\tilde{Z}_s)_{s\geqslant 0}$ is a complex $(\tilde{\mathcal{F}}_s)_{s\geqslant 0}$-Brownian motion starting from $z' = \phi(z)$. Now $\tilde{B}_t = \tilde{Z}_t$ for $t < \tilde{T}$ and, since ϕ is a bijection, $\tilde{T} = \inf\{t \geqslant 0 : \tilde{Z}_t \notin D'\}$. So we have shown the claimed identity of distributions.

In the cases where D is not bounded or ϕ fails to have a C^1 extension to $\bar{D}$, choose a sequence of bounded open sets $D_n \uparrow D$ with $\bar{D}_n \subseteq D$ for all n. Set $D_n' = \phi(D_n)$ and set

$$T_n = \inf\{t \geqslant 0 : B_t \notin D_n\}, \quad T_n' = \inf\{t \geqslant 0 : B_t' \notin D_n'\}.$$

Set $\tilde{T}_n = \int_0^{T_n} |\phi'(B_t)|^2 dt$. Then $\tilde{T}_n \uparrow \tilde{T}$ and $T_n' \uparrow T'$ almost surely as $n \to \infty$. Since ϕ is C^1 on $\bar{D}_n$, we know that $(\tilde{T}_n, (\tilde{B}_t)_{t<\tilde{T}_n})$ and $(T_n', (B_t')_{t<T_n'})$ have the same distribution for all n, which implies the desired result on letting $n \to \infty$. $\square$

Note that $\phi(B)$ would be a time-change of Brownian motion even if ϕ was not assumed to be one-to-one. However, in that case, stopping B at the time it leaves D might not correspond to stopping $\phi(B)$ at the time it leaves $\phi(D)$ (i.e., it would not necessarily be true that $\tilde{T} = \inf\{t \geqslant 0 : \tilde{Z}_t \notin D'\}$ in the above proof).

1.3 Harmonic measure

Let D be a proper domain of $\mathbb{C}$. Given a point $z_0 \in D$, and a Brownian motion B starting from z_0, let $T_D = \inf\{t > 0 : B_t \notin D\}$ be the first time that B leaves D. Suppose that $T_D < \infty$ almost surely. (It is not hard to see this property depends only on D, and not on z_0).

The random variable B_{T_D} is an element of ∂D almost surely. By definition, the **harmonic measure** in D viewed from z_0, is the law of B_{T_D}.

Example 1.7. *The harmonic measure in the unit disc $D = \mathbb{D}$, viewed from 0, is the uniform distribution on the unit circle $\partial\mathbb{D}$.*

Proof. We start with the observation that Brownian motion is rotationally invariant: that is, if B is a Brownian motion, then $e^{i\theta}B_t$ is also a Brownian motion. Consequently, if μ denote the harmonic measure in $\mathbb{D}$ viewed from zero, then μ is also rotationally invariant: that is, $\mu(I) = \mu(e^{i\theta}I)$ for any circular arc $I \subset \partial\mathbb{D}$. If $F(\theta) = \mu(I_\theta)$, where I_θ is the circular arc between 1 and $e^{i\theta}$ for $\theta \in (0, 2\pi)$, then $F(\theta + \theta') = F(\theta) + F(\theta')$ whenever $0 \leqslant \theta + \theta' \leqslant 2\pi$. From this and the right continuity of F it follows that F is linear, and the result follows. $\square$

Before our next example it will be useful to review some properties of the Cauchy distribution.

Definition 1.8. *A random variable C with values in $\mathbb{R}$ has the* ***Cauchy distribution****, if it has a density with respect to Lebesgue measure given by*

$$f_C(x) = \frac{1}{\pi(1+x^2)}; x \in \mathbb{R}. \tag{4}$$

Note that $f_C(x) = f_C(-x)$, i.e., the law of C is symmetric about zero (C and $-C$ have the same law). The expectation of C is however not well defined. In order to prepare for our next example, let us compute the Fourier transform of C.

Lemma 1.9. *Let C have the Cauchy distribution. Then for any $t \in \mathbb{R}$, $\mathbb{E}(e^{itC}) = e^{-|t|}$.*

Proof. Observe that

$$\begin{aligned}\mathbb{E}(e^{itC}) &= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} dx \\ &= \lim_{R\to\infty} \int_{-R}^{R} e^{itx} \frac{1}{\pi(1+x^2)} dx, .\end{aligned}$$

by the dominated convergence theorem. Suppose without loss of generality that $t \geqslant 0$. The integral in the right hand side can be computed using the residue theorem: let γ_R denote the closed contour consisting of the interval from $-R$ to R, and the upper-half circle of radius R. Consider the holomorphic function $f(z) = e^{itz}/[\pi(1+z^2)]$. This has a pole of order 1 at $z = i$: more precisely,

$$f(z) = \frac{e^{itz}}{\pi(z+i)(z-i)}.$$

Hence

$$\oint_{\gamma_R} f(z)dz = 2i\pi \mathrm{Res}(f)|_{z=i} = 2\pi i \times \frac{e^{-t}}{2i\pi} = e^{-t}$$

However it is easy to check that the contribution of the upper half circle tends to zero as $R \to \infty$, hence $\mathbb{E}(e^{itC}) = e^{-t}$ for $t \geqslant 0$. By symmetry this concludes the proof of the lemma. $\square$

Lecture 3: Friday 17 March

Example 1.10. *Let $D = \mathbb{H}$ be the upper half plane, and let $z_0 = x_0 + iy_0 \in \mathbb{H}$. Then the harmonic measure in $\mathbb{H}$ viewed from z_0, is the measure on $\mathbb{R}$ with density with respect to Lebesgue measure given by*

$$h_{\mathbb{H}}(x) = \frac{y_0}{\pi}\frac{1}{(x-x_0)^2 + y_0^2}$$

In other words, μ has the law of a $x_0 + y_0 C$, where C has the Cauchy distribution.

Proof. There are several ways to see this, but we opt for the following simple argument. By translation and scale invariance, we suppose without loss of generality that $z_0 = i$, in which case we aim to show that the harmonic measure μ is the Cauchy distribution. We do so by identifying the Fourier transforms of the two distributions and invoking the Fourier inversion theorem (which says that two measures with the same Fourier transform must be identical).

We have already computed the Fourier transform of the Cauchy distribution. Let us compute that of the harmonic measure in $\mathbb{H}$ viewed from $z_0 = i$. We do so by considering a suitable complex-valued martingale. Let $t \geqslant 0$, and consider the function $f(z) = e^{itz}$, which is clearly holomorphic, and bounded on $\mathbb{H}$ since $t \geqslant 0$. By Kakutani's formula (i.e., by (3)),

$$\mathbb{E}_{z_0}(M_{T(\mathbb{H})}) = \mathbb{E}_{z_0}(M_0) = e^{itz_0} = e^{-t}.$$

On the other hand left hand side is simply $\mathbb{E}_i(e^{itB_{T(\mathbb{H})}})$ which is the desired Fourier transform. By symmetry, we deduce for all $t \in \mathbb{R}$,

$$\mathbb{E}_i(e^{itB_{T(\mathbb{H})}}) = e^{-|t|},$$

so the Fourier transform of the harmonic measure in $\mathbb{H}$ viewed from i is the same as that of the Cauchy distribution (see Lemma 1.9). □

A fundamental fact about harmonic measure is that they are conformally invariant. This is a direct consequence of conformal invariance of Brownian motion, in the following sense.

Theorem 1.11. *Let $f : D \to D'$ be a conformal isomorphism. Suppose that f extends continuously to a continuous homeomorphism from $\bar{D}$ to $\bar{D}'$. Let $z_0 \in D$ and let μ denote the harmonic measure in D viewed from z_0. Let μ' denote the harmonic measure in D' viewed from $z_0' = f(z_0)$. Then*

$$\mu' = \mu \circ f^{-1},$$

i.e., μ' is the image of μ through the map f.

Proof. This is a direct consequence of Theorem 1.6, and the fact that the location of particle when it first leaves a domain is independent of the time-parametrisation of the Brownian curve. □

Example 1.12. *Sometimes a conformal isomorphism between two domains can be found explicitly (we will study this in much more detail soon). For instance, a conformal isomorphism from the upper half plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ is given by*

$$\psi_{\mathbb{H},\mathbb{D}}(z) = \frac{z-i}{z+i}$$

(A justification will be given later). From this, Example 1.7 and Theorem 1.11, we can get another proof of Example 1.10.

1.4 Example: Spitzer's law for winding of Brownian motion

Let $(B_{t\wedge T}, t \geqslant 0)$ denote a Brownian motion, started from $z_0 = \varepsilon \in (0,1)$, and stopped at the time $T = T(\mathbb{D})$ when it first leaves the unit disc. By elementary properties of planar Brownian motion, almost surely this path does not hit zero. We may therefore uniquely write B in polar coordinates as

$$B_t = r(t)e^{i\theta(t)}; 0 \leqslant t \leqslant T,$$

where $r(0) = \varepsilon$, $\theta(0) = 0$, and r and θ are *continuous* real-valued functions on $[0,T]$. The value $\theta(T)$ is called the (topological) **winding** of B around zero, and $\lfloor \theta(T)/(2\pi) \rfloor$ represents the net number of full turns that B makes around the origin, counted with a sign (positive for every counterclockwise turn, negative for every clockwise turn).

What can be said about the law of $\theta(T)$ when $\varepsilon \to 0$? Since the Brownian paths starts close to zero, we expect that it may have made many turns. The following (in a slightly different form) is Spitzer's law.

Theorem 1.13. *We have the following identity in law under $\mathbb{P}_\varepsilon$.*

$$\frac{\theta(T)}{\log(1/\varepsilon)} = C,$$

where C has the Cauchy distribution.

Proof. Consider the exponential map $z \in \mathbb{C} \mapsto f(z) = e^z$. This maps the left half of the plane $D = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ to the unit disc. The starting point $z_0 = \varepsilon$ in the unit disc corresponds in D to starting the Brownian motion at $w_0 = -\log 1/\varepsilon$. Let $(W_t)_{t\geqslant 0}$ denote a Brownian motion starting from w_0, so $B_t = f(W_t)$ is a (time-changed) Brownian motion starting from z_0. Furthermore, stopping B when it leaves $\mathbb{D}$ corresponds to stopping W when it leaves D. Moreover,

$$\theta(T) = \mathrm{Im}(W_{T(D)}),$$

where $T(D) = \inf\{t \geqslant 0 : W_t \notin D\}$ is the first time W leaves D. The result follows from Example 1.10. ∎

1.5 Green's function

Let $p_t^{\mathbb{C}}(x,y)$ denote the transition probability of a Brownian motion B in $\mathbb{R}^2$. Thus,

$$p_t^{\mathbb{C}}(x,y) = (2\pi t)^{-d/2} \exp(-|x-y|^2/(2t)), \tag{5}$$

and recall that this is nothing else but the density of the law of B_t (starting from x), with respect to Lebesgue measure on $\mathbb{R}^2$. For $D \subset \mathbb{R}^2$ an open set, we define $p_t^D(x,y)$ to be the transition probability of Brownian motion, killed when leaving D, which is defined as the density, with respect to Lebesgue measure on $\mathbb{R}^2$, of the law of B_t, but restricted to the event $\{\tau_D > t\}$. In other words, by definition, for any Borel set A in $\mathbb{R}^2$,

$$\mathbb{P}_x(B_t \in A, \tau_D > t) = \int_{\mathbb{R}^2} 1_A(y) p_t^D(x,y)\, \mathrm{d}y. \tag{6}$$

The (almost everywhere, for a fixed $t \geqslant 0$ and a fixed $x \in \mathbb{R}^2$) existence of a function satisfying (6) follows directly from the Radon–Nikodym derivative theorem, since it is clear that if A has zero Lebesgue measure, then $\mathbb{P}_x(B_t \in A, \tau_D > t) \leqslant \mathbb{P}_x(B_t \in A) = 0$.

By conditioning on the position at time t of B_t, it is not hard to check that $p_t^D(x,y)$ can be expressed rather simply in terms of the whole space transition probabilities in (5) and the so-called Brownian bridge $(b_s)_{0\leqslant s\leqslant t}$ of duration t from x to y, which describes the law of B, conditionally given $B_0 = x$ and $B_t = y$. Namely, if we denote by $\mathbb{P}_{x\to y;t}$ this law, then we see that

$$\begin{aligned}\mathbb{P}_x(B_t \in A; \tau_D > t) &= \int_{\mathbb{R}^2} \mathbb{P}_{x\to y;t}(b_t \in A; \tau_D > t) p_t^{\mathbb{C}}(x,y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^2} 1_A(y) \mathbb{P}_{x\to y;t}(\tau_D > t) p_t^{\mathbb{C}}(x,y)\, \mathrm{d}y.\end{aligned}$$

Comparing with (6), we deduce that, for every fixed $t \geqslant 0$ and almost every y,

$$p_t^D(x,y) = \pi_t^D(x,y) p_t^{\mathbb{C}}(x,y); \text{ where } \pi_t^D(x,y) = \mathbb{P}_{x\to y;t}(\tau_D > t). \tag{7}$$

The right hand side is easily seen to be a jointly continuous function in $t > 0$ and $x, y \in \bar{D}$, as this is clearly satisfied by both $\pi_t^D(x,y)$ and $p_t^{\mathbb{C}}(x,y)$ separately. For the full plane transition probabilities, this is immediate from the explicit formula. For the term $\pi_t^D(x,y)$, the continuity is provided by the following lemma.

Lemma 1.14. *The quantity $\pi_t^D(x,y)$ is symmetric in $x, y \in D$, i.e.: $\pi_t^D(x,y) = \pi_t^D(y,x)$. Furthermore, it is jointly continuous in $(t > 0, x \in D, y \in D)$.*

Proof. There are various ways to construct the Brownian bridge. In one dimension, a Brownian bridge $(\omega_t)_{0\leqslant t\leqslant 1}$ from 0 to 0 of duration 1 is obtained by setting

$$\omega_t = W_t - tW_1, \quad 0 \leqslant t \leqslant 1$$

where W is a standard one-dimensional Brownian motion. (That this describes the law of W conditioned on $W_1 = 0$ follows from the Gaussian structure of the process W.)

Alternatively, ω can be described as a centered Gaussian process with covariance $\mathbb{E}(\omega_s\omega_t) = s \wedge t - st$. We obtain a Brownian bridge in $\mathbb{R}^2$ from 0 to 0 of unit duration by setting $X_s = (\omega_s^1, \omega_s^2)$ for $0 \leqslant s \leqslant 1$, where ω^1 and ω^2 are independent one-dimensional Brownian bridges of unit duration from 0 to 0. Finally, we obtain the law of a Brownian bridge from x to y of duration t by Brownian scaling and translation: namely,

$$b_s = \sqrt{t} X_{\frac{s}{t}} + (1 - \tfrac{s}{t})x + \tfrac{s}{t}y, \quad 0 \leqslant s \leqslant t. \tag{8}$$

The symmetry is easy to deduce from the fact that $(\omega_s)_{0\leqslant s\leqslant 1}$ and $(\omega_{1-s})_{0\leqslant s\leqslant 1}$ have the same law (i.e., a one-dimensional standard Brownian bridge is reversible). Furthermore, the law $\mathbb{P}_{x,y:t}$ (described by (8)), viewed as a function of $x, y \in \mathbb{R}^d$ and $t > 0$, is clearly jointly continuous in all three parameters (for the Prokhorov metric on path space induced by uniform convergence). The joint continuity of $\pi_t^D(x,y)$ follows from the portmanteau theorem and the observation that there is probability zero for a Brownian bridge to visit $\bar{D}$ but not D^c. □

The continuity of π^D and $p^{\mathbb{C}}$ in all three arguments defines the transition probability function $p_t^D(x,y)$ of Brownian motion killed when leaving D uniquely.

Example 1.15. *If $D = \mathbb{H}$ is the upper half plane, then $p_t^D(x,y) = p_t^{\mathbb{C}}(x,y) - p_t^{\mathbb{C}}(x,\bar{y})$ by the reflection principle.*

Clearly, by the Markov property of Brownian motion, the transition probabilities satisfy the Chapman–Kolmogorov equation:

$$p_{t+s}^D(x,y) = \int_{\mathbb{R}^2} p_t^D(x,z)p_s^D(z,y)\,\mathrm{d}z. \tag{9}$$

Note also immediately for future reference that, by definition of $p_t^D(x,y)$ and the monotone class theorem, if ϕ is any nonnegative Borel function, then

$$\mathbb{E}_x(\phi(B_t)1_{\tau_D>t}) = \int_{\mathbb{R}^2} \phi(y)p_t^D(x,y)\,\mathrm{d}y.$$

Consequently, by Fubini's theorem,

$$\begin{aligned}
\mathbb{E}_x(\int_0^{\tau_D} \phi(B_s)\,\mathrm{d}s) &= \mathbb{E}_x(\int_0^\infty \phi(B_s)1_{\tau_D>s}\,\mathrm{d}s) \\
&= \int_0^\infty \int_{\mathbb{R}^2} \phi(y)p_s^D(x,y)\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_{\mathbb{R}^2} \phi(y)(\int_0^\infty p_s^D(x,y)\,\mathrm{d}s)\,\mathrm{d}y. \qquad (10)
\end{aligned}$$

The time integral, in brackets in the above right hand side, is (up to a factor 1/2) called the Green function:

Definition 1.16 (Continuous Green function)**.** *The Green function $G(x, y) = G_D(x, y)$ is defined by*

$$G_D(x, y) = \frac{1}{2} \int_0^\infty p_t^D(x, y)\, \mathrm{d}t \tag{11}$$

for $x \neq y$ in D.

The factor $1/2$ is for normalisation only. Intuitively, the Green function measures the expected amount of time spent "at" a point y (i.e., near y) before exiting the domain D. Note in particular that, combining our definition of the Green function with (10), we obtain:

$$\mathbb{E}_x\Big(\int_0^{\tau_D} \phi(B_s)\, \mathrm{d}s\Big) = \frac{1}{2} \int_{\mathbb{R}^2} G_D(x, y)\phi(y)\, \mathrm{d}y. \tag{12}$$

This agrees with our intuition that the Green function measures the expected amount of time spent near a point y before leaving the domain D.

Remark 1.17. *Different authors choose to normalise the Green function differently. Our choice is such that (as we will soon see), $G_D(x, y) = -1/(2\pi) \log |x - y| + O(1)$. This is the "natural" choice from an analytic point of view, as G_D is then the integral kernel corresponding to the inverse of (minus) the Laplacian with Dirichlet boundary conditions on D, without further normalising constants.*

On the diagonal it is always the case that $G_D(x, x) = \infty$ because of the nonintegrability of the function $1/t$ near $t = 0$. Even away from the diagonal, the function might not be finite. Consider for instance the case where $D = \mathbb{C}$, or even $D = \mathbb{C} \setminus \{0\}$ (in which case the process never leaves D, so $p_t^D(x, y) = p_t^{\mathbb{C}}(x, y)$, which is not integrable because the function $1/t$ is not integrable either near $t = +\infty$). However, it is not hard to see that if D is *regular*, then $G_D(x, y) < \infty$ for $x \neq y$. Here, regular means that $\partial D \neq \emptyset$, and starting from any point $z \in \partial D$, $\mathbb{P}_z(\tau_D = 0) = 1$, i.e., a Brownian motion is guaranteed to leave D immediately.

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Proposition 1.18. *Let D be regular. Then the Green function G_D satisfies $G_D(x, y) < \infty$ for all $x \neq y \in D$. Furthermore, the estimate:*

$$G_D(x, y) = -\frac{1}{2\pi}(1 + o(1)) \log |x - y|,$$

holds as $y \to x$.

Remark 1.19. *It is possible to obtain more precise estimates on the behaviour of the Green function near the diagonal. See exercises.*

Proof. It is possible to see that because D is a regular domain,

$$p_t^D(x, y) \leqslant ct^{-1}(\log t)^{-2} \tag{13}$$

for some constant $c = c(x, y) > 0$, see Lemma 2.32 in [13] (in fact, the constant c can be chosen uniformly depending only on the distances from x and y to the boundary.

Since this is an integrable function near $t = +\infty$, we see that for some constant $c' > 0$,

$$G_D(x, y) \leqslant c' + \frac{1}{2} \int_0^1 p_t^D(x, y)\,\mathrm{d}t.$$

Furthermore, clearly, $p_t^D(x, y) \leqslant p_t^{\mathbb{C}}(x, y)$, hence, writing $r = |x - y|$, and making a change of variables $u = r^2/t$,

$$\begin{aligned} G_D(x, y) &\leqslant c' + \int_0^1 e^{-r^2/(2t)} \frac{\mathrm{d}t}{4\pi t} \\ &= c' + \frac{1}{4\pi} \int_{r^2}^\infty e^{-u/2} \frac{\mathrm{d}u}{u} \\ &= O(1) - \frac{\log(r^2)}{4\pi} = -\frac{1}{2\pi} \log|x - y| + O(1). \end{aligned}$$

This gives the desired upper bound. For the lower bound, we simply observe that $\pi_t^D(x, y) \to 1$ as $t \to 0$, and truncate the infinite integral to get a lower bound of the form

$$G_D(x, y) \geqslant \frac{1}{2} \int_0^\varepsilon p_t^{\mathbb{C}}(x, y) \pi_t^D(x, y)\,\mathrm{d}t,$$

where ε is arbitrary. We conclude as above. □

Proposition 1.20. *Let D be regular. Then G_D is symmetric: $G_D(x, y) = G_D(y, x)$. Furthermore, suppose we fix $x \in D$. Then $G_D(x, \cdot)$ is harmonic on $D \setminus \{x\}$. Finally, $G_D(x, y)$ converges to 0 as $y \in D$ converges to a point on the boundary of D.*

Proof. The symmetry of G_D is a direct consequence of Lemma 1.14. For the harmonicity, we observe that by symmetry, $G_D(x, y) = u(y) = G_D(y, x)$ for $y \in D^* = D \setminus \{x\}$. Applying the strong Markov property of Brownian motion at the first hitting time of the sphere $\partial B(y, r)$, where $0 < r < \mathrm{dist}(y, \partial D^*)$, we get

$$u(y) = \frac{1}{2\pi} \int_0^{2\pi} u(y + re^{i\theta})\,\mathrm{d}\theta.$$

Thus u satisfies the circle average property on D^* and is thus harmonic by Proposition 1.2. For the final point, we recall that by symmetry, $p_t^D(x, y) = p_t^D(y, x)$. Suppose $y_n \to b \in \partial D$. Then $p_t^D(y_n, x) \to p_t^D(b, x)$ by continuity. However the latter is zero since D is regular. We conclude by an application of the dominated convergence theorem and (13). □

It turns out that these properties characterise the Green function.

Proposition 1.21. *Let D be a regular domain and fix $x \in D$. Suppose that ϕ is harmonic in $D^* = D \setminus \{x\}$, converges to 0 near the boundary, and satisfies $\phi(y) = -1/(2\pi)(1 + o(1) \log|x - y|$ as $y \to x$. Then $\phi(y) = G_D(x, y)$.*

Proof. Let $\varepsilon > 0$. Let B be a Brownian motion starting from some arbitrary $y \in D^*$, and for $\varepsilon > 0$ smaller than $|y - x|$, let

$$T_\varepsilon = \inf\{t > 0 : |B_t - x| \leqslant \varepsilon\}.$$

Then set $\tau_{\varepsilon,D} = \tau_D \wedge T_\varepsilon$. Since $G_D(x, \cdot)$ and $\phi(\cdot)$ are both harmonic functions in D^*, we deduce that

$$M_t := G_D(x, B_{t\wedge\tau_{\varepsilon,D}}) - \phi(B_{t\wedge\tau_{\varepsilon,D}})$$

is a martingale. It is furthermore bounded. We can thus apply the optional stopping theorem:

$$\mathbb{E}_y(M_0) = \mathbb{E}_y(M_{T_\varepsilon\wedge\tau_D}).$$

The left hand side equals $G_D(x, y) - \phi(y)$. It thus suffices to show that the right hand side converges to 0. Since $G_D(x, \cdot)$ and $\phi(\cdot)$ both satisfy the same boundary conditions on ∂D, the only contribution to the right hand side comes from the event $T_\varepsilon < \tau_D$. On that event, by assumption on ϕ and Proposition 1.18, $M_{T_\varepsilon\wedge\tau_D} = o(\log 1/\varepsilon)$. Hence

$$G_D(x, y) - \phi(y) = o(\log 1/\varepsilon)\mathbb{P}_y(T_\varepsilon < \tau_D).$$

We conclude the proof by noting that the probability in the right hand side is at most $O(1/\log(1/\varepsilon))$. $\square$

An immediate corollary of the proof is the following formula:

Corollary 1.22. *For a regular domain D and $z_0 \in D$, we have*

$$G_D(z_0, z) = \frac{1}{2\pi}\left(-\log|z - z_0| + \mathbb{E}_z[\log|B_{T(D)} - z_0|]\right)$$

This follows since the right hand side is clearly harmonic away from z_0, has zero boundary conditions, and blows logarithmically with the right constant as $z \to z_0$.

We now come to a fundamental property of the Green function in two dimensions, which is its invariance under conformal maps. In a sense this is a consequence of the invariance of Brownian motion under conformal maps, up to a change of time. Yet the Green function seems at first sight sensitive to the time-parametrisation of the Brownian curve, since informally it measures the expected time spent near a point before leaving the domain. From that perspective the result below is at first a little surprising.

Theorem 1.23 (Conformal invariance of the Green function)**.** *Let $D, D' \subset \mathbb{C}$ be two domains. Suppose that $f : D \to D' = f(D)$ is a conformal isomorphism (i.e., holomorphic and one-to-one). Then*

$$G_{f(D)}(f(x), f(y)) = G_D(x, y).$$

Proof. The proof is a simple application of the change of variable formula. Let ϕ be a smooth test function supported in D, and let $x' = f(x)$. Then, by (12),

$$\int_{D'} G_{D'}(x', y')\phi(y')\,\mathrm{d}y' = \mathbb{E}_{x'}(\int_0^{\tau'} \phi(B'_s)\,\mathrm{d}s)$$

where B' is a Brownian motion and τ' is its exit time from $D' = f(D)$. On the other hand, the change of variable formula applied to the left hand side gives us, letting $y' = f(y)$ (a change of variable whose Jacobian derivative evaluates to $\mathrm{d}y' = |f'(y)|^2\,\mathrm{d}y$):

$$\int_{D'} G_{D'}(x', y')\phi(y')\,\mathrm{d}y' = \int_D G_{D'}(f(x), f(y))\phi(f(y))|f'(y)|^2\,\mathrm{d}y. \tag{14}$$

Now let us compute the left hand side in a different way, using the conformal invariance of Brownian motion discussed above. This allows us to write $B'_s = f(B_{F^{-1}(s)})$; moreover, in this correspondence one has $\tau' = F^{-1}(\tau_D)$. We apply the change of variable formula, but now to the time parameter $t = F^{-1}(s)$, or (since F^{-1} is the inverse of F), $s = F(t)$. The Jacobian derivative is thus

$$\mathrm{d}s = F'(t)\,\mathrm{d}t = |f'(B_t)|^2\,\mathrm{d}t,$$

by definition of F and the fundamental theorem of calculus. Thus,

$$\begin{aligned}\mathbb{E}_{x'}(\int_0^{\tau'} \phi(B'_s)\,\mathrm{d}s) &= \mathbb{E}_x(\int_0^{F^{-1}(\tau)} \phi(f(B_{F^{-1}(s)}))\,\mathrm{d}s)\\ &= \mathbb{E}_x(\int_0^{\tau} \phi(f(B_t))|f'(B_t)|^2\,\mathrm{d}t)\\ &= \int_D G_D(x, y)\phi(f(y))|f'(y)|^2\,\mathrm{d}y. \end{aligned}\tag{15}$$

Identifying the right hand sides of (14) and (15), since the test function ϕ is arbitrary, we conclude that

$$G_{D'}(f(x), f(y)))|f'(y)|^2 = G_D(x, y)|f'(y)|^2$$

first as distributions, thus as functions. The result follows by cancelling the factors of $|f'(y)|^2$ on both sides. ☐

Remark 1.24. *Having done the proof, it is now* a posteriori *easier explain the invariance of the Green function under conformal maps. When we apply the change of variables spatially, we pick up a term $|T'(y)|^2$ from the change of variable, because we are in dimension $d = 2$. When we apply it temporally, we pick up another term $|T'(y)|^2$ from Itô's formula. The fact that these two factors match exactly is what gives the conformal invariance of the Green function.*

From this perspective the conformal invariance of the Green function is unique to the case of dimension $d = 2$. In other dimensions, the Green function would not even be invariant under scalings $z \mapsto rz$ (even though this leaves Brownian motion invariant up to time change in any dimension).

Example 1.25. *In the unit disc $D = \mathbb{D}$, if $x = 0$ and $y \in \mathbb{D}$ with $y \neq x$, then $G_D(0, y)$ depends only on the modulus $|y|$ of y, since $G_D(0, \cdot)$ is invariant under rotation (which is a conformal map).*

In fact, we can compute the Green function in the unit disc explicitly:

Proposition 1.26. *We have*

$$G_{\mathbb{D}}(0, y) = -\frac{1}{2\pi} \log |y|.$$

Proof. This either follows from Corollary 1.22 and rotational symmetry, or the following scale invariance argument. Set $f(r) = G_{\mathbb{D}}(0, y)$ for $|y| = r$. We will check that for every $0 < r, s \leqslant 1$:

$$f(rs) = f(r) + f(s). \tag{16}$$

To see this, fix $0 \leqslant r \leqslant 1$. Then for $z \in B(0, r)$,

$$\phi(z) = G_{\mathbb{D}}(0, z) - f(r)$$

is a harmonic function in $B(0, r) \setminus \{0\}$, converges to 0 at the boundary of $\partial B(0, r)$, and satisfies $\phi(z) = -1/(2\pi)(1 + o(1)) \log(1/|z|)$ as $z \to 0$ by Proposition 1.18. Thus by Proposition 1.21, $\phi(z)$ is the Green function in $B(0, r)$. But this is scale invariant. Thus $\phi(z) = f(|z|/r)$ by Theorem 1.23. Hence if $|z| = rs$, $f(rs) - f(r) = f(s)$, which is (16).

From (16) and the continuity of f (a consequence of Proposition 1.20), we deduce that

$$f(r) = \alpha \log r; \quad 0 < r \leqslant 1,$$

for some $\alpha \in \mathbb{R}$. We deduce $\alpha = -1/(2\pi)$ by considering the behaviour near $r = 0$ (and Proposition 1.18). $\square$

Lecture 5: Friday 24 March

2 Riemann Mapping Theorem

We review the notion of conformal isomorphism of complex domains and discuss the question of existence and uniqueness of conformal isomorphisms between proper simply connected complex domains. Then we illustrate, by a simple special case, Loewner's idea of encoding the evolution of complex domains using a differential equation.

2.1 Statement

We shall be concerned with certain sorts of subsets of the complex plane $\mathbb{C}$ and mappings between them. Recall that a set $D \subseteq \mathbb{C}$ is a *domain* if it is non-empty, open and connected. We say that D is *simply connected* if every continuous map γ of the circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ into D is the restriction of a continuous map ψ of the disc $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ into D. In fact it suffices that this property holds for piecewise smooth (or even polygonal) *simple* loops (i.e. for every injective map γ from the circle to D).

A convenient criterion for a domain $D \subseteq \mathbb{C}$ to be simply connected is that its complement in the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is connected. A domain is *proper* if it is not the whole of $\mathbb{C}$.

Example 2.1. *The open unit disc $\mathbb{D} = \{|z| < 1\}$, the open upper half-plane $\mathbb{H} = \{\mathrm{Re}(z) > 0\}$, and the open infinite strip $S = \{0 < \mathrm{Im}(z) < 1\}$ are all examples of proper simply connected domains (note that in the latter case, the complement S^c is connected in $\mathbb{C} \cup \{\infty\}$ but not in $\mathbb{C}$). The annulus $A = B(0, R) \setminus B(0, r)$ is connected but not simply connected.*

The following elementary property of simply connected domains will be useful in our proof of the Riemann mapping theorem:

Lemma 2.2. *Every proper simply connected domain D is regular, in the sense that if $b \in \partial D$ and $T(D) = \inf\{t > 0 : B_t \notin D\}$, then $\mathbb{P}_b(T(D) = 0) = 1$.*

Proof. This comes from a simple zero-one argument. Let $b \in \partial D$. Note that D^c can not be reduced to b (otherwise $D = \mathbb{C} \setminus \{b\}$ and is not simply connected). Thus there exists $\zeta \neq b$ with $\zeta \in D^c$. Since D^c is connected we can find a curve (in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ connecting b and ζ. By cutting this curve if necessary, we see that there exists a curve γ in $\mathbb{C} \setminus D$ such that one end point is b. Let $\varepsilon > 0$ be such that $\mathrm{diam}(\gamma) \geqslant \varepsilon$. Let us now check that starting a Brownian motion from b, $\mathbb{P}_b(T(D) = 0) = 1$. Note that by Blumenthal's zero-one law, it suffices to show that this probability is positive. Consider the event E_n that the Brownian motion makes a loop in the ball $B_n = B(b, 2^{-n})$ which disconnects its center b from its boundary, before leaving this ball. Then letting $E = \liminf_{n\to\infty} E_n = \cap_{n\geqslant 1} \cup_{m\geqslant n} E_n$ be the event that E_n occurs infinitely often,

$$\{T(D) = 0\} \supset E,$$

since if the Brownian motion makes a loop around b in B_n it is guaranteed to hit the curve γ and thus to have left D before leaving B_n. On the other hand, by monotonicity,

$$\mathbb{P}_b(E) = \lim_{n\to\infty} \mathbb{P}_b(\cup_{m\geqslant n} E_m) \geqslant \liminf_{n\to\infty} \mathbb{P}_b(E_n).$$

But by scale invariance, $\mathbb{P}_b(E_n)$ does not depend on n and is positive. This concludes the proof. □

Recall that a holomorphic function f on a domain D is a *conformal map* if $f'(z) \neq 0$ for all $z \in D$. We call a bijective conformal map $f : D \to D'$ a *conformal isomorphism*. In this case, the image $D' = f(D)$ is also a domain and the inverse map $f^{-1} : D' \to D$ is also a conformal map. Every conformal map is locally a conformal isomorphism. The function $z \mapsto e^z$ is conformal on $\mathbb{C}$ but is not a conformal isomorphism on $\mathbb{C}$ because it is not injective. The following result, Riemann's mapping theorem, is fundamental to geometric function theory as well as to SLE theory.

Theorem 2.3 (Riemann mapping theorem)**.** *Let D be a proper simply connected domain. Then there exists a conformal isomorphism $\phi : D \to \mathbb{D}$. In fact, given $z_0 \in D$, we can find such a conformal isomorphism such that $\phi(z_0) = 0$.*

This theorem can be used to work out properties of conformally invariant objects (harmonic measure, harmonic functions, Brownian motion, Green function) on arbitrary proper simply connected domains from the analogous properties on given concrete simply connected domains such as the upper half plane or the unit disc, where these objects are often more concrete.

Proof. Let D be a proper simply connected domain, and $z_0 \in D$. We will prove that there is a conformal isomorphism ϕ sending z_0 to 0 and get an "explicit" expression for ϕ. Let B be a Brownian motion starting from z_0, and as usual let $T(D)$ denote the exit time of D by B. By Lemma 2.2, we know that D is regular, i.e., if $b \in \partial D$ then $\mathbb{P}_b(T(D) = 0) = 1$. We will show that we can find a conformal isomorphism sending z_0 to 0. We define the following function

$$u(z) = \mathbb{E}_z[\log |B_{T(D)} - z_0|],$$

which we already encountered in Corollary 1.22. Note that u is harmonic in all of D by Kakutani's formula (Theorem 1.1). Since D is simply connected, there exists a (unique up to additive constant) harmonic function v on D such that $f = u + iv$ is holomorphic.

Now consider

$$\phi(z) = (z - z_0)e^{-f(z)}.$$

Let us check that ϕ satisfies the required properties. First, ϕ is holomorphic since f is. Let us check that if $z \in D$ then $\phi(z) \in \mathbb{D}$. To see this, recall that by the maximum modulus principle (Theorem 1.5), the maximum modulus of the harmonic function ϕ is attained on the boundary. But by Corollary 1.22, $|\phi(z)| = e^{-2\pi G_D(z_0,z)}$. Thus as $z \to b \in \partial D$, $G_D(z_0, z) \to 0$ and $|\phi(z)| \to 1$. In particular, $f(z) \in \mathbb{D}$ for all $z \in D$.

Let us check that ϕ is one-to-one. For this we will use Rouché's theorem:

Lemma 2.4. *Suppose f and g are analytic functions in a simply connected domain U. Let $\mathcal{N}(f;U)$ denote the number of zeros of f (counted with multiplicity) in U. Suppose $|f(z)| < |g(z)|$ for $z \in \partial U$. Then $\mathcal{N}(f;U) = \mathcal{N}(f+g;U)$.*

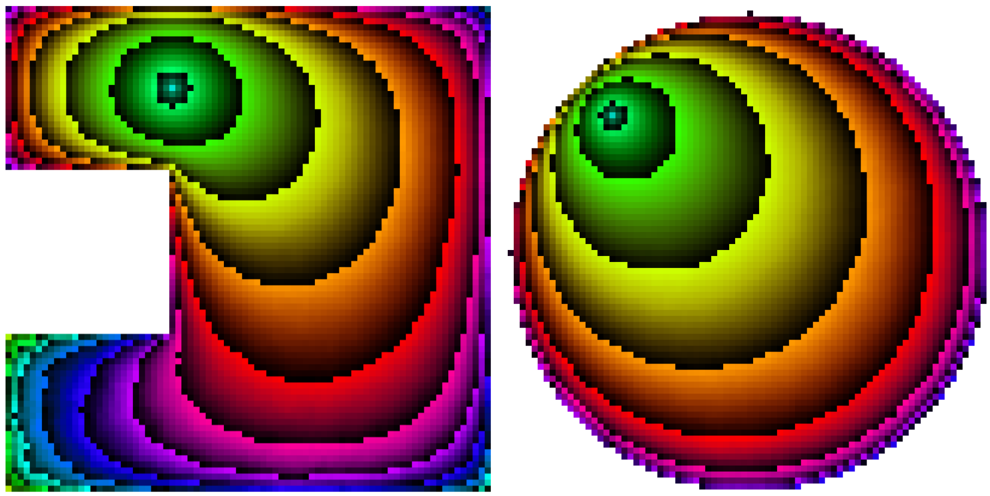

Figure 2: Occupation measure (in two different domains) by many Brownian motions, approximating the Green function. The level lines are the equipotentials, which describe the Riemann map. Simulations by Oskar Koiner.

Furthermore we have $\mathcal{N}(\phi; D) = 1$. Indeed, $e^{f(z)}$ can never equal zero so the only zero of ϕ, even counting multiplicity, is attained at z_0. Now fix $w \in \mathbb{D}$. Since $|w| < 1$ and $|\phi| = 1$ on ∂D, we deduce that by Rouché's theorem that $\mathcal{N}(\phi; D) = \mathcal{N}(\phi - w; D) = 1$. We deduce that $\phi^{-1}(w)$ consists of exactly a single point. In other words, since $w \in \mathbb{D}$ was arbitrary, we see that ϕ is one-to-one. $\square$

Remark 2.5. *Let D be a proper simply connected domain. For a given $z_0 \in D$ and $r > 0$, the level sets of the Green function, namely*

$$\gamma_r = \{z \in D : G_D(z_0, z) = r\}$$

are called the equipotential curves. If we imagine that a point charge is placed at z_0, this will induce an electric potential at every point in space. We can wire the outside of D^c to maintain the electric potential fixed (say equal to 0) on ∂D. Then γ_r corresponds to the regions where the electrostatic potential is constant equal to some fixed value (this is because in two dimensions, the electric potential induced by a point charge is proportional to $\log |z_0 - \cdot|$, whereas in three dimensions it would be given by the harmonic function $1/|z_0 - \cdot|$). The conformal map constructed in the proof of the preceding theorem has the property that it takes the equipotentials of (D, z_0) to those of $(\mathbb{D}, 0)$, namely, concentric circles. This corresponds to Riemann's original intuition (the result was part of his 1851 PhD dissertation). However Riemann's original proof was considered flawed; modern proofs typically presented in undergraduate courses today tend to be rather different.

We shall discuss ways to specify a unique choice of conformal isomorphism $\phi : D \to \mathbb{D}$ or $\phi : D \to \mathbb{H}$ in the next two sections. In general, there is no exact usable formula for ϕ in terms of D. Nevertheless, we shall want to derive certain properties of ϕ from properties of D. We shall see that Brownian motion provides a useful tool for this.

2.2 Möbius transformations

A *Möbius transformation* is any function f on $\mathbb{C} \cup \{\infty\}$ of the form

$$f(z) = \frac{az + b}{cz + d} \tag{17}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Here $f(-d/c) = \infty$ and $f(\infty) = a/c$. Hence f is nonconstant (whereas if $ad - bc = 0$ then f is clearly constant); in fact, it is easy to check that f is a bijection from $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself, hence f is a homeomorphism from $\hat{\mathbb{C}}$ to itself, and is clearly analytic on $\mathbb{C} \setminus \{-d/c\}$.

Möbius transformations form a group under composition. This group is generated by translations ($z \mapsto z + c$ for some $c \in \mathbb{C}$), rotations ($z \mapsto e^{i\theta} z$ for some $\theta \in \mathbb{R}$), scalings ($z \mapsto az$ for some $a > 0$), and the inversion map $z \mapsto 1/z$. A Möbius map transforms every generalised circle *exactly* to a generalised circle (where generalised circle is either a circle or an infinite line). This is straightforward for translations, scaling and rotations, and must be checked by calculation for the inversion map. (Möbius maps can thus be thought of as "perfect" conformal maps).

Example 2.6. *Define the Möbius map* Ψ *by*

$$\Psi(z) = \frac{i - z}{i + z}.$$

Note that Ψ *maps* $\mathbb{H}$ *to* $\mathbb{D}$*, since if* $x \in \mathbb{R}$*,* $|\psi(x)| = 1$*. Thus the restriction of* Ψ *to* $\mathbb{H}$ *is a conformal isomorphism from* $\mathbb{H}$ *to* $\mathbb{D}$*.*

Example 2.7. *A Möbius transformation* f *restricts to a conformal automorphism of* $\mathbb{H}$ *if and only if we can write (17) with* $a, b, c, d \in \mathbb{R}$*, and* $ad - bc > 0$ *(we may assume without loss of generality* $ad - bc = 1$*).*

To see this, note that when $a, b, c, d \in \mathbb{R}$ then f maps $\mathbb{R}$ to $\mathbb{R}$ hence $\mathbb{H}$ to $\mathbb{H}$ or $-\mathbb{H}$. Considering the image of iy we see that when $ad - bc > 0$ we are in the former case. Conversely, if f is such a map then since it sends the real line to itself we must have $f(0) = b/d \in \mathbb{R}$, $f(\infty) = a/c \in \mathbb{R}$ and $f^{-1}(0) = -b/a \in \mathbb{R}$ and $f^{-1}(\infty) = -d/c \in \mathbb{R}$, so a, b, c and d are all real. The result follows.

The following lemma is a basic result of complex analysis.

Lemma 2.8 (Schwarz lemma)**.** *Let* $f : \mathbb{D} \to \mathbb{D}$ *be a holomorphic function with* $f(0) = 0$*. Then* $|f(z)| \leqslant |z|$ *for all* z*. Moreover, if* $|f(z)| = |z|$ *for some* $z \neq 0$*, then* $f(w) = e^{i\theta} w$ *for all* w*, for some* $\theta \in [0, 2\pi)$*.*

Proof. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Consider the function $g(z) = f(z)/z$. By Taylor's theorem, g is analytic and hence holomorphic in $\mathbb{D}$. Fix $z \in \mathbb{D}$ and $r \in (|z|, 1)$. Then by the maximum principle

$$|g(z)| \leqslant \sup_{|w|=r} |g(w)| \leqslant \frac{1}{r}.$$

Letting $r \to 1$, we get $|g(z)| \leqslant 1$ and hence $|f(z)| \leqslant |z|$ for all $z \in \mathbb{D}$. If $|f(z)| = |z|$ for some $z \neq 0$, then $|g(z)| = 1$, say $g(z) = e^{i\theta}$. Then g is constant on $\mathbb{D}$ by the maximum modulus principle, so $f(w) = e^{i\theta} w$ for all $w \in \mathbb{D}$. □

For $\theta \in [0, 2\pi)$ and $w \in \mathbb{D}$, define $\Phi_{\theta,w}$ on $\mathbb{D}$ by

$$\Phi_{\theta,w}(z) = e^{i\theta} \frac{z - w}{1 - \bar{w}z}. \tag{18}$$

Then $\Phi_{\theta,w}$ is a conformal automorphism of $\mathbb{D}$ and is the restriction of a Möbius transformation to $\mathbb{D}$. Indeed, for $|z| = 1$ one can check (by expanding the terms) that $|z - w|^2 = |1 - \bar{w}z|^2$ and so $|\Phi_{\theta,w}(z)| = 1$.

Corollary 2.9. *Let ϕ be a conformal automorphism of $\mathbb{D}$. Set $w = \phi^{-1}(0)$ and $\theta = \arg \phi'(w)$. Then $\phi = \Phi_{\theta,w}$. In particular ϕ is the restriction of a Möbius transformation to $\mathbb{D}$ and extends to a homeomorphism of $\bar{\mathbb{D}}$.*

Proof. Set $f = \phi \circ \Phi_{0,w}^{-1}$. It suffices to check that f is a rotation, i.e., $f(z) = e^{i\alpha} z$ for some $\alpha \in [0, 2\pi)$. Note that f is a conformal automorphism of $\mathbb{D}$ and $f(0) = 0$. Pick $u \in \mathbb{D} \setminus \{0\}$ and set $v = f(u)$. Note that $v \neq 0$. Now, either $|f(u)| = |v| \geqslant |u|$ or $|f^{-1}(v)| = |u| \geqslant |v|$. In any case, by the Schwarz lemma, either f is a rotation or f^{-1} is a rotation. Either way, f is a rotation. □

We now explain how Schwarz's lemma gives us a first form of uniqueness for the Riemann mapping theorem.

Corollary 2.10. *Let D be a proper simply connected domain and let $w \in D$. Then there exists a unique conformal isomorphism $\phi : D \to \mathbb{D}$ such that $\phi(w) = 0$ and $\arg \phi'(w) = 0$.*

Proof. By the Riemann mapping theorem there exists a conformal isomorphism $\phi_0 : D \to \mathbb{D}$. Set $v = \phi_0(w)$ and $\theta = -\arg \phi_0'(w)$ and take $\phi = \Phi_{\theta,v} \circ \phi_0$. Then $\phi : D \to \mathbb{D}$ is a conformal isomorphism with $\phi(w) = 0$ and $\arg \phi'(w) = 0$. If ψ is another such conformal isomorphism, then $f = \psi \circ \phi^{-1}$ is a conformal automorphism of $\mathbb{D}$ with $f(0) = 0$ and $\arg f'(0) = 0$, so $f = \Phi_{0,0}$ which is the identity function. Hence ϕ is unique. □

Lecture 6: Monday 27 March

The next corollary identifies the conformal automorphisms of $\mathbb{H}$ fixing ∞.

Corollary 2.11. *Let ϕ be a conformal automorphism of $\mathbb{H}$. If $\phi(\infty) = \infty$, then $\phi(z) = \sigma z + \mu$ for all $z \in \mathbb{H}$, for some $\sigma > 0$ and $\mu \in \mathbb{R}$. If $\phi(\infty) = \infty$ and $\phi(0) = 0$, then $\phi(z) = \sigma z$ for all $z \in \mathbb{H}$, for some $\sigma > 0$.*

Proof. Set $\mu = \phi(0)$ and $\sigma = \phi(1) - \phi(0)$. Since $\Psi \circ \phi \circ \Psi^{-1}$ is a conformal automorphism of $\mathbb{D}$, and hence by Corollary 2.9 a Möbius transform. Thus ϕ itself is a Möbius transformation of $\mathbb{H}$, so $\phi(z) = (az + b)/(cz + d)$ for all $z \in \mathbb{H}$, for some $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$, and ϕ extends to a homeomorphism of $\hat{\mathbb{C}}$. Since $\phi(\infty) = \infty$ we must have $c = 0$, and we obtain the result with $\mu = b/d$ and $\sigma = a/d > 0$. □

2.3 Martin boundary

Given a simply connected domain D, the Riemann mapping theorem gives us the existence of a conformal isomorphism from D to $\mathbb{D}$. Often the behaviour of this map near the boundary of D will be of interest to us, or the behaviour of its inverse $g = f^{-1}$ near the boundary of $\mathbb{D}$. Note that there are examples (which arise frequently in practice and in particular in these notes) of simply connected domains D such that f cannot be extended continuously to ∂D; and examples where $g = f^{-1}$ cannot be extended continuously. The boundary behaviour of conformal isomorphisms is a topic which has been analysed in considerable depth, and is the subject of the reference book by Pommerenke, [16]. We will not develop the full theory here, and instead quickly develop the notion of Martin boundary, which is a useful way of addressing some problems.

The Martin boundary is a general object of potential theory[2]. We shall however limit our discussion to the case of harmonic functions in a proper simply connected complex domain D. In this case, the Riemann mapping theorem, combined with the conformal invariance of harmonic functions, allows a very simple approach. Make a choice of conformal isomorphism $\phi : D \to \mathbb{D}$. We can define a metric d_ϕ on D by $d_\phi(z, z') = |\phi(z) - \phi(z')|$. Then d_ϕ is locally equivalent to the original metric but possibly not uniformly so. Say that a sequence $(z_n : n \in \mathbb{N})$ in D is D**-Cauchy** if it is Cauchy for d_ϕ. Since every conformal automorphism of $\mathbb{D}$ extends to a homeomorphism of $\bar{\mathbb{D}}$, this notion does not depend on the choice of ϕ.

Definition 2.12. *Write $\hat{D}$ for the completion of D with respect to the metric*[3] *and define the* **Martin boundary** *$\delta D = \hat{D} \setminus D$.*

The set $\hat{D}$ does not depend on the choice of ϕ and nor does its topology. This construction ensures that the map ϕ extends uniquely to a homeomorphism $\hat{D} \to \bar{\mathbb{D}}$. It follows then that every conformal isomorphism ψ of proper simply connected domains $D \to D'$ has a unique extension as a homeomorphism $\hat{D} \to \hat{D}'$. We abuse notation in writing $\phi(z)$ for the value of this extension at points $z \in \delta D$. Write ∂D for the boundary of D as a subset of $\mathbb{C}$, that is the set of limit points of D in $\mathbb{C}$, which in general is not identifiable with δD. For $b \in \delta D$, we say that a simply connected subdomain $N \subseteq D$ is a **neighbourhood of** b **in** D if $\{z \in \mathbb{D} : |z - \phi(b)| < \varepsilon\} \subseteq \phi(N)$ for some $\varepsilon > 0$.

Example 2.13. *A sequence $(z_n : n \in \mathbb{N})$ in $\mathbb{H}$ is $\mathbb{H}$-Cauchy if either it converges in $\mathbb{C}$ or $|z_n| \to \infty$ as $n \to \infty$. Thus we identify $\delta\mathbb{H}$ with $\mathbb{R} \cup \{\infty\}$.*

[2] See for example [5]

[3] This is the set of equivalence classes of D-Cauchy sequences $z = (z_n : n \in \mathbb{N})$, where $z \sim z'$ if $(z_1, z_1', z_2, z_2', \dots)$ is also a D-Cauchy sequence.

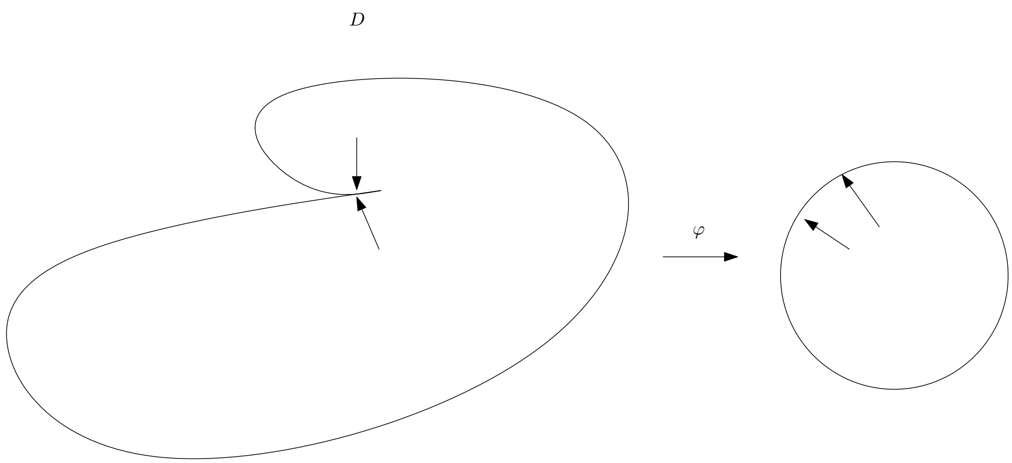

Figure 3: Two distinct points of δD and their images under φ.

Example 2.14. *For the slit domain $D = \mathbb{H} \setminus (0, i]$ and, for $z \in [0, i)$, the sequences $(z + (1+i)/n : n \in \mathbb{N})$ and $(z + (-1+i)/n : n \in \mathbb{N})$ are D-Cauchy but are not equivalent, so their equivalence classes z^+ and z^- are distinct Martin boundary points.*

Example 2.15. *Let D be a simply connected domain, and let $z \in D$. Consider a Brownian motion B starting from z. Then almost surely, as $t \uparrow T(D) = \inf\{t > 0 : B_t \notin D\}$, $B_t \to b =: \hat{B}_T \in \delta D$.*

This makes it possible to view the harmonic measure defined in Section 1.3 as a measure on δD instead of on ∂D. For instance, we then have conformal invariance of harmonic measure (Theorem 1.11) without assuming that the map extends to a continuous homeomorphism between the closures of the two domains.

Corollary 2.16. *Let D be a proper simply connected domain and let $b_1, b_2, b_3 \in \delta D$, ordered anticlockwise. Then there exists a unique conformal isomorphism $\phi : D \to \mathbb{H}$ such that $\phi(b_1) = \infty$, $\phi(b_2) = 0$ and $\phi(b_3) = 1$.*

Proof. Let us start by uniqueness. Suppose that ϕ_1, ϕ_2 are two such conformal isomorphisms. Then $f = \phi_1 \circ \phi_2^{-1} : \mathbb{H} \to \mathbb{H}$ is a conformal automorphism, fixing $0, 1$ and ∞. By Corollary 2.11, $f(z) = \sigma z$ for some $\sigma > 0$. But since $f(1) = 1$ we have $f(z) = z$ or $\phi_1 = \phi_2$.

Now let us check existence. Let ϕ be a conformal isomorphism from D to $\mathbb{H}$. We will transform ϕ by applying successively Möbius maps until b_1, b_2 and b_3 are mapped to their respective targets. Let $x_1 = \phi(b_1)$, with $x_1 \in \mathbb{R} \cup \{\infty\}$. If $x_1 \neq \infty$ we apply a translation $z \mapsto z - x_1$, followed by the inversion $z \mapsto 1/z$; in other words we let $\phi_1 = T_1 \circ \phi$ with $T_1(z) = 1/(z - x_1)$. Then $\phi_1(b_1) = \infty$. Now we move to b_2; let $x_2 = \phi_1(b_2)$ and note that $x_2 \in \mathbb{R}$ (and in particular $x_2 \neq \infty$). Consider the translation $T_2(z) = z - x_2$, and set $\phi_2 = T_2 \circ \phi_1$. Then $\phi_2(b_2) = 0$, and $\phi_2(b_1) = \infty$. It remains to deal with b_3. Since b_1, b_2 and b_3 are ordered anticlockwise we have $\phi_2(b_3) = x_3 \in (0, \infty)$. Let $T_3(z) = z/x_3$, and set

$\phi_3 = T_3 \circ \phi_2$. Then ϕ_3 is a conformal isomorphism from D to $\mathbb{H}$, with the desired boundary conditions. □

Note that in both Corollary 2.10 and Corollary 2.16, we obtain uniqueness of the conformal map by the imposition of three real-valued constraints. An intuitive way to understand this is that any two conformal isomorphisms from $D \to \mathbb{H}$ (say) differ by a conformal automorphism from $\mathbb{H}$ to $\mathbb{H}$. This is parameterised by four (real) parameters $a, b, c, d \in \mathbb{R}$ but one is redundant by scaling (i.e., we can always assume $ad - bc = 1$ without loss of generality).

Finally, although we will not use this, we note that the boundary behaviour is somewhat simpler if the domain D is a so-called Jordan domain. A **Jordan curve** is a continuous injective map $\gamma : \partial\mathbb{D} \to \mathbb{C}$. Say D is a **Jordan domain** if ∂D is the image of a Jordan curve. It can be shown in this case that any conformal isomorphism $D \to \mathbb{D}$ extends to a homeomorphism $\bar{D} \to \bar{\mathbb{D}}$, so we can identify δD with ∂D (see Theorem 2.6 in [16]).

2.4 SLE$_0$

This section and the next are for orientation only, and do not form part of the theoretical development. We will explain Loewner's method for describing a particularly simple path γ in the upper half plane, namely the straight line going from 0 to ∞ (a nicer and more useful way of describing this path is as a geodesic connecting 0 and ∞ for the hyperbolic metric in the upper-half plane). While the method initially appears complicated for such a simple path, this method will generalise seamlessly to the complicated case of SLE.

Consider the (deterministic) process $(\gamma_t)_{t\geqslant 0}$ in the closed upper half-plane $\bar{\mathbb{H}}$ given by

$$\gamma_t = 2i\sqrt{t}.$$

This process belongs to the family of processes $(\mathrm{SLE}_\kappa : \kappa \in [0, \infty))$ to which these notes are devoted, corresponding to the parameter value $\kappa = 0$. Think of $(\gamma_t)_{t\geqslant 0}$ as progressively eating away the upper half-plane so that the subdomain $H_t = \mathbb{H} \setminus K_t$ remains at time t, where $K_t = \gamma(0, t] = \{\gamma_s : s \in (0, t]\}$. There is a conformal isomorphism $g_t : H_t \to \mathbb{H}$ given by

$$g_t(z) = \sqrt{z^2 + 4t}$$

which has the following asymptotic behaviour as $|z| \to \infty$

$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}).$$

In particular $g_t(z) - z \to 0$ as $|z| \to \infty$. As we shall show in Proposition 3.3, there is only one conformal isomorphism $H_t \to \mathbb{H}$ with this last property. Thus we can think of the family of maps $(g_t)_{t\geqslant 0}$ as a canonical encoding of the path $(\gamma_t)_{t\geqslant 0}$.

Consider the vector field b on $\bar{\mathbb{H}} \setminus \{0\}$ defined by

$$b(z) = \frac{2}{z} = \frac{2(x - iy)}{x^2 + y^2}.$$

Fix $z \in \bar{\mathbb{H}} \setminus \{0\}$ and define

$$\zeta(z) = \inf\{t \geqslant 0 : \gamma_t = z\} = \begin{cases} y^2/4, & \text{if } z = iy \\ \infty, & \text{otherwise.} \end{cases}$$

Then $\zeta(z) > 0$, and $z \in \bar{K}_t$ if and only if $\zeta(z) \leqslant t$. Set $z_t = g_t(z)$. Then for $t < \zeta(z)$

$$\dot{z}_t = \frac{2}{\sqrt{z_t^2 + 4t}} = b(z_t) \tag{19}$$

and, if $\zeta(z) < \infty$, then $z_t \to 0$ as $t \to \zeta(z)$. Thus $(g_t(z) : z \in \bar{\mathbb{H}} \setminus \{0\}, t < \zeta(z))$, the conformal map characterising $\gamma[0,t]$, is also the (unique) maximal flow of the vector field b in $\bar{\mathbb{H}} \setminus \{0\}$. By *maximal* we mean that $(z_t : t < \zeta(z))$ cannot be extended to a solution of the differential equation on a longer time interval.

2.5 Loewner evolutions

Think of SLE_0 as obtained via the associated flow $(g_t)_{t\geqslant 0}$ by iterating continuously a map $g_{\delta t}$, which nibbles an infinitesimal piece $(0, 2i\sqrt{\delta t}]$ of $\mathbb{H}$ near 0. Charles Loewner, in the 1920's, studied complex domains $H_t = \mathbb{H} \setminus \gamma(0,t]$ for more general curves $(\gamma_t)_{t\geqslant 0}$, by a similar continuous iteration of conformal maps, obtained now by considering the flow of a time-dependent vector field $\bar{\mathbb{H}}$ of the form

$$b(t,z) = \frac{2}{z - \xi_t}, \quad t \geqslant 0, \quad z \in \mathbb{H}.$$

Here, $(\xi_t : t \geqslant 0)$ is a given continuous real-valued function, which is called the *driving function* or *Loewner transform* of the curve γ. We shall study this flow in detail below, showing that it always provides a construction of a family of domains $(H_t : t \geqslant 0)$, and sometimes also a path γ. Note that the flow lines $(g_t(z))_{t\geqslant 0}$ for SLE(0) separate, left and right, each side of the singularity at 0, with the path $(\gamma_t)_{t\geqslant 0}$ growing up between the left-moving flow lines and the right-moving ones. In the general case, assuming that the qualitative picture remains the same, when we move the singularity point ξ_t to the left, we may expect that some left-moving flow lines are deflected to the right, so the curve $(\gamma_t)_{t\geqslant 0}$ turns to the left. Moreover, the wilder the fluctuations of $(\xi_t)_{t\geqslant 0}$, the more convoluted we may expect the resulting path $(\gamma_t)_{t\geqslant 0}$ to be.

Oded Schramm, in 1999, realized that for some conjectured conformally invariant scaling limits $(\gamma_t)_{t\geqslant 0}$ of planar random processes, with a certain spatial Markov property, the process $(\xi_t)_{t\geqslant 0}$ would have to be a Brownian motion, of some diffusivity κ. The associated processes $(\gamma_t)_{t\geqslant 0}$ were at that time totally new and have since revolutionized our understanding of conformally invariant planar random processes.

Lecture 7: Monday 17 April

3 Compact $\mathbb{H}$-hulls and their mapping-out functions

A subset K of the upper half-plane $\mathbb{H}$ is called a *compact* $\mathbb{H}$*-hull* if K is bounded and $H = \mathbb{H} \setminus K$ is a simply connected domain. We shall associate to K a canonical conformal isomorphism $g_K : H \to \mathbb{H}$, the *mapping-out function* of K. At the same time we associate to K a real constant a_K, which we will identify as the *half-plane capacity* of K. These are all basic objects of Loewner's theory, or more precisely of its *chordal* variant, where we consider evolution of hulls in a given domain towards a chosen boundary point. We shall see later that the theory has a property of conformal invariance which allows us to reduce the general case to the study of the special domain $\mathbb{H}$ with ∞ as the boundary point, which is mathematically most tractable.

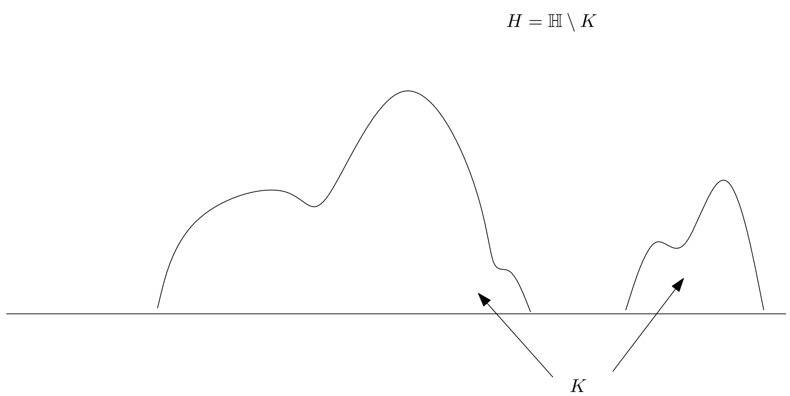

Figure 4: A compact $\mathbb{H}$-hull.

3.1 Extension of conformal maps by reflection

We start by explaining how a conformal isomorphism $\phi : D \to \mathbb{H}$ can be extended analytically to suitably regular parts of the boundary ∂D. We have already seen that ϕ extends continuously to the Martin boundary but now we want more regularity. The idea is to reflect the domain across the boundary. Given a proper simply connected domain $D \subseteq \mathbb{H}$, define

$$D^0 = \{x \in \mathbb{R} : D \text{ is a neighbourhood of } x \text{ in } \mathbb{H}\}, \quad D^* = D \cup D^0 \cup \{\bar{z} : z \in D\}.$$

More generally, for any open set $U \subseteq D^0$, define

$$D^*_U = D \cup U \cup \{\bar{z} : z \in D\}.$$

As U varies, the sets D_U^* are exactly the open sets which are invariant under conjugation and whose intersection with $\mathbb{H}$ is D. Say that a function $f^* : D_U^* \to \mathbb{C}$ is *reflection-invariant* if

$$f^*(\bar{z}) = \overline{f^*(z)}, \quad z \in D_U^*.$$

Given a continuous function f on D, there is at most one continuous, reflection-invariant function f^* on D_U^* extending f. Then f^* is the *continuous extension by reflection* of f. Such an extension f^* exists exactly when f has a continuous extension to $D \cup U$ which is real-valued on U. Any continuous extension by reflection of a holomorphic function is holomorphic, by an application of Morera's theorem. This is called the Schwarz reflection principle.

Proposition 3.1. *Let $D \subseteq \mathbb{H}$ be a simply connected domain. Let I be a proper open subinterval of $\mathbb{R}$ with $I \subseteq D^0$ and let $x \in I$. Then there exists a unique conformal isomorphism $\phi : D \to \mathbb{H}$ which extends to a homeomorphism $D \cup I \to \mathbb{H} \cup (-1,1)$ taking x to 0. In particular I is naturally identified with an interval of the Martin boundary δD. Moreover ϕ extends further to a reflection-invariant conformal isomorphism $\phi^* : D_I^* \to \mathbb{H}_{(-1,1)}^*$.*

Proof $(\star)$. Note that D_I^* and $\mathbb{H}_{(-1,1)}^*$ are proper simply connected domains. By the Riemann mapping theorem, there exists a unique conformal isomorphism $\phi^* : D_I^* \to \mathbb{H}_{(-1,1)}^*$ with $\phi^*(x) = 0$ and $\arg(\phi^*)'(x) = 0$. Define $\rho : D_I^* \to \mathbb{H}_{(-1,1)}^*$ by $\overline{\rho(z)} = \phi^*(\bar{z})$. Then ρ is a conformal isomorphism with $\rho(x) = 0$ and $\arg \rho'(x) = 0$. Hence $\rho = \phi^*$ and so ϕ^* is reflection-invariant. Then $\phi^*(I) \subseteq (-1,1)$ and $(\phi^*)^{-1}(-1,1) \subseteq I$, so $\phi^*(I) = (-1,1)$. Now $\phi^*(D)$ is connected and does not meet $(-1,1)$. Since $\arg(\phi^*)'(x) = 0$, by considering a neighbourhood of x, we must have $\phi^*(D) \subseteq \mathbb{H}$. The same argument shows that $(\phi^*)^{-1}(\mathbb{H}) \subseteq D$, so $\phi^*(D) = \mathbb{H}$. Hence ϕ^* restricts to a conformal isomorphism $\phi : D \to \mathbb{H}$ with the required properties.

On the other hand, any map ψ with these properties has a continuous extension ψ^* by reflection to D_I^*, which is a bijection to $\mathbb{H}_{(-1,1)}^*$ and is holomorphic by the Schwarz reflection principle. Moreover $\psi^*(x) = 0$, and $\arg(\psi^*)'(x) = 0$ since $\psi^*(I) = (-1,1)$. Hence $\psi^* = \phi^*$ and so $\psi = \phi$. □

Proposition 3.2. *Let $D \subseteq \mathbb{H}$ be a simply connected domain and let $\phi : D \to \mathbb{H}$ be a conformal isomorphism. Suppose that ϕ is bounded on bounded sets. Then ϕ extends by reflection to a conformal isomorphism ϕ^* on D^*.*

Proof $(\star)$. Fix $x \in D^0$ and a bounded open interval $I \subseteq D^0$ containing x. Write $\phi_{x,I}$ for the conformal isomorphism obtained in Proposition 3.1. Then $f = \phi \circ \phi_{x,I}^{-1} : \mathbb{H} \to \mathbb{H}$ is a Möbius transformation which is bounded, and hence continuous, on a neighbourhood of $(-1,1) = \phi_{x,I}(I)$ in $\mathbb{H}$. Hence $\phi = f \circ \phi_{x,I}$ extends by reflection to a conformal isomorphism $\phi_I^* = f^* \circ \phi_{x,I}^*$ on D_I^*. The maps ϕ_I^* must be consistent, and hence extend to a conformal map ϕ^* on D^*. Now ϕ^* can only fail to be injective on D^0 but, as a conformal map, can only fail to be injective on an open set in $\mathbb{C}$. Hence ϕ^* is a conformal isomorphism. □

3.2 Construction of the mapping-out function

Given any compact $\mathbb{H}$-hull K, we now specify a particular conformal isomorphism $g = g_K : \mathbb{H} \setminus K \to \mathbb{H}$. This will give us a convenient way to encode the geometry of K. We get uniqueness by requiring that g_K looks like the identity at ∞.

Theorem 3.3. *Let K be a compact $\mathbb{H}$-hull and set $H = \mathbb{H} \setminus K$. There exists a unique conformal isomorphism $g_K : H \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $|z| \to \infty$. Moreover $g_K(z) - z$ is bounded uniformly in $z \in H$. Moreover, for some $a_K \in \mathbb{R}$, we have*

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad |z| \to \infty. \tag{20}$$

Moreover g_K extends by reflection to a conformal isomorphism g_K^* on H^*.

The notation g_K will be used throughout. The function g_K takes $\mathbb{H} \setminus K$ to the standard domain $\mathbb{H}$, so K no longer appears as a defect of the domain. Thus we call g_K the *mapping-out function* of K. The condition $g_K(z) - z \to 0$ at ∞ which makes g_K unique is sometimes called the *hydrodynamic normalization*. The constant a_K, which we will see later is nonnegative, will be called the *half-plane capacity* and is a measure of the size (seen from infinity) of the set K.

Proof. Set $D = \{z : -z^{-1} \in H\}$. Then $D \subseteq \mathbb{H}$ is a simply connected domain which is a neighbourhood of 0 in $\mathbb{H}$. Choose a bounded open interval $I \subseteq D^0$ containing 0. By Proposition 3.1, there exists a conformal isomorphism $\phi : D \to \mathbb{H}$ which extends to a reflection-invariant conformal isomorphism ϕ^* on D_I^*, with $\phi^*(0) = 0$ and $\arg(\phi^*)'(0) = 0$. Consider the Taylor expansion of ϕ^* at 0. Since ϕ^* maps I into $\mathbb{R}$, the coefficients must all be real. So, as $z \to 0$, we have

$$\phi^*(z) = az + bz^2 + cz^3 + O(|z|^4)$$

for some $a \in (0, \infty)$ and $b, c \in \mathbb{R}$. Define g_K on H by $g_K(z) = -a\phi(-z^{-1})^{-1} - (b/a)$. It is a straightforward exercise to check that g_K is a conformal isomorphism to $\mathbb{H}$ and that g_K has the claimed expansion at ∞, with $a_K = (b/a)^2 - (c/a)$. In particular, $g_K(z) - z$ is bounded near ∞. Now ϕ^* is a homeomorphism of neighbourhoods of 0, so g_K can only take bounded sets to bounded sets. Hence $g_K(z) - z$ is uniformly bounded on H and, by Proposition 3.2, g_K extends by reflection to a conformal isomorphism on H^*.

Finally, if $g : H \to \mathbb{H}$ is any conformal isomorphism such that $g(z) - z \to 0$ as $|z| \to \infty$, then $f = g \circ g_K^{-1}$ is a conformal automorphism of $\mathbb{H}$ with $f(z) - z \to 0$ as $|z| \to \infty$. Then $f(\infty) = \infty$, so $f(z) = \sigma z + \mu$ for some $\sigma \in (0, \infty)$ and $\mu \in \mathbb{R}$ by Corollary 2.11, and then $f(z) = z$ for all z, showing that $g = g_K$. □

Example 3.4. *The mapping-out function has a simple form for the half-disc $\bar{\mathbb{D}} \cap \mathbb{H}$ and for the slit $(0, i] = \{iy : y \in (0, 1]\}$:*

$$g_{\bar{\mathbb{D}} \cap \mathbb{H}}(z) = z + 1/z, \quad g_{(0,i]}(z) = \sqrt{z^2 + 1} = z + 1/(2z) + O(|z|^{-2}). \tag{21}$$

The first example can be checked using the fact that the image of the unit circle under the map $z \mapsto z + z^{-1}$ is the segment $[-2, 2]$ and symmetry.

Definition 3.5. *The half-plane capacity of the compact $\mathbb{H}$-hull K is the quantity*

$$\operatorname{hcap}(K) = \lim_{z\to\infty} z(g_K(z) - z) = a_K.$$

Example 3.6. *For the upper semidisc $K = \bar{\mathbb{D}} \cap \mathbb{H}$, we have* $\operatorname{hcap}(K) = 1$*, while for the segment $K = (0, i]$ we have* $\operatorname{hcap}(K) = 1/2$.

For a Brownian motion $B = (B_t)_{t\geqslant 0}$ starting from some $z \in H$, let $T = T(H) = \inf\{t > 0 : B_t \notin H\}$ (the first hitting time of $K \cup \mathbb{R}$).

Proposition 3.7. *We have*

$$\operatorname{hcap}(K) = \lim_{y\to\infty} y\mathbb{E}_{iy}[\operatorname{Im}(B_T)].$$

In particular $\operatorname{hcap}(K) \geqslant 0$.

Proof. Fix $z \in H$ and consider a complex Brownian motion $(B_t)_{t\geqslant 0}$ starting from z. For $t < T = T(H)$, $g_K(B_t)$ is a time-changed Brownian motion in the upper-half plane $\mathbb{H}$, starting from $g_K(z)$, and so converges a.s. to a random a random variable, denoted earlier in Example 2.15 by $g_K(\hat{B}_T) \in \mathbb{R}$, as $t \uparrow T$ (here $\hat{B}_T \in \delta H$, the Martin boundary). Recall also that $g_K(z) - z$ is a bounded holomorphic function on H, by definition of g_K. If we set $M_t = g_K(B_t) - B_t$ for $t < T$ then $(M_t)_{t<T}$ is therefore a bounded martingale which converges to $g_K(\hat{B}_T) - B_T$ as $t \to T$. Hence, by optional stopping,

$$g_K(z) - z = \mathbb{E}_z(g_K(\hat{B}_T) - B_T). \tag{22}$$

Recall also $z(g_K(z) - z) \to \operatorname{hcap}(K)$ as $z \to \infty$. Take $z = iy$, hence we have

$$\operatorname{hcap}(K) = \lim_{y\to\infty} iy\mathbb{E}_{iy}[g_K(\hat{B}_T) - B_T].$$

On the other hand, $\operatorname{hcap}(K)$ is known to be real so we can take the real part on both sides. Noting that $g_K(\hat{B}_T) \in \mathbb{R}$ by definition, we deduce that

$$\lim_{y\to\infty} y\mathbb{E}_{iy}[\operatorname{Im}(B_T)] = \operatorname{hcap}(K)$$

as desired (note that the existence of the limit is a consequence of the argument). □

3.3 Properties of the mapping-out function

The following scaling and translation properties may be deduced from the defining characterization of the mapping-out function. The details are left as an exercise.

Proposition 3.8. *Let K be a compact $\mathbb{H}$-hull. Let $r \in (0, \infty)$ and $x \in \mathbb{R}$. Set*

$$rK = \{rz : z \in K\}, \quad K + x = \{z + x : z \in K\}.$$

Then rK and $K + x$ are compact $\mathbb{H}$-hulls and we have

$$g_{rK}(z) = rg_K(z/r), \quad g_{K+x}(z) = g_K(z - x) + x.$$

Thus $\operatorname{hcap}(rK) = r^2 \operatorname{hcap}(K)$*,* $\operatorname{hcap}(K + x) = \operatorname{hcap}(K)$*.*

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Nested compact $\mathbb{H}$-hulls $K_0 \subseteq K$ may be encoded by the composition of mapping-out functions.

Proposition 3.9. *Let K_0 and K_1 be compact $\mathbb{H}$-hulls. Set $K = K_0 \cup g_{K_0}^{-1}(K_1)$. Then K is a compact $\mathbb{H}$-hull K containing K_0 and we have*

$$g_K = g_{K_1} \circ g_{K_0}, \quad \mathrm{hcap}(K) = \mathrm{hcap}(K_0) + \mathrm{hcap}(K_1). \tag{23}$$

Moreover we obtain all compact $\mathbb{H}$-hulls K containing K_0 in this way.

Proof. Set $H_0 = \mathbb{H} \backslash K_0$ and $H = \mathbb{H} \backslash K$. We can define a conformal isomorphism $g : H \to \mathbb{H}$ by $g = g_{K_1} \circ g_{K_0}$. In particular H is a simply connected domain. Consider a sequence of points (z_n) in H_0 with $|z_n| \to \infty$. Then $g_{K_0}(z_n)/z_n \to 1$ and $|g_{K_0}(z_n)| \to \infty$. Hence there exists N such that for all $n \geqslant N$ we have $g_{K_0}(z_n) \notin K_1$ and then

$$z_n(g(z_n) - z_n) = z_n(g_{K_1}(g_{K_0}(z_n)) - g_{K_0}(z_n)) + z_n(g_{K_0}(z_n) - z_n) \to a_{K_1} + a_{K_0}.$$

Hence K is bounded and $g = g_K$ and $a_K = a_{K_0} + a_{K_1}$.

On the other hand, suppose K is any compact $\mathbb{H}$-hull containing K_0. Define $K_1 = g_{K_0}(K \setminus K_0)$ and $H_1 = g_{K_0}(\mathbb{H} \setminus K)$. Then $K = K_0 \cup g_{K_0}^{-1}(K_1)$ and $H_1 = \mathbb{H} \setminus K_1$. Also, K_1 is bounded and H_1 is a simply connected domain, so K_1 is a compact $\mathbb{H}$-hull, as required. $\square$

3.4 Boundary and continuity estimates

Recall from the proof of Proposition 3.7 (or Example 2.15) the Brownian limit $\hat{B}_{T(H)}$, which is a random variable in the Martin boundary δH. Recall also that g_K extends to a homeomorphism from δH to $\delta\mathbb{H} = \mathbb{R} \cup \{\infty\}$.

Proposition 3.10. *Let $S \subseteq \delta H$ be measurable. Then*

$$\lim_{y \to \infty,\, x/y \to 0} \pi y \mathbb{P}_{x+iy}(\hat{B}_{T(H)} \in S) = \mathrm{Leb}(g_K(S)). \tag{24}$$

Proof. Write $g_K(x + iy) = u + iv$. Then $u/y \to 0$ and $v/y \to 1$ as $y \to \infty$ with $x/y \to 0$. By conformal invariance of Brownian motion, and using the known form (Example 1.10) for the density of harmonic measure in $\mathbb{H}$, we have

$$\mathbb{P}_{x+iy}(\hat{B}_{T(H)} \in S) = \mathbb{P}_{u+iv}(B_{T(\mathbb{H})} \in g_K(S)) = \int_{g_K(S)} \frac{v}{\pi((t-u)^2 + v^2)} dt$$

On multipying by πy and letting $y \to \infty$ and $x/y \to 0$ we obtain the desired formula. $\square$

For an interval $(a, b) \subseteq H^0$, we can take $S = (a, b)$ and $x = 0$ in Proposition 3.10 to obtain

$$g_K(b) - g_K(a) = \lim_{y \to \infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \in (a, b)). \tag{25}$$

On the other hand, we can also take $S = \delta H \setminus H^0$ to obtain

$$\lim_{y\to\infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \in K) = \lim_{y\to\infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \notin H^0) = \mathrm{Leb}(\mathbb{R} \setminus g_K(H^0)). \tag{26}$$

Here we used the fact that $\partial H \setminus (K \cup H^0)$ is countable for the first equality. The left hand side is known as the capacity (simply) of the the compact $\mathbb{H}$-hull K, see Section 3.6 for additional details (it is another measure of the size of K seen from infinity).

Define

$$\mathrm{rad}(K) = \inf\{r \geqslant 0 : K \subseteq r\bar{\mathbb{D}} + x \text{ for some } x \in \mathbb{R}\}.$$

Proposition 3.11. *Let K be a compact $\mathbb{H}$-hull and let $x \in \mathbb{R}$. Suppose that the interval $[x, \infty)$ does not intersect $\bar{K}$. Then $g_K(x) \geqslant x$. If also $K \subseteq \mathbb{D}$ and $x \in (1, \infty)$, then $g_K(x) \leqslant x + 1/x$.*

Proof. For $b > x$ and $y > \mathrm{rad}(K)$, we have

$$\mathbb{P}_{iy}(B_{T(H)} \in (x, b)) \leqslant \mathbb{P}_{iy}(B_{T(\mathbb{H})} \in (x, b)).$$

Multiply by πy and let $y \to \infty$, using Proposition 3.10, to obtain $g_K(b) - g_K(x) \leqslant b - x$. Subtract b and let $b \to \infty$ to see that $g_K(x) \geqslant x$. If $K \subseteq \mathbb{D}$ and $x \in (1, \infty)$, then also

$$\mathbb{P}_{iy}(B_{T(\mathbb{H}\setminus\bar{\mathbb{D}})} \in (x, b)) \leqslant \mathbb{P}_{iy}(B_{T(H)} \in (x, b)).$$

Multiply by πy and let $y \to \infty$, using Proposition 3.10 again and the known form (21) of the mapping-out function for $\bar{\mathbb{D}} \cap \mathbb{H}$, to obtain

$$(b + 1/b) - (x + 1/x) \leqslant g_K(b) - g_K(x).$$

Then subtract b and let $b \to \infty$ to see that $g_K(x) \leqslant x + 1/x$. □

We already know (by definition of g_K) that $g_K(z) - z$ is bounded uniformly on H; the next result gives us a quantitative bound which will be useful later on where it will give us the continuity of the Loewner transform. For this reason we refer to this as the continuity estimate.

Proposition 3.12. *Let K be a compact $\mathbb{H}$-hull. Then*

$$|g_K(z) - z| \leqslant 3\,\mathrm{rad}(K), \quad z \in H. \tag{27}$$

Proof. By a scaling and translation argument, using Proposition 3.8, it will suffice to consider the case where $K \subseteq \bar{\mathbb{D}}$ and $\mathrm{rad}(K) = 1$. Recall that by (22) we have

$$g_K(z) - z = \mathbb{E}_z(g_K(\hat{B}_T) - B_T).$$

Note that $\{|x| > 1\} \subseteq H^0$ and $\{|x| > 2\} \subseteq g_K(\{|x| > 1\})$. If $|B_T| > 1$, then $B_T \in H^0$, so, by Proposition 3.11, $|g_K(\hat{B}_T) - B_T| \leqslant 1/|B_T| \leqslant 1$. On the hand, if $|B_T| \leqslant 1$, then $g_K(\hat{B}_T) \notin g_K(\{|x| > 1\})$, so $|g_K(\hat{B}_T)| \leqslant 2$. In any case $|g_K(\hat{B}_T) - B_T| \leqslant 3$. Hence $|g_K(z) - z| \leqslant 3$. □

3.5 Differentiability estimate

The expansion (20) at ∞ for mapping-out functions states that, for every compact $\mathbb{H}$-hull K, there are constants $C(K) < \infty$ and $R(K) < \infty$ such that,

$$\left| g_K(z) - z - \frac{a_K}{z} \right| \leqslant \frac{C(K)}{|z|^2}, \quad |z| \geqslant R(K).$$

The next result strengthens this estimate, stating that, if $K \subseteq \bar{\mathbb{D}}$, then we can take $C(K) = Ca_K$ and $R(K) = 2$, where $C < \infty$ does not depend on K. This will be used to show that the Loewner transform is differentiable.

Proposition 3.13. *There is an absolute constant $C < \infty$ with the following properties. For all $r \in (0, \infty)$ and all $\xi \in \mathbb{R}$, for any compact $\mathbb{H}$-hull $K \subseteq r\bar{\mathbb{D}} + \xi$,*

$$\left| g_K(z) - z - \frac{a_K}{z - \xi} \right| \leqslant \frac{Cra_K}{|z - \xi|^2}, \quad |z - \xi| \geqslant 2r. \tag{28}$$

Proof. We shall prove the result in the case $r = 1$ and $\xi = 0$, when $K \subseteq \bar{\mathbb{D}}$. The general case then follows by scaling and translation. We will use the following lemma about harmonic functions,

Lemma 3.14. *Let u be a harmonic function in a domain D and let $z \in D$. Then*

$$\left| \frac{\partial u}{\partial x}(z) \right| \leqslant \frac{4\|u\|_\infty}{\pi \operatorname{dist}(z, \partial D)}.$$

Proof. It will suffice to show that, for all $\varepsilon > 0$, the estimate holds with 4 replaced by $4(1 + \varepsilon)$. Fix $\varepsilon > 0$. By scaling and translation, we reduce to the case where $z = 0$ and $\operatorname{dist}(0, \partial D) = 1 + \varepsilon$. Then u is continuous on $\bar{\mathbb{D}}$ so, for $z \in \mathbb{D}$, by Kakutani's formula,

$$u(z) = \int_0^{2\pi} u(e^{i\theta}) h_{\mathbb{D}}(z, \theta) d\theta,$$

where $h_{\mathbb{D}}(z, \theta)$ denotes the density of harmonic measure in $\mathbb{D}$, viewed from z, evaluated at θ. Using the explicit form (given in (18)) of the Möbius map $\phi_{z,0}$ from $\mathbb{D}$ to $\mathbb{D}$ and which sends z to 0, it is not hard to get an exact formula for $h_{\mathbb{D}}(z, \theta)$: we find

$$h_{\mathbb{D}}(z, \theta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} = \frac{1}{2\pi} \frac{1 - x^2 - y^2}{(\cos\theta - x)^2 + (\sin\theta - y)^2}, \quad 0 \leqslant \theta < 2\pi.$$

Differentiating this formula with respect to z, we see that $\nabla h_{\mathbb{D}}(\cdot, \theta)$ is bounded on a neighbourhood of 0, uniformly in θ, with

$$\nabla h_{\mathbb{D}}(0, \theta) = \frac{1}{\pi} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Hence we may differentiate under the integral sign to obtain

$$\nabla u(0) = \frac{1}{\pi}\int_0^{2\pi} u(e^{i\theta})\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} d\theta.$$

Then

$$\left|\frac{\partial u}{\partial x}(0)\right| \leqslant \frac{\|u\|_\infty}{\pi}\int_0^{2\pi}|\cos\theta| d\theta = \frac{4\|u\|_\infty}{\pi} = \frac{4(1+\varepsilon)\|u\|_\infty}{\pi\,\mathrm{dist}(0,\partial D)},$$

as desired. □

We return to the proof of Proposition 28. Let $D = \mathbb{H}\setminus\bar{\mathbb{D}} = \{z\in\mathbb{H} : |z|>1\}$. Write $T = T(H)$ and define for $\theta\in[0,\pi]$

$$a(\theta) = \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_T)).$$

For $z\in D$, using (22) and then the strong Markov property, we have

$$\mathrm{Im}(z - g_K(z)) = \mathbb{E}_z(\mathrm{Im}(B_T)) = \int_0^\pi h_D(z,\theta)a(\theta)d\theta,$$

where $h_D(z,\theta)$ denotes the density of harmonic measure in D viewed from z, evaluated at θ. Consider the conformal isomorphism $g : D\to\mathbb{H}$ given by $g(z) = z + z^{-1}$. Note that $g(e^{i\theta}) = 2\cos\theta$. Then, for $z\in D$ and $w = g(z)$,

$$h_D(z,\theta) = h_{\mathbb{H}}(w, 2\cos\theta)\frac{d}{d\theta}g(e^{i\theta}) = \mathrm{Im}\left(\frac{1}{2\cos\theta - w}\right)\frac{2\sin\theta}{\pi}$$

by the chain rule. Hence

$$\mathrm{Im}(z - g_K(z)) = \int_0^\pi \mathrm{Im}\left(\frac{1}{2\cos\theta - w}\right)\frac{2\sin\theta}{\pi}a(\theta)d\theta.$$

Set

$$a = \int_0^\pi \frac{2\sin\theta}{\pi}a(\theta)d\theta. \tag{29}$$

Consider the holomorphic function f on $H^*\setminus\{0\}$ given by

$$f(z) = g_K^*(z) - z - a/z,$$

and set $v(z) = \mathrm{Im}(f(z))$. Observe that there is a constant $C<\infty$ such that, for all $|z|\geqslant 3/2$ and $\theta\in[0,\pi]$,

$$\left|\frac{1}{w-2\cos\theta} - \frac{1}{z}\right| = \frac{|2\cos\theta - z^{-1}|}{|z||z+z^{-1}-2\cos\theta|} \leqslant \frac{C}{|z|^2}.$$

and hence, for $z\in\mathbb{H}$ with $|z|\geqslant 3/2$,

$$|v(z)| \leqslant \int_0^\pi\left|\frac{1}{w-2\cos\theta} - \frac{1}{z}\right|\frac{2\sin\theta}{\pi}a(\theta)d\theta \leqslant \frac{Ca}{|z|^2}.$$

Since $v(\bar{z}) = -v(z)$, the same bound holds without the restriction $z \in \mathbb{H}$. Then, for $|z| \geqslant 2$, we can apply Lemma 3.14 in the domain $D_z = \{w \in \mathbb{C} : |w| > (3/4)|z|\}$ to obtain, for a new constant $C < \infty$,

$$\left|\frac{\partial v}{\partial x}(z)\right|, \left|\frac{\partial v}{\partial y}(z)\right| \leqslant \frac{Ca}{|z|^3}.$$

By the Cauchy–Riemann equations, the same bound holds for $|f'(z)|$ for all $|z| \geqslant 3/2$. Now $f(z) \to 0$ as $|z| \to \infty$ so, for $|z| \geqslant 2$ we have,

$$|f(z)| = \left|\int_1^\infty f'(tz)zdt\right| \leqslant \frac{Ca}{|z|^2}\int_1^\infty t^{-3}dt = \frac{Ca}{|z|^2}. \tag{30}$$

Hence $zf(z) \to 0$ as $|z| \to \infty$, so $a = a_K$ and (30) is the desired estimate. $\square$

Note from the proof of Proposition 3.14 the formula (29) gives us

$$\mathrm{hcap}(K) = \frac{2}{\pi}\int_0^\pi \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_{T(H)}))\sin\theta d\theta.$$

This shows in particular that $\mathrm{hcap}(K) > 0$ for all non-empty compact $\mathbb{H}$-hulls.

The next result is deeper, relying on Beurling's estimate, which is proved in the appendix as Theorem A.3. It may be considered as a continuity estimate for half-plane capacity and may be skipped on a first reading.

Proposition 3.15. ($\star$) *Suppose $K \subset K'$ are two compact $\mathbb{H}$-hulls, and that* $\mathrm{dist}(z, \partial K \cup \mathbb{R}) \leqslant \varepsilon$ *for all $z \in \partial K'$ and some $\varepsilon > 0$. Then*

$$\mathrm{hcap}(K') \leqslant \mathrm{hcap}(K) + \frac{16}{\pi}\,\mathrm{rad}(K')^{3/2}\varepsilon^{1/2}.$$

Proof. We reduce to the case where $K' \subseteq \mathbb{D}$ by scaling and translation. Let B be a complex Brownian motion starting from $z \in H'$. Write $T = T(H)$ and $T' = T(H')$ and note that $T \geqslant T'$. By Beurling's estimate, for $z \in \partial K'$ and $r > 0$,

$$\mathbb{P}_z(|B_T - z| \geqslant r) \leqslant \mathbb{P}_z(T \geqslant T(z + r\mathbb{D})) \leqslant 2\sqrt{\varepsilon/r}$$

so, using the strong Markov property at T', for $z = e^{i\theta}$ and $\theta \in (0, \pi)$, we have

$$\mathbb{P}_{e^{i\theta}}(|B_T - B_{T'}| \geqslant r) \leqslant 2\sqrt{\varepsilon/r}.$$

Now $|\mathrm{Im}(B_T) - \mathrm{Im}(B_{T'})| \leqslant |B_T - B_{T'}| \wedge 1$, so

$$\mathbb{E}_{e^{i\theta}}|\mathrm{Im}(B_T) - \mathrm{Im}(B_{T'})| = \int_0^1 \mathbb{P}_{e^{i\theta}}(|B_T - B_{T'}| \geqslant r)dr \leqslant 4\sqrt{\varepsilon}.$$

Then, using (29),

$$\begin{aligned}\mathrm{hcap}(K') &= \int_0^\pi \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_{T'}))\frac{2\sin\theta}{\pi}d\theta \\ &\leqslant \int_0^\pi \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_T))\frac{2\sin\theta}{\pi}d\theta + \int_0^\pi 4\sqrt{\varepsilon}\frac{2\sin\theta}{\pi}d\theta = \mathrm{hcap}(K) + \frac{16}{\pi}\sqrt{\varepsilon}.\end{aligned}$$

This concludes the proof. $\square$

3.6 Capacity and half-plane capacity ($\star$)

We discuss a related notion of capacity which is sometimes useful; this may be skipped on a first reading. Define for a compact $\mathbb{H}$-hull K the *capacity from* ∞ *in* $\mathbb{H}$ by

$$\operatorname{cap}(K) = \lim_{y\to\infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \in K).$$

The existence of this limit was shown in (26). It is clear from the definition that

$$\operatorname{cap}(K) \leqslant \operatorname{cap}(K') \quad \text{whenever} \quad K \subseteq K'.$$

We use (26) together with known properties of mapping-out functions to obtain

$$\operatorname{cap}(\bar{\mathbb{D}} \cap \mathbb{H}) = 4, \quad \operatorname{cap}((0, i]) = 2$$

and, for $r \in (0, \infty)$ and $x \in \mathbb{R}$,

$$\operatorname{cap}(rK) = r \operatorname{cap}(K), \quad \operatorname{cap}(K + x) = \operatorname{cap}(K).$$

Proposition 3.16. *Let K be a compact $\mathbb{H}$-hull such that $\bar{K}$ is connected. Then*

$$\operatorname{rad}(K) \leqslant \operatorname{cap}(K) \leqslant 4 \operatorname{rad}(K).$$

Proof. Set $r = \operatorname{rad}(K)$. Then $K \subseteq r\bar{\mathbb{D}} \cap \mathbb{H} + x$ for some $x \in \mathbb{R}$. So (without using connectedness)

$$\operatorname{cap}(K) \leqslant \operatorname{cap}(r\bar{\mathbb{D}} \cap \mathbb{H} + x) = 4r.$$

By translation and scaling we may assume that $r = 1$ and that there exist $s \in (0, 1]$ and $c \in [0, 1]$ such that $s^2 + c^2 = 1$ and $is \in K$ and either $c \in \bar{K}$ or $-c \in \bar{K}$. Set

$$K_0 = (0, is], \quad \rho(K) = \{-x + iy : x + iy \in K\}, \quad \sigma(K) = K \cup \rho(K).$$

Fix $y \in (1, \infty)$ and consider a complex Brownian motion B starting from iy. Note that B cannot hit $S = K_0 \cup [-c, c]$ without first hitting $\overline{\sigma(K)}$. Hence, by symmetry,

$$\mathbb{P}_{iy}(B_{T(H_0)} \in S) \leqslant 2\mathbb{P}_{iy}(B_{T(H)} \in \bar{K}) = 2\mathbb{P}_{iy}(B_{T(H)} \in K).$$

If $c > 0$ then $g_{K_0}(\pm c) = \pm\sqrt{s^2 + c^2} = \pm 1$, whilst if $c = 0$ then $g_{K_0}(0\pm) = \pm 1$. Hence, by Proposition 3.10, in both cases, on multiplying by πy and letting $y \to \infty$, we obtain

$$2 \leqslant 2 \operatorname{cap}(K).$$

$\square$

Proposition 3.17. *Let A and K be disjoint compact $\mathbb{H}$-hulls. Then*

$$\operatorname{cap}(g_A(K)) \leqslant \operatorname{cap}(K).$$

Proof. Write $g_A(iy) = u + iv$ and recall that $v/y \to 1$ and $u \to 0$ as $y \to \infty$. By conformal invariance of Brownian motion, we have

$$\mathbb{P}_{u+iv}(B \text{ hits } g_A(K) \text{ before } \mathbb{R}) = \mathbb{P}_{iy}(B \text{ hits } K \text{ before } A \cup \mathbb{R}) \leqslant \mathbb{P}_{iy}(B \text{ hits } K \text{ before } \mathbb{R}).$$

Now multiply by πy and let $y \to \infty$, using Proposition 3.10, to obtain the desired inequality.

$\square$

4 Chordal Loewner theory

We establish a one-to-one correspondence between continuous real-valued paths $(\xi_t)_{t\geqslant 0}$ and increasing families $(K_t)_{t\geqslant 0}$ of compact $\mathbb{H}$-hulls having a certain local growth property. The null path $\xi_t \equiv 0$ corresponds to $K_t = (0, 2i\sqrt{t}]$. For smooth paths $(\xi_t)_{t\geqslant 0}$ starting from 0, it is known that $K_t = \{\gamma_s : 0 < s \leqslant t\}$ for some continuous simple path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ starting from 0 and such that $\gamma_t \in \mathbb{H}$ for all $t > 0$. In the absence of smoothness, the situation can be more complicated, as we shall see later. In this *chordal* version of the theory, the boundary point ∞ plays a special role as the point towards which the hulls evolve. In the alternative *radial* theory, which we will not discuss, an interior point of the domain plays this special role instead.

4.1 Local growth property and Loewner transform

Let $(K_t)_{t\geqslant 0}$ be a family of compact $\mathbb{H}$-hulls. Say that $(K_t)_{t\geqslant 0}$ is *increasing* if K_s is strictly contained in K_t whenever $s < t$. Assume that $(K_t)_{t\geqslant 0}$ is increasing. Set $K_{t+} = \cap_{s>t} K_s$ and, for $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$.

Definition 4.1. *Say that* $(K_t)_{t\geqslant 0}$ *has the* local growth property *if*

$$\mathrm{rad}(K_{t,t+h}) \to 0 \quad \textit{as } h \downarrow 0 \textit{ uniformly on compacts in } t.$$

This is a type of continuity condition for the growth of $(K_t)_{t\geqslant 0}$ but note that $K_t \setminus K_s$ can be large even when $K_{s,t}$ is small. See Figure 5 for an illustration.

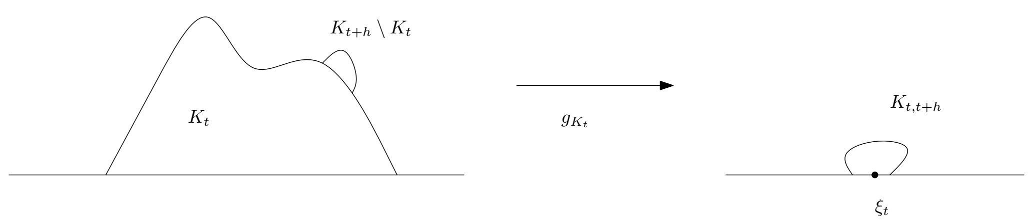

Figure 5: The local growth property and the Loewner transform.

Proposition 4.2. *Let* $(K_t)_{t\geqslant 0}$ *be an increasing family of compact* $\mathbb{H}$*-hulls having the local growth property. Then* $K_{t+} = K_t$ *for all* t*. Moreover, the map* $t \mapsto \mathrm{hcap}(K_t)$ *is continuous and strictly increasing on* $[0, \infty)$*. Moreover, for all* $t \geqslant 0$ *there is a unique* $\xi_t \in \mathbb{R}$ *such that* $\xi_t \in \overline{K_{t,t+h}}$ *for all* $h > 0$*, and the process* $(\xi_t)_{t\geqslant 0}$ *is continuous.*

Proof. Set $K_{t,t+} = g_{K_t}(K_{t+} \setminus K_t)$. For all $t \geqslant 0$ and $h > 0$, we have

$$\mathrm{hcap}(K_{t+h}) = \mathrm{hcap}(K_t) + \mathrm{hcap}(K_{t,t+h}).$$

Now $\mathrm{hcap}(K_{t,t+}) \leqslant \mathrm{hcap}(K_{t,t+h}) \leqslant \mathrm{rad}(K_{t,t+h})^2$. Hence, by the local growth property, $t \mapsto \mathrm{hcap}(K_t)$ is continuous and $\mathrm{hcap}(K_{t,t+}) = 0$, so $K_{t,t+} = \emptyset$ and so $K_{t+} = K_t$. On the other hand $K_{t,t+h} \neq \emptyset$ so $\operatorname{hcap}(K_{t,t+h}) > 0$ and so $t \mapsto \operatorname{hcap}(K_t)$ is strictly increasing on $[0, \infty)$.

For fixed $t \geqslant 0$, the sets $\overline{K_{t,t+h}}$ are compact and decreasing in $h > 0$ so, using the local growth property, they have a unique common element $\xi_t \in \mathbb{R}$. For $t \geqslant 0$ and $h > 0$, choose $z \in K_{t+2h} \setminus K_{t+h}$ and set $w = g_{K_t}(z)$ and $w' = g_{K_{t+h}}(z)$. Then $w \in K_{t,t+2h}$ and $w' \in K_{t+h,t+2h}$, with $w' = g_{K_{t,t+h}}(w)$. Hence

$$|\xi_t - w| \leqslant 2\,\mathrm{rad}(K_{t,t+2h}), \quad |\xi_{t+h} - w'| \leqslant 2\,\mathrm{rad}(K_{t+h,t+2h}), \quad |w - w'| \leqslant 3\,\mathrm{rad}(K_{t,t+h})$$

where we used the continuity estimate (27) for the last inequality. Hence

$$|\xi_{t+h} - \xi_t| \leqslant 2\,\mathrm{rad}(K_{t+h,t+2h}) + 3\,\mathrm{rad}(K_{t,t+h}) + 2\,\mathrm{rad}(K_{t,t+h}) \to 0$$

as $h \to 0$, uniformly on compacts in t. □

Definition 4.3. *The process $(\xi_t)_{t\geqslant 0}$ is called the* **Loewner transform** *of $(K_t)_{t\geqslant 0}$.*

We shall see in the next two subsections that the family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ can be reconstructed from its Loewner transform.

We shall sometimes be presented with a family of compact $\mathbb{H}$-hulls parametrized not by $[0, \infty)$ but by $[0, T)$ for some $T \in (0, \infty)$. The preceding definitions and results transfer immediately to this case. The following result is left as an exercise.

Proposition 4.4. *Let $T, T' \in (0, \infty]$ and let $\tau : [0, T') \to [0, T)$ be a homeomorphism. Let $(K_t)_{t\in[0,T)}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property and having Loewner transform $(\xi_t)_{t\in[0,T)}$. Set $K'_t = K_{\tau(t)}$ and $\xi'_t = \xi_{\tau(t)}$. Then $(K'_t)_{t\in[0,T')}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property and having Loewner transform $(\xi'_t)_{t\in[0,T')}$.*

By Proposition 4.2, the map $t \mapsto \operatorname{hcap}(K_t)/2$ is a homeomorphism on $[0, T)$. On choosing τ as the inverse homeomorphism we obtain a family $(K'_t)_{t\in[0,T')}$ such that $\operatorname{hcap}(K'_t) = 2t$ for all t. We say in this case that $(K'_t)_{t\in[0,T')}$ is **parametrized by half-plane capacity**. The 2 is standard in the literature and is present because of a relation with the radial Loewner theory, which we will not discuss.

Lecture 9; Monday 28 April

4.2 Loewner's differential equation

We now come to Loewner's crucial observation: the local growth property implies that the mapping-out functions satisfy a differential equation.

Proposition 4.5. *Let $(K_t)_{t\geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls, satisfying the local growth property and parametrized by half-plane capacity, and let $(\xi_t)_{t\geqslant 0}$ be its Loewner* *transform. Set $g_t = g_{K_t}$ and $\zeta(z) = \inf\{t \geqslant 0 : z \in K_t\}$. Then, for all $z \in \mathbb{H}$, the function $(g_t(z) : t \in [0, \zeta(z)))$ is differentiable, and satisfies* Loewner's differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}. \tag{31}$$

Moreover, if $\zeta(z) < \infty$, then $g_t(z) - \xi_t \to 0$ as $t \to \zeta(z)$.

Proof. Let $0 \leqslant s < t < \zeta(z)$ and set $z_t = g_t(z)$. Note that $\operatorname{hcap}(K_s) + \operatorname{hcap}(K_{s,t}) = \operatorname{hcap}(K_t)$, so $\operatorname{hcap}(K_{s,t}) = 2(t-s)$. Also, $g_{K_{s,t}}(z_s) = z_t$ and $K_{s,t} \subseteq \xi_s + 2\operatorname{rad}(K_{s,t})\bar{\mathbb{D}}$. We apply Propositions 3.12 and 3.13 to the compact $\mathbb{H}$-hull $K_{s,t}$ to obtain

$$|z_t - z_s| \leqslant 3\operatorname{rad}(K_{s,t}), \tag{32}$$

so $(z_t)_{0 \leqslant t < \zeta(z)}$ is continuous, using the local growth property.

Now if $\zeta(z) < \infty$, then for $s < \zeta(z) < t$ we have $z \in K_t \setminus K_s$, so $z_s \in K_{s,t}$, so $|z_s - \xi_s| \leqslant 2\operatorname{rad}(K_{s,t})$, and so by the local growth property $|z_s - \xi_s| \to 0$ as $s \to \zeta(z)$.

Let $z \in \mathbb{H}$ and suppose $\zeta > t \geqslant s$, hence $\delta = \inf\{|z_u - \xi_u| : u \in [0,t]\} > 0$ by continuity. Choosing $t > s$ close to enough to s such $\operatorname{rad}(K_{s,t}) \leqslant \delta/8$, we see that $|z_s - \xi_s| \geqslant 4\operatorname{rad}(K_{s,t})$, hence

$$\left| z_t - z_s - \frac{2(t-s)}{z_s - \xi_s} \right| \leqslant \frac{4C\operatorname{rad}(K_{s,t})(t-s)}{|z_s - \xi_s|^2}. \tag{33}$$

Then (33) and the local growth property show that $(z_t : t \in [0, \zeta(z)))$ is differentiable with $\dot{z}_t = 2/(z_t - \xi_t)$. □

4.3 Understanding the Loewner transform

This section aims to develop understanding of how the geometry of a curve $(\gamma_t)_{t \geqslant 0}$ is reflected in the Loewner transform $(\xi_t)_{t \geqslant 0}$ of the hulls $(K_t)_{t \geqslant 0}$ given by $K_t = \gamma((0,t])$. Anticipating Section 4.4, where we shall see that the transform determines the hulls, this also sheds some light on how a given choice of transform affects the geometry of any resulting curve. Thus the section is for orientation only, and thus not part of the theoretical development, though it is useful for the intuition.

Fix $\alpha \in (0, \pi/2)$ and take $\gamma(t) = r(t)e^{i\alpha}$, where $r(t)$ is chosen so that $\operatorname{hcap}(K_t) = 2t$. Note that the scaling map $z \mapsto \lambda z$ takes H_t to $H_{\lambda^2 t}$, so the mapping-out functions $g_t = g_{K_t}$ satisfy $g_{\lambda^2 t}(z) = \lambda g_t(z/\lambda)$. Hence, by Loewner's equation, we have $\xi_{\lambda^2 t} = \lambda \xi_t$, so $\xi_t = c_\alpha \sqrt{t}$, where $c_\alpha = \xi_1$. The value of c_α is known, but we shall be content to see:

Proposition 4.6. *We have $c_\alpha > 0$.*

Proof. To see this, fix τ so that $\operatorname{rad}(K_\tau) = 1$ and note that, given $\varepsilon > 0$, we can find $b > 1$ such that $g_\tau(b) \leqslant b + \varepsilon$ and $g_\tau(-b) \geqslant -b - \varepsilon$. Write δ^- for the interval of δH_τ from $-b$ to γ_τ and δ^+ for the interval of δH_τ from γ_τ to b. Then, for $y > 1$

$$\mathbb{P}_{iy}(\hat{B}_{T(H_\tau)} \in \delta^-) \geqslant \mathbb{P}_{iy}(\hat{B}_{T(H_\tau)} \in \delta^+).$$

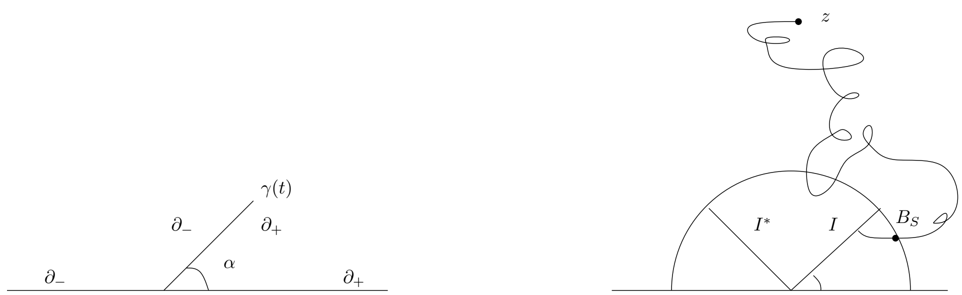

Figure 6: a. ∂_- and ∂_+. b. If $\arg(B_S) \leqslant \alpha$ then $B_T \in \partial_+$.

This is left as an exercise (see Figure 4.3 for the main idea). Now multiply by πy and let $y \to \infty$. By Proposition 3.10, we deduce that

$$g_\tau(\gamma_\tau) - g_\tau(-b) \geqslant g_\tau(b) - g_\tau(\gamma_t).$$

Now $g_\tau(\gamma_\tau) = \xi_\tau = c_\alpha\sqrt{\tau}$, so $2c_\alpha\sqrt{\tau} = 2\xi_\tau \geqslant g_\tau(b) + g_\tau(-b) \geqslant 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies that $c_\alpha \geqslant 0$. But we cannot have $c_\alpha = 0$, since this corresponds to the case $\alpha = \pi/2$. (In fact, c_α is decreasing in α with $c_\alpha \to \infty$ as $\alpha \to 0$.) □

Note the infinite initial velocity required for the Loewner transform needed to achieve a "turn to the right" with greater angle of turn for greater c_α. For a "turn to the left", we take $\xi_t = -c_\alpha\sqrt{t}$. The term "driving function" is sometimes used for the Loewner transform, which may be thought as referring not only to the fact that it drives Loewner's differential equation (31), but also to the fact that it is, literally, a function which indicates how to "turn the wheel".

4.4 Inversion of the Loewner transform

Loewner's differential equation offers the prospect that we might recover the family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ from its Loewner transform $(\xi_t)_{t\geqslant 0}$ by solving the equation, indeed that we might construct such a family $(K_t)_{t\geqslant 0}$ starting from any continuous real-valued function $(\xi_t)_{t\geqslant 0}$. We now show this is true.

Fix a continuous real-valued function $(\xi_t)_{t\geqslant 0}$, which we call the *driving function*. Define for $t \geqslant 0$ and $z \in \mathbb{C} \setminus \{\xi_t\}$

$$b(t, z) = \frac{2}{z - \xi_t} = \frac{2(\bar{z} - \xi_t)}{|z - \xi_t|^2}.$$

Note that $b(t, .)$ is holomorphic on $\mathbb{C} \setminus \{\xi_t\}$ and, for $|z - \xi_t|, |z' - \xi_t| \geqslant 1/n$,

$$|b(t, z) - b(t, z')| \leqslant 2n^2|z - z'|.$$

The following proposition is then a straightforward application of general properties of differential equations. For reasons that will become clear later, while we are mainly interested in solving the differential equation in the upper half-plane, it is convenient to solve it in the entire complex plane.

Proposition 4.7. *For all $z \in \mathbb{C} \setminus \{\xi_0\}$, there is a unique $\zeta(z) \in (0, \infty]$ and a unique continuous map $t \mapsto g_t(z) : [0, \zeta(z)) \to \mathbb{C}$ such that, for all $t \in [0, \zeta(z))$, we have $g_t(z) \neq \xi_t$ and*

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - \xi_s} ds \tag{34}$$

and such that $|g_t(z) - \xi_t| \to 0$ as $t \to \zeta(z)$ whenever $\zeta(z) < \infty$. Set $\zeta(\xi_0) = 0$ and define $C_t = \{z \in \mathbb{C} : \zeta(z) > t\}$. Then, for all $t \geqslant 0$, C_t is open, and $g_t : C_t \to \mathbb{C}$ is holomorphic.

The process $(g_t(z) : t \in [0, \zeta(z)))$ is the *maximal solution starting from* z, and $\zeta(z)$ is its *lifetime*. Define

$$K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}, \quad H_t = \{z \in \mathbb{H} : \zeta(z) > t\} = \mathbb{H} \setminus K_t.$$

Fix $z \in \mathbb{H}$ and $s \leqslant t < \zeta(z)$, set $y_s = \operatorname{Im} g_s(z)$ and $\delta = \inf_{s \leqslant t} |z_s - \xi_s|$. Then $\delta > 0$ and $\dot{y}_s \geqslant -2y_s/\delta^2$ so $y_t \geqslant e^{-2t/\delta^2} y_0 > 0$. Hence $g_t(H_t) \subseteq \mathbb{H}$. Although we have defined the functions ζ and g_t on $\mathbb{C}$ and C_t respectively, *it is convenient to agree from now on that ζ and g_t refer to the restrictions of these functions to $\mathbb{H}$ and H_t*, except where we make explicit reference to a larger domain. The family of maps $(g_t)_{t \geqslant 0}$ is then called the *Loewner flow (in $\mathbb{H}$) with driving function $(\xi_t)_{t \geqslant 0}$*.

Proposition 4.8. *The family of sets $(K_t)_{t \geqslant 0}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Moreover $\operatorname{hcap}(K_t) = 2t$ and $g_{K_t} = g_t$ for all t (in particular g_t is a conformal isomorphism). Moreover the driving function $(\xi_t)_{t \geqslant 0}$ is the Loewner transform of $(K_t)_{t \geqslant 0}$.*

Proof. For $t \geqslant 0$ and $z \in \mathbb{H}$, we have $\operatorname{Im}(b(t, z)) < 0$, so given $T \geqslant 0$ and $w \in \mathbb{H}$, Loewner's differential equation has a unique solution $(z_s : s \in [0, T])$ in $\mathbb{H}$ with given terminal value $z_T = w$. Let us check why. For $0 \leqslant t \leqslant T$, set $\hat{\xi}_t = \xi_{T-t}$ and consider the reverse Loewner equation

$$\dot{w}_t = \frac{-2}{w_t - \hat{\xi}_t} \text{ for } 0 \leqslant t \leqslant T; \qquad w_0 = w. \tag{35}$$

Note the $-$ in the numerator which comes from reversing the direction of time. The vector field $\hat{b}(t, z) = -2/(z - \hat{\xi}_t)$ now satisfies $\operatorname{Im}(\hat{b}(t, w)) \geqslant 0$ hence the differential equation (35) can be solved for all times including on $[0, T]$. Let $z = w_T$ and let us check that $z_t := w_{T-t}$ satisfies the (forward) Loewner equation, starting from z and ends in w at time T. To see this, note that we can write

$$w_t = w + \int_0^t \frac{-2}{w_s - \hat{\xi}_s} ds$$

so $z = w + \int_0^T \frac{-2}{w_s - \hat{\xi}_s} ds$ and

$$z_t = w_{T-t} = z - \int_{T-t}^T \frac{-2}{w_s - \hat{\xi}_s} ds = z + \int_0^T \frac{2}{z_u - \xi_u} du$$

after change of variable $u = T - s$.

Thus $\zeta(z) > T$ and $g_T(z) = w$ and z is the unique point in $\mathbb{H}$ with these properties (by uniqueness of the Cauchy problem for the vector field $\hat{b}$). Hence $g_T : H_T \to \mathbb{H}$ is a bijection. We know that g_T is holomorphic by Proposition 4.7, so g_T is a conformal isomorphism. In particular H_T is simply connected.

We next show that K_T is bounded by obtaining some basic estimates for the Loewner flow. Fix $T \geqslant 0$ and set $r = \sup_{t \leqslant T} |\xi_t - \xi_0| \vee \sqrt{T}$. Fix $R \geqslant 4r$ and take $z \in \mathbb{H}$ with $|z - \xi_0| \geqslant R$. Define

$$\tau = \inf\{t \in [0, \zeta(z)) : |g_t(z) - z| \geqslant r\}.$$

If this set is empty we take $\tau = \zeta(z)$ by convention. Note that τ could in principle be infinity, but at least $0 < \tau \leqslant \zeta(z)$. Furthermore for $t \leqslant \tau$ and $t < \zeta(z)$ we have $|g_t(z) - z| \leqslant r$. Hence

$$|g_t(z) - \xi_t| = |(g_t(z) - z) + (z - \xi_0) + (\xi_0 - \xi_t)| \geqslant R - 2r,$$

thus $\zeta(z) > \tau$. Furthermore,

$$g_t(z) - z = \int_0^t \frac{2}{g_s(z) - \xi_s} ds,$$

so

$$|g_t(z) - z| \leqslant \frac{2t}{R - 2r} \leqslant \frac{t}{r}.$$

If $\tau < T$, then the first estimate implies that $|g_\tau(z) - z| \leqslant \tau/r < T/r \leqslant r$, a contradiction. Hence $\tau \geqslant T$ and then $\zeta(z) > T$ so $z \in H_T$. Since we may choose $R = 4r$, this implies

$$|z - \xi_0| \leqslant 4r \text{ for all } z \in K_T \tag{36}$$

so K_T is bounded and hence is a compact $\mathbb{H}$-hull.

Now let us check that $g_t = g_{K_t}$ and that $\operatorname{hcap}(K_t) = 2t$. We also have:

$$z(g_t(z) - z) - 2t = 2\int_0^t \frac{z - g_s(z) + \xi_s}{g_s(z) - \xi_s} ds,$$

hence

$$|z(g_t(z) - z) - 2t| \leqslant \frac{(4r + 2|\xi_0|)t}{R - 2r}.$$

By letting $R \to \infty$ we see that $z(g_t(z) - z) \to 2t$ as $|z| \to \infty$, for all $t \geqslant 0$. In particular $g_t(z) - z \to 0$ as $|z| \to \infty$, so $g_t = g_{K_t}$ and then $\operatorname{hcap}(K_t) = 2t$ for all t.

It remains to prove the local growth property and identify the Loewner transform. Fix $s \geqslant 0$. Define for $t \geqslant 0$

$$\tilde{\xi}_t = \xi_{s+t}, \quad \tilde{H}_t = g_s(H_{s+t}), \quad \tilde{K}_t = \mathbb{H} \setminus \tilde{H}_t, \quad \tilde{g}_t = g_{s+t} \circ g_s^{-1}.$$

We can differentiate in t to see that $(\tilde{g}_t)_{t \geqslant 0}$ is the Loewner flow driven by $(\tilde{\xi}_t)_{t \geqslant 0}$, $\tilde{H}_t$ is the domain of $\tilde{g}_t$, and $\tilde{K}_t = g_s(K_{s+t} \setminus K_s) = K_{s,s+t}$. The estimate (36) applies to give

$$|z - \xi_s| \leqslant 4\left(\sup_{s \leqslant u \leqslant s+t} |\xi_u - \xi_s| \vee \sqrt{t}\right) \text{ for all } z \in K_{s,s+t}. \tag{37}$$

Hence $(K_t)_{t \geqslant 0}$ has the local growth property and has Loewner transform $(\xi_t)_{t \geqslant 0}$. $\square$

Lecture 10: Friday 28 April

4.5 The Loewner flow on $\mathbb{R}$ characterizes $\bar{K}_t \cap \mathbb{R}$ $(\star)$

By Proposition 3.3, for all $t \geqslant 0$, the map $g_t : H_t \to \mathbb{H}$ extends to a reflection-invariant conformal isomorphism g_t^* on the reflected domain H_t^*. We now show that this is exactly the extended Loewner flow g_t from Proposition 4.7. Later analysis of properties of SLE relies on this property, while the fact that this requires proof has sometimes been overlooked.

For $z \in \mathbb{C}$, define

$$\zeta^*(z) = \inf\{t \geqslant 0 : z \notin H_t^*\}.$$

Proposition 4.9. *We have $\zeta^* = \zeta$ on $\mathbb{C}$. Moreover, $H_t^* = C_t$ and $g_t^* = g_t$ on C_t for all $t > 0$.*

Proof. By taking complex conjugates in (34) and using uniqueness we see that $\zeta(\bar{z}) = \zeta(z)$ on $\mathbb{C}$ and $g_t(\bar{z}) = \overline{g_t(z)}$ for all $z \in \mathbb{C}$ and all $t \in [0, \zeta(z))$. In particular C_t is invariant under conjugation for all t, and $g_t : C_t \to \mathbb{C}$ is a holomorphic extension by reflection of its restriction to H_t for all t. Hence $C_t \subseteq H_t^*$ and g_t is the restriction of g_t^* to C_t for all t.

It remains to show for $t > 0$ and $x \in H_t^0 = H_t^* \cap \mathbb{R}$ that $\zeta(x) > t$. Note first that, for $z \in H_t$ and $r < s \leqslant t$, we have

$$|g_r^*(z) - g_s^*(z)| \leqslant 3 \operatorname{rad}(K_{r,s}) \tag{38}$$

and this estimate extends to H_t^0 by continuity. We will show further that for $x \in H_t^0$

$$\inf_{s \leqslant t} |g_s^*(x) - \xi_s| > 0.$$

This then allows us to pass to the limit $z \to x$ with $z \in H_t$ in (34), to see that $(g_s^*(x) : s \leqslant t)$ satisfies (34), so $\zeta(x) > t$.

Now, for $x \in H_t^0$ and $s < t$, we have $g_s^*(x) \neq \xi_s$. To see this, note that $x \in H_s^0$ so g_s^* is conformal at x, and there is a sequence (w_n) in K_t such that $g_s^*(w_n) \to \xi_s$; then $g_s^*(x) = \xi_s$ would imply $w_n \to x$, which is impossible. The function $s \mapsto |g_s^*(x) - \xi_s|$ is thus continuous on $[0, t]$ and positive on $[0, t)$. It remains to show that it is also positive at t.

Write I for the interval of H_t^0 containing x. Then $g_t^*(I)$ is an open interval containing $g_t^*(x)$. Consider the intervals

$$J_s = \cap_{r \in [s,t]} g_r^*(I), \quad s < t.$$

For s sufficiently close to t, by (38), J_s contains a neighbourhood of $g_t^*(x)$. Hence, if $g_t^*(x) = \xi_t$, then for some $s < t$, we would have $\xi_s \in J_s$, so $\xi_s = g_s^*(y)$ for some $y \in H_t^0$, which we have shown is impossible. $\square$

An immediate corollary is the following characterization of the set of limit points of K_t in $\mathbb{R}$ in terms of the lifetime ζ of the Loewner flow on $\mathbb{R}$.

Proposition 4.10. *For all $x \in \mathbb{R}$ and all $t > 0$, we have*

$$x \in \bar{K}_t \quad \textit{if and only if} \quad \zeta(x) \leqslant t. \tag{39}$$

4.6 Loewner–Kufarev theorem

Write $\mathcal{K}$ for the set of all compact $\mathbb{H}$-hulls. Fix a metric d of uniform convergence on compacts for $C(\mathbb{H}, \mathbb{H})$. We make $\mathcal{K}$ into a metric space using the *Carathéodory metric*

$$d_{\mathcal{K}}(K_1, K_2) = d(g_{K_1}^{-1}, g_{K_2}^{-1}).$$

Write $\mathcal{L}$ for the set of increasing families of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ having the local growth property and such that $\operatorname{hcap}(K_t) = 2t$ for all t. Then $\mathcal{L} \subseteq C([0,\infty), \mathcal{K})$. We fix on $C([0,\infty), \mathcal{K})$ a metric of uniform convergence on compact time intervals.

Theorem 4.11. *There is a bi-adapted homeomorphism $L : C([0,\infty), \mathbb{R}) \to \mathcal{L}$ given by*

$$L((\xi_t)_{t\geqslant 0}) = (K_t)_{t\geqslant 0}, \quad K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}$$

where $\zeta(z)$ is the lifetime of the maximal solution to Loewner's differential equation

$$\dot{z}_t = 2/(z_t - \xi_t)$$

starting from z. Moreover,

$$\bar{K}_t \cap \mathbb{R} = \{x \in \mathbb{R} : \zeta(x) \leqslant t\}$$

where $\zeta(x)$ is the lifetime of the maximal solution to $\dot{x}_t = 2/(x_t - \xi_t)$ starting from x.

Moreover $(\xi_t)_{t\geqslant 0}$ is then the Loewner transform of $(K_t)_{t\geqslant 0}$, given by

$$\{\xi_t\} = \cap_{s>t}\overline{K_{t,s}}, \quad K_{t,s} = g_{K_t}(K_s \setminus K_t) \tag{40}$$

where g_{K_t} is the mapping-out function for K_t.

We call L the *Loewner map*. The proof that L and its inverse are continuous and adapted is left as an exercise. The rest of the theorem recapitulates the results of the preceding two sections.

5 Schramm–Loewner evolutions

We review the arguments which led Schramm to use a Brownian motion as the driving function in Loewner's theory. Then we state the fundamental result of Rohde and Schramm that associates to the resulting family of compact $\mathbb{H}$-hulls a unique continuous path.

5.1 Schramm's theorem

We say that a random variable $(K_t)_{t\geqslant 0}$ in $\mathcal{L}$ is a *Schramm–Loewner evolution*[4] if its Loewner transform is a Brownian motion of some diffusivity $\kappa \in [0,\infty)$. We will refer to such a random family of compact $\mathbb{H}$-hulls as an *SLE*(κ). The Loewner–Kufarev theorem allows us to construct SLE(κ) as $K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}$, where $\zeta(z)$ is the lifetime of the maximal solution to Loewner's differential equation

$$\dot{z}_t = 2/(z_t - \xi_t)$$

starting from z, and where $(\xi_t)_{t\geqslant 0}$ is a Brownian motion of diffusivity κ (i.e., $\xi_t = \sqrt{\kappa} B_t$ for some standard one-dimensional Brownian motion $(B_t)_{t\geqslant 0}$).

Schramm's revolutionary observation was that these processes offered the unique possible scaling limits for a range of lattice-based planar random systems at criticality, such as loop-erased random walk, Ising model, percolation and self-avoiding walk. Such limits had been conjectured but without a candidate for the limit object. Any scaling limit is scale invariant. In fact it was widely conjectured that there would be limit objects, associated to some class of planar domains, with a stronger property of invariance under conformal maps. Moreover, the local determination of certain paths in the lattice models suggested a form of 'domain Markov property'.

There is a natural scaling map on $\mathcal{L}$. For $\lambda \in (0,\infty)$ and $(K_t)_{t\geqslant 0} \in \mathcal{L}$, define $K_t^\lambda = \lambda K_{\lambda^{-2}t}$. Recall that $\operatorname{hcap}(\lambda K_t) = \lambda^2 \operatorname{hcap}(K_t)$. We have rescaled time so that $(K_t^\lambda)_{t\geqslant 0} \in \mathcal{L}$. We say that a random variable $(K_t)_{t\geqslant 0}$ in $\mathcal{L}$ is *scale invariant* if $(K_t^\lambda)_{t\geqslant 0}$ has the same distribution as $(K_t)_{t\geqslant 0}$ for all $\lambda \in (0,\infty)$.

There is also a natural time-shift map on $\mathcal{L}$. For $s \in [0,\infty)$ and $(K_t)_{t\geqslant 0} \in \mathcal{L}$, define $K_t^{(s)} = g_{K_s}(K_{s+t} \setminus K_s) - \xi_s$. Then $(K_t^{(s)})_{t\geqslant 0} \in \mathcal{L}$. We say that a random variable $(K_t)_{t\geqslant 0}$ in $\mathcal{L}$ has the *domain Markov property* if $(K_t^{(s)})_{t\geqslant 0}$ has the same distribution as $(K_t)_{t\geqslant 0}$ and is independent of $\mathcal{F}_s = \sigma(\xi_r : r \leqslant s)$ for all $s \in [0,\infty)$.

Theorem 5.1. *Let $(K_t)_{t\geqslant 0}$ be a random variable in $\mathcal{L}$. Then $(K_t)_{t\geqslant 0}$ is an SLE if and only if $(K_t)_{t\geqslant 0}$ is scale invariant and has the domain Markov property.*

Proof. Write $(\xi_t)_{t\geqslant 0}$ for the Loewner transform of $(K_t)_{t\geqslant 0}$ and note that $(\xi_t)_{t\geqslant 0}$ is continuous. For $\lambda \in (0,\infty)$ and $s \in [0,\infty)$, define $\xi_t^\lambda = \lambda \xi_{\lambda^{-2}t}$ and $\xi_t^{(s)} = \xi_{s+t} - \xi_s$. Then $(K_t^\lambda)_{t\geqslant 0}$ has Loewner transform $(\xi_t^\lambda)_{t\geqslant 0}$ and $(K_t^{(s)})_{t\geqslant 0}$ has Loewner transform $(\xi_t^{(s)})_{t\geqslant 0}$.

[4] In Schramm's papers, SLE stood for *stochastic Loewner evolution*. As usual, our default assumption is that Brownian motion starts at 0.

Hence $(K_t)_{t\geqslant 0}$ has the domain Markov property if and only if $(\xi_t)_{t\geqslant 0}$ has stationary independent increments. Also $(K_t)_{t\geqslant 0}$ is scale invariant if and only if the law of $(\xi_t)_{t\geqslant 0}$ is invariant under Brownian scaling. By the Lévy–Khinchin Theorem[5] , $(\xi_t)_{t\geqslant 0}$ has both these properties if and only if it is a Brownian motion of some diffusivity $\kappa \in [0,\infty)$, that is to say, if and only if $(K_t)_{t\geqslant 0}$ is an SLE. $\square$

5.2 Rohde–Schramm theorem

A continuous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ is said to generate an increasing family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ if $H_t = \mathbb{H}\setminus K_t$ is the unbounded component of $\mathbb{H}\setminus\gamma[0,t]$ for all t, where $\gamma[0,t] = \{\gamma_s : s\in[0,t]\}$. Rohde and Schramm proved the following fundamental and hard result, except for the case $\kappa = 8$, which was then added by Lawler, Schramm and Werner. We refer to the original papers [14, 18] for the proof.

Theorem 5.2. *Let $(K_t)_{t\geqslant 0}$ be an SLE(κ) for some $\kappa\in[0,\infty)$. Write $(g_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$ for the associated Loewner flow and transform. The map $g_t^{-1} : \mathbb{H}\to H_t$ extends continuously to $\bar{\mathbb{H}}$ for all $t\geqslant 0$, almost surely. Moreover, if we set $\gamma_t = g_t^{-1}(\xi_t)$, then $(\gamma_t)_{t\geqslant 0}$ is continuous and generates $(K_t)_{t\geqslant 0}$, almost surely.*

We call $(\gamma_t)_{t\geqslant 0}$ an *SLE(κ) path*, or simply an SLE(κ), allowing the notation to signal that we mean the path rather than the hulls.

5.3 SLE as a random chord

By a *two-pointed domain* we mean a triple $\boldsymbol{D} = (D, z_0, z_\infty)$, where D is a proper simply connected planar domain and z_0 and z_∞ are distinct points in the Martin boundary δD. Write $\mathcal{D}$ for the set of all two-pointed domains. By a *conformal isomorphism of two-pointed domains* $(D, z_0, z_\infty)\to(D', z_0', z_\infty')$, we mean a conformal isomorphism $\phi : D\to D'$ such that $\phi(z_0) = z_0'$ and $\phi(z_\infty) = z_\infty'$. We call any conformal isomorphism $\sigma : \boldsymbol{D}\to(\mathbb{H}, 0,\infty)$ a *scale* for $\boldsymbol{D}$. By Corollary 2.16, such a scale σ exists for all $\boldsymbol{D}\in\mathcal{D}$. Moreover, for all $\lambda\in(0,\infty)$, the map $z\mapsto\lambda\sigma(z)$ is also a scale for $\boldsymbol{D}$ and, by Corollary 2.11, these are all the scales for $\boldsymbol{D}$.

Fix $\boldsymbol{D} = (D, z_0, z_\infty)\in\mathcal{D}$ and a scale σ for $\boldsymbol{D}$. We call a subset $K\subseteq D$ a ***D**-hull* if $D\setminus K$ is a simply connected neighbourhood of z_∞ in D. Write $\mathcal{K}(\boldsymbol{D})$ for the set of all $\boldsymbol{D}$-hulls. Note that $\mathcal{K}(\mathbb{H}, 0,\infty)$ is simply the set $\mathcal{K}$ of compact $\mathbb{H}$-hulls. The Carathéodory topology on $\mathcal{K}$ is scale invariant. For each choice of scale σ, the map $K\mapsto\sigma(K)$ is a bijection $\mathcal{K}(\boldsymbol{D})\to\mathcal{K}$. We use this bijection to define the Carathéodory topology on $\mathcal{K}(\boldsymbol{D})$, which is then independent of the choice of scale. Similarly, we extend to increasing families of $\boldsymbol{D}$-hulls the notion of the local growth property.

Definition 5.3. *Write $\mathcal{L}(\boldsymbol{D}) = \mathcal{L}(\boldsymbol{D},\sigma)$ for the set of increasing families $(K_t)_{t\geqslant 0}$ of $\boldsymbol{D}$-hulls having the local growth property and such that* $\operatorname{hcap}(\sigma(K_t)) = 2t$ *for all t. We call*

[5] From the Lévy-Khinchin representation, the only continuous Lévy processes are scaled Brownian motions with constant drift, and the scaling invariance forces the drift to vanish.

$\mathcal{L}(\boldsymbol{D})$ *the set of* ***chords*** *in D from z_0 to z_∞ (parametrised by the half-plane capacity induced by the choice of the scale σ).*

The set $\mathcal{L}$ defined in Section 4.6 corresponds to the case $\boldsymbol{D} = (\mathbb{H}, 0, \infty)$ and $\sigma(z) = z$, to which we default unless $\boldsymbol{D} \in \mathcal{D}$ and a scale σ on $\boldsymbol{D}$ are explicitly mentioned. We use on $\mathcal{L}(\boldsymbol{D}, \sigma)$ the topology of uniform convergence on compact time intervals.

Let $(K_t)_{t\geqslant 0} \in \mathcal{L}(\boldsymbol{D}, \sigma)$ be a chord in $\boldsymbol{D}$ with respect to the scale σ, and for $t \geqslant 0$ set $D_t = D \setminus K_t$, which by definition is again a simply connected domain. There is a natural scale on D_t inherited from σ which is simply g_t, where $g_t = g_{\sigma(K_t)} \circ \sigma$ is the Loewner flow (from D to $\mathbb{H}$) of $(K_s)_{s\geqslant 0}$. We can also turn D_t into a two-pointed domain $\boldsymbol{D}_t$ by adjoining the two Martin boundary points, z_t and z_∞, where $z_t = (g_t)^{-1}(\xi_t)$ (here $\xi = (\xi_s)_{s\geqslant 0}$ is the Loewner transform of $(K_s)_{s\geqslant 0}$), and z_∞ is the target of the original chord $(K_s)_{s\geqslant 0}$. Then note that $(K_{t+s})_{s\geqslant 0}$ is a chord in $\boldsymbol{D}_t$ with respect to the scale $\sigma_t := \sigma \circ g_t$.

For $\kappa \in [0, \infty)$, we say that a random variable $(K_t)_{t\geqslant 0}$ in $\mathcal{L}(\boldsymbol{D}, \sigma)$ is an *SLE(κ) in $\boldsymbol{D}$ of scale σ* if the Loewner transform $(\xi_t)_{t\geqslant 0}$ of$(\sigma(K_t))_{t\geqslant 0}$ is a Brownian motion of diffusivity κ. We we will write $\mathcal{F}_t = \sigma(\xi_s : s \leqslant t)$; and write $\mu_{\boldsymbol{D},\sigma}$ for the law of SLE(κ) in the two-pointed domain $\boldsymbol{D} = (D, z_0, z_\infty)$ with respect to the scale σ.

In this language the conformal invariance and domain Markov property of SLE take a striking form, which may be used to characterise the law of SLE.

Theorem 5.4. *Let $(K_t, t \geqslant 0)$ be an SLE(κ) in the two-pointed domain $\boldsymbol{D}$ with respect to scale σ. Let $f : \boldsymbol{D} \to \boldsymbol{D}'$ be a conformal isomorphism of two-pointed domains. Then $f(K_t)_{t\geqslant 0}$ is an SLE(κ) in $\boldsymbol{D}'$ with respect to the scale $\sigma \circ f^{-1}$. In other words,*

$$\mu_{\boldsymbol{D},\sigma} \circ f^{-1} = \mu_{f(\boldsymbol{D}),\sigma\circ f^{-1}}. \tag{41}$$

Furthermore, for any $t \geqslant 0$, given $(K_s)_{0\leqslant s\leqslant t}$, the conditional law of the chord $(K_{t+s})_{s\geqslant 0} \in \mathcal{L}(\boldsymbol{D}_t, \sigma_t)$ is an SLE(κ) in $\boldsymbol{D}_t$ with respect to the scale $\sigma_t = \sigma \circ g_t$. In particular, for every nonnegative Borel function F on the space $\mathcal{L} = \mathcal{L}(\mathbb{H}, 0, \infty)$ of chords in $\mathbb{H}$ with respect to the identity scale ι,

$$\mathbb{E}[F((g_t(K_{t+s}))_{s\geqslant 0})|\mathcal{F}_t] = \int_{\mathcal{L}} F d\mu_{\mathbb{H},\iota}. \tag{42}$$

The first identity (41) encapsulates the conformal invariance of SLE, and the second (42) encapsulates its domain (or conformal) Markov property. Using the strong Markov property of Brownian motion, it is not hard to extend (42) to stopping times T which are a.s. finite.

As already alluded to, these properties also characterise SLE and we get the following result (implicit in Schramm's original paper [20]):

Theorem 5.5. *Suppose given a family $\mu_{\boldsymbol{D},\sigma}$ of laws for each two-pointed domain $\boldsymbol{D}$ and scale σ, and suppose that these measures $\mu_{\boldsymbol{D},\sigma}$ satisfy conformal invariance in the sense of (41), and conformal Markov property in the sense of (42). Then there exists $\kappa \geqslant 0$ such that $\mu_{\boldsymbol{D},\sigma}$ is the law of SLE(κ) in $\boldsymbol{D}$ with respect to scale σ, for any two-pointed domain $\boldsymbol{D}$ and scale σ.*

5.4 SLE and Bessel flow

We now begin an analysis of the properties of SLE. This is achieved not directly but by establishing first some properties of the associated Loewner flow – an approach which will recur below. A particularly important observation will be that, along the real line, the Loewner flow reduces to the flow of Bessel processes (whose definition is explained below) associated with a dimension d depending on κ via the relation $d = 1 + 4/\kappa$. This will be a crucial part of the description of the **phases of SLE** in subsequent sections.

We first recall what is the Bessel **stochastic differential equation** (SDE). Fix $d \in [1, \infty)$, and for a standard one-dimensional Brownian motion $(B_t)_{t \geqslant 0}$ and $x \in (0, \infty)$ consider the SDE on $\mathbb{R}$ described by

$$X_0 = x; \qquad dX_t = dB_t + \frac{d-1}{2X_t} dt; \tag{43}$$

until the time $\tau = \tau(x)$ where the solution X touches zero for the first time (which may be infinite if X never touches zero). Equivalently, X is the unique stochastic process adapted to the (augmented) filtration generated by B such that

$$X_t = x + B_t + \int_0^t \frac{d-1}{2X_s} ds$$

for all $t < \tau(x)$. The existence and pathwise uniqueness of solutions to this equation is guaranteed by the classical theory of stochastic differential equations. If we want to emphasise the dependence of the solution on the starting point x then we call $X_t^{(x)}$ the corresponding unique solution.

Example 5.6. *A classical example of a process satisfying the Bessel SDE is the norm of a d-dimensional Brownian motion. Thus suppose for now $d \geqslant 1$ is an integer, let $x \in \mathbb{R}^d \setminus \{0\}$ and let $B = (B^1, \ldots, B^d)$ be a d-dimensional Brownian motion, and let $X_t = \|B_t\| = \sqrt{(B_t^1)^2 + \ldots + (B_t^d)^2}$. Then X_t is a Bessel process starting from $\|x\|$ (apply Itô's formula).*

In the general case, (43) makes sense even when $d \geqslant 1$ is not an integer (we take $d \geqslant 1$ for simplicity since then the drift is nonnegative). We still call $d \geqslant 1$ the "dimension" of the Bessel process. As may be expected from Example 5.6, the value of d has a considerable impact on the properties of X and in particular on whether X ever hits zero (i.e., whether $\tau < \infty$); the value $d = 2$ is (as we will soon see) critical in that sense (corresponding to the well-known dichotomy between transience and recurrence of Brownian motion).

The notion of solution to the equation (43) can be usefully augmented by considering the all possible solutions to this equation when we vary the starting point $x \in (0, \infty)$, using the same driving Brownian motion. The family $(X^{(x)})_{x>0} = (X_t^{(x)}, t \geqslant 0)_{x>0}$, defined outside of measure zero set for all $x > 0$ and $t \geqslant 0$ simultaneously and jointly measurable in $t \geqslant 0$ and $x > 0$, is called the **flow of solutions** to the Bessel SDE (43) or, more simply, the Bessel flow. It is not hard to see that the flow is ordered: thus if $x < y$ then $\tau(x) \leqslant \tau(y)$ and $X^{(x)} \leqslant X_t^{(y)}$ for all $t \leqslant \tau(x)$. Moreover if $t < \tau(x)$ then $X_t^{(x)} < X_t^{(y)}$ (this follows from solving the Bessel SDE in the reverse direction of time).

Now let us come back to SLE and explain how Bessel flows arise naturally there. Let $\kappa \geqslant 0$ and consider the Loewner flow $(g_t(x) : t \in [0, \zeta(x)), x \in \mathbb{R} \setminus \{0\})$ on $\mathbb{R}$ associated to SLE(κ). Recall that the Loewner transform $(\xi_t)_{t\geqslant 0}$ is a Brownian motion of diffusivity κ. Recall also that, for each $x \in \mathbb{R} \setminus \{0\}$, for all $t \in [0, \zeta(x))$, we have $g_t(x) \neq \xi_t$ and

$$g_t(x) = x + \int_0^t \frac{2}{g_s(x) - \xi_s} ds$$

with $g_t(x) - \xi_t \to 0$ as $t \to \zeta(x)$ whenever $\zeta(x) < \infty$. Set

$$a = \frac{2}{\kappa}, \quad d = 2a + 1 = 1 + \frac{4}{\kappa}, \quad B_t = -\frac{\xi_t}{\sqrt{\kappa}}, \quad \tau(x) = \zeta(x\sqrt{\kappa})$$

and for $t \in [0, \tau(x))$ set

$$X_t(x) = \frac{g_t(x\sqrt{\kappa}) - \xi_t}{\sqrt{\kappa}}.$$

Then $(B_t)_{t\geqslant 0}$ is a standard Brownian motion starting from 0. Moreover, from Loewner's equation we that for all $x \in \mathbb{R} \setminus \{0\}$ and $t \in [0, \tau(x))$, we have $X_t(x) \neq 0$ and

$$X_t(x) = x + B_t + \int_0^t \frac{a}{X_s(x)} ds \tag{44}$$

with $X_t(x) \to 0$ as $t \to \tau(x)$ whenever $\tau(x) < \infty$. Thus $(X_t(x), t \geqslant 0)_{x>0}$ is the **Bessel flow of parameter a and dimension $d = 2a + 1 = 1 + 4/\kappa$ driven by $(B_t)_{t\geqslant 0}$**.

In this context we note once again two simple properties. First, by considering uniqueness of solutions in reversed time, we obtain the following monotonicity property: for $x, y \in (0, \infty)$ with $x < y$, we have $\tau(x) \leqslant \tau(y)$ and $X_t(x) < X_t(y)$ for all $t < \tau(x)$. Second, there is a scaling property. Fix $\lambda \in (0, \infty)$ and set

$$\tilde{B}_t = \lambda B_{\lambda^{-2}t}, \quad \tilde{\tau}(x) = \lambda^2 \tau(\lambda^{-1}x), \quad \tilde{X}_t(x) = \lambda X_{\lambda^{-2}t}(\lambda^{-1}x).$$

Then $(\tilde{B}_t)_{t\geqslant 0}$ is a Brownian motion. Moreover the family of processes $(\tilde{X}_t(x) : t \in [0, \tilde{\tau}(x)), x \in \mathbb{R} \setminus \{0\})$ is the Bessel flow of parameter a driven by $(\tilde{B}_t)_{t\geqslant 0}$, and hence has the same distribution as $(X_t(x) : t \in [0, \tau(x)), x \in \mathbb{R} \setminus \{0\})$.

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The next proposition shows that the behaviour of the Bessel flow depends considerably on the dimension d; in addition to the critical value $d = 2$ which separates transience from recurrence and which we have already mentioned, we find an additional critical value at $d = 3/2$ such that for $3/2 < d < 2$ the solutions reach zero in "clumps", whereas for $1 < d \leqslant 3/2$ the solutions from different starting points hit zero but only one at a time.

Proposition 5.7. *Let $x, y \in (0, \infty)$ with $x < y$. Then*

(a) *for $a \in (0, 1/4]$ (equivalently $1 < d \leqslant 3/2$ and $\kappa \geqslant 8$), we have*

$$\mathbb{P}(\tau(x) < \tau(y) < \infty) = 1;$$

(b) *for $a \in (1/4, 1/2)$ (equivalently $3/2 < d < 2$ and $4 < \kappa < 8$), we have*

$$\mathbb{P}(\tau(x) < \infty) = 1, \quad \mathbb{P}(\tau(x) < \tau(y)) = \phi\left(\frac{y-x}{y}\right)$$

where ϕ is given by

$$\phi(\theta) \propto \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}, \quad \phi(1) = 1; \tag{45}$$

(c) *for $a \in [1/2, \infty)$ (equivalently $d \geqslant 2$ and $\kappa \leqslant 4$, we have*

$$\mathbb{P}(\tau(x) < \infty) = 0.$$

Furthermore, for $a \in (1/2, \infty)$, we have $X_t(x) \to \infty$ as $t \to \infty$ almost surely.

Proof. Fix $x > 0$ and write $X_t = X_t(x)$ and $\tau = \tau(x)$. For $r \in (0, \infty)$ define a stopping time

$$T(r) = \inf\{t \in [0, \tau) : X_t = r\}.$$

Fix $r, R \in (0, \infty)$ and assume that $0 < r < x < R$. Write $S = T(r) \wedge T(R)$ and M^S for the stopped process $M^S = (M_{t \wedge S}, t \geqslant 0)$. Note that $T(r) < \tau$ on $\{\tau < \infty\}$. Also, $X_t \geqslant B_t + x$ for all $t < \tau$, so $T(R) < \infty$ almost surely on $\{\tau = \infty\}$. In particular, $S < \infty$ almost surely.

Assume for now that $a \neq 1/2$. Set $M_t = X_t^{1-2a}$ for $t < \tau$. Note that M^S is uniformly bounded. By Itô's formula

$$dM_t = (1-2a)X_t^{-2a} dX_t - a(1-2a)X_t^{-2a-1} dt = (1-2a)X_t^{-2a} dB_t.$$

Hence M^S is a bounded martingale and by optional stopping

$$x^{1-2a} = M_0 = \mathbb{E}(M_S) = r^{1-2a}\mathbb{P}(X_S = r) + R^{1-2a}\mathbb{P}(X_S = R).$$

Hence

$$\mathbb{P}(X_S = R) = \frac{x^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}}. \tag{46}$$

Note that as $r \downarrow 0$ we have $\{X_S = R\} \uparrow \{T(R) < \tau\}$ and so $\mathbb{P}(X_S = R) \to \mathbb{P}(T(R) < \tau)$. Similarly, $\mathbb{P}(X_S = r) \to \mathbb{P}(T(r) < \infty)$ as $R \to \infty$. For $a \in (0, 1/2)$, we can let $r \to 0$ in (46) to obtain

$$\mathbb{P}(T(R) < \tau) = (x/R)^{1-2a}.$$

Then, letting $R \to \infty$, we deduce that $\mathbb{P}(\tau = \infty) = 0$.

For $a \in (1/2, \infty)$, we consider the limit $r \to 0$. Then (46) forces $\mathbb{P}(X_S = R) \to 1$, so $\mathbb{P}(T(R) < \tau) = 1$ for all R and hence $\mathbb{P}(\tau = \infty) = 1$. Let us now check that $X_t \to \infty$. Note that M is positive and, as a continuous local martingale, M is also a time-change of Brownian motion. Hence $M_t = X_t^{1-2a}$ must converge almost surely as $t \to \infty$, and the total quadratic variation $[M]_\infty = (2a-1)^2 \int_0^\infty X_t^{-4a} dt$ must be finite almost surely. This forces $X_t \to \infty$ as $t \to \infty$ almost surely (since the limit of X_t is known to exist in $[0, \infty]$).

In the case $a = 1/2$, we instead set $M_t = \log X_t$ and argue as above to obtain

$$\log x = \mathbb{P}(X_S = r) \log r + \mathbb{P}(X_S = R) \log R.$$

The same argument as for $a \in (1/2, \infty)$ can then be used to see that $\mathbb{P}(\tau = \infty) = 1$.

Assume from now on that $a \in (0, 1/2)$. It remains to show for $0 < x < y$ that

$$\mathbb{P}(\tau < \tau(y)) = \begin{cases} 1, & \text{if } a \leqslant 1/4 \\ \phi(\frac{y-x}{y}), & \text{if } a > 1/4. \end{cases}$$

Define for $\theta \in [0, 1]$

$$\chi(\theta) = \int_\theta^1 \frac{du}{u^{2-4a}(1-u)^{2a}}.$$

Note that χ is continuous on $[0, 1]$ as a map into $[0, \infty]$, with $\chi(0) < \infty$ for $a \in (1/4, 1/2)$ and $\chi(0) = \infty$ for $a \in (0, 1/4]$. Note also that χ is C^2 on $(0, 1)$, with

$$\chi''(\theta) + 2\left(\frac{1-2a}{\theta} - \frac{a}{1-\theta}\right)\chi'(\theta) = 0. \tag{47}$$

Fix $y > x$ and write $Y_t = X_t(y)$. For $t < \tau$, define $R_t = Y_t - X_t$, $\theta_t = R_t/Y_t$ and $N_t = \chi(\theta_t)$. By Itô's formula

$$\begin{aligned} dR_t &= dY_t - dX_t \\ &= -\frac{aR_t dt}{X_t Y_t}, \\ d(\frac{1}{Y_t}) &= -\frac{dY_t}{Y_t^2} + \frac{1}{Y_t^3} d[Y]_t \\ &= -\frac{dB_t}{Y_t^2} + \frac{1-a}{Y_t^3} dt. \end{aligned}$$

Therefore,

$$\begin{aligned} d\theta_t = d(\frac{R_t}{Y_t}) &= R_t d(\frac{1}{Y_t}) + \frac{1}{Y_t} dR_t + d[\frac{1}{Y_t}, R_t] \\ &= -\frac{R_t}{Y_t^2} dB_t + \frac{R_t}{Y_t^3}(1-a)dt - \frac{aR_t}{X_t Y_t^2} dt \text{ (as } R \text{ is of finite variation)} \\ &= -\frac{\theta_t}{Y_t} dB_t + \frac{\theta_t}{Y_t}(\frac{(1-a)}{Y_t} - \frac{a}{X_t})dt \\ &= -\frac{\theta_t}{Y_t} dB_t + \left(\frac{\theta_t}{Y_t}\right)^2 \left(\frac{1-2a}{\theta_t} - \frac{a}{1-\theta_t}\right) dt \end{aligned}$$

so using again Itô's formula and the differential equation (47) satisfied by χ we finally find

$$dN_t = \chi'(\theta_t)d\theta_t + \frac{1}{2}\chi''(\theta_t)d[\theta_t] = -\frac{\chi'(\theta_t)\theta_t dB_t}{Y_t}.$$

Hence $(N_t : t < \tau)$ is a local martingale. Now N is non-negative and is a time-change of Brownian motion, so N_t must converge to some limit as $t \to \tau$. Since χ is strictly decreasing, it follows that θ_t converges to some limit θ_τ as $t \to \tau$.

Let us check that θ_τ is either 1 (when $\tau < \tau(y)$) or 0 (when $\tau = \tau(y)$). Suppose first that $\tau < \tau(y)$. Then $\theta_\tau = 1$ so $N_\tau = 0$. Conversely suppose that $\tau = \tau(y)$, and let us check that almost surely on that event we have $\theta_\tau = 0$. Indeed, since N is a continuous nonnegative martingale, it converges a.s. hence note that we necessarily have $[N]_\tau < \infty$ almost surely. This quadratic variation can be evaluated as

$$[N]_t = \int_0^t \frac{\chi'(\theta_s)^2\theta_s^2}{Y_s^2} ds.$$

Let A be the event $A = \{\tau = \tau(y)\} \cap \{\theta_\tau > 0\}$ and let us show $\mathbb{P}(A) = 0$. On A, we have $\theta_s \to \theta_\tau > 0$ and $\chi'(\theta_s) \to \chi'(\theta_\tau) > 0$ as $s \uparrow \tau$ so we also have

$$\int_0^{\tau(y)} \frac{ds}{Y_s^2} < \infty.$$

Now consider the random variables

$$A(y) = \int_0^{\tau(y)} \frac{1}{Y_t^2} dt, \quad A_n(y) = \int_{T(2^{-n+1}y)}^{T(2^{-n}y)} \frac{1}{Y_t^2} dt, \quad n \geqslant 1,$$

where $T(r) = \inf\{t > 0 : Y_t = r\}$ (as before, although we had used X instead of Y). By the strong Markov property (of the driving Brownian motion), the random variables $(A_n(y) : n \in \mathbb{N})$ are independent. By the scaling property, they all have the same distribution. Hence, since $A_1(y) > 0$ almost surely, we must have $A(y) = \sum_n A_n(y) = \infty$ almost surely, by the strong law of large numbers. This means that A is contained in an event of probability zero, as desired. We conclude that if $\tau = \tau_y$ then $\theta_\tau = 0$, a.s..

In the case $a \in (0, 1/4]$, $\tau = \tau_y$ would thus imply that $N_t = \chi(\theta_t) \to \infty$ as $t \uparrow \tau$, a contradiction, so $\mathbb{P}(\tau < \tau(y)) = 1$. On the other hand, for $a \in (1/4, 1/2)$, the process N^τ is a bounded martingale so by optional stopping

$$\chi\left(\frac{y-x}{y}\right) = N_0 = \mathbb{E}(N_\tau) = \chi(0)\mathbb{P}(\tau = \tau(y)).$$

This concludes the proof. □

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A variation of the calculation for $\mathbb{P}(\tau(x) < \tau(y))$ allows us to compute $\mathbb{P}(\tau(x) < \tau(-y))$.

Proposition 5.8. *Let $x, y \in (0, \infty)$. Then for $a \in (0, 1/2)$ we have*

$$\mathbb{P}(\tau(x) < \tau(-y)) = \psi\left(\frac{y}{x+y}\right)$$

where ψ is given by

$$\psi(\theta) \propto \int_0^\theta \frac{du}{u^{2a}(1-u)^{2a}}, \quad \psi(1) = 1. \tag{48}$$

Proof. Note that ψ is continuous and increasing on $[0, 1]$ with $\psi(0) = 0$ and $\psi(1) = 1$. Also ψ is C^2 on $(0, 1)$ with

$$\psi''(\theta) + 2a\left(\frac{1}{\theta} - \frac{1}{1-\theta}\right)\psi'(\theta) = 0.$$

Write $X_t = X_t(x)$ and $Y_t = -X_t(-y)$ and set $T = \tau(x) \wedge \tau(-y)$. For $t \leqslant T$ set $R_t = X_t + Y_t$ and $\theta_t = Y_t/R_t$. By Itô's formula, for $t \leqslant T$,

$$dR_t = \frac{aR_t}{X_t Y_t} dt,$$

in particular $t \mapsto R_t$ is increasing. Thus either $\tau(x) < \tau(-y)$, or $\tau(-y) < \tau(x)$ (but we cannot have $\tau(x) = \tau(-y)$; this can also be seen geometrically directly by considering the Loewner evolution).

Define a process $Q = (Q_t)_{t \geqslant 0}$ by setting $Q_t = \psi(\theta_{T \wedge t})$. Then Q is continuous and uniformly bounded. Note that $\theta_T = 1$ if $\tau(x) < \tau(-y)$ and $\theta_T = 0$ if $\tau(-y) < \tau(x)$, and that $Q_T = \theta_T$. By Itô's formula, for $t \leqslant T$,

$$d\theta_t = \frac{a}{R_t^2}\left(\frac{1}{\theta_t} - \frac{1}{1-\theta_t}\right) dt - \frac{dB_t}{R_t}$$

so

$$dQ_t = \psi'(\theta_t) d\theta_t + \frac{1}{2}\psi''(\theta_t) d[\theta_t] = -\frac{\psi'(\theta_t) dB_t}{R_t}.$$

Hence Q is a bounded martingale. By optional stopping

$$\mathbb{P}(\tau(x) < \tau(-y)) = \mathbb{P}(\theta_T = 1) = \mathbb{E}(Q_T) = Q_0 = \psi(\theta_0) = \psi\left(\frac{y}{x+y}\right).$$

This concludes the proof. □

5.5 Hitting probabilities for SLE(κ) on the real line

We translate the results for the Bessel flow back in terms of the path γ of an SLE(κ).

Proposition 5.9. *Let γ be an SLE(κ). Then*

(a) *for $\kappa \in (0, 4]$, we have $\gamma[0, \infty) \cap \mathbb{R} = \{0\}$ almost surely;*

(b) *for* $\kappa \in (4,8)$ *and all* $x, y \in (0,\infty)$, γ *hits both* $[x,\infty)$ *and* $(-\infty,-y]$ *almost surely, and*

$$\mathbb{P}(\gamma \text{ hits } [x,x+y)) = \phi\left(\frac{y}{x+y}\right), \quad \mathbb{P}(\gamma \text{ hits } [x,\infty) \text{ before } (-\infty,-y]) = \psi\left(\frac{y}{x+y}\right)$$

where ϕ *and* ψ *are given by (45) and (48) respectively;*

(c) *for* $\kappa \in [8,\infty)$, *we have* $\mathbb{R} \subseteq \gamma[0,\infty)$ *almost surely.*

Proof. Fix $x, y \in (0,\infty)$ and $t > 0$. If $\gamma[0,t] \cap [x,\infty) = \emptyset$ then by compactness there is a neighbourhood of $[x,\infty)$ in $\mathbb{H}$ disjoint from $\gamma[0,t]$ which is then contained in H_t, so $x \notin \bar{K}_t$, and so $\zeta(x) > t$ by Proposition 4.10. On the other hand, if $\gamma_s \in [x,\infty)$ for some $s \in [0,t]$, then $\gamma_s \in \bar{K}_t$ so $\zeta(x) \leqslant \zeta(\gamma_s) \leqslant t$, also by Proposition 4.10. Hence

$$\{\gamma[0,t] \text{ hits } [x,\infty)\} = \{\zeta(x) \leqslant t\}, \quad \{\gamma \text{ hits } [x,x+y)\} = \{\zeta(x) < \zeta(x+y)\}.$$

Recall that $\zeta(x) = \tau(x/\sqrt{\kappa})$, where τ is the lifetime of the Bessel flow of parameter $a = 2/\kappa$. Thus

$$\{\gamma \text{ hits } [x,\infty)\} = \{\tau(x/\sqrt{\kappa}) < \infty\}, \quad \{\gamma \text{ hits } [x,x+y)\} = \{\tau(x/\sqrt{\kappa}) < \tau((x+y)/\sqrt{\kappa})\}$$

and similarly

$$\{\gamma \text{ hits } [x,\infty) \text{ before } (-\infty,-y]\} = \{\tau(x/\sqrt{\kappa}) < \tau(-y/\sqrt{\kappa})\}.$$

Hence, from Proposition 5.7 we deduce:

(a) if $\kappa \in (0,4]$ then $a \in [1/2,\infty)$, so $\mathbb{P}(\gamma \text{ hits } [x,\infty)) = 0$;

(b) if $\kappa \in (4,8)$ then $a \in (1/4,1/2)$, so

$$\mathbb{P}(\gamma \text{ hits } [x,\infty)) = 1, \quad \mathbb{P}(\gamma \text{ hits } [x,x+y)) = \phi\left(\frac{y}{x+y}\right)$$

and

$$\mathbb{P}(\gamma \text{ hits } [x,\infty) \text{ before } (-\infty,-y]) = \psi\left(\frac{y}{x+y}\right);$$

(c) if $\kappa \in [8,\infty)$ then $a \in (0,1/4)$, so $\mathbb{P}(\gamma \text{ hits } [x,x+y)) = 1$.

Hence, in case (a),

$$\mathbb{P}(\gamma \text{ hits } \mathbb{R} \setminus \{0\}) = \lim_{n\to\infty} \mathbb{P}(\gamma \text{ hits } (-\infty,-1/n] \cup [1/n,\infty)) = 0$$

and, in case (c), we see that, almost surely, for all rationals $x, y \in (0,\infty)$, we have $\gamma_t \in [x,x+y)$ for some $t \geqslant 0$. Since γ is continuous, this implies that $[0,\infty) \subseteq \gamma[0,\infty)$ almost surely, and then $\mathbb{R} \subseteq \gamma[0,\infty)$ almost surely by symmetry. $\square$

5.6 Phases of SLE

Recall that one can scale a standard Brownian motion, either in time or space, to obtain a Brownian motion of any diffusivity. Thus "all Brownian motions look the same". In contrast, as the parameter κ is varied, SLE(κ) runs through three phases where it exhibits markedly different behaviour. The following results are proved in [14]. We will present proofs below for some of the easier cases.

Theorem 5.10. *Let $(\gamma_t)_{t\geqslant 0}$ be an SLE. Then $|\gamma_t| \to \infty$ as $t \to \infty$, almost surely.*

Theorem 5.11. *Let $(\gamma_t)_{t\geqslant 0}$ be an SLE(κ). Then*

(a) *for $\kappa \in [0,4]$, $(\gamma_t)_{t\geqslant 0}$ is a simple path almost surely;*

(b) *for $\kappa \in (4,8)$, $\cup_{t\geqslant 0} K_t = \mathbb{H}$ almost surely, but for each given $z \in \bar{\mathbb{H}} \setminus \{0\}$, $(\gamma_t)_{t\geqslant 0}$ does not hit z almost surely;*

(c) *for $\kappa \in [8,\infty)$, $\gamma[0,\infty) = \bar{\mathbb{H}}$ almost surely.*

The behaviour in case (b) is called *swallowing*, while in (c) we see that $(\gamma_t)_{t\geqslant 0}$ is a space-filling curve. We already saw in Proposition 5.9 that $\mathbb{R} \subseteq \gamma[0,\infty)$ almost surely when $\kappa \in [8,\infty)$ but will not prove the stronger statement (c) in these notes.

Using Proposition 5.9 and the form of the function ϕ introduced from Proposition 5.7 it is not hard to formulate a guess for the so-called **Hausdorff dimension** (or in fact Minkowski dimension, to be more precise) of the intersection of the SLE trace with the real line.

Example 5.12. *Let γ be an SLE(κ) curve with $\kappa \in (4,8)$. Fix $x \in (0,1)$ and $\varepsilon > 0$, and break the interval $I = (x,1)$ into intervals of size ε, so $I_i = [x + i\varepsilon, x + (i+1)\varepsilon)$, for $i = 0,\dots,\lfloor (1-x)/\varepsilon \rfloor$. Let $N_\varepsilon = \#\{i : I_i \cap \gamma[0,\infty) \neq \emptyset\}$ denote the number of such intervals hit by the SLE curve.*

Then

$$\mathbb{E}(N_\varepsilon) \asymp \varepsilon^{-h}, \text{ where } h = 2 - \frac{8}{\kappa} \in (0,1).$$

This suggests that $\dim(\gamma[0,\infty) \cap \mathbb{R}) = h = 2 - 8/\kappa$, a.s. This result was proved by Alberts and Sheffield [1]; the above observation essentially provides the upper bound in this result; a complementary lower bound is obtained from a second moment argument which requires controlling the probability that the SLE curves intersects two such intervals. This should be compared with a celebrated result of Beffara [2], which states that the dimension of the SLE trace in $\mathbb{H}$ is given by $\min(1+\kappa/8, 2)$:

$$\dim(\gamma[0,\infty)) = \min(1+\kappa/8, 2),$$

almost surely. This result is however much harder to prove.

5.7 Simple phase

Proposition 5.13. *Let $(\gamma_t)_{t\geqslant 0}$ be an SLE(κ), with $\kappa \in (0,4]$. Then $(\gamma_t)_{t\geqslant 0}$ is a simple path almost surely.*

Proof. Recall the notation $K_{s,s+t} = g_s(K_{s+t} \setminus K_s)$ and $K_t^{(s)} = K_{s,s+t} - \xi_s$. By the domain Markov property, $(K_t^{(s)})_{t\geqslant 0}$ is an SLE(κ). By the Rohde–Schramm theorem, almost surely, for all rational $s \geqslant 0$ and all $t \geqslant 0$, $g^{-1}_{K_{s,s+t}}$ extends continuously to $\bar{\mathbb{H}}$ and $g^{-1}_{K_{s,s+t}}(z) \to \gamma_t^{(s)} + \xi_s$ as $z \to \xi_{s+t}$ with $z \in \mathbb{H}$. Let E be the event that for all such $s \geqslant \mathbb{Q} \cap [0,\infty)$ and for all $t > 0$, $\gamma_t^{(s)} \in \mathbb{H}$. Then by Proposition 5.9, since $\kappa \leqslant 4$, E has probability 1.

Let us suppose that E holds and fix $0 \leqslant r < r'$ arbitrary; let us show that $\gamma_r \neq \gamma_{r'}$.

Since $\mathrm{hcap}(K_t) = 2t$ for all $t \geqslant 0$, almost surely, there is no non-degenerate interval on which $(\gamma_t)_{t\geqslant 0}$ is constant. Therefore (using the continuity of γ) we can find a rational $s \in (r, r')$ such that $\gamma_s \neq \gamma_r$. Take $t = r' - s > 0$. Thus $\gamma_t^{(s)} \in \mathbb{H}$ on the event E. Hence, by Rohde–Schramm again,

$$\gamma_{s+t} = \lim_{z\to\xi_{s+t}, z\in\mathbb{H}} g_s^{-1}(g^{-1}_{K_{s,s+t}}(z)) = g_s^{-1}(\gamma_t^{(s)} + \xi_s).$$

so $\gamma_{r'} \in H_s \subseteq H_r$ and therefore $\gamma_{r'} \neq \gamma_r$, as desired. □

Lemma 5.14. *Let $(\gamma_t)_{t\geqslant 0}$ be a simple path in $\mathbb{H} \cup \{0\}$ starting from 0. Write $(\xi_t)_{t\geqslant 0}$ and $(g_t)_{t\geqslant 0}$ for the Loewner transform and flow associated to $(\gamma(0,t])_{t\geqslant 0}$, as usual. Fix $r \in (0,1)$, set $\tau = \inf\{t \geqslant 0 : |\gamma_t - 1| = r\}$ and suppose that $\tau < \infty$. Then*

$$|g_\tau(1) - \xi_\tau| \leqslant r.$$

Proof. Write $\gamma_\tau = a + ib$ and consider the line segments $I = (a, a+ib]$ and $J = [a \wedge 1, 1]$. Now g_τ extends continuously to $\mathbb{R}\setminus\{0\}$ and to γ_τ, with $g_\tau(\gamma_\tau) = \xi_\tau$. So the image $g_\tau(I\cup J)$ is a continuous path in $\bar{\mathbb{H}}$ joining ξ_τ and $[g_\tau(1), \infty)$. So, by conformal invariance of Brownian motion,

$$\begin{aligned}\mathbb{P}_{g_\tau(iy)}(B_{T(\mathbb{H})} \in [\xi_\tau, g_\tau(1)]) &\leqslant \mathbb{P}_{g_\tau(iy)}(B_{T(\mathbb{H}\setminus g_\tau(I))} \in g_\tau(I \cup J)) \\ &= \mathbb{P}_{iy}(B_{T(H_\tau\setminus I)} \in I \cup J) \leqslant \mathbb{P}_{iy}(\hat{B}_{T(\mathbb{H}\setminus I)} \in I^+ \cup J)\end{aligned}$$

where I^+ denotes the right side of I. Note that $g_I(a+ib) = a$ and $g_I(a+) = a+b$, and $g_I(1) = a+r$ when $a \leqslant 1$, so $\mathrm{Leb}(g_I(I^+ \cup J)) \leqslant r$. Recall that $g_\tau(iy) - iy \to 0$ as $y \to \infty$. Then, by Proposition 3.10, on multiplying by πy and letting $y \to \infty$, we obtain the desired estimate. □

Proposition 5.15. *Let γ be an SLE(κ), with $\kappa \in (0,4)$. Then $|\gamma_t| \to \infty$ as $t \to \infty$, almost surely.*

Proof. By Proposition 5.7, we know that $\inf_{t\geqslant 0}(g_t(1) - \xi_t) > 0$ almost surely. So, by the lemma, we must have, $\inf_{t\geqslant 0}|\gamma_t - 1| > 0$ almost surely. We know that g_1 extends continuously to $\mathbb{R} \setminus \{0\}$ and that $\gamma_1 \in \mathbb{H}$. Set $a^\pm = \lim_{x \downarrow 0} g_1(\pm x)$. Then $a^- < \xi_1 = g_1(\gamma_1) < a^+$. Set $r^\pm = \inf_{t \geqslant 0} |\gamma_t^{(1)} + \xi_1 - a^\pm|$ and set

$$N^\pm = \{z \in H_1 : |g_1(z) - a^\pm| < r^\pm\}, \quad N = N^- \cup \gamma(0,1] \cup N^+.$$

Then $\gamma_t \notin N$ for all $t \geqslant 0$. By scaling and the Markov property, $r^\pm > 0$ almost surely. Since $[0,1] \cup \gamma(0,1]$ and $[-1,0] \cup \gamma(0,1]$ are simple paths, N is a neighbourhood of 0 in $\mathbb{H}$. Then $\liminf_{t \to \infty} |\gamma_t|$ is almost surely positive, and hence infinite, by scaling. $\square$

5.8 Swallowing phase

Proposition 5.16. *Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(κ), with $\kappa \in (4,8)$. Then $(\gamma_t)_{t \geqslant 0}$ is not a simple curve, nor a space-filling curve, almost surely.*

Proof. By Lemma 5.7, for any $x > 0$,

$$\mathbb{P}(\gamma \text{ hits } [x,\infty)) = 1,$$

and

$$\mathbb{P}(\gamma \text{ hits } [x,y]) = \mathbb{P}(\zeta(x) < \zeta(y)) = \phi\left(\frac{y-x}{y}\right) \in (0,1).$$

Hence $\gamma_{\zeta(x)} \in (x,\infty)$ almost surely. Moreover, for $y > x$, we have $\{\gamma_{\zeta(x)} < y\} = \{\zeta(y) > \zeta(x)\}$ and $\{\gamma_{\zeta(x)} \geqslant y\} = \{\zeta(y) = \zeta(x)\}$ and both events have positive probability. In particular, we see that γ hits any given interval in $\mathbb{R}$ of positive length with positive probability. Now if S_1 is the set of all limit points of $g_1(\partial K_1 \cap \mathbb{H})$, then S_1 is an interval of positive length containing ξ_1. Thus we can find a subinterval $I \subset S_1$ such that $d(\xi_1, I) > 0$. Then by the above observation $g_1(\gamma(1,\infty)) \cap I$ is nonempty with positive probability. On the other hand, some topological considerations show that $\partial K_1 \cap \mathbb{H} \subseteq \gamma[0,1]$, so γ has double points with positive probability and hence almost surely by a zero-one argument (see below).

On the other hand, on $\{\gamma_{\zeta(x)} > y\}$, there is a neighbourhood of $[x,y]$ in $\mathbb{H}$ which does not meet γ and $\operatorname{dist}([x,y], H_{\zeta(x)}) > 0$. In particular, γ is not space-filling, with positive probability, and then almost surely.

Here is an elaboration of the zero-one argument for double points. Define, for $t > 0$, $A_t = \{\gamma_s = \gamma_{s'} \text{ for some distinct } s, s' \in [0,t]\}$. Then the sets A_t are non-decreasing in t and all have the same probability, p say, by scaling. But then $p = \mathbb{P}(\cap_t A_t)$ and $\cap_t A_t \in \mathcal{F}_{0+}$, where $\mathcal{F}_{0+} = \cap_{t>0} \sigma(\xi_s : s \leqslant t)$. But, by Blumenthal's zero-one law, $\mathcal{F}_{0+}$ contains only null sets and their complements. Hence $p \in \{0,1\}$. $\square$

Proposition 5.17. *Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(κ), with $\kappa \in (4,8)$. Then $\operatorname{dist}(0, H_t) \to \infty$, in particular $|\gamma_t| \to \infty$, as $t \to \infty$, almost surely.*

Proof. The set S of limit points of $g_{\zeta(1)}(z)$ as $z \to 0$, $z \in \mathbb{H}$ is a compact (possibly empty) subset of $(-\infty, \xi_{\zeta(1)})$. Pick $y < \inf S$. With positive probability, $\operatorname{dist}(S, g_{\zeta(1)}(H_{\zeta(y)})) > 0$, so $\text{dist}(0, H_{\zeta(y)}) > 0$, so $\mathbb{P}(\text{dist}(0, H_t) > 0) = \delta$ for some $t > 0$ and $\delta > 0$. This extends to all t by scaling, with the same δ. So $\mathbb{P}(\text{dist}(0, H_t) > 0 \text{ for all } t > 0) = \delta$ and then $\delta = 1$ by a zero-one argument. Finally $\text{dist}(0, H_t)$ is non-decreasing and, for all $r < \infty$, as $t \to \infty$,

$$\mathbb{P}(\text{dist}(0, H_t) \leqslant r) = \mathbb{P}(\text{dist}(0, H_1) \leqslant r/\sqrt{t}) \to 0.$$

$\square$

6 Locality and restriction

6.1 Conformal transformations of Loewner evolutions

A conformal isomorphism ϕ of initial domains in $\mathbb{H}$ takes one family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ to another $(\phi(K_t))_{t<T}$, defined up to the time T when $(K_t)_{t\geqslant 0}$ leaves the initial domain. We show that the local growth property is preserved under such a transformation, and obtain formulae for the half-plane capacity and Loewner transform of $(\phi(K_t))_{t<T}$.

6.1.1 Initial domains

By an *initial domain* (in $\mathbb{H}$) we mean a set $N\cup I$ where $N \subseteq \mathbb{H}$ is a simply connected domain and $I \subseteq \mathbb{R}$ is an open interval, such that N is a neighbourhood of I in $\mathbb{H}$. Thus $I \subseteq N^0$ in the notation of Section 3.2. An *isomorphism of initial domains* is a homeomorphism $\phi : N\cup I \to \tilde{N}\cup\tilde{I}$ which restricts to a conformal isomorphism $N \to \tilde{N}$. By Proposition 3.1, if $I \neq \mathbb{R} \neq \tilde{I}$, then, given points $x \in I$ and $\tilde{x} \in \tilde{I}$, there is a unique such isomorphism with $\phi(x) = \tilde{x}$, which then extends to a reflection-invariant conformal isomorphism $\phi^* : N_I^* \to \tilde{N}_{\tilde{I}}^*$. In this section, we suppose given an isomorphism of initial domains $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ and a compact $\mathbb{H}$-hull K with $\bar{K} \subseteq N \cup I$. Write $I = (x^-, x^+)$. Define

$$\tilde{K} = \phi(K), \quad \tilde{H} = \mathbb{H} \setminus \tilde{K}, \quad N_K = g_K(N \setminus K), \quad I_K = (g_K^*(x^-), g_K^*(x^+)).$$

Note that $\tilde{H}$ is not the image of $H = \mathbb{H} \setminus K$ under ϕ, nor is I_K the image of I under g_K^*. Nevertheless, we now show that $\tilde{H}$ is simply connected and N_K is a neighbourhood of I_K in $\mathbb{H}$. You are advised to sketch an example as you follow the results in this section. The proofs could be skipped in a first reading.

Proposition 6.1. *The set $\tilde{K}$ is a compact $\mathbb{H}$-hull with $\bar{\tilde{K}} \subseteq \tilde{N} \cup \tilde{I}$ and the set $N_K \cup I_K$ is an initial domain.*

Proof $(\star)$. Since ϕ^* is a homeomorphism and $\bar{K} \subseteq N \cup I$, we have $\bar{\tilde{K}} = \phi^*(\bar{K}) \subseteq \tilde{N} \cup \tilde{I}$. Since $\bar{K}$ is compact, this also shows that $\tilde{K}$ is bounded.

Pick $x \in I$ and consider the conformal isomorphism $\psi : \mathbb{D} \to N_I^*$ such that $\psi(0) = x$ and $\psi'(0) > 0$. Fix $r \in (0, 1)$ and for $\theta \in [0, \pi]$ define $p(\theta) = \psi(re^{i\theta})$. Then $p = (p(\theta) : \theta \in (0, \pi))$ is a simple curve in N and $p(0), p(\pi) \in I$. We can and do choose r so that $p(\theta) \in H^*$ for all $\theta \in [0, \pi]$. Then $\phi(p)$ and $g_K(p)$ are simple curves in $\mathbb{H}$ which each disconnect $\mathbb{H}$ in two components. Write D_0 for the bounded component of $\mathbb{H} \setminus g_K(p)$ and D_1 for the unbounded component of $\mathbb{H} \setminus \phi(p)$. Then $D_1 \cup \phi(p)$ is simply connected and $D_1 \subseteq \tilde{H}$. On the other hand $D_0 \cup g_K(p)$ is also simply connected and $\phi \circ g_K^{-1}$ is a homeomorphism $D_0 \cup g_K(p) \to \tilde{H} \setminus D_1$. Hence $\tilde{H} = \phi(g_K^{-1}(D_0)) \cup \phi(p) \cup D_1$ is simply connected.

Finally, given $y^-, y^+ \in I \setminus \bar{K}$ with $y^- < y^+$ we can choose r so that $p(0) > y^+$ and $p(\pi) < y^-$. Then D_0 is a neighbourhood of $(g_K^*(y^-), g_K^*(y^+))$ in $\mathbb{H}$. But $D_0 \subseteq N_K$. Hence N_K is a neighbourhood of I_K in $\mathbb{H}$. $\square$

Define $\tilde{N}_{\tilde{K}}$ and $\tilde{I}_{\tilde{K}}$ analogously to N_K and I_K and define $\phi_K : N_K \to \tilde{N}_{\tilde{K}}$ by

$$\phi_K = g_{\tilde{K}} \circ \phi \circ g_K^{-1}.$$

Proposition 6.2. *The map ϕ_K extends to an isomorphism $N_K \cup I_K \to \tilde{N}_{\tilde{K}} \cup \tilde{I}_{\tilde{K}}$ of initial domains.*

Proof ($\star$). Write I^- and I^+ for the leftmost and rightmost component intervals of the open set $I \setminus \bar{K} \subseteq \mathbb{R}$. Set $J^\pm = g_K^*(I^\pm)$ and $J = J^- \cup J^+$. Define similarly $\tilde{J}^\pm$ and $\tilde{J}$ starting from $\tilde{I}$ and $\tilde{K}$. Then $J \subseteq I_K$ and $I_K \setminus J$ is a compact subset of I_K. A similar statement holds for $\tilde{J}$ and $\tilde{I}_{\tilde{K}}$. Define $\psi : (N_K)^*_J \to (\tilde{N}_{\tilde{K}})^*_{\tilde{J}}$ by $\psi = g^*_{\tilde{K}} \circ \phi^* \circ (g^*_K)^{-1}$. Then ψ is a holomorphic extension of ϕ_K which takes J^- to $\tilde{J}^-$ and J^+ to $\tilde{J}^+$. Since N_K is a neighbourhood of I_K in $\mathbb{H}$, we have $I_K \subseteq \hat{N}_K$ by Proposition 3.1, and similarly $\tilde{I}_{\tilde{K}} \subseteq \hat{\tilde{N}}_{\tilde{K}}$. Write $\hat{\phi}_K$ for the extension of ϕ_K as a homeomorphism $\hat{N}_K \to \hat{\tilde{N}}_{\tilde{K}}$. Then $\hat{\phi}_K = \psi$ on J, so we must have $\hat{\phi}_K(I_K) = \tilde{I}_{\tilde{K}}$, and so ϕ extends to a homeomorphism $N_K \cup I_K \to \tilde{N}_{\tilde{K}} \cup \tilde{I}_{\tilde{K}}$ as required. $\square$

Recall from Proposition 3.8 the scaling property $\operatorname{hcap}(rK) = r^2 \operatorname{hcap}(K)$. This makes it plausible, for a conformal isomorphism ϕ of some initial domain $N \cup I$ and for a small hull K near $\xi \in I$, that $\phi'(\xi)^2 \operatorname{hcap}(K)$ is a good approximation for $\operatorname{hcap}(\phi(K))$. We now prove such an estimate, in a normalized form.

Proposition 6.3. *There is an absolute constant $C < \infty$ with the following property. Let $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ be an isomorphism of initial domains. Assume that $0 \in I$ and $\phi(0) = 0$ and $\phi'(0) = 1$. Let $K \subseteq N$ be a compact $\mathbb{H}$-hull. Suppose that for some $0 < r < \varepsilon < R < \infty$ we have*

$$K \cup \phi(K) \subseteq r\mathbb{D}, \quad (\varepsilon\mathbb{D}) \cap \mathbb{H} \subseteq N \cup \tilde{N} \subseteq R\mathbb{D}.$$

Then

$$1 - CrR/\varepsilon^2 \leqslant \frac{\operatorname{hcap}(\phi(K))}{\operatorname{hcap}(K)} \leqslant 1 + CrR/\varepsilon^2.$$

Proof ($\star$). It will suffice to prove the upper bound. The lower bound then follows by interchanging the roles of $N \cup I$ and $\tilde{N} \cup \tilde{I}$. Recall the formula (29), valid for $K \subseteq \mathbb{D}$,

$$\operatorname{hcap}(K) = \int_0^\pi \mathbb{E}_{e^{i\theta}}(\operatorname{Im} B_{T(H)}) \frac{2\sin\theta}{\pi} d\theta.$$

Fix $\alpha \geqslant 1$. Since $K \subseteq r\mathbb{D}$, we can apply this to $\sigma^{-1}K$ for $\sigma \in [r, \alpha r]$ and use the scale invariance of Brownian motion to obtain

$$\sigma \operatorname{hcap}(K) = \int_0^\pi \mathbb{E}_{\sigma e^{i\theta}}(\operatorname{Im} B_{T(H)}) \frac{2\sigma\sin\theta}{\pi} \sigma d\theta.$$

Next, integrate over σ to obtain

$$\frac{(\alpha^2-1)r^2}{2} \operatorname{hcap}(K) = \int_{S(r,\alpha r)} \mathbb{E}_z(\operatorname{Im} B_{T(H)}) \frac{2\operatorname{Im} z}{\pi} A(dz), \tag{49}$$

where $A(dz)$ denotes area measure and $S(r, \alpha r)$ is the half-annulus $\{z \in \mathbb{H} : r \leqslant |z| \leqslant \alpha r\}$. Set $\psi = \phi^{-1}$. By conformal invariance of Brownian motion,

$$\mathbb{E}_w(\operatorname{Im} B_{T(\tilde{H})}) = \mathbb{E}_{\psi(w)}(\operatorname{Im} \phi(B_{T(H)})).$$

Apply the identity (49) to $\phi(K)$, replacing r by $\rho \geqslant r$ and taking $\alpha = 2$ to obtain

$$\begin{aligned} \frac{3\rho^2}{2} \operatorname{hcap}(\phi(K)) &= \int_{S(\rho,2\rho)} \mathbb{E}_{\psi(w)}(\operatorname{Im} \phi(B_{T(H)})) \frac{2 \operatorname{Im} w}{\pi} A(dw) \\ &= \int_{\psi(S(\rho,2\rho))} \mathbb{E}_z(\operatorname{Im} \phi(B_{T(H)})) \frac{2 \operatorname{Im} \phi(z)}{\pi} |\phi'(z)|^2 A(dz) \end{aligned} \tag{50}$$

where we made the change of variable $z = \psi(w)$ for the second equality.

We apply Cauchy's integral formula to ϕ^* and ψ^* to see that, for $|z| \leqslant 1/2$, we have $|\phi''(z)| \leqslant 8R$ and $|\psi''(z)| \leqslant 8R$. Then, by Taylor's theorem, using $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0) = 1$ and the fact that ϕ is real on I, we obtain for $|z| \leqslant 1/2$

$$|\phi'(z)| \leqslant 1 + 8R|z|, \quad \operatorname{Im} \phi(z) \leqslant (1 + 16|z|R) \operatorname{Im} z, \quad |\psi(z) - z| \leqslant 4R|z|^2.$$

Assume that $48rR \leqslant 1$ and take $\alpha = 2(1 + 48rR)$ then $\alpha \leqslant 4$. Note that $r \leqslant 2r - 4R(2r)^2$. Set $\rho = \inf\{s \geqslant r : r = s - 4Rs^2\}$. Then $\rho \leqslant 2r \leqslant 1/4$. Hence, for $z \in S(\rho, 2\rho)$, we have $|\psi(z)| \geqslant \rho - 4R\rho^2 = r$ and $|\psi(z)| \leqslant 2\rho + 16R\rho^2 = 2r + 24R\rho^2 \leqslant \alpha r \leqslant 4r \leqslant 1/2$ so $\psi(S(\rho, 2\rho)) \subseteq S(r, \alpha r)$. A comparison of (49) and (50) then yields

$$\operatorname{hcap}(\phi(K)) \leqslant (1 + 16rR)(1 + 64rR)(1 + 32rR)^2(1 + 192rR) \operatorname{hcap}(K)$$

which in turns yields the claimed estimate for a suitable choice of the constant C. $\square$

More generally, for any isomorphism of initial domains $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$, any $\xi \in I$, and any compact $\mathbb{H}$-hull $K \subseteq N$, the preceding estimate can be applied to the map $\bar{\phi}(z) = \phi'(\xi)^{-1}(\phi(z + \xi) - \phi(\xi))$ to obtain the estimate

$$(1 - \bar{C}rR/\varepsilon^2)\phi'(\xi)^2 \operatorname{hcap}(K) \leqslant \operatorname{hcap}(\phi(K)) \leqslant (1 + \bar{C}rR/\varepsilon^2)\phi'(\xi)^2 \operatorname{hcap}(K) \tag{51}$$

where $\bar{C} = C \max\{\phi'(\xi)^2, \phi'(\xi)^{-2}\}$, whenever $K \subseteq \xi + r\mathbb{D}$ and $\phi(K) \subseteq \phi(\xi) + r\mathbb{D}$ and

$$\xi + (\varepsilon\mathbb{D}) \cap \bar{\mathbb{H}} \subseteq N \cup I \subseteq \xi + R\mathbb{D}, \quad \phi(\xi) + (\varepsilon\mathbb{D}) \cap \bar{\mathbb{H}} \subseteq \tilde{N} \cup \tilde{I} \subseteq \phi(\xi) + R\mathbb{D}.$$

The details are left as an exercise.

6.1.2 Loewner evolution and isomorphisms of initial domains

Let $(K_t)_{t \geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls with the local growth property. Write $(\xi_t)_{t \geqslant 0}$ for the Loewner transform of $(K_t)_{t \geqslant 0}$. Let $N \cup I$ and $\tilde{N} \cup \tilde{I}$ be initial domains, with $\xi_0 \in I$ and let $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ be an isomorphism. Set $T = \inf\{t \geqslant 0 : \bar{K}_t \not\subseteq N \cup I\}$.

For $t < T$, we consider the compact $\mathbb{H}$-hull $\tilde{K}_t = \phi(K_t)$ and other associated objects, as in the preceding section, writing now

$$g_t = g_{K_t}, \quad \tilde{g}_t = g_{\tilde{K}_t}, \quad \phi_t = \phi_{K_t} = \tilde{g}_t \circ \phi \circ g_t^{-1}, \quad \tilde{\xi}_t = \phi_t(\xi_t)$$

and

$$N_t = N_{K_t}, \quad I_t = I_{K_t}, \quad \tilde{N}_t = \tilde{N}_{\tilde{K}_t}, \quad \tilde{I}_t = \tilde{I}_{\tilde{K}_t}.$$

Proposition 6.4. *The increasing family of compact $\mathbb{H}$-hulls $(\tilde{K}_t)_{t<T}$ has the local growth property and has Loewner transform $(\tilde{\xi}_t)_{t<T}$.*

Proof $(\star)$. Fix $t_0 \in [0, T)$. Let ψ be as in the proof of Proposition 6.1 and choose $r \in (0, 1)$ so that $K_{t_0} \subseteq \psi(r\mathbb{D})$. It will suffice to prove the proposition with $N \cup I$ replaced by $\psi(r\mathbb{D} \cap \bar{\mathbb{H}})$, which is the bounded component of $\bar{\mathbb{H}} \setminus \{\psi(re^{i\theta}) : \theta \in [0, \pi]\}$, and with ϕ replaced by its restriction to $\psi(r\mathbb{D} \cap \bar{\mathbb{H}})$. Hence we may assume without loss that $N \cup I$ is the the bounded component of $\bar{\mathbb{H}} \setminus p$, for some simple curve $p = (p(\theta) : \theta \in [0, \pi])$ with $p(0), p(\pi) \in \mathbb{R}$ and $p(\theta) \in \mathbb{H}$ for all $\theta \in (0, \pi)$, and that ϕ extends to a homeomorphism $\bar{N} \to \bar{\tilde{N}}$.

For $t \leqslant t_0$ and $z, z' \in N \setminus K_t$, we have

$$|g_t(z) - g_t(z')| \leqslant |g_t(z) - z| + |z - z'| + |z' - g_t(z')| \leqslant 6\,\mathrm{rad}(K_t) + 2\,\mathrm{rad}(N) \leqslant 8\,\mathrm{rad}(N).$$

Hence, using a similar estimate for $\tilde{N}$ and reflection symmetry, we have[6]

$$N_t^* \subseteq \xi_t + R\mathbb{D}, \quad \tilde{N}_t^* \subseteq \tilde{\xi}_t + R\mathbb{D}. \tag{52}$$

where $R = 8 \max\{\mathrm{rad}(N), \mathrm{rad}(\tilde{N})\} < \infty$. The maps

$$(t, \theta) \mapsto |g_t^*(p(\theta)) - \xi_t|, \quad (t, \theta) \mapsto |\tilde{g}_t^*(\phi(p(\theta))) - \tilde{\xi}_t|$$

are continuous and positive on $[0, t_0] \times [0, \pi]$, hence are bounded below, by $\varepsilon > 0$ say. Then, for all $t \leqslant t_0$, we have

$$\xi_t + \varepsilon\mathbb{D} \subseteq N_t^*, \quad \tilde{\xi}_t + \varepsilon\mathbb{D} \subseteq \tilde{N}_t^*. \tag{53}$$

Since $\phi_t^* : N_t^* \to \tilde{N}_t^*$ is a conformal isomorphism, it follows by Cauchy's integral formula that

$$|\phi_t'(z)| \leqslant 2R/\varepsilon, \quad z \in \xi_t + (\varepsilon/2)\mathbb{D} \cap \mathbb{H}. \tag{54}$$

Now, for all $r \in (0, \varepsilon/2]$, we can find $h > 0$ such that, for all $t \leqslant t_0$, we have

$$K_{t,t+h} \subseteq \xi_t + r\mathbb{D} \tag{55}$$

and then, setting $\rho = 2R/\varepsilon$,

$$\tilde{K}_{t,t+h} = \phi_t(K_{t,t+h}) \subseteq \tilde{\xi}_t + \rho r\mathbb{D}. \tag{56}$$

Hence $(\tilde{K}_t)_{t \leqslant t_0}$ has the local growth property and has Loewner transform $(\tilde{\xi}_t)_{t \leqslant t_0}$. $\square$

[6] Only one of the two inclusions in (52) and one of those in (53) are used in this proof. We shall need all of them for the proof of Proposition 6.5.

Proposition 6.5. *For all $t \in [0, T)$, we have*[7]

$$\operatorname{hcap}(\tilde{K}_t) = \int_0^t \phi_s'(\xi_s)^2 d(\operatorname{hcap}(K_s)). \tag{57}$$

Proof ($\star$). Fix $t_0 \in [0, T)$ and follow the same reduction as in Proposition 6.4, introducing constants R, ε and $\rho = 2R/\varepsilon$. For $t \leqslant t_0$, from (54), we see that $|\phi_t'(\xi_t)| \leqslant \rho$. On the other hand, by considering the inverse map $\psi_t^* : \tilde{N}_t^* \to N_t^*$, we obtain similarly $|\phi_t'(\xi_t)| = |\psi_t'(\tilde{\xi}_t)|^{-1} \geqslant 1/\rho$. Given $\delta \in (0, 1]$, choose $r > 0$ so that $CrR\rho^3 \leqslant \varepsilon^2\delta$. There exists an $h > 0$ such that, for all $t \leqslant t_0$,

$$K_{t,t+h} \subseteq \xi_t + r\mathbb{D}.$$

Then, using the estimates (52), (53), (55) and (56), for $s \in (0, h)$, we can apply the estimate (51) to the isomorphism $\phi_t : N_t \cup I_t \to \tilde{N}_t \cup \tilde{I}_t$ and the compact $\mathbb{H}$-hull $K_{t,t+s}$ to obtain

$$(1 - \delta)\, \phi_t'(\xi_t)^2 \operatorname{hcap}(K_{t,t+s}) \leqslant \operatorname{hcap}(\tilde{K}_{t,t+s}) \leqslant (1 + \delta)\, \phi_t'(\xi_t)^2 \operatorname{hcap}(K_{t,t+s}).$$

Now, for all $n \in \mathbb{N}$, setting $s = t_0/n$, we have

$$\operatorname{hcap}(\tilde{K}_{t_0}) = \sum_{j=0}^{n-1} \operatorname{hcap}(\tilde{K}_{js,(j+1)s}).$$

For $n > t_0/h$, we can apply the bounds just obtained with $t = js$ and sum over j to obtain

$$(1 - \delta) \sum_{j=0}^{n-1} \phi_{js}'(\xi_{js})^2 \operatorname{hcap}(K_{js,(j+1)s}) \leqslant \operatorname{hcap}(\tilde{K}_{t_0}) \leqslant (1 + \delta) \sum_{j=0}^{n-1} \phi_{js}'(\xi_{js})^2 \operatorname{hcap}(K_{js,(j+1)s}).$$

Let $n \to \infty$ and then $\delta \to 0$ to obtain the claimed identity. $\square$

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Proposition 6.6. *The set $S = \{(t, z) : t \in [0, T), z \in N_t \cup I_t\}$ is open in $[0, \infty) \times \bar{\mathbb{H}}$. The function $(t, z) \mapsto \phi_t(z)$ on S is differentiable in t for all z, with derivative given by*

$$\dot{\phi}_t(z) = \frac{2\phi_t'(\xi_t)^2}{\phi_t(z) - \phi_t(\xi_t)} - \phi_t'(z)\frac{2}{z - \xi_t}, \quad z \in N_t \cup I_t \setminus \{\xi_t\} \tag{58}$$

and

$$\dot{\phi}_t(\xi_t) = -3\phi_t''(\xi_t). \tag{59}$$

[7] A shorter proof of this formula is possible using Proposition 3.15 to compare $\operatorname{hcap}(\tilde{K}_{t,t+h})$ and $\operatorname{hcap}(\phi_t'(\xi_t)K_{t,t+h})$, provided $\operatorname{rad}(K_{t,t+h})^{5/2}/\operatorname{hcap}(K_{t,t+h}) \to 0$ as $h \to 0$ uniformly on compacts in t. The estimate (37) shows this condition holds provided $(\xi_t)_{t \geqslant 0}$ is Hölder of exponent greater than $2/5$, so this covers the case of SLE. We have given the longer argument to avoid any spurious condition and because it is also more elementary, in that it does not rely on Beurling's estimate, used for Proposition 3.15.

Moreover, $\dot{\phi}_t$ is holomorphic on $N_t \cup I_t$, with derivative given by

$$\dot{\phi}_t'(z) = 2\left(-\frac{\phi_t'(\xi_t)^2\phi_t'(z)}{(\phi_t(z)-\phi_t(\xi_t))^2} + \frac{\phi_t'(z)}{(z-\xi_t)^2} - \frac{\phi_t''(z)}{z-\xi_t}\right), \quad z \in N_t \cup I_t \setminus \{\xi_t\} \tag{60}$$

and

$$\dot{\phi}_t'(\xi_t) = \frac{1}{2}\frac{\phi_t''(\xi_t)^2}{\phi_t'(\xi_t)} - \frac{4}{3}\phi_t'''(\xi_t). \tag{61}$$

Proof. By Propositions 4.11 and 6.2, when reparametrized by hcap, $(\tilde{g}_t)_{t<T}$ satisfies Loewner's equation driven by $(\tilde{\xi}_t)_{t<T}$. So, by Proposition 6.5, we obtain

$$\dot{\tilde{g}}_t(z) = 2\phi_t'(\xi_t)^2/(\tilde{g}_t(z) - \tilde{\xi}_t), \quad z \in \tilde{H}_t.$$

Set $f_t = g_t^{-1}$ and differentiate the equation $f_t(g_t(z)) = z$ in t to obtain

$$\dot{f}_t(z) = -2f_t'(z)/(z-\xi_t), \quad z \in \mathbb{H}.$$

For $z \in N_t$ we have $\phi_t(z) = \tilde{g}_t(\phi(f_t(z)))$. By the chain rule, for $t \in [0,T)$ and $z \in N_t$, we see that $\phi_t(z)$ is differentiable in t, with derivative given by (58), which is then holomorphic in z with derivative given by (60). Note that the functions on the right hand sides of (58) and (60) are continuous in $z \in N_t \cup I_t \setminus \{\xi_t\}$. It is straightforward to check using l'Hôpital's rule that they extend continuously to ξ_t with the values given in (59) and (61). Then for $x \in I_t$ and $z \in N_t$, the functions $\phi_s(z)$ and $\dot{\phi}_s(z)$ and $\dot{\phi}_s'(z)$ converge as $z \to x$ locally uniformly for s near t. The result follows by standard arguments. □

6.2 SLE(6), locality and percolation

6.2.1 Locality of SLE(6)

SLE(6) has a special invariance property called *locality* which can be understood informally as meaning that, in its general formulation as a measure on chords in (D, z_0, z_1), it does not know what domain it is in beyond the fact that, each time it hits the boundary δD, it turns towards its endpoint z_1, as it must do in order to satisfy the non-crossing property. By Smirnov's theorem SLE(6) is the scaling limit of critical site percolation on the planar hexagonal lattice. Thus, if the upper half-plane is tiled with yellow and blue hexagons, with the colours at each site independent and equally likely, and if we place blue hexagons along the positive real axis and yellow ones along the negative real axis, then the unique blue/yellow interface joining 0 and ∞ converges weakly to an SLE(6) in the limit of small lattice spacing. The lattice model has its own obvious locality property, so the fact that locality implies $\kappa = 6$ for SLE was an early clue towards Smirnov's result.

Theorem 6.7. *Let $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ be an isomorphism of initial domains with $0 \in I$ and $0 = \phi(0) \in \tilde{I}$. Let $(\gamma_t)_{t\geqslant 0}$ be an SLE(κ) or some $\kappa \geqslant 0$. Set*

$$T = \inf\{t \geqslant 0 : \gamma_t \notin N \cup I\}, \quad \tilde{T} = \inf\{t \geqslant 0 : \gamma_t \notin \tilde{N} \cup \tilde{I}\}.$$

Then $(\phi(\gamma_t))_{t<T}$ in its canonical reparametrization has the same distribution as $(\gamma_t)_{t<\tilde{T}}$ if and only if $\phi(z) = \sigma z + c$ for some $\sigma > 0, c \in \mathbb{R}$, or $\kappa = 6$.

Proof. Suppose ϕ is not of the form $\sigma z + c$ for any $\sigma > 0, c \in \mathbb{R}$. Write $(K_t)_{t \geqslant 0}$ for the family of compact $\mathbb{H}$-hulls generated by $(\gamma_t)_{t \geqslant 0}$ and write $(\xi_t)_{t \geqslant 0}$ for its Loewner transform, which is a Brownian motion of diffusivity κ. For $t < T$, set $\tilde{K}_t = \phi(K_t)$ and $\phi_t = g_{\tilde{K}_t} \circ \phi \circ (g_{K_t})^{-1}$. By Propositions 6.2 and 6.5, $(\tilde{K}_t)_{t<T}$ is a family of compact $\mathbb{H}$-hulls having the local growth property, whose Loewner transform $(\tilde{\xi}_t)_{t<T}$ and half-plane capacity are given by

$$\tilde{\xi}_t = \phi_t(\xi_t), \quad \operatorname{hcap}(\tilde{K}_t) = 2 \int_0^t \phi_s'(\xi_s)^2 ds.$$

The set $S_0 = \{(t, x) : t \in [0, T), x \in I_{K_t}\}$ is open in $[0, \infty) \times \mathbb{R}$ and $\xi_t \in I_{K_t}$ for all $t < T$. By Proposition 6.6, the adapted random map $(t, x) \mapsto \phi_t(x) : S_0 \to \mathbb{R}$ is $C^{1,2}$ with $\dot{\phi}_t(\xi_t) = -3\phi_t''(\xi_t)$ for all $t < T$. By the generalized Itô formula, we have

$$d\tilde{\xi}_t = \dot{\phi}_t(\xi_t)dt + \phi_t'(\xi_t)d\xi_t + \frac{1}{2}\phi_t''(\xi_t)d[\xi]_t.$$

Since $d[\xi]_t = \kappa dt$,

$$d\tilde{\xi}_t = (\frac{\kappa}{2} - 3)\phi_t''(\xi_t)dt + \phi_t'(\xi_t)d\xi_t.$$

Hence $(\tilde{\xi})_{t<T}$ is a local martingale if and only if $\kappa = 6$ (else $\phi'' = 0$ on some nontrivial interval of the real line where it is analytic, which contradicts our asumption on ϕ). Furthermore in that case, we see that its quadratic variation is $[\tilde{\xi}]_t = 3 \operatorname{hcap}(\tilde{K}_t)$. The canonical reparametrization $(\tilde{K}_{\tau(s)})_{s<S}$ of $(\tilde{K}_t)_{t<T}$ and its Loewner transform $(\eta_s)_{s<S}$ are given by

$$\operatorname{hcap}(\tilde{K}_{\tau(s)}) = 2s, \quad \operatorname{hcap}(\tilde{K}_T) = 2S, \quad \eta_s = \tilde{\xi}_{\tau(s)}.$$

Now (by optional stopping) $(\eta_s)_{s<S}$ is a continuous local martingale (in its own filtration) and its quadratic variation is given by $[\eta]_s = [\tilde{\xi}]_{\tau(s)} = 6s$. Hence, by Lévy's characterization, $(\eta_s)_{s<S}$ extends[8] to a Brownian motion $(\eta_s)_{s \geqslant 0}$ of diffusivity 6. Write $\tilde{\gamma}$ for the SLE(6) driven by $(\eta_s)_{s \geqslant 0}$, then $\phi(\gamma_{\tau(s)}) = \tilde{\gamma}_s$ for $s < S$ and $S = \inf\{s \geqslant 0 : \tilde{\gamma}_s \notin \tilde{N} \cup \tilde{I}\}$. Hence $(\phi(\gamma_{\tau(s)}))_{s<S}$ and $(\gamma_s)_{s<\tilde{T}}$ have the same distribution as required. □

The significance of the above result is perhaps easier to appreciate when we consider SLE(6) as a random chord in a general two pointed domain $\mathbf{D} = (D, z_0, z_1)$ as in Section 5.3. By an *initial domain* $N \cup I$ in such a two-pointed domain $\mathbf{D} = (D, z_0, z_1)$ we mean a simply connected subdomain $N \subseteq D$ along with an interval I of $\delta D \setminus \{z_1\}$ containing z_0 such that N is a neighbourhood of I in D.

If we choose a *scale* σ from (D, z_0, z_1) to $(\mathbb{H}, 0, \infty)$ (which we recall is just a conformal isomorphism from D to $\mathbb{H}$ mapping z_0 to 0 and z_1 to ∞), then $\sigma(N) \cup \sigma(I)$ is an initial domain in $(\mathbb{H}, 0, \infty)$, which is just an initial domain in $\mathbb{H}$ in the sense of Section 6.1.1 such that $0 \in \sigma(I)$. We can now give a precise version of the informal account of locality which began this section.

[8] We know that $T < \infty$ almost surely, so we can do this here without extending the probability space, using $(\xi_{T+t} - \xi_T)_{t \geqslant 0}$.

Corollary 6.8. *Suppose that the two-pointed domains $\mathbf{D} = (D, z_0, z_1)$ and $\tilde{\mathbf{D}} = (\tilde{D}, \tilde{z}_0, \tilde{z}_1)$ sharing an initial domain $N_0 \cup I_0$, and such that $z_0 = \tilde{z}_0$. Fix two scales σ and $\tilde{\sigma}$ on $\mathbf{D}$ and $\tilde{\mathbf{D}}$ respectively.*

Let γ be an SLE(6) in $\mathbf{D}$ with respect to scale σ and let $\tilde{\gamma}$ be an SLE(6) in $\tilde{\mathbf{D}}$ with respect to scale $\tilde{\sigma}$. Let

$$T = \inf\{t \geqslant 0 : \gamma_t \notin N_0 \cup I_0\}, \quad \tilde{T} = \inf\{t \geqslant 0 : \tilde{\gamma}_t \notin N_0 \cup I_0\}.$$

Then $(\gamma_t)_{t<T}$ and $(\tilde{\gamma}_t)_{t<\tilde{T}}$ have the same distribution.

Proof. Set $\psi = \tilde{\sigma}^{-1} \circ \sigma$. Then ψ is a conformal isomorphism mapping D to $\tilde{D}$ and z_0 to $\tilde{z}_0 = z_0$, z_1 to $\tilde{z}_1$. Furthermore if we write $\phi = \psi|_{N_0 \cup I_0}$ then ϕ maps $N_0 \cup I_0$ to $\tilde{N}_0 \cup \tilde{I}_0$, where $\tilde{N}_0 = \psi(N_0)$ and $\tilde{I}_0 = \psi(I_0)$. In fact ϕ is an isomorphism of initial domains.

By definition of SLE(κ) as a random chord with respect to scales σ and $\tilde{\sigma}$, if $\gamma = (\gamma_t)_{t \geqslant 0}$ is an SLE(κ) in (D, z_0, z_1) with respect to scale σ, then $\tilde{\gamma}_t = (\psi(\gamma_t))_{t \geqslant 0}$ is an SLE(κ) in $(\tilde{D}, z_0, \tilde{z}_1)$ with respect to scale $\tilde{\sigma}$. The question is therefore whether $\phi(\gamma_t)_{t<T}$ has the same law as $(\gamma_t)_{t<T}$. Since ϕ is an isomorphism of initial domains, this is indeed the case by Theorem 6.7. □

6.2.2 SLE(6) in an equilateral triangle

While physicists investigated critical percolation using nonrigorous methods, Cardy established a formula for the limiting crossing probabilities of a rectangle: that is, given $L > 0$, Cardy's formula gave a concrete prediction for the limiting probability $p(L)$ (as the mesh size tends to zero) that there exists a path of a single colour crossing from left to right a rectangle $R_L = (0, L) \times (0, 1)$.

Carleson observed that, assuming conformal invariance, this formula became considerably simpler on a triangle Δ with vertices (a, b, c) where $a = 0$, $b = 1$ and $c = e^{i\pi/3}$. Namely, the limiting probability $q(x)$ to observe a crossing between (a, c) and $(b, b+x(c-b))$ is just $q(x) = x$. (Note that, if we denote by d the point $d = b + x(c - b) = 1 + xe^{2i\pi/3}$, there is a single $L > 0$ such that if R is the rectangle $R_L = (0, L) \times (0, 1)$, and if ψ is the unique conformal map sending from Δ to R and mapping a to i, b to 0, and c to $L + i$, then ψ also maps d to L. The assumption of conformal invariance forces $p(L) = q(x)$; thus Carleson's version of Cardy's formula, namely $q(x) = x$, comes from the explicit form of $p(L)$ and that of the conformal map sending R_L to Δ.)

If we imagine that the side (a, c) consists of blue hexagons and the side (a, b) of yellow hexagons, the formula can be rephrased in terms of hitting probability for the interface starting from a which keeps blue hexagons on its left and yellow hexagons on its right: namely, the probability it hits the side (b, c) below the point $(c, b+x(c-b))$ should converge to x. The corresponding formula can be stated as a theorem directly for SLE(6). In turn, since Smirnov proved that Cardy's formula holds in the limit for critical percolation, this provides another way of identifying SLE(6) as the unique possible limit for the scaling limit of cluster interface exploration process in critical percolation (and indeed can be used to prove this fact).

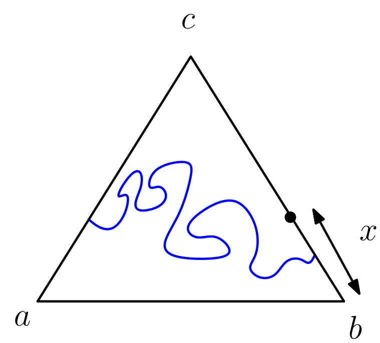

Figure 7: Cardy's formula in an equilateral triangle: the probability of the crossing event in the figure (i.e., to find a connected path of blue hexagons between (a, c) and $(b, b + xe^{2i\pi/3})$) converges to x in the limit of fine mesh size.

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Let Δ be the equilateral triangle with vertices $a = 0$, $b = 1$, $c = e^{i\pi/3}$.

Theorem 6.9. *Let γ be SLE(6) in $(\Delta, 0, 1)$, with respect to some arbitrary scale σ. Then the point X at which γ hits the edge $[b, c] = [1, e^{\pi i/3}]$ is uniformly distributed.*

Proof. Note that the law of the position X does not depend on the choice of scale σ, since changing σ just changes the time parameterisation of the curve.

A conformal map sending $(\mathbb{H}, 0, 1, \infty)$ to $(\Delta, 0, 1, e^{\pi i/3})$ is known explicitly in complex analysis, and is given by the so-called **Schwarz–Christoffel** transformation

$$f(z) = c \int_0^z \frac{dw}{w^{2/3}(1-w)^{2/3}}, \quad c = \frac{\Gamma(2/3)}{\Gamma(1/3)^2}.$$

(To understand this formula consider how the argument of f changes close to the real line, especially near $w = 0$ and 1.) Consider the map $z \mapsto \varphi(z) = 1/(1-z)$. This is a conformal automorphism which cyclically permutes $0, 1, \infty$. The map $z \mapsto g(z) = 1 + e^{2i\pi/3}z$ is a conformal automorphism of Δ which cyclically permutes a, b, c. Thus

$$f(\varphi(z)) = g(f(z)),$$

by uniqueness of the Riemann map. Thus, composing by $\varphi^{-1}(z) = (z-1)/z$, we deduce

$$f(z) = 1 + e^{2i\pi/3} f((z-1)/z)$$

for all $z \in \mathbb{H}$. This identity extends by continuity when $z \to x \in \mathbb{R}$.

Let $x \in (0, 1)$ and choose $y > 0$ so that $f(y/(1+y)) = x$. Then, by conformal invariance and Proposition 5.9 (letting ϕ be the function giving the hitting probabilities in that proposition),

$$\mathbb{P}(X \in [1, 1 + xe^{2i\pi/3}]) = \mathbb{P}(\text{SLE}(6) \text{ in } (\mathbb{H}, 0, \infty) \text{ hits } [1, 1+y]) = \phi\left(\frac{y}{1+y}\right) = x.$$

Thus X is uniform on $[1, e^{i\pi/3}]$. $\square$

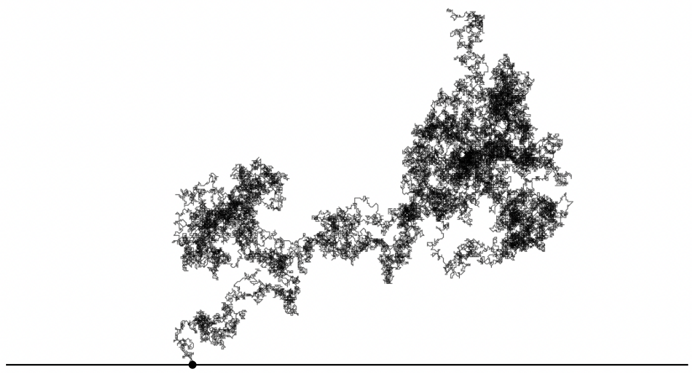

Figure 8: Simulation of a Brownian excursion (by Lawler–Schramm–Werner [11])

6.3 SLE(8/3) and restriction

6.3.1 Brownian excursion in the upper half-plane

Let $z = x + iy \in \bar{\mathbb{H}}$. Let $(X_t)_{t \geqslant 0}$ be a Brownian motion in $\mathbb{R}$, starting from x. Let $(W_t)_{t \geqslant 0}$ be a Brownian motion in $\mathbb{R}^3$, starting from $(y, 0, 0)$, and independent of $(X_t)_{t \geqslant 0}$. Set $R_t = |W_t|$. Then $(R_t)_{t \geqslant 0}$ is a Bessel process of dimension 3 starting from y. Set $E_t = X_t + iR_t$. Thus E satisfies the stochastic differential equation:

$$dE_t = dZ_t + i\frac{1}{\operatorname{Im}(E_t)}dt$$

where Z is a complex Brownian motion.

The process $(E_t : t \geqslant 0)$ is called a *Brownian excursion in* $\mathbb{H}$ *starting from* z, and can be thought of (as we will see below) as a planar Brownian motion starting at $z = 0$ at time $t = 0$, conditioned to remain in $\mathbb{H}$ for all $t > 0$ (even more appropriately, we can think of it as a Brownian motion starting from 0, and conditioned to enter $\mathbb{H}$ immediately and leave $\mathbb{H}$ via ∞).

Whilst this process is of interest in its own right, we introduce it here primarily as a means to study SLE(8/3), in particular using the following formula for the derivative of the mapping-out function. For a compact $\mathbb{H}$-hull A with $0 \notin \bar{A}$, we write ϕ_A for the shifted mapping-out function, given by

$$\phi_A(z) = g_A(z) - g_A(0). \tag{62}$$

Proposition 6.10. *Let A be a compact $\mathbb{H}$-hull with $0 \notin \bar{A}$. Let $(E_t)_{t \geqslant 0}$ be a Brownian excursion in $\mathbb{H}$ starting from 0. Then*

$$\mathbb{P}_0((E_t)_{t \geqslant 0} \textit{ does not hit } A) = \phi_A'(0).$$

Proof. Let $(Z_t)_{t \geqslant 0}$ be a complex Brownian motion starting from $z = x + iy \in \mathbb{H}$. Let $(E_t)_{t \geqslant 0}$ be a Brownian excursion in $\mathbb{H}$, also starting from z. Write $Z_t = X_t + iY_t$, and suppose it is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(\mathcal{F}_t)_{t\geqslant 0}$ denote the associated filtration. Define for $r \geqslant 0$

$$T_r = \inf\{t \geqslant 0 : Y_t = r\}, \quad S_r = \inf\{t \geqslant 0 : \operatorname{Im} E_t = r\}.$$

Lemma 6.11. *Suppose $r > y$. Define a new measure on $(\Omega, \mathcal{F}_{T_r \wedge T_0})$ by setting*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{Y_{T_0 \wedge T_r}}{y} = \frac{r}{y} 1_{\{T_r < T_0\}}. \tag{63}$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Under $\tilde{\mathbb{P}}$, $(Z_t)_{0\leqslant t\leqslant T_0\wedge T_r}$ has the law of a Brownian motion conditioned so that $T_r < T_0$. Furthermore this coincides with the law of $(E_t)_{0\leqslant t\leqslant S_r}$ starting from $z = x + iy$.

Proof. The second equality in (63) is obvious since if $T_0 < T_r$ then $Y_{T_0\wedge T_r} = 0$, so only its complement contributes to the Radon-Nikodym derivative $Y_{T_0\wedge T_r}/y$, in which case it is equal to r/y. To see that $\tilde{\mathbb{P}}$ is a probability measure, simply note that

$$\mathbb{E}_{\mathbb{P}}(\frac{Y_{T_0\wedge T_r}}{y}) = \frac{r}{y}\mathbb{P}(T_r < T_0) = 1$$

by the standard gambler's ruin estimate for one-dimensional Brownian motion. Hence $\tilde{\mathbb{P}}$ has total mass equal to one and is thus a probability measure on $(\Omega, \mathcal{F})$. This also makes it clear that $\tilde{\mathbb{P}}$ is the law of Z conditioned so that $T_r < T_0$; in particular, $\tilde{\mathbb{P}}(T_r < T_0) = 1$.

Set $M_t = y^{-1} Y_{T_0\wedge T_r\wedge t}$. Then $(M_t)_{t\geqslant 0}$ is a bounded non-negative $\mathbb{P}-$martingale with $M_0 = 1$ and with final value $y^{-1}Y_{T_0\wedge T_r} = (r/y)1_{\{T_0 > T_r\}}$. Consequently the Radon-Nikodym derivative of the restriction $\tilde{\mathbb{P}}|_{\mathcal{F}_t}$ with respect to $\mathbb{P}|_{\mathcal{F}_t}$ coincides with M_t.

Under $\tilde{\mathbb{P}}$, the processes $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ remain independent and $(X_t)_{t\geqslant 0}$ is a Brownian motion. It remains to identify the law of Y (under $\tilde{\mathbb{P}}$), which we do using Girsanov's theorem. To do so, note that M is the exponential martingale $M = \mathcal{E}(N)$ associated to the local martingale $dN_t = dM_t/M_t$ with $N_0 = 0$; that is, $M_t = \exp(N_t - \frac{1}{2}[N]_t)$. Thus if we define a process $(B_t)_{t\geqslant 0}$ by

$$\begin{aligned} dB_t &= dY_t - d[Y, N]_t \\ &= dY_t - M_t^{-1} d[M, Y]_t \\ &= dY_t - 1_{\{t\leqslant T_r\}} Y_t^{-1} dt, \quad B_0 = 0, \end{aligned} \tag{64}$$

we deduce from Girsanov's theorem that, under $\tilde{\mathbb{P}}$, $(B_t)_{t\geqslant 0}$ is a local martingale and hence, having the same quadratic variation as $(Y_t)_{t\geqslant 0}$, is a Brownian motion, by Lévy's characterization. Rewriting the equation (64) defining B differently we have

$$dY_t = dB_t + 1_{\{t\leqslant T_r\}} Y_t^{-1} dt,$$

where B is a Brownian motion starting from 0 under $\tilde{\mathbb{P}}$. This is the same stochastic differential equation as that of the three-dimensional Bessel process $(R_t)_{t\leqslant T_r}$.

By the Yamada–Watanabe theorem $(Y_t)_{0\leqslant t\leqslant T_r}$ under $\tilde{\mathbb{P}}$ has the same law as $(\operatorname{Im}(E_t))_{0\leqslant t\leqslant T_r}$ under $\mathbb{P}$. So $(X_t + i\tilde{Y}_t)_{0\leqslant t\leqslant T_r}$ under $\tilde{\mathbb{P}}$ has the same law as $(E_t)_{0\leqslant t\leqslant S_r}$ under $\mathbb{P}$. Hence $(Z_t)_{t\leqslant T_r}$ under $\tilde{\mathbb{P}}$ has the same law as $(E_t)_{t\leqslant S_r}$ under $\mathbb{P}$. This concludes the proof of Lemma 6.11. $\square$

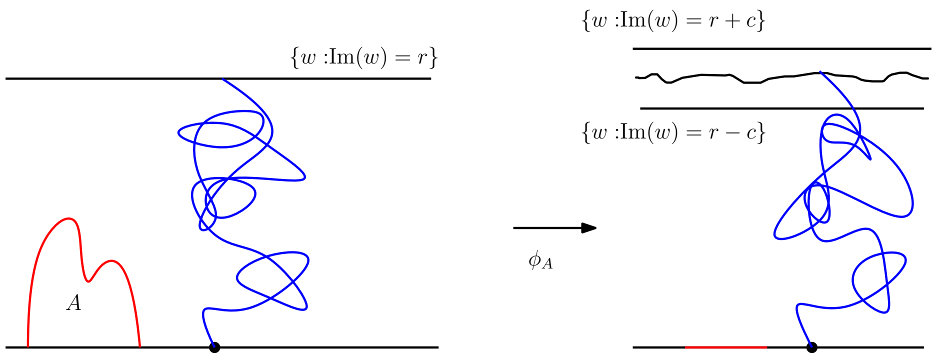

Figure 9: Proof of Proposition 6.10.

Now let us return to the proof of Proposition 6.10. Suppose $z \in \mathbb{H} \setminus A$ and set

$$p_r(z) = \mathbb{P}_z((E_t)_{t \leqslant S_r} \text{ does not hit } A).$$

Then

$$p_r(z) = \tilde{\mathbb{P}}_z((Z_t)_{t \leqslant T_r} \text{ does not hit } A) = \mathbb{E}_z(y^{-1} Y_{T_0 \wedge T_r} 1_{\{T_A > T_r\}}) = (r/y)\mathbb{P}_z(T_r < T_0 \wedge T_A),$$

where $T_A = \inf\{t \geqslant 0 : Z_t \in A\}$. Now $g_A(w) - w \to 0$ as $|w| \to \infty$ and in fact by Proposition 3.12, we know that $|g_A(w) - w| \leqslant 3\,\mathrm{rad}(A)$. Hence

$$|\operatorname{Im} g_A(w) - r| \leqslant c = 3\,\mathrm{rad}(A) \quad \text{whenever} \quad \mathrm{Im}(w) = r$$

and hence, by conformal invariance of Brownian motion and the gambler's ruin formula for one-dimensional Brownian motion,

$$\frac{\operatorname{Im} g_A(z)}{r + c} = \mathbb{P}_{g_A(z)}(T_{r+c} < T_0) \leqslant \mathbb{P}_z(T_r < T_0 \wedge T_A) \leqslant \mathbb{P}_{g_A(z)}(T_{r-c} < T_0) = \frac{\operatorname{Im} g_A(z)}{r - c}.$$

So

$$\mathbb{P}_z((E_t)_{t \geqslant 0} \text{ does not hit } A) = \lim_{r \to \infty} p_r(z) = \operatorname{Im} g_A(z)/y.$$

Note that $\operatorname{Im} g_A(z)/y \to g'_A(0) > 0$ as $z \to 0$ in $\mathbb{H}$. Take now $z = 0$, fix $\varepsilon > 0$ with $A \cap \varepsilon\mathbb{D} = \emptyset$, and set

$$S = \inf\{t \geqslant 0 : |E_t| = \varepsilon\},$$

then $|E_S| = \varepsilon$ and $\operatorname{Im} E_S > 0$ almost surely. Hence, by the strong Markov property of $(E_t)_{t \geqslant 0}$ and bounded convergence, as $\varepsilon \to 0$,

$$\mathbb{P}_0((E_t)_{t \geqslant 0} \text{ does not hit } A) = \mathbb{E}(\operatorname{Im} g_A(E_S)/\operatorname{Im}(E_S)) \to g'_A(0) = \phi'_A(0),$$

which concludes the proof. $\square$

6.3.2 Restriction measures

Lecture 15 from Friday 19 May

We shift attention from compact $\mathbb{H}$-hulls to a different class of subsets in $\mathbb{H}$. A *filling* is any connected set K in $\mathbb{H}$ having 0 and ∞ as limit points in $\hat{\mathbb{H}}$ and such that $\mathbb{H} \setminus K$ is the union of two simply connected domains D^- and D^+ which are neighbourhoods of $(-\infty, 0)$ and $(0, \infty)$ in $\mathbb{H}$ respectively. Write S for the set of all such fillings.

Let us introduce a suitable σ-algebra on the set of fillings. Write $\mathcal{N}$ for the set of simply connected domains which are neighbourhoods of both 0 and ∞ in $\mathbb{H}$. For $D \in \mathcal{N}$, define $S_D = \{K \in S : K \subseteq D\}$. Set $\mathcal{A} = \{S_D : D \in \mathcal{N}\}$ and write $\mathcal{S}$ for the σ-algebra on S generated by $\mathcal{A}$.

Definition 6.12. *A random filling K is an $(S, \mathcal{S})$-random variable.*

Example 6.13. *Given a Brownian excursion $(E_t)_{t\geqslant 0}$, consider the set K which is the union of $K_0 = \{E_t : t \in (0, \infty)\}$ and all the bounded components of $\mathbb{H} \setminus K_0$. (In words, we "fill all the holes" of E).*

Then the sets $\{K \subseteq D\}$ for $D \in \mathcal{N}$ are all measurable. Hence K is a random filling.

By definition of the σ-algebra on S and a π-system argument, the law of a random filling is entirely determined by probabilities $\mathbb{P}(K \in S_D)$ for $D \in \mathcal{N}$. In other words, it is determined by $\mathbb{P}(K \cap A = \emptyset)$ for A an arbitrary compact $\mathbb{H}$-hull such that $0 \notin \bar{A}$ (write $\mathcal{Q}_0$ for the set of such hulls). Recall from (62) that given $A \in \mathcal{Q}_0$, we denote $\phi_A(z) = g_A(z) - g_A(0)$.

Definition 6.14. *Let $\alpha > 0$. A random filling K (or rather its law) is called a* restriction measure of exponent α *if there exists $\alpha > 0$ such that $\mathbb{P}(K \cap A = \emptyset) = \phi'_A(0)^\alpha$ for any $A \in \mathcal{Q}_0$.*

Note that if a restriction measure of exponent α exists, its law is unique.

Example 6.15. *By Proposition 6.10, the law of a Brownian excursion is a restriction measure with exponent $\alpha = 1$.*

The reason we care about restriction measures is because of the following lemma.

Lemma 6.16. *Let K be sampled from a restriction measure with exponent $\alpha > 0$. Let $A \in \mathcal{Q}_0$. Then conditionally on $K \cap A = \emptyset$, the law of $\phi_A(K)$ is again a restriction measure with exponent α (and so has the same law as the unconditional law of K).*

Proof. Fix $B \in \mathcal{Q}_0$. Then

$$\begin{aligned}
\mathbb{P}(K \cap A = \emptyset, \phi_A(K) \cap B = \emptyset) &= \mathbb{P}(K \bigcap (A \cup \phi_A^{-1}(B)) = \emptyset) \\
&= \phi'_{\tilde{A}}(0)^\alpha
\end{aligned}$$

where $\tilde{A} = A \cup \phi_A^{-1}(B)$, since K is a restriction measure with exponent $\alpha > 0$. However, $\phi_{\tilde{A}} = \phi_B \circ \phi_A$ so $\phi'_{\tilde{A}}(0) = \phi'_B(0)\phi'_A(0)$ (as $\phi_A(0) = 0$). Therefore

$$\begin{aligned}\mathbb{P}(K \cap A = \emptyset, \phi_A(K) \cap B = \emptyset) &= \phi'_B(0)^\alpha \phi'_A(0)^\alpha \\ &= \mathbb{P}(K \cap A = \emptyset)\phi'_B(0)^\alpha.\end{aligned}$$

Thus dividing by $\mathbb{P}(K \cap A = \emptyset)$, we get

$$\mathbb{P}(\phi_A(K) \cap B = \emptyset | K \cap A = \emptyset) = \phi'_B(0)^\alpha,$$

as desired for a restriction measure of exponent α. $\square$

We say that restriction samples enjoy the *restriction property*.

6.3.3 Restriction property of SLE(8/3)

We first show that the law of SLE(8/3) is a restriction measure.

Theorem 6.17. *The law of SLE(8/3) is a restriction measure with exponent $\alpha = 5/8$. In other words, let A be a compact $\mathbb{H}$-hull with $0 \notin \bar{A}$. Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(8/3). Then*

$$\mathbb{P}((\gamma_t)_{t \geqslant 0} \text{ does not hit } A) = \phi'_A(0)^{5/8}.$$

Proof. Set $K_t = \{\gamma_s : s \in (0, t]\}$ and $T = \inf\{t \geqslant 0 : \gamma_t \in A\}$. The Loewner transform $(\xi_t)_{t \geqslant 0}$ of $(K_t)_{t \geqslant 0}$ is a Brownian motion of diffusivity $\kappa = 8/3$. For $t < T$, set $\tilde{K}_t = \phi_A(K_t)$ and $\phi_t = g_{\tilde{K}_t} \circ \phi_A \circ (g_{K_t})^{-1}$. Then $\phi_t : \mathbb{H} \setminus g_{K_t}(A) \to \mathbb{H}$ is a conformal isomorphism and $\phi_t(z) - z + g_A(0) \to 0$ as $|z| \to \infty$, so ϕ_t is a shift of the mapping-out function for $g_{K_t}(A)$. Set $\Sigma_t = \phi'_t(\xi_t)$. The set $S_0 = \{(t, x) : t \in [0, T), x \in I_{K_t}\}$ is open in $[0, \infty) \times \mathbb{R}$ and $\xi_t \in I_{K_t}$ for all $t < T$. By Proposition 6.6, the adapted random map $(t, x) \mapsto \phi'_t(x) : S_0 \to \mathbb{R}$ is $C^{1,2}$ and

$$\dot{\phi}'_t(\xi_t) = \frac{1}{2}\frac{\phi''_t(\xi_t)^2}{\phi'_t(\xi_t)} - \frac{4}{3}\phi'''_t(\xi_t)$$

for all $t < T$. By the generalized Itô formula, we have

$$d\Sigma_t = \dot{\phi}'_t(\xi_t)dt + \phi''_t(\xi_t)d\xi_t + \frac{1}{2}\phi'''_t(\xi_t)d[\xi]_t.$$

Since $d[\xi]_t = \kappa dt$, this simplifies to give

$$d\Sigma_t = \phi''_t(\xi_t)d\xi_t + \frac{1}{2}\frac{\phi''_t(\xi_t)^2}{\phi'_t(\xi_t)}dt.$$

Fix $\alpha \in (0, 1]$ and set $M_t = \Sigma_t^\alpha$. By Itô's formula,

$$dM_t = \alpha\Sigma_t^{\alpha-1}d\Sigma_t + \frac{1}{2}\alpha(\alpha - 1)\Sigma_t^{\alpha-2}d\Sigma_t d\Sigma_t = \alpha M_t dY_t,$$

where

$$dY_t = \frac{d\Sigma_t}{\Sigma_t} + \frac{1}{2}(\alpha - 1)\frac{\phi_t''(\xi_t)^2}{\Sigma_t^2}\kappa dt = \frac{\phi_t''(\xi_t)}{\Sigma_t}d\xi_t + \frac{1}{2}(1 + (\alpha - 1)\kappa)\frac{\phi_t''(\xi_t)^2}{\Sigma_t^2}dt.$$

We choose $\alpha = 5/8$ so the final term vanishes. Then $(Y_t)_{t<T}$ and hence also $(M_t)_{t<T}$ is a continuous local martingale.

By Proposition 6.10, conditional on γ, we have

$$\phi_t'(\xi_t) = \mathbb{P}_{\xi_t}((E_s)_{s\geqslant 0} \text{ does not hit } g_{K_t}(A)). \tag{65}$$

In particular $M_t \in [0, 1]$ for all $t < T$, so M_t has an almost sure limit, M_T say, as $t \uparrow T$ and then by optional stopping

$$\mathbb{E}(M_T) = M_0 = \phi_A'(0)^{5/8}.$$

We shall show that $M_T = 1_{\{T=\infty\}}$ almost surely, so $\mathbb{P}(T = \infty) = \mathbb{E}(M_T) = \phi_A'(0)^{5/8}$, as required.

Consider first the case where $T = \infty$. We want to show that

$$\lim_{t\to\infty} \mathbb{P}_{\xi_t}((E_s)_{s\geqslant 0} \text{ hits } g_{K_t}(A)) = 0.$$

There exist connected compact $\mathbb{H}$-hulls A^- and A^+ such that $A \subseteq A^- \cup A^+$ and which $(\gamma_t)_{t\geqslant 0}$ does not hit. Hence we may reduce to the case where A is connected. By Propositions 3.17 and 3.16, we have

$$\mathrm{rad}(g_t(A)) \leqslant \mathrm{cap}(g_t(A)) \leqslant \mathrm{cap}(A) \leqslant 4\,\mathrm{rad}(A).$$

Fix $x \in \bar{A} \cap \mathbb{R}$. By Proposition 5.7, we have $|g_t(x) - \xi_t| \to \infty$ as $t \to \infty$. Hence, as $t \to \infty$,

$$\mathbb{P}_{\xi_t}((E_s)_{s\geqslant 0} \text{ hits } g_{K_t}(A)) \leqslant \mathbb{P}_0((E_s)_{s\geqslant 0} \text{ hits } g_{K_t}(x) - \xi_t + 8r\bar{\mathbb{D}}) \to 0.$$

Consider now the case where $T < \infty$. Write A_0 for the component of A containing γ_T and assume for now that the boundary of A_0 in $\mathbb{H}$ may be parametrized as a simple smooth curve $(\beta(u) : u \in \mathbb{R})$, with $\beta(0) = \gamma_T$. By symmetry it will suffice to consider the case where A_0 is based on $(0, \infty)$. Write A_0^o for the interior of A_0. Then

$$\lim_{t\uparrow T} \mathbb{P}_{\xi_t}((E_s)_{s\geqslant 0} \text{ hits } g_{K_t}(A_0)) \geqslant \mathbb{P}_{\xi_T}((E_s)_{s\geqslant 0} \text{ hits } g_{K_T}(A_0^o)).$$

We will need the following symmetry estimate.

Lemma 6.18. *Let $\beta : \mathbb{R} \to \mathbb{C}$ be a simple curve, differentiable at 0 with $\beta(0) = 0$ and $\dot{\beta}(0) \neq 0$. Set $A = \beta((-\infty, 0])$ and $D = \mathbb{D} \setminus A$, and assume that D is simply connected. Write $A^{\pm}$ for the left and right sides of $A \cap \mathbb{D}$ in δD. Then (using the notation $\hat{B}_{T(D)}$ from Section 2.3)*

$$\lim_{t\downarrow 0} \mathbb{P}_{\beta_t}(\hat{B}_{T(D)} \in A^+) = \lim_{t\downarrow 0} \mathbb{P}_{\beta_t}(\hat{B}_{T(D)} \in A^-) = 1/2.$$

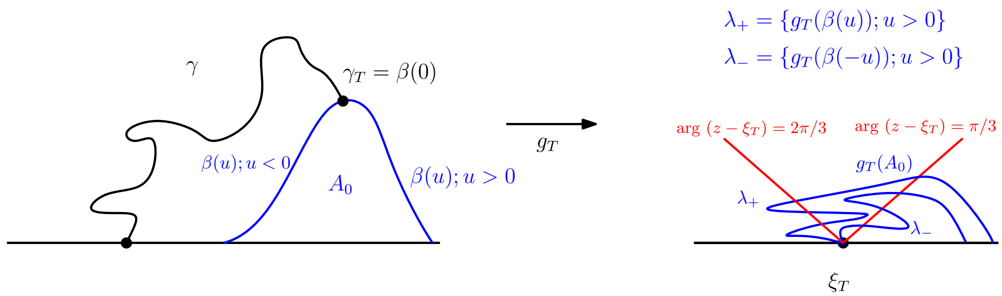

Figure 10: Proof of Proposition 6.10.

Proof. By rotation invariance, it will suffice to consider the case where $\dot{\beta}(0) \in (0, \infty)$. For $r \in (0, 1]$, set $\tau(r) = \inf\{t \geqslant 0 : |\beta(-t)| = r\}$. We deduce from the hypothesis that D is simply connected that $\tau(1) < \infty$ and $|\gamma(-t)| \geqslant 1$ for all $t \geqslant \tau(1)$. Given $\varepsilon > 0$, there exists $r_0 \in (0, 1]$ such that, for all $t \in (0, \tau(r_0))$, we have $|\arg \beta(t)| \leqslant \varepsilon$ and $|\arg \beta(-t) - \pi| \leqslant \varepsilon$. Then there exists $r \in (0, r_0)$ such that $|\gamma(-t)| \geqslant r$ for all $t \in [\tau(r_0), \tau(1))$. Define $A(r) = \beta((-\tau(r), 0])$ and $D(r) = (r\mathbb{D}) \setminus A(r)$. Then $D(r)$ is simply connected. Write $A^+(r)$ for the right side of $A(r)$ in $\delta D(r)$. Then, for $t \in (0, r)$,

$$\mathbb{P}_{\beta_t}(\hat{B}_{T(D)} \in A^+) \geqslant \mathbb{P}_{\beta_t}(\hat{B}_{T(D(r))} \in A^+(r))$$

and

$$\liminf_{t \downarrow 0} \mathbb{P}_{\beta_t}(\hat{B}_{T(D(r))} \in A^+(r)) \geqslant \mathbb{P}_{e^{-2i\varepsilon}}(B \text{ hits } (-\infty, 0] \text{ from above}) = \frac{1}{2} - \frac{\varepsilon}{\pi}$$

where we used a scaling argument for the inequality and the fact that $\arg(B)$ is a local martingale for the equality. By symmetry, and since $\varepsilon > 0$ was arbitrary, this proves the result. $\square$

As a consequence of Lemma 6.18, we deduce

$$\liminf_{u \downarrow 0} \mathbb{P}_{\beta(u)}((B_s)_{s \geqslant 0} \text{ hits } \gamma(0, T] \text{ on the left side}) \geqslant 1/2.$$

Hence, by conformal invariance,

$$\liminf_{u \downarrow 0} \mathbb{P}_{g_T(\beta(u))}((B_s)_{s \geqslant 0} \text{ hits } \mathbb{R} \text{ to the left of } \xi_T) \geqslant 1/2.$$

and so, since $1/3 < 1/2$,

$$\liminf_{u \downarrow 0} \arg(g_T(\beta(u)) - \xi_T) \geqslant \pi/3.$$

That is to say, if λ_+ is the curve $\lambda_+(u) = g_T(\beta(u))$ for $u > 0$ then we can choose $\varepsilon > 0$ small enough that $\lambda_+[0, \varepsilon]$ is above (to the left of) the ray $\{z : \arg(z - \xi_T) = \pi/3\}$, see Figure 10.

For the same reasons,

$$\limsup_{u\uparrow 0} \arg(g_T(\beta(u)) \leqslant 2\pi/3,$$

i.e. if λ_- denotes the curve $\lambda_-(u) = g_T(\beta(-u))$ for $u > 0$ then $\lambda_-[0, \varepsilon]$ lies to the right of the ray $\{z : \arg(z - \xi_T) = 2\pi/3\}$ for small enough $\varepsilon > 0$. We deduce that

$$\mathbb{P}_{\xi_T}((E_s)_{s\geqslant 0} \text{ hits } g_{K_T}(A_0^o)) \geqslant \mathbb{P}_0(\Omega_+ \cap \Omega_-) \tag{66}$$

where

$$\Omega_+ = \cap_{n\in\mathbb{N}}\{\arg(E_s) \in (0, \pi/3) \text{ for some } s \in (0, 1/n)\};$$
$$\Omega_- = \cap_{n\in\mathbb{N}}\{\arg(E_s) \in (2\pi/3, \pi) \text{ for some } s \in (0, 1/n)\};.$$

Indeed consider a time $t_n \leqslant 1/n$ such that $\arg(z_n) \leqslant \pi/3$ where $z_n = E_{t_n}$. Then (if n is large enough) z_n lies to the right of λ_+. If z_n lies to the left λ_- then z_n lies in between λ_+ and λ_- so is in $g_T(A_0)$. If however z_n also lies to the right of λ_-, then consider a time $s_n < t_n \leqslant 1/n$ such that $\arg(w_n) \geqslant 2\pi/3$ where $w_n = E_{s_n}$. Then by continuity $E[s_n, t_n]$ intersects λ_- and hence intersects $g_T(A_0)$ too since $g_T(A_0)$ is smooth near that intersection. Altogether, (66) holds.

To conclude (in the case where A is smooth) it therefore suffices to prove that $\mathbb{P}_0(\Omega_+ \cap \Omega_-) = 1$. Recall the representation $E_s = X_s + i|W_s|$, where $(X_s)_{s\geqslant 0}$ and $(W_s)_{s\geqslant 0}$ are Brownian motions in $\mathbb{R}$ and $\mathbb{R}^3$ respectively. Then, by a scaling argument, $\mathbb{P}_0(\Omega_+) > 0$ and so $\mathbb{P}_0(\Omega_+) = 1$ by Blumenthal's zero-one law. Therefore $\mathbb{P}_0(\Omega_+ \cap O_-) = 1$.

For general A, there is a sequence of compact $\mathbb{H}$-hulls $A_n \downarrow A$ such that the boundary in $\mathbb{H}$ of every component of every A_n is a simple smooth curve. Then, using Proposition 6.10,

$$\mathbb{P}((\gamma_t)_{t\geqslant 0} \text{ does not hit } A_n) = \mathbb{P}((E_t)_{t\geqslant 0} \text{ does not hit } A_n)^{5/8}$$

for all n. On letting $n \to \infty$ and using Proposition 6.10 again, we obtain the desired result for A. □

The proposition just proved shows that the SLE(8/3) enjoys the **restriction property**. Using general considerations and Lemma 6.16, this can be used to show the following result:

Theorem 6.19. *Let A be a compact $\mathbb{H}$-hull with $0 \notin \bar{A}$. Let $(\gamma_t)_{t\geqslant 0}$ be an SLE$(8/3)$. Then, conditional on the event $\{(\gamma_t)_{t\geqslant 0}$ does not hit $A\}$, the process $(\phi_A(\gamma_t))_{t\geqslant 0}$ in its canonical reparametrization is also an SLE$(8/3)$.*

Proof of Theorem 6.19. Let $S_{A^c} = \{(\gamma_t)_{t\geqslant 0}$ does not hit $A\}$. Theorem 6.17 and Lemma 6.16 imply that, conditional on the event S_{A^c}, the law of $(\phi_A(\gamma_t))_{t\geqslant 0}$ is identical to that of an SLE(8/3), when both are viewed as random fillings.

Write S_0 for the set of fillings in $\mathbb{H}$ of the form $K = \{\gamma_t : t \in (0, \infty)\}$, where $(\gamma_t)_{t\geqslant 0}$ is a simple path in $\bar{\mathbb{H}}$ parametrized so that $\text{hcap}(\gamma(0, t]) = 2t$ for all t. It is straightforward to see that S_0 is $\mathcal{S}$-measurable and we shall show also that the map $\theta : S_0 \to C([0, \infty), \bar{\mathbb{H}})$

given by $\theta(K) = \gamma$ is $\mathcal{S}$-measurable. For $(\gamma_t)_{t\geqslant 0}$ an SLE(8/3) and $K = \{\gamma_t : t \in (0,\infty)\}$, for A a compact $\mathbb{H}$-hull with $0 \notin \bar{A}$ and $D = \mathbb{H} \setminus A$, we have

$$\{(\gamma_t)_{t\geqslant 0} \text{ does not hit } A\} = \{K \subseteq D\}$$

and on this event the canonical reparametrization $(\tilde{\gamma}_t)_{t\geqslant 0}$ of $(\phi_A(\gamma_t))_{t\geqslant 0}$ is given by $\theta(\phi_A(K))$. Then, for B a measurable set in $C([0,\infty),\bar{\mathbb{H}})$, the set $\theta^{-1}(B)$ is $\mathcal{S}$-measurable and we obtain

$$\begin{aligned}\mathbb{P}((\tilde{\gamma}_t)_{t\geqslant 0} \in B | \{(\gamma_t)_{t\geqslant 0} \text{ does not hit } A\}) \\ = \mathbb{P}(\phi_A(K) \in \theta^{-1}(B) | K \subseteq D) = \mathbb{P}(K \in \theta^{-1}(B)) = \mathbb{P}((\gamma_t)_{t\geqslant 0} \in B).\end{aligned}$$

We now complete the proof of the theorem by showing the measurability of the map θ. For $n \geqslant 0$, write L_n for the dyadic lattice $2^{-n}\{j + ik : j \in \mathbb{Z},\ k \in \mathbb{Z}^+\}$. For each $p \in L_n$, consider the set $Q = Q(p) = p + \{x + iy : x, y \in [0, 2^{-n}]\}$ and write $\mathcal{N}_Q$ for the countable set of domains $D = \mathbb{H} \setminus A \in \mathcal{N}$ where $A = A^- \cup A^+$ and A^- and A^+ are disjoint simple paths which are unions of horizontal and vertical dyadic line segments, and which together with some boundary interval $I(D)$ of Q and some interval of $\mathbb{R}$ containing 0, form a simple closed curve in $\overline{\mathbb{H} \setminus Q}$. For $D \in \mathcal{N}_Q$, write $K(D)$ for the hull whose boundary in $\mathbb{H}$ consists of $A^- \cup A^+ \cup I(D)$. For $K \in S_0$, set $\gamma = \theta(K)$ and define

$$\tau_Q(K) = \inf\{t \geqslant 0 : \gamma_t \in Q\}, \quad h_Q(K) = \operatorname{hcap}(\gamma(0, \tau_Q(K)]), \quad e_Q(K) = \gamma_{\tau_Q(K)}.$$

By Proposition 3.15, given $t > 0$, we have $h_Q(K) < t$ if and only if $K \subseteq D$ for some $D \in \mathcal{N}_Q$ with $\operatorname{hcap}(K(D)) < t$. Also, given an open boundary interval I of Q, we have $e_Q(K) \in I$ if and only if $K \subseteq D$ for some $D \in \mathcal{N}_Q$ with $I(D) \subseteq I$. Hence $h_Q : S_0 \to [0,\infty]$ and $e_Q : S_0 \to \bar{\mathbb{H}}$ are both $\mathcal{S}$-measurable. For each $n \geqslant 0$, choose an enumeration $(p_m : m \geqslant 0)$ of L_n so that so that $h_m = h_{Q(p_m)}(K)$ is non-decreasing in m and set $e_m = e_{Q(p_m)}(K)$. Note that if $h_m = h_{m'}$ then $e_m = e_{m'}$. Set $h_m = 2t_m$. Define a path $(\theta_t^{(n)}(K))_{t\geqslant 0}$ by linear interpolation of $((t_m, e_m) : m \geqslant 0)$. Then $\theta^{(n)} : S_0 \to C([0,\infty),\bar{\mathbb{H}})$ is measurable for all n. Now $\theta_t(K) = \theta_t^{(n)}(K)$ for all $t \in T(n) = \{t_m : m \geqslant 0\}$ and, since $\theta(K)$ is simple, $\cup_n M(n)$ is dense in $[0,\infty)$. Hence, by uniform continuity, the paths $\theta^{(n)}(K)$ converge to $\theta(K)$ uniformly on compacts, so θ is also measurable, as required. $\square$

Another corollary of Theorem 6.17 is the following remarkable identity. Suppose α is a nonnegative integer and A is a compact $\mathbb{H}$-hull such that $0 \notin \bar{A} \cap \mathbb{R}$. Then $\Phi_A'(0)^\alpha$ is the probability that α independent Brownian excursions avoid A, by Proposition 6.10. Hence this is the probability that the hull generated by α independent Brownian excursions does not intersect A. Thus one way to informally interpret the result of Theorem 6.17 is to say that the SLE(8/3) chord can be thought of as 5/8 of a Brownian excursion. More precisely, we have the following result as an immediate corollary to Proposition 6.10 and Theorem 6.17:

Theorem 6.20. *The compact hull generated by 8 independent SLE(8/3) chords and the compact hull generated by 5 independent Brownian excursions have the same distribution.*

One of the particularly striking aspects of this result is that the curves themselves (SLE(8/3) and Brownian excursions) are very different from one another.

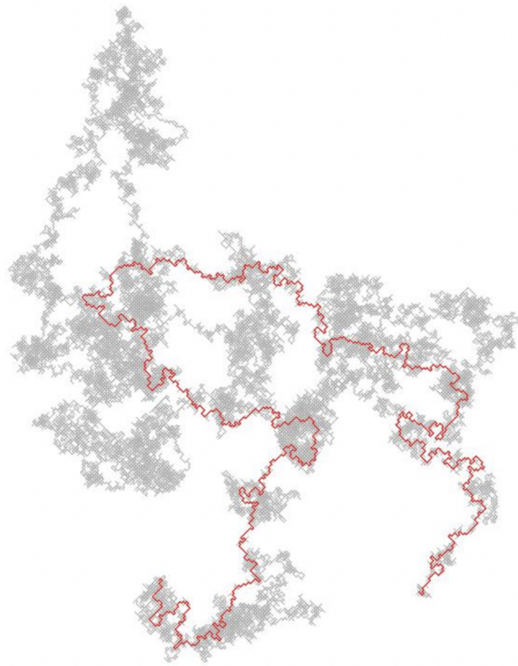

Figure 11: A random walk and its loop-erasure (picture F. Viklund).

Lecture 16: Friday 26 May

7 Loop-erased random walk

Suppose G is a finite graph. Let a be a vertex and U be a set of vertices; suppose that $T_U < \infty$, $\mathbb{P}_a$-a.s, where $\mathbb{P}_a$ denotes the law of simple random walk $(\Gamma_n)_{n \geqslant 0}$ on G starting from a, and T_U the hitting time of U for that walk.

Definition 7.1. *A* ***loop-erased random walk*** *from a to U is the process obtained from $(\Gamma_n)_{0 \leqslant n \leqslant T_U}$ by chronologically erasing the loops from Γ. More precisely, the loop-erasure $\beta = (\beta_0, \ldots, \beta_\ell)$ is defined inductively as follows: $\beta_0 = a$. If $\beta_n \in U$ then $n = \ell$, else $\beta_{n+1} = \Gamma_L$, where $L = 1 + \max\{m \leqslant T_U : \Gamma(m) = \beta_n\}$.*

From the definition, a loop-erased random walk (LERW) is a random simple curve starting from a and ending in U. They were introduced by Lawler [12] initially as a tractable toy model for self-avoiding walk. It turns out that (especially in low dimensions such as $d = 2$ which will be the case of interest here) LERW and self-avoiding walks are quite different models; yet LERW is central to many other models of statistical mechanics, including in particular the **Uniform Spanning Tree** to which it is related via a fundamental relation known as **Wilson's algorithm**.

Using another connection to a model of statistical mechanics known as the **dimer model**, Kenyon [9, 10] was the first to give proofs of results for LERW in two dimensions consistent with the conformal invariance prediction. Notably, he showed that the limiting (rescaled) probability that a given edge belongs to the loop-erasure is asymptotically conformally invariant, and he showed that the expected length ℓ of the path is $n^{5/4+o(1)}$ as $n \to \infty$.

In his original article introducing SLE, Schramm [20] predicted convergence of planar LERW to SLE(2). This was confirmed in the landmark result [14] by Lawler, Schramm and Werner. In this section we give arguments which single out SLE(2) as the only plausible candidate for the scaling limit (these arguments are different from the original line of reasoning of Schramm, who instead relied on the connection with the dimer model and Kenyon's results).

7.1 Discrete lemmas

We start with a few background lemmas on LERW which will be useful in identifying martingales. From now on we suppose given a simply connected, bounded domain D. Given a small $\delta > 0$, we associate a graph $G = G_\delta$ to D as follows. First, we define the "internal" vertices V^0 to be $V^0 = D \cap \delta\mathbb{Z}^2$. Given two internal vertices $u, v \in V^0$, say that there is an edge $e = (u, v)$ if and only if u and v are neighbours in $\delta\mathbb{Z}^2$ and the straight segment $[u, v]$ is completely contained in D. In addition, given an internal vertex $v \in V^0$, if a straight segment of length δ starting from v encounters ∂D, then we identify this portion of segment to a vertex and call it a boundary vertex u. We call U the set of boundary vertices. We may now define the graph G: we define its vertex set V to be $V = V^0 \cup U$. The edge set of G is as described above between internal vertices $u, v \in V^0$. In addition, given a boundary vertex $u \in U$ and an internal vertex $v \in V^0$ we put an edge between u and v if u is identified with the edge emanating from v encountering ∂D.

Note that the resulting graph G is a finite graph (since D is bounded) and $T_U < \infty$ a.s. for any starting point $a \in V$. The first lemma below says that it doesn't matter if the loops are erased in the forward or backward direction.

Lemma 7.2. *Let Γ be a random walk from a, stopped at the time $T = T_U$ when it hits U. Let β denote the loop-erasure of Γ, and let γ denote the loop-erasure of the time-reversal $\hat{\Gamma} = (\Gamma_T, \Gamma_{T-1}, \dots, \Gamma_0)$. Then γ has the same law as the time-reversal of β.*

The same result holds when we condition on $\Gamma_T = u \in U$.

Note that deterministically it is not always the case that a path and its time-reversal produce the same loop-erasure. Instead this is proved by summing over all paths $[s_0, \dots, s_n]$ which might produce a given $[\gamma_0, \dots, \gamma_\ell]$ as a loop-erasure, and noticing that there is a way of rerouting the loops of $[s_0, \dots, s_n]$ to obtain a new path $[s'_0, \dots, s'_n]$ in such a way that the antichronological loop-erasure of s' produces the same simple path γ.

It turns out that the antichronological loop-erasure is easier to handle. In the rest of this section we fix $a \in V^0$ and have in mind the situation where a is a neighbour of some boundary vertex. We also fix $u \in U$, and consider a random walk $\Gamma = [\Gamma_0, \dots, \Gamma_T]$ starting from a, conditioned so that $\Gamma_T = u$. We also consider the loop-erasure $\gamma = [\gamma_0, \dots, \gamma_\ell]$ of its time reversal $\hat{\Gamma}$, so that $\gamma_0 = u$ and $\gamma_\ell = a$.

Note that if $n_j = \inf\{m \geqslant: \Gamma_m = \gamma_j\}$ then n_j is not a stopping time. However, fix $j \geqslant 0$ and vertices $(u_0, \dots, u_j)$ such that the event $E = \{[\gamma_0, \dots, \gamma_j] = [u_0, \dots, u_j]\}$ has positive probability (in particular, $u_0 \in U$ is a boundary vertex, and $u_1, \dots, u_j \in V^0$ are internal vertices). Set D_j to be the "slit domain" $D_j = V^0 \setminus \{u_0, \dots, u_j\}$.

For $j \geqslant 0$ and $v \in D_j$, consider a walk Γ starting from v, and let $T_j = \inf\{n \geqslant 0 : \Gamma_n \notin D_j\}$. (Thus T_0 is the first hitting time of the boundary U).

Lemma 7.3. *Fix $j \geqslant 0$ and vertices $(u_0, \dots, u_j)$ such that the event $E = \{[\gamma_0, \dots, \gamma_j] = [u_0, \dots, u_j]\}$ has positive probability. Then E can be written as*

$$E = \left\{ \begin{array}{cccc} \Gamma_{T_j} = u_j, & \Gamma_{T_{j-1}} = u_{j-1}, & \dots, & \Gamma_{T_1} = u_1, \quad \Gamma_{T_0} = u_0 \\ \Gamma_{T_{j-1}-1} = u_j, & \Gamma_{T_{j-2}-1} = u_{j-1}, & \dots, & \Gamma_{T_0-1} = u_1 \end{array} \right\}.$$

Proof. Let E' be the event above. E' ensures that the walk Γ leaves D_j via u_j at time T_j. It then makes loop(s) based at u_j, until $T_{j-1} - 1$ (so these loops cannot touch $[u_{j-1}, \dots, u_0]$), then moves to u_{j-1}, and so on and so forth. Let $\hat{\Gamma}$ denote the time-reversal of $\Gamma[0, T_0]$. Thus $\hat{\Gamma}_n = \Gamma_{T_0 - n}$ for $0 \leqslant n \leqslant T_0$. Let $\hat{n} = T_0 - n$.

Then it is easy to check (e.g. by induction) that the event E' corresponds to the following in terms of the time-reversal: first, $\hat{\Gamma}_0 = u_0$, moreover for each $0 \leqslant i \leqslant j$, the last visit by $\hat{\Gamma}$ to $[u_0, \dots, u_i]$ is at time $\hat{T}_i$, when $\hat{\Gamma}_{\hat{T}_i} = u_i$; after that (if $i < j$) $\hat{\Gamma}$ moves to u_{i+1}, i.e., $\hat{\Gamma}_{\hat{T}_i + 1} = u_{i+1}$ and $\hat{\Gamma}$ never returns to $[u_0, \dots, u_i]$.

Using the definition of loop-erased random walk (Definition 7.1) it is therefore clear that these conditions ensure (indeed, are equivalent to) $[u_0, \dots u_j] = [\gamma_0, \dots, \gamma_j]$. $\square$

Now let

$$H_{D_j}(v, u) = \mathbb{P}_v(\Gamma_{T_j} = u)$$

be the (discrete) harmonic measure in D_j viewed from v. A corollary from the above lemma is the following simple observation.

Corollary 7.4. *Fix $j \geqslant 0$ and $[u_0, \dots, u_j]$ such that $\mathbb{P}_a([\gamma_0, \dots, \gamma_j] = [u_0, \dots, u_j]) > 0$. Then the same holds under $\mathbb{P}_v$ for any other $v \in D_j$. Furthermore,*

$$F_v([u_0, \dots, u_j]) := \frac{\mathbb{P}_v([\gamma_0, \dots, \gamma_j] = [u_0, \dots, u_j])}{\mathbb{P}_a([\gamma_0, \dots, \gamma_j] = [u_0, \dots, u_j])}$$

satisfies

$$F_v([u_0, \dots, u_j]) = \frac{H_{D_j}(v, u_j)}{H_{D_j}(a, u_j)}. \tag{67}$$

We can now identify the desired "martingale observable":

Lemma 7.5. *Fix $a \in V^0$. Let $(\gamma_n)_{n \geqslant 0}$ be the (antichronological) loop-erasure of a random walk $(\Gamma_n)_{0 \leqslant n \leqslant T_U}$. Fix any other vertex $v \in V^0$, and let $\Lambda = \Lambda_v = \inf\{n \geqslant 0 : \gamma_n \in \{a, v\}\}$ be the hitting time of a or v by the loop-erased walk. Then the process*

$$M = M^v = (M_n^v)_{0 \leqslant n \leqslant \Lambda_v}; \text{ where } M_n^v = F_v([\gamma_0, \ldots, \gamma_n]) \text{ for } 0 \leqslant n \leqslant \Lambda_v \tag{68}$$

is a martingale.

Proof. The lemma is immediate from general considerations on Radon–Nikodym derivatives. Suppose that $(\Omega, \mathcal{F})$ is a measurable space with a filtration $(\mathcal{F}_n)_{n \geqslant 0}$, and suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two probability measures such that $\mathbb{Q}|_{\mathcal{F}_n}$ is absolutely continuous with respect to $\mathbb{P}|_{\mathcal{F}_n}$ for each n. Let

$$M_n = \frac{d\mathbb{Q}|_{\mathcal{F}_n}}{d\mathbb{P}|_{\mathcal{F}_n}}$$

be the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{F}_n$. In other words, M_n is $\mathcal{F}_n$-measurable and for any $A \in \mathcal{F}_n$, $\mathbb{E}_{\mathbb{Q}}(1_A) = \mathbb{E}_{\mathbb{P}}(1_A M_n)$. Then M_n is a $\mathbb{P}$-martingale with respect to the filtration $\mathcal{F}_n$: indeed, $(M_n)_{n \geqslant 0}$ is adapted by definition, $M_n \geqslant 0$ and $\mathbb{E}_{\mathbb{P}}(M_n) = 1$ so M_n is integrable, and for any $A \in \mathcal{F}_n$, since $A \in \mathcal{F}_n \subset \mathcal{F}_{n+1}$,

$$\mathbb{E}_{\mathbb{P}}(M_{n+1} 1_A) = \mathbb{E}_{\mathbb{Q}}(1_A) = \mathbb{E}_{\mathbb{P}}(1_A M_n)$$

so $\mathbb{E}(M_{n+1} | \mathcal{F}_n) = M_n$ since $A \in \mathcal{F}_n$ is arbitrary.

The definition of F_v in Corollary 7.4 means that $F_v([\gamma_0, \ldots, \gamma_j])$ is the Radon–Nikodym derivative of the law of the (antichronological) loop-erasure of a random walk started from v, compared to the law of that same loop-erasure but started from a. Therefore M is a $\mathbb{P}_a$-martingale with respect to the filtration generated by $(\gamma_0, \ldots, \gamma_{\Lambda_v})$. $\square$

7.2 The Poisson kernel ratio

Let D be a simply connected domain. Let $a \in D$ and $z \in D$; let $b \in D$ be point on boundary of D and suppose for now that ∂D is smooth near b. Hence if $H_D(z; \cdot)$ denotes the (continuous) harmonic measure in D viewed from z, then $H_D(z, \cdot)$ has a density with respect to arclength measure on ∂D. This density (call it $h_D(z, \cdot)$) is sometimes called the **Poisson kernel** in D viewed from z.

Let

$$F_{D,a,b}(z) = \frac{h_D(z, b)}{h_D(a, b)} \tag{69}$$

be the ratio of Poisson kernels at b viewed from z vs. a. Then $F_{D,a,b}(z)$ should be viewed as a continuum analogue of the quantity defined in (67).

While the Poisson kernel is not well behaved under conformal transformation (indeed, even the Poisson kernel itself is only well defined when ∂D is smooth near b), we will see however that the ratio of Poisson kernels $F_{D,a,b}(z)$ is conformally invariant whenever it is defined.

Lemma 7.6. *Let $\phi : D \to D'$ be a conformal isomorphism. Let $a, z \in D$ and $b \in \partial D$; suppose that ∂D is smooth near b and ϕ extends to a diffeomorphism in a neighbourhood of b. Set $a' = \phi(a), z' = \phi(z), b' = \phi(b)$. Then $\partial D'$ is smooth near b', and*

$$F_{D,a,b}(z) = F_{D',a',b'}(z').$$

Proof. By definition

$$h_D(z, b) = \lim_{I \subset \partial D, I \downarrow b} \frac{H_D(z; I)}{|I|}; \quad h_{D'}(z', b') = \lim_{I' \subset \partial D', I' \downarrow b'} \frac{H_{D'}(z'; I')}{|I'|}$$

Set $I' = \phi(I)$ and suppose $I \downarrow b$. When we apply the conformal isomorphism ϕ, we have by conformal invariance $H_D(z, I) = H_{D'}(z', \phi(I))$. We also have that $|\phi(I)| \sim k|I|$ as $I \downarrow b$, where $k = \|(D\phi)u\|$, u is the unit vector describing the direction of ∂D near b, and $D\phi$ is the Jacobian matrix of ϕ at b. Thus

$$k h_D(z, b) = h_{D'}(z', b').$$

This is true for all z including also for $z = a$. Thus taking ratios, we obtain

$$\frac{h_D(z, b)}{h_D(a, b)} = \frac{h_{D'}(z', b')}{h_{D'}(a', b')},$$

as desired. $\square$

As a result, the ratio of Poisson kernels $F_{D,a,b}(z)$ can be defined even when D is not smooth near b, and b is instead just a point on the Martin boundary. Indeed, let ϕ denote any conformal map from D to $\mathbb{H}$. Then we define

$$F_{D,a,b}(z) = F_{\mathbb{H},a',b'}(z')$$

where $a' = \phi(a), z' = \phi(z)$, and b' is the point of $\mathbb{R}$ identified with b under ϕ. This definition coincides with the earlier one in case of a smooth domain.

7.3 Identification of SLE(2) as scaling limit

Let D be simply connected domain, and let γ_δ denote the (antichronological) loop-erasure of a random walk Γ starting a vertex a_δ such that $a_\delta \to a \in \partial D$ as $\delta \to 0$, and conditioned to to touch the boundary U at u_δ. Let us also assume that $u_\delta \to u \in \partial D$ as the mesh size $\delta \to 0$.

When the mesh size δ tends to zero, we expect that γ_δ will converge to a curve γ from u to a. We will want to identify this curve as an SLE(κ) curve for some parameter κ to be identified. Fix a conformal isomorphism ϕ from D to $\mathbb{H}$, sending u to 0 and a to ∞ (i.e., ϕ is a choice of scale for the two-pointed domain $\mathbf{D} = (D, u, a)$).

Let $(\xi_t)_{t \geqslant 0}$ be the driving function of the curve $(\phi(\gamma_t))_{t \geqslant 0}$ in $\mathbb{H}$ from 0 to ∞, (reparametrised by half-plane capacity), and let g_t denote the associated Loewner flow. As a consequence of the conformal invariance of Poisson kernel ratio and the fact that this Poisson kernel ratio defines a martingale at the discrete level via Lemma 7.5, we expect that for every $z \in \mathbb{H}$, $(M^z)_{0 \leqslant t \leqslant \zeta(z)}$ is a martingale, where

$$M_t^z = h_{\mathbb{H}}(g_t(z), \xi_t); 0 \leqslant t \leqslant \zeta(z). \tag{70}$$

To see (70), note that if the target of the loop-erasure was a point $a \in \mathbb{H}$, then we would expect

$$M_t^{a,z} := \frac{h_{H_t}(z, \gamma_t)}{h_{H_t}(a, \gamma_t)}$$

to give a martingale. Applying the conformal isomorphism, g_t, we have

$$M_t^{a,z} = \frac{h_{\mathbb{H}}(g_t(z), \xi_t)}{h_{\mathbb{H}}(g_t(a), \xi_t)}.$$

We deduce that also $h_{\mathbb{H}}(a, 0) M_t^{a,z}$ is martingale for every $a \in \mathbb{H}$. Letting $a \to \infty$ and using the fact that for every $t > 0$,

$$\frac{h_{\mathbb{H}}(a, 0)}{h_{\mathbb{H}}(g_t(a), \xi_t)} \to 1$$

as $a \to \infty$ (itself a consequence of the fact that $g_t(a) - a \to 0$, and the form of $h_{\mathbb{H}}$ computed explicitly in Example 1.10), we deduce that M^z should be a martingale. This explains (70).

Proposition 7.7. *Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(κ) in $\mathbb{H}$. For $z \in \mathbb{H}$, set*

$$M_t^z = h_{\mathbb{H}}(g_t(z), \xi_t); \quad 0 \leqslant t < \zeta(z).$$

Then M^z is a local martingale if and only if $\kappa = 2$.

Proof. We recall that if $z = x + iy$, then by Example 1.10,

$$h_{\mathbb{H}}(z, x_0) = \frac{y}{\pi[(x - x_0)^2 + y^2]} = \frac{-1}{\pi} \operatorname{Im} \left(\frac{1}{z - x_0} \right). \tag{71}$$

Set $Z_t = g_t(z) - \xi_t$, so M^z is a martingale for all z if and only if

$$N_t = \operatorname{Im} \left(\frac{1}{Z_t} \right)$$

is a martingale. By Loewner's equation,

$$\begin{aligned} dZ_t &= \dot{g}_t(z) dt - d\xi_t \\ &= \frac{2}{Z_t} dt - d\xi_t, \end{aligned}$$

so by Itô's formula,

$$
\begin{aligned}
dN_t &= \operatorname{Im}\left(-\frac{dZ_t}{Z_t^2} + \frac{1}{2}\frac{2d[Z]_t}{Z_t^3}\right) \\
&= \operatorname{Im}\left(\frac{d\xi_t}{Z_t^2} - \frac{2}{Z_t^3}dt + \frac{\kappa}{Z_t^3}dt\right) \\
&= \operatorname{Im}\left(\frac{1}{Z_t^2}\right)d\xi_t + \operatorname{Im}\left(\frac{1}{Z_t^3}\right)(\kappa - 2)dt.
\end{aligned}
$$

The first term gives a local martingale, and the second one is of finite variation, so M is a local martingale if and only if $\kappa = 2$. $\square$

Lecture 17, Friday 2 June

8 SLE(4) and the Gaussian free field

We define the planar Gaussian free field and review some of its properties. Then we prove a relation with SLE(4) due to Schramm and Sheffield which suggests that SLE(4) can be interpreted as a fracture line of this Gaussian process. Since the free field is distribution-valued, we begin with a quick review of some classical material on function spaces and distributions.

8.1 Conformal invariance of function spaces

Let D be a domain. A *test-function* on D is an infinitely differentiable function on D which is supported on some compact subset of D. The set of all such test-functions is denoted $\mathcal{D}(D)$. The set $\mathcal{D}(D)$ is made into a locally convex topological vector space[9] in which convergence is characterized as follows. A sequence $f_n \to 0$ in $\mathcal{D}(D)$ if and only if there is a compact set $K \subseteq D$ such that $\operatorname{supp} f_n \subseteq K$ for all n and f_n and all its derivatives converge to 0 uniformly on D. A continuous linear map $u : \mathcal{D}(D) \to \mathbb{R}$ is called a *distribution*[10] on D. Thus, the set of distributions on D is the dual space of $\mathcal{D}(D)$. It is denoted by $\mathcal{D}'(D)$ and is given the weak-$*$ topology. Thus $u_n \to u$ in $\mathcal{D}'(D)$ if and only if $u_n(\rho) \to u(\rho)$ for all $\rho \in \mathcal{D}(D)$. In this context, we think of each $\rho \in \mathcal{D}(D)$ as specifying a suitably regular signed measure on D, given by $\rho(x)dx$, and of each $u \in \mathcal{D}'(D)$ as a generalized function on D, which can be viewed through its 'averages' $u(\rho)$ with respect to test-functions. We will freely identify ρ with the signed measure $\rho(x)dx$. Note that, since $\mathcal{D}(D)$ is separable, the Borel σ-algebra on $\mathcal{D}'(D)$ is generated by the coordinate functions $u \mapsto u(\rho)$. We specialize from this point on to the planar case, where we note the following result.

Proposition 8.1. *Let $\phi : D_0 \to D$ be a conformal isomorphism of planar domains.*

(a) *The map $f \mapsto f \circ \phi^{-1}$ is a linear homeomorphism $\mathcal{D}(D_0) \to \mathcal{D}(D)$.*

(b) *For $\rho \in \mathcal{D}(D)$, the image measure of $\rho(x)dx$ by ϕ^{-1} is given by $\rho_0(x)dx$, where $\rho_0 = (\rho \circ \phi)|\phi'|^2$, and the map $\rho \mapsto \rho_0$ is a linear homeomorphism $\mathcal{D}(D) \to \mathcal{D}(D_0)$.*

(c) *For $u_0 \in \mathcal{D}'(D_0)$, consider the distribution u on D given by $u(\rho) = u_0(\rho_0)$. The map $u_0 \mapsto u$ is a linear homeomorphism $\mathcal{D}'(D_0) \to \mathcal{D}'(D)$.*

The proof, which is left as an exercise, rests on the fact that ϕ and all its derivatives are bounded on compact subsets of D_0, and ϕ^{-1} and all its derivatives are bounded on

[9]See [19] for more details.

[10]This conflicts with the usage of distribution to mean the law of a random variable but is standard and should not cause confusion.

compact subsets of D. We use for the second assertion the Jacobian formula

$$\int_D f(\phi^{-1}(x))\rho(x)dx = \int_{D_0} f(y)\rho(\phi(y))|\phi'(y)|^2 dy.$$

In (c), in the case where u_0 is a function, then u is also a function, given by $u = u_0 \circ \phi^{-1}$. We will continue to write $u_0 \circ \phi^{-1}$ to signify u, even when u_0 is a distribution and the composition cannot be understood literally.

Let D be a proper simply connected domain. The Green function $(G_D(x, y) : x, y \in D)$ was introduced in Section 1.5 and more specifically in Definition 1.16; we showed in Proposition 1.18 that for a regular domain we have $G_D(x, y) < \infty$ for all $x \neq y \in D$. Write $\mathcal{M}_D^+$ for the set of Borel nonnegative measures μ on D having finite (logarithmic) energy

$$\mathcal{E}_D(\mu) = \int_{D^2} G_D(x, y)\mu(dx)\mu(dy). \tag{72}$$

We will also write $\mathcal{M}_D$ for the set of signed Borel measures $\mu = \mu^+ - \mu^-$ where $\mu^+, \mu^- \in \mathcal{M}_D^+$. The energy has a conformal invariance property which it inherits from the Green function. The proof is left as an exercise.

Proposition 8.2. *Let $\phi : D_0 \to D$ be a conformal isomorphism of proper simply connected domains. Let μ_0 be a Borel measure on D_0 and write μ for the image measure $\mu_0 \circ \phi^{-1}$ on D. Then*

$$\mathcal{E}_D(\mu) = \mathcal{E}_{D_0}(\mu_0).$$

8.2 Gaussian free field

Let us turn to the definition of the Gaussian free field (as a stochastic process, in the terminology of [3]). Recall that if I is an index set, a **stochastic process indexed by** I is just a collection of random variables $(X_i)_{i \in I}$, defined on some given probability space. The **law** of the process is a measure on $\mathbb{R}^I$, endowed with the product topology. It is uniquely characterised by its finite-dimensional marginals, via Kolmogorov's extension theorem.

Given $n \geqslant 1$, a random vector $X = (X_i)_{1 \leqslant i \leqslant n}$ is called **Gaussian** if any linear combination of its entries is real Gaussian; i.e., if $\langle \lambda, X \rangle$ is a real Gaussian random variable. The law of X is entirely specifed by its mean vector $\mu = \mathbb{E}(X) \in \mathbb{R}^n$, i.e., $\mu_i = \mathbb{E}(X_i)$ for each $1 \leqslant i \leqslant n$, and its covariance matrix $\Sigma \in \mathcal{M}(\mathbb{R}^n)$ given by $\Sigma_{i,j} = \mathrm{Cov}(X_i, X_j)$, $1 \leqslant i, j \leqslant n$. Conversely, given a vector $\mu \in \mathbb{R}^n$ and a symmetric, nonnegative[11] matrix $\Sigma \in \mathcal{M}(\mathbb{R}^n)$, there exists a (unique) law on $\mathbb{R}^n$ which is that of a Gaussian vector with mean μ and covariance matrix Σ.

Fix a set I and suppose we are given a function $C : I \times I \to \mathbb{R}$, symmetric nonnegative in the sense that for every $n \geqslant 1$, every $t_1, \ldots, t_n \in I$, and every $\lambda_1, \ldots, \lambda_n$,

[11]Here nonnegative is in the sense of matrices, i.e., $\sum_{i,j} \lambda_i \lambda_j \Sigma_{i,j} \geqslant 0$ for each $\lambda \in \mathbb{R}^n$, or equivalently the eigenvalues of Σ are all nonnegative.

$\sum_{i,j=1}^n \lambda_i\lambda_j C(t_i,t_j) \geqslant 0$. Then associated to this function C we can define a centered Gaussian vector $(X_{t_1},\dots,X_{t_n})$ with covariance matrix $\Sigma_{i,j} = C(t_i,t_j)$. The resulting laws are automatically consistent in the sense of Kolmogorov as the parameter $t_1,\dots,t_n \in I$ and $n \geqslant 1$ are varied, so by Kolmogorov's extension theorem the function C defines a unique law on $\mathbb{R}^I$ of a stochastic process $(X_t)_{t\in I}$ indexed by I such that the restriction of $(X_t)_{t\in I}$ to the n-tuple of indices $t_1,\dots,t_n \in I$ gives us a centered Gaussian vector $(X_{t_1},\dots,X_{t_n})$ with the above covariance matrix. The process $(X_t)_{t\in I}$ is called the (centered) **Gaussian stochastic process on I with covariance function C**. Given a real-valued function $(\mu(t), t \in I)$ we can also define a Gaussian stochastic process on I with mean function μ and covariance function $(C(s,t))_{s,t\in I}$ simply by shifting the previous centred stochastic process by $\mu(t)$ at each $t \in I$.

Let D be a regular planar domain (where regular is in the sense of Section 1.5; i.e., starting from any point on the boundary of D, a Brownian motion would leave D almost surely instantaneously). We will define the Gaussian free field in D (with zero boundary conditions) as a centered Gaussian stochastic process indexed by the set $\mathcal{M}_D$ of (signed) Borel measures with finite logarithmic energy (see (72)). Essentially, our definition will be that the Gaussian free field on D with zero boundary conditions is the centered Gaussian stochastic process $(\Gamma_\rho)_{\rho\in\mathcal{M}_D}$ indexed by $\mathcal{M}_D$ such that for $\rho_1,\rho_2 \in \mathcal{M}_D$ we have

$$\mathrm{Cov}(\Gamma_{\rho_1},\Gamma_{\rho_2}) = \mathcal{E}_D(\rho_1,\rho_2) := \int_{D^2} G_D(x,y)\rho_1(dx)\rho_2(dy).$$

However in order to do so a few things need to be checked:

- $\mathcal{E}_D(\rho_1,\rho_2)$ is well defined assuming only that $\rho_1,\rho_2 \in \mathcal{M}_D$. In fact this is not obvious even if we assume $\rho_1,\rho_2 \in \mathcal{M}_D^+$.
- The function $\mathcal{E}_D(\cdot,\cdot)$ is symmetric nonnegative on $\mathcal{M}_D^2$ (so is a valid covariance function).

These properties are however relatively easy to prove, as seen below (see also Section 1.3 of [3] where they were to our knowledge first discussed).

Lemma 8.3. *If $\rho_1,\rho_2 \in \mathcal{M}_D^+$ then $\mathcal{E}_D(\rho_1,\rho_2) < \infty$. Furthermore $\rho_1+\rho_2 \in \mathcal{M}_D^+$.*

Proof. By the Chapman–Kolmogorov equations (9) we have

$$p_t^D(x,y) = \int_D p_{t/2}^D(x,z)p_{t/2}^D(z,y)dz,$$

so

$$G_D(x,y) = \int_D dz \int_0^\infty p_u^D(x,z)p_u^D(y,z)du.$$

(Note that there is no factor 1/2 on account of the change of variable $t/2 = u$.) Consequently, if $\rho_1 \in \mathcal{M}_D^+$ and $\rho_2 \in \mathcal{M}_D^+$ are arbitrary,

$$\begin{aligned}
\mathcal{E}_D(\rho_1, \rho_2) &= \int_{D^2} G_D(x,y)\rho_1(dx)\rho_2(dy) \\
&= \int_D dz \int_0^\infty \int_{D^2} \rho_1(dx)\rho_2(dy) p_u^D(x,z) p_u^D(y,z) du \\
&= \int_D dz \int_0^\infty \left(\int_D \rho_1(dx) p_u^D(x,z) \right) \times \left(\int_D \rho_2(dx) p_u^D(x,z) \right) du. \qquad (73)
\end{aligned}$$

In particular, if $\rho_1 = \rho_2 \in \mathcal{M}_D^+$ then

$$\mathcal{E}_D(\rho,\rho) = \int_D dz \int_0^\infty \left(\int_D \rho(dx) p_u^D(x,z) \right)^2 du, \qquad (74)$$

for any $\rho \in \mathcal{M}_D^+$. Hence using the inequality $2ab \leqslant a^2 + b^2$, valid for any real numbers a and b, we deduce that $\mathcal{E}_D(\rho_1, \rho_2) < \infty$ whenever $\rho_1, \rho_2 \in \mathcal{M}_D^+$.

Now let us check that $\rho_1 + \rho_2 \in \mathcal{M}_D^+$. A priori $\mathcal{E}_D(\rho_1+\rho_2) = \mathcal{E}_D(\rho_1) + 2\mathcal{E}_D(\rho_1,\rho_2) + \mathcal{E}_D(\rho_2)$. This is an equality between terms which are positive but might be infinite. Nevertheless, from what we have just seen if $\rho_1, \rho_2 \in \mathcal{M}_D^+$ all three terms on the right hand side are finite. Thus the left hand side is finite too and the lemma is proved. □

Lemma 8.3 allows us to extend the notion of energy $\mathcal{E}_D(\rho_1,\rho_2)$ onto $\mathcal{M}_D^2$ and not just $(\mathcal{M}_D^+)^2$; writing $\rho_i = \rho_i^+ - \rho_i^-$ for $i = 1,2$ we then have

$$\mathcal{E}_D(\rho_1,\rho_2) = \mathcal{E}_D(\rho_1^+,\rho_2^+) + \mathcal{E}_D(\rho_1^-,\rho_2^-) - \mathcal{E}_D(\rho_1^+,\rho_2^-) - \mathcal{E}_D(\rho_1^-,\rho_2^+);$$

where the finiteness of all four terms on the right hand side comes from Lemma 8.3. Note furthermore that $\mathcal{M}_0^D$ is then a vector space (again by Lemma 8.3) and then $\mathcal{E}_D$ is a bilinear form on $\mathcal{M}_0^D$.

Lemma 8.4. *The bilinear form $\mathcal{E}_D$ is symmetric and nonnegative on $\mathcal{M}_D^2$. That is, for every $n \geqslant 1$ and every $\rho_1, \ldots, \rho_n \in \mathcal{M}_D$, for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$,*

$$\sum_{i,j=1}^n \lambda_i \lambda_j \mathcal{E}_D(\rho_i, \rho_j) \geqslant 0.$$

In particular $\mathcal{E}_D$ is a valid covariance function for a Gaussian stochastic process on $\mathcal{M}_D$.

Proof. Since $\mathcal{E}_D$ is a bilinear form, we have:

$$\sum_{i,j=1}^n \lambda_i \lambda_j \mathcal{E}_D(\rho_i, \rho_j) = \mathcal{E}_D(\rho)$$

where

$$\rho = \sum_{i=1}^n \lambda_i \rho_i \in \mathcal{M}_D.$$

The desired nonnegativity therefore follows directly from (74). □

Lecture 18, Monday 5 June

By Lemma 8.4 the following definition is therefore meaningful:

Definition 8.5. *Let D be a regular domain. The **Gaussian free field** in D with zero (or Dirichlet) boundary conditions is the centered Gaussian stochastic process $(\Gamma_\rho)_{\rho \in \mathcal{M}_D}$ indexed by $\mathcal{M}_D$ with covariance function the bilinear form $\mathcal{E}_D(\cdot, \cdot)$ on $\mathcal{M}_D^2$.*

Furthermore, suppose D is a simply connected domain and let f be a bounded measurable function f on the Martin boundary δD of D. Then the Gaussian free field with boundary conditions f is the law of a Gaussian stochastic process with same covariance as above, and mean function $\bar{u}$ given by $(\bar{u}, \rho) = \int_D u(x)\rho(dx)$ where u is the harmonic extension of f to D, i.e., $u(x) = \mathbb{E}_x(f(B_T))$. In other words, we have $\Gamma = \Gamma_0 + u$, where Γ_0 has zero boundary conditions and u is as above.

If no boundary conditions are specified then we always mean zero boundary conditions. In the following we write GFF for Gaussian free field. Let Γ be a zero boundary conditions GFF with boundary conditions f. Because $\mathcal{E}_D$ is a bilinear form, it is in fact not hard to see that $(\Gamma_\rho)_{\rho \in \mathcal{M}_D}$ is a.s. linear in $\rho \in \mathcal{M}_D$: more precisely:

Lemma 8.6. *Let $\rho_1, \rho_2 \in \mathcal{M}_D$ and $\lambda, \mu \in \mathbb{R}$. Then $\Gamma_{\lambda\rho_1 + \mu\rho_2} = \lambda\Gamma_{\rho_1} + \mu\Gamma_{\rho_2}$, a.s.*

The proof, left to the reader, consists in checking that the variance of the difference is zero. The linearity relation of Lemma 8.6 makes it sensible to abuse of notations, and denote

$$(\Gamma, \rho) := \Gamma_\rho$$

as if Γ was a random distribution. In fact, it is possible to show that the restriction of h to $\mathcal{D}'(D)$ coincides as a stochastic process with a random distribution (in fact, a random series which converges in a Sobolev space of negative index, see Chapter 1.4 in [3]; however this fact will not be needed in what follows). In any case note that Γ may not be evaluated pointwise (because $\rho = \delta_x$ does not lie in $\mathcal{M}_D$), however it may be integrated against smooth, compactly supported test functions $\rho \in \mathcal{D}_0(D)$. In fact the integral of Γ against relatively more singular measures is well defined: for instance, the integral of Γ along a segment or a circle are all well-defined, since the Lebesgue measure on such a one-dimensional smooth curve is an element of $\mathcal{M}_D$ (indeed, this boils down to the fact that the divergence of the Green function is only logarithmic, and the fact that in one dimension $\int_0^1 \log(r^{-1})dr < \infty$.

The Gaussian free field inherits from the energy and harmonic measure a property of conformal invariance. The proof is left as an exercise.

Proposition 8.7. *Let $\phi : D \to D'$ be a conformal isomorphism of regular domains. Suppose that Γ is a Gaussian free field on D with zero boundary values. Then $\Gamma \circ \phi^{-1}$ is a Gaussian free field on D' with zero boundary values.*

Suppose now that D is simply connected, and that Γ is a Gaussian free field on D with boundary value f. Then $\Gamma \circ \phi^{-1}$ is a Gaussian free field on D' with boundary value $f \circ \phi^{-1}$.

The Gaussian free field has the following Markov property; this is not the most general statement possible but will be sufficient for our purposes (e.g., U does not in general need to be a simply connected subdomain).

Proposition 8.8. *Let U be a simply connected subdomain of a proper simply connected domain D. Let Γ_D be a Gaussian free field (with zero boundary conditions) on D. Then Γ_D has an almost surely unique decomposition $\Gamma_D = \Gamma_U + \Phi$ such that:*

- *Γ_U, Φ are independent;*
- *Γ_U is a Gaussian free on U (with zero boundary conditions), and is extended to be zero outside of U;*
- *The restriction of Φ to U coincides with a harmonic function u, i.e., almost surely there exists a harmonic function u such that $(\Phi, \rho) = (u, \rho)$ for all $\rho \in \mathcal{D}(U)$ almost surely.*

Remark 8.9. *If U^c has a nonempty interior V, then since Γ_U is identically zero outside of U, $\Phi|_V$ coincides with $(\Gamma_D)|_V$. In other words, Φ encodes all the information about Γ_D outside of U. Inside U, Φ is harmonic so may be viewed as the harmonic extension of the boundary values determined by the restriction of Γ_D to V. It is in this sense that the above may be viewed as a spatial or domain Markov property: when we condition on the restriction of Γ_D to V (call $\mathcal{F}_V$ the corresponding σ-algebra), the conditional law inside U can be decomposed as the sum of two independent terms: the harmonic term Φ, which is determined only by the "boundary values" of Γ_D, and which gives us the conditional expectation of Γ_D given $\mathcal{F}_V$, and an independent fluctuation term Γ_U which happens to have the law of a GFF with zero boundary conditions in U. Note that in the end, for the description of the conditional law of the restriction of Γ_D to U given $\mathcal{F}_V$, only the information about the "boundary values" of Γ_D along ∂U are relevant.*

Proof. In [3] the proof of this Markov property is based on the random series representation of Γ_D and a Hilbertian decomposition of a certain Sobolev space.

We offer here a different and more pedestrian proof. Since U is simply connected we can approximate it by smooth simply connected subdomains; hence we can without loss of generality assume that U is analytic, i.e., U is the image of the unit disc by a map which is an analytic in a neighbourhood of the unit disc. Then $V = D \setminus \bar{U}$ is open.

For $x \in D$, let B be a Brownian motion starting from x, and let $T = \inf\{t \geqslant 0 : B_t \notin U\}$. Let H_x denote the law of B_T, so that if $x \notin U$, then $H_x = \delta_x$, while if $x \in U$, H_x is simply the harmonic measure on $\delta U = \partial U$ (since U is analytic) viewed from x. Note that since ∂U is smooth, if $x \in U$, H_x is absolutely continuous with respect to Lebesgue measure on ∂U and hence H_x has finite logarithmic energy, i.e., $H_x \in \mathcal{M}_D$.

Furthermore, given a test function $f \in \mathcal{D}(U)$, if we define $\rho_f(dw) = \int_{x \in U} f(x) H_x(dw)$ then for the same reason $\rho_f \in \mathcal{M}_D$.

Lemma 8.10. *For $x, y \in D$ let*

$$f_U(x, y) = \iint_{z,w} G_D(z, w) H_x(dz) H_y(dw).$$

(Note that if $x, y \in U$ then the domain of integration is really $z, w \in \partial U$.) Then for all $x, y \in D$ we have

$$G_D(x, y) = G_U(x, y) + f_U(x, y).$$

(Here we extend $G_U(x, y)$ to be zero if at least one of x or y are in $D \setminus U$).

Proof of Lemma 8.10. If both $x, y \notin U$ there is nothing to prove. Suppose that $x \in U$, say. We can rewrite $f_U(x, y)$ using two independent Brownian motions B and B' starting from x and y respectively, in the form

$$f_U(x, y) = \mathbb{E}_{x,y}[G_D(B_T, B'_{T'})],$$

where T and T' are the respective exit times from U. It is then easy to check that $f_U(x, y)$ satisfies the mean value property so by Proposition 1.2 is harmonic in each variable over all U (including at $y = x$). If we consider the difference $g(y) = G_D(x, y) - f_U(x, y)$, viewed as a function of the variable $y \in U \setminus \{x\}$. Then it is clear that g_y is harmonic in $U \setminus \{x\}$, tends to zero at ∂U, and $g(y) = -(2\pi)^{-1} \log |x - y| + O(1)$ as $y \to x$. By Proposition 1.21 we see that $g(y) = G_U(x, y)$ over $y \in U$ as desired. It is not hard to see this also holds for $y \notin U$ using harmonicity of $G_D(\cdot, y)$ in all of U in that case. □

We return to the proof of Proposition 8.8. For a test function $f \in \mathcal{D}(U)$ set $\rho_f = \int_{x \in U} f(x) H_x(dw)$ and note that $\rho_f \in \mathcal{M}_D$ as mentioned above. Define

$$(\Phi, f) = (\Gamma_D, \rho_f),$$

and note that Φ is a centered Gaussian stochastic process (for now indexed by test functions $f \in \mathcal{D}(U)$) and that

$$\begin{aligned}
\mathrm{Var}(\Phi, f) = \mathcal{E}_D(\rho_f, \rho_f) &= \iint G_D(z, w) \rho_f(dz) \rho_f(dw) \\
&= \iint_{z,w} \iint_{x,y} G_D(z, w) f(x) H_x(dz) dx f(y) H_y(dw) dy \\
&= \iint_{x,y} f(x) f(y) f_U(x, y) dx dy.
\end{aligned}$$

In particular when plugging $\mathrm{Var}(\Phi, \Delta f) = 0$ by integration by part, and hence $(\Phi, \Delta f) = 0$ a.s. Elliptic regularity arguments going beyond the scope of these notes imply that Φ coincides with the restriction to U of a harmonic function on U. In particular (Φ, ρ) is defined for arbitrary $\rho \in \mathcal{M}_U$ and in fact, as can be checked, for all $\rho \in \mathcal{M}_D$. Let Γ_U be independent from Φ and have the law of a Gaussian free field on U. Let $\tilde{\Gamma} = \Gamma_U + \Phi$. We aim to show that $\tilde{\Gamma}$ is has the law of a Gaussian free field in D. Note that it is a.s. linear, and a Gaussian process indexed by $\mathcal{M}_D$. To conclude it therefore suffices to check that $\mathrm{Var}((\tilde{\Gamma}, \rho)) = \mathrm{Var}((\Gamma_D, \rho))$ for all $\rho \in \mathcal{M}_D$.

We have for such $\rho \in \mathcal{M}_U$, using Lemma 8.10:

$$\begin{aligned}\mathrm{Var}(\Gamma_D, \rho) = \mathcal{E}_D(\rho, \rho) &= \iint G_D(x,y)\rho(dx)\rho(dy) \\ &= \iint [G_U(x,y) + f_U(x,y)]\rho(dx)\rho(dy) \\ &= \mathrm{Var}(\Gamma_U, \rho) + \mathrm{Var}(\Phi, \rho) \\ &= \mathrm{Var}(\Gamma_U + \Phi, \rho) = \mathrm{Var}(\tilde{\Gamma}, \rho)\end{aligned}$$

as desired. □

8.3 Angle martingales for SLE(4)

Lecture 19, Friday 16 June

We study a family of martingales for SLE(4) and their relation to the Green function. Then by integrating with respect to a test-function we obtain a splitting identity for the characteristic function of a certain Gaussian free field in $\mathbb{H}$.

Define s_0 on $\delta\mathbb{H}$ by $s_0(\pm x) = \pm 1$ for $x \in (0, \infty)$ and $s_0(0) = s_0(\infty) = 0$. Write σ_0 for the harmonic extension of s_0 in $\mathbb{H}$. Then (writing $h_{\mathbb{H}}(z, dx)$ for the harmonic measure in $\mathbb{H}$ viewed from z),

$$\sigma_0(z) = \int_{\delta\mathbb{H}} s_0(x) h_{\mathbb{H}}(z, dx) = 1 - (2/\pi)\arg(z), \quad z \in \mathbb{H}.$$

Let γ be an SLE(4). Write $(g_t(z) : z \in \mathbb{H}, t < \zeta(z))$ and $(\xi_t)_{t \geqslant 0}$ for the associated Loewner flow and Loewner tranform and set $H_t = \{z \in \mathbb{H} : t < \zeta(z)\}$. Define $s_t(x) = s_0(g_t(x) - \xi_t)$ for $x \in \delta H_t$. The harmonic extension σ_t of s_t in H_t is then given by $\sigma_t(z) = \sigma_0(g_t(z) - \xi_t)$.

Lemma 8.11. *For all $z \in \mathbb{H}$, the process $(\sigma_t(z) : t < \zeta(z))$ is a continuous local martingale and $\zeta(z) = \infty$ almost surely. Moreover, for all $w \in \mathbb{H} \setminus \{z\}$,*

$$d[\sigma_t(z), \sigma_t(w)] = \frac{16}{\pi^2} \mathrm{Im}\left(\frac{1}{g_t(z) - \xi_t}\right) \mathrm{Im}\left(\frac{1}{g_t(w) - \xi_t}\right) dt.$$

Proof. Write $Z_t = g_t(z) - \xi_t$ and $W_t = g_t(w) - \xi_t$. From Loewner's equation, we have $dZ_t = (2/Z_t)dt - d\xi_t$ for $t < \zeta(z)$. Then, by Itô's formula,

$$d \log Z_t = \frac{dZ_t}{Z_t} - \frac{d[Z]_t}{2Z_t^2} = -\frac{d\xi_t}{Z_t} + \left(2 - \frac{\kappa}{2}\right)\frac{dt}{Z_t^2}.$$

Since $\kappa = 4$, this shows that the real and imaginary parts of $(\log Z_t : t < \zeta(z))$ are continuous local martingales. Now $Z_t \to 0$ as $t \to \zeta(z)$ when $\zeta(z) < \infty$, so $\log|Z_t| = \mathrm{Re} \log Z_t \to -\infty$. This is impossible for a continuous local martingale, so $\zeta(z) = \infty$

almost surely. Also $\sigma_t(z) = 1 - (2/\pi)\,\mathrm{Im}\log Z_t$, so $(\sigma_t(z) : t < \zeta(z))$ is a continuous local martingale, with

$$d\sigma_t(z) = \frac{2}{\pi}\,\mathrm{Im}\left(\frac{1}{Z_t}\right) d\xi_t. \tag{75}$$

Then, for $t < \zeta(z) \wedge \zeta(w)$,

$$d[\sigma_t(z)\sigma_t(w)] = \frac{4}{\pi^2}\,\mathrm{Im}\left(\frac{1}{Z_t}\right)\mathrm{Im}\left(\frac{1}{W_t}\right) d[\xi]_t.$$

Since $d[\xi]_t = 4dt$, this concludes the lemma. □

Remark 8.12. *In (75), the expression of the martingale $d\sigma_t(z)$ involves the density of harmonic measure in $\mathbb{H}$ (i.e., the Poisson kernel) viewed from Z_t at zero (recall that $h_{\mathbb{H}}(z, dx) = (-1/\pi)\,\mathrm{Im}(1/(z-x))$, an observation already used in (71), see Example 1.10 for the derivation). Heuristically, this can be understood from the fact that in order to find a change in the value of $\sigma_t(z)$ between times t and $t+dt$, the Brownian motion starting from $Z_t = g_t(z) - \xi_t$ must touch the real line in an interval of size $|d\xi_t|$ near zero, which is where the boundary conditions for σ_0 (or s_0) change; the amplitude of the change is then of size 2, which explains the factor $2/\pi$ in (75).*

We now show that the above quadratic covariation between $\sigma_t(z)$ and $\sigma_t(w)$ is related to the Green function

Lemma 8.13 (Hadamard's identity)**.** *With the same notations as above,*

$$d[\sigma_t(z), \sigma_t(w)] = -(8/\pi) dG_{H_t}(z, w) : \quad t < \zeta(z) \wedge \zeta(w)$$

Proof. By conformal invariance of the Green function, for $t < \zeta(z) \wedge \zeta(w)$,

$$G_{H_t}(z, w) = G_{\mathbb{H}}(g_t(z), g_t(w)) = \frac{1}{2\pi}\log\left|\frac{Z_t - \bar{W}_t}{Z_t - W_t}\right|.$$

Now $d(Z_t - W_t) = 2(W_t - Z_t)dt/(Z_t W_t)$, so

$$d\log(Z_t - W_t) = \frac{-2dt}{Z_t W_t}, \quad d\log(Z_t - \bar{W}_t) = \frac{-2dt}{Z_t \bar{W}_t}$$

so

$$\begin{aligned} dG_{H_t}(z, w) &= d\,\mathrm{Re}\left(\frac{1}{2\pi}\log\left(\frac{Z_t - \bar{W}_t}{Z_t - W_t}\right)\right) \\ &= \mathrm{Re}\left(\frac{1}{\pi Z_t}\left(\frac{1}{W_t} - \frac{1}{\bar{W}_t}\right)\right) dt = -\frac{2}{\pi}\,\mathrm{Im}\left(\frac{1}{Z_t}\right)\mathrm{Im}\left(\frac{1}{W_t}\right) dt. \end{aligned}$$

Hence $d[\sigma_t(z), \sigma_t(w)] = -(8/\pi) dG_{H_t}(z, w)$, as desired. □

Remark 8.14. *Note that* $\operatorname{Im}(1/Z_t) < 0$ *so* $dG_{H_t}(x,y) < 0$ *(in particular* $G_{H_t}(z,w)$ *is of finite variation). This can be seen from the fact that the domains* H_t *are monotone decreasing in* t*; so that if* $t_1 \leqslant t_2$ *and we write* $H_i = H_{t_i}$ *we have* $H_2 \subset H_1$ *and for all* $s \geqslant 0$, $p_s^{H_2}(x,y) \leqslant p_s^{H_1}(x,y)$ *for all* $x, y \in H_2$*, hence* $G_{H_2}(x,y) \leqslant G_{H_1}(x,y)$.

The fact that $dG_{H_t}(z,w)$ *involves the product of the Poisson kernel in* $\mathbb{H}$ *viewed from* Z_t *and* W_t *at 0 can be understood heuristically as follows.* $G_{H_t}(z,w)$ *counts the time spent at* w *by all trajectories starting from* z *and remaining in* H_t*. When* t *increases by a small amount* dt*, the trajectories that are lost are those that go through the small portion of the curve that is added between* t *and* $t + dt$*. After mapping out, these are the trajectories that go from* Z_t *to* W_t *via (approximately) 0 in the upper-half plane. Such trajectories can be factored out into a portion from* Z_t *to 0 and another one from* W_t *to 0.*

The identity

$$dG_{H_t}(z,w) = -\frac{2}{\pi} \operatorname{Im}\left(\frac{1}{Z_t}\right) \operatorname{Im}\left(\frac{1}{W_t}\right) dt$$

*first appeared in a paper by Makarov and Smirnov [**?**] and is called by them* ***Hadamard's identity****.*

Proposition 8.15. *Set* $\lambda = \sqrt{\pi/8}$ *and* $\gamma^* = \gamma[0,\infty]$*. Write* D^- *and* D^+ *for the left and right components of* $\mathbb{H} \setminus \gamma^*$*. Then, for all* $\rho \in \mathcal{D}(\mathbb{H})$*, we have*

$$\exp\left\{i\lambda \mathcal{H}_{\mathbb{H}}(s_0,\rho) - \frac{\mathcal{E}_{\mathbb{H}}(\rho)}{2}\right\} = \mathbb{E}\left(\exp\left\{i\lambda\rho(D^+) - \frac{\mathcal{E}_{D^+}(\rho)}{2}\right\} \exp\left\{-i\lambda\rho(D^-) - \frac{\mathcal{E}_{D^-}(\rho)}{2}\right\}\right),$$

where $\mathcal{H}_{\mathbb{H}}(s_0,\rho) = (\sigma_0,\rho) = \int_{z\in\mathbb{H}} \rho(z)\sigma_0(z)dz$.

Proof. The martingale $\sigma_t(z), t < \zeta(z)$ is bounded and so has a limit at $t \to \zeta(z)$. Note that this allows us (for a fixed $z \in \mathbb{H}$) to extend its definition up to and including $t = \zeta(z)$.

Fix $\rho \in \mathcal{D}(\mathbb{H})$ and set

$$M_t = \int_{\mathbb{H}\setminus\gamma^*} \lambda\sigma_t(z)\rho(z)dz.$$

For $z \in \mathbb{H} \setminus \gamma^*$, the map $t \mapsto \sigma_t(z)$ is continuous on $[0,\infty)$ and $|\sigma_t(z)| \leqslant 1$ so, by dominated convergence, $t \mapsto M_t$ is continuous on $[0,\infty)$, almost surely.

Furthermore, for all $z \in \mathbb{H}$, we have $\zeta(z) = \infty$ almost surely, hence $z \in \mathbb{H} \setminus \gamma^*$ almost surely. Thus (by Fubini's theorem) the trace $\gamma^* = \{z \in \mathbb{H} : \zeta(z) < \infty\}$ has zero planar Lebesgue measure almost surely.

It follows from Fubini's theorem that M is a martingale: indeed, for $s \leqslant t$ and $A \in \mathcal{F}_s$,

$$\mathbb{E}(M_t 1_A) = \int_{\mathbb{H}} \mathbb{E}(1_{\{\zeta(z)=\infty\}}\lambda\sigma_t(z)1_A)\rho(z)dz = \int_{\mathbb{H}} \mathbb{E}(\lambda\sigma_s(z)1_A)\rho(z)dz = \mathbb{E}(M_s 1_A).$$

since $\sigma_t(z)$ is a martingale by Lemma 8.11.

Its quadratic variation may be identified by Hadamard's identity (Lemma 8.13): we guess

$$d[M]_t = \lambda^2 \iint \rho(z)\rho(w)d[\sigma_t(z),\sigma_t(w)]dzdw = -d\mathcal{E}_{H_t}(\rho) \tag{76}$$

using the value of $\lambda = \sqrt{\pi/8}$ and the definition of the logarithmic energy. To justify (76), we use again Fubini's theorem (using again that $\zeta(z) = \infty$ almost surely for a fixed $z \in \mathbb{H}$) as follows:

$$\mathbb{E}((M_t^2 + \mathcal{E}_{H_t}(\rho))1_A) = \int_{\mathbb{H}^2} \mathbb{E}((\lambda^2 \sigma_t(z)\sigma_t(w) + G_{H_t}(z,w))1_A)\rho(z)\rho(w)dzdw$$
$$= \int_{\mathbb{H}^2} \mathbb{E}((\lambda^2 \sigma_s(z)\sigma_s(w) + G_{H_s}(z,w))1_A)\rho(z)\rho(w)dzdw = \mathbb{E}((M_s^2 + \mathcal{E}_{H_s}(\rho))1_A).$$

Hence $(M_t : t \geqslant 0)$ and $(M_t^2 + \mathcal{E}_{H_t}(\rho) : t \geqslant 0)$ are continuous martingales. Thus $(M_t : t \geqslant 0)$ has quadratic variation process $[M]_t = \mathcal{E}_{\mathbb{H}}(\rho) - \mathcal{E}_{H_t}(\rho)$. Set $E_t = \exp\{iM_t - \mathcal{E}_{H_t}(\rho)/2\}$. By Itô's formula, $(E_t : t \geqslant 0)$ is a local martingale, which is moreover bounded. So

$$\mathbb{E}\left(\exp\{iM_t - \mathcal{E}_{H_t}(\rho)/2\}\right) = \mathbb{E}(E_t) = \mathbb{E}(E_0) = \exp\{iM_0 - \mathcal{E}_{\mathbb{H}}(\rho)/2\}. \tag{77}$$

Now, we know that $\gamma_t \to \infty$ as $t \to \infty$ almost surely, so $\sigma_t \to \pm 1$ on $D^\pm$, and so

$$M_t \to \lambda\rho(D^+) - \lambda\rho(D^-).$$

Also, $G_{H_t} \to G_{D^\pm}$ on $D^\pm \times D^\pm$ and $G_{H_t} \to 0$ on $D^\pm \times D^\mp$ almost surely, so

$$\mathcal{E}_{H_t}(\rho) \to \mathcal{E}_{D^-}(\rho) + \mathcal{E}_{D^+}(\rho).$$

On letting $t \to \infty$ in (77), using bounded convergence, we obtain the claimed identity. $\square$

8.4 Schramm–Sheffield theorem

Proposition 8.15 can be interpreted in terms of the characteristic functions of certain Gaussian free fields, and then implies immediately the following result of Schramm and Sheffield, which expresses an identity in law for the corresponding fields.

Theorem 8.16. *Let γ be an SLE(4) and let D^- and D^+ be the left and right components of $\mathbb{H} \setminus \gamma^*$. Conditional on γ, let Γ^- and Γ^+ be independent Gaussian free fields with zero boundary values, on D^- and D^+ respectively. Write $\bar{\Gamma}^\pm$ for their extensions as random variables in $\mathcal{D}'(\mathbb{H})$. Set $\lambda = \sqrt{\pi/8}$ and define*

$$\Gamma = (\bar{\Gamma}^+ + \lambda 1_{D^+}) - (\bar{\Gamma}^- + \lambda 1_{D^-})$$

Then Γ is a Gaussian free field on $\mathbb{H}$ with boundary values $-\lambda$ and λ on the left and right half-lines respectively.

In other words, Theorem 8.16 above, which is due to Schramm and Sheffield [22], shows the existence of a **coupling** between a Gaussian free field Γ on $\mathbb{H}$ with boundary values $\pm\lambda$ on the real line, and a curve γ with the law of an SLE(4) such that the values of Γ to the left of γ are $-\lambda$, and to the right of γ, $+\lambda$; it is in this sense that γ can be viewed as a level line of the field Γ. In fact, it can be shown [22, 8] that γ is a measurable *function*

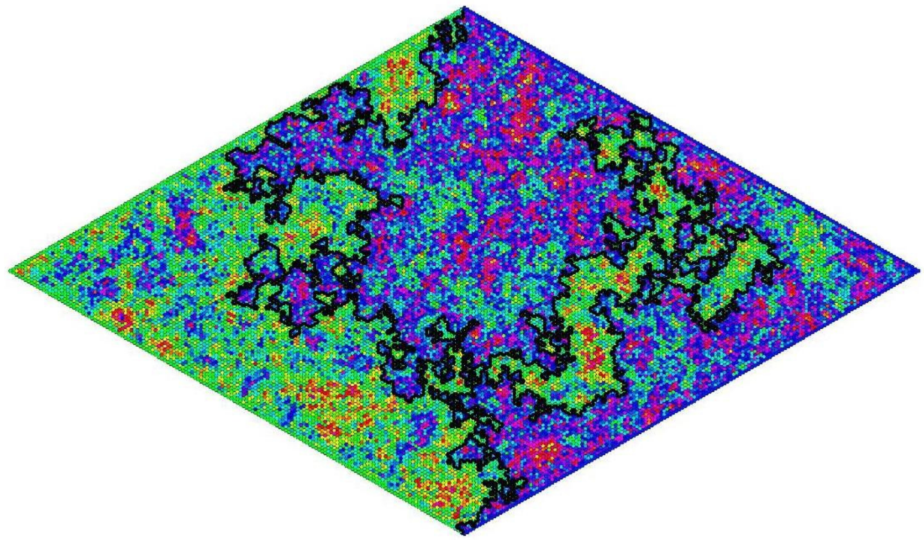

Figure 12: Coupling between Gaussian free field and SLE(4). Picture courtesy of S. Sheffield.

of the field Γ, as one would expect of a curve describing the level line of a field. However, the argument to show this property is significantly more subtle than the proof of Theorem 8.16.

Augmenting this intuition, Schramm and Sheffield [21] showed (in fact before [22]) the remarkable result that a level line of the discrete Gaussian free field on the triangular lattice with boundary conditions $\pm\lambda$ converge to SLE(4).

Before giving the proof of Theorem 8.16, here is a motivating argument, which is not rigorous. *'Suppose we can find a simple chord $\gamma = (\gamma_t : t \geqslant 0)$ in $(\mathbb{H}, 0, \infty)$, parametrized by half-plane capacity, along which there is a cliff in Γ, with value λ to the right and $-\lambda$ to the left. Indeed, suppose we can find γ without looking at the values of Γ away from the cliff. Then, by the Markov property and conformal invariance of the free field, conditional on $\mathcal{F}_t = \sigma(\gamma_s : s \leqslant t)$, $\tilde{g}_t(\Gamma|_{H_t})$ has the original distribution of Γ, and so γ has the domain Markov property. Moreover, by conformal invariance of the free field, γ is also scale invariant, so γ is an SLE(κ) for some $\kappa \in [0, \infty)$. Consider the function $\phi_t(z) = \mathbb{E}(\Gamma(z)|\mathcal{F}_t)$. Then for fixed t, ϕ_t must be the harmonic extension in H_t of the boundary values of Γ on δH_t. Thus $\phi_t = \lambda\sigma_t(z)$. Now $(\phi_t(z) : t < \zeta(z))$ appears to be a martingale. Hence, as we saw in the proof of Proposition 8.11, we must have $\kappa = 4$.'* Note that the theorem turns the construction backwards and does not state that γ is a measurable function of Γ.

Proof of Theorem 8.16. By Proposition 8.15, for all $\rho \in \mathcal{D}(\mathbb{H})$,

$$\mathbb{E}(\exp\{i\Gamma(\rho)\}) = \exp\{i\mathcal{H}_{\mathbb{H}}(\lambda s_0, \rho) - \mathcal{E}_{\mathbb{H}}(\rho)/2\}$$

so $\Gamma(\rho)$ is Gaussian of mean $\mathcal{H}_{\mathbb{H}}(\lambda s_0, \rho)$ and variance $\mathcal{E}_{\mathbb{H}}(\rho)$ by uniqueness of characteristic functions, and so Γ is a Gaussian free field on $\mathbb{H}$ with boundary value λs_0, as required.

Using the linearity in ρ of $\mathcal{H}_{\mathbb{H}}(\lambda s_0, \rho)$ and the bilinearity in ρ of $\mathcal{E}_{\mathbb{H}}(\rho)$ we deduce that the finite-dimensional distributions of $(\Gamma(\rho_1), \ldots, \Gamma(\rho_n))$ match those of a Gaussian free field in $\mathbb{H}$ with boundary values λs_0. This completes the proof. $\square$

The finite-time identity (77) can be interpreted similarly. *Conditional on $(\gamma_s : s \leqslant t)$, let Γ_t be a Gaussian free field on H_t with boundary value λs_t and let $\bar{\Gamma}_t$ be its extension as a random variable in $\mathcal{D}'(\mathbb{H})$. Then Γ_t is a Gaussian free field on $\mathbb{H}$ with boundary value λs_0.*

9 Additional topics

The following section is meant to give a brief overview of some topics closely related to SLE that have come play to an important role in recent developments of the theory. We do not go in any depth and do not include any proofs, but focus on some main ideas.

9.1 Radial SLE

In a few natural discrete models of statistical mechanics, the natural curves do not go from boundary to boundary, as in the chordal theory of SLE we have developed, but rather from an interior point to the boundary (think of a loop-erased random walk, or a self-avoiding walk, starting from an interior point). In this case the scaling limits will involve a variant of SLE known as radial SLE.

To describe this, our first task is to parameterise the curves appropriately. While in the chordal case this was given by the half-plane capacity (which is natural as the curves are targeted at infinity), here it is natural to take as reference domain the unit disc $\mathbb{D}$, and view the curve as being oriented from a boundary point b (say $b = 1$) to some interior target z (say $z = 0$). We then measure the size of a compact $\mathbb{D}$-hull K (i.e., K is such that $K \subset \mathbb{D}$ and $\mathbb{D} \setminus K$ is simply connected) via its so-called **conformal radius**. More precisely, given such a compact $\mathbb{D}$-hull note that there exists a unique conformal isomorphism, denoted by g_K, such that $g_K : \mathbb{D} \setminus K \to \mathbb{D}$, such that $g_K(0) = 0$ and $g_K'(0) > 0$. The number

$$R_K = 1/g_K'(0)$$

is called the conformal radius of $\mathbb{D} \setminus K$ viewed from 0. It is a (conformal) measure of the distance from 0 to the boundary of $\mathbb{D} \setminus K$.[12] By Schwarz's lemma (Lemma 2.8), note that $R_K \leqslant 1$. The conformal radius also behaves multiplicatively under composition of conformal maps, hence if $K_1 \subset K_2$ are two compact $\mathbb{D}$-hulls then $R_{K_2} \leqslant R_{K_1}$.

We leave it to the reader to formulate a notion of local growth for families of compact $\mathbb{D}$-hulls $(K_t)_{t \geqslant 0}$, analogous to Definition 4.1. Associated to such a growing family of compact $\mathbb{D}$-hulls with local growth, there is a well-defined **Loewner transform** U_t which is now continuous function on the unit circle (the Martin boundary of $\mathbb{D}$). By continuity, we may write $U_t = e^{i\xi_t}$ for a unique continuous real-valued function $(\xi_t)_{t \geqslant 0}$ with $\xi_0 = 0$ so that $e^{i\xi_0} = U_0 = b = 1$. It is more convenient to think of $(\xi_t)_{t \geqslant 0}$ as the Loewner transform.

Thanks to the above monotonicity, any such family can be parametrised so that $R_{K_t} = e^{-t}$, i.e.,

$$-\log R_{K_t} = t.$$

If that is the case, we say that $(K_t)_{t \geqslant 0}$ is parameterised by its **log-conformal radius** or capacity. There is a form of Loewner's theorem for such a growing family of compact $\mathbb{D}$-hulls. A simplified version of this theorem (which conveys the most relevant information) is as follows:

[12] Indeed, by the so-called Koebe 1/4 theorem, it can be proved that $(d/4) \leqslant R_K \leqslant d$, where $d = \mathrm{dist}(0, \partial(\mathbb{D} \setminus K))$.

Theorem 9.1. *Let $(K_t)_{t\geqslant 0}$ be as above and write $g_t = g_{K_t}$. Then g_t satisfies a differential equation, namely*

$$\frac{d}{dt}g_t(z) = g_t(z)\frac{e^{i\xi_t} + g_t(z)}{e^{i\xi_t} - g_t(z)}, \tag{78}$$

which is valid until a time $\tau(z)$ such that $|g_t(z) - e^{i\xi_t}| \to 0$ as $t \uparrow \zeta(z)$.

Conversely, given a continuous real-valued function $(\xi_t)_{t\geqslant 0}$, solving the ordinary differential equation (78)defines a unique growing family of compact $\mathbb{D}$-hulls satisfying local growth and parameterised by capacity having $(\xi_t)_{t\geqslant 0}$ as its Loewner transform.

Radial SLE_κ is obtained by setting $\xi_t = \sqrt{\kappa}B_t$, as usual. The properties of radial SLE_κ are easiest to describe when we express it as an evolution in the upper-half plane. More precisely, consider the map $\psi(z) = e^{iz}$, which takes the upper half plane $\mathbb{H}$ to the unit disc $\mathbb{D} \setminus \{0\}$, and takes 0 and ∞ respectively to 1 and the *interior* point 0. Thus for $z \in \bar{\mathbb{H}}$, let

$$h_t(z) := \psi^{-1} \circ g_t \circ \psi(z) = -i \log g_t(e^{iz}),$$

which is defined either locally or as a multivalued function, but describes the growth in $\mathbb{H}$ instead of $\mathbb{D}$. Then note that h_t satisfies the equation

$$\begin{aligned}
\frac{d}{dt}h_t(z) &= -i\frac{\dot{g}_t(z)}{g_t(z)} \\
&= -i\frac{e^{i\xi_t} + g_t(e^{iz})}{e^{i\xi_t} - g_t(e^{iz})} \\
&= -i\frac{e^{i\xi_t} + e^{ih_t(z)}}{e^{i\xi_t} - e^{ih_t(z)}} \\
&= \cot\left(\frac{h_t(z) - \xi_t}{2}\right).
\end{aligned}$$

Recall that as $x \to 0$, $\cot(x) = \cos(x)/\sin(x) \sim 1/x$, so as $(h_t(z) - \xi_t) \to 0$ (i.e., as $t \uparrow \zeta(z)$),

$$\frac{d}{dt}h_t(z) \sim \frac{2}{h_t(z) - \xi_t}.$$

We recover (approximately) the chordal form of Loewner's differential equation.[13]

This can be used to show that the **phases** of radial SLE_κ are the same as those of chordal SLE_κ. Hence radial SLE_κ is simple for $\kappa \leqslant 4$, space-filling for $\kappa \geqslant 8$, and neither of those things for $\kappa \in (4, 8)$.

In fact, arguments similar to those used in the proof of locality (Theorem 6.7) could be used to show that a radial SLE_6, observed in the upper half plane and up to a certain time, is nothing but a chordal SLE_6 (up to a time change). More precisely, if γ_t is a chordal SLE_6 in $\mathbb{H}$ from 0 to ∞, then the curve $(e^{i\gamma_t})_{t\leqslant T}$ in $\bar{\mathbb{D}}$, considered up to the first time T such that $0 \not\leftrightarrow \partial\mathbb{D}$ (the first time that 0 is disconnected from the boundary of the unit disc by $e^{i\gamma}$), is a radial SLE_6, up to a time-change. This is a form of locality for SLE_6.

[13] The traditional factor 2 in the chordal form of Loewner's equation actually originates from this calculation.

9.2 SLE$_\kappa(\rho)$.

In some natural problems (including many models of statistical mechanics), conformal invariance and/or domain Markov property may only be expected up to the specification of an additional point, typically on the boundary of the domain, which plays a special role in the model. This could reflect a "change in the boundary conditions" of the model, or some other more subtle distinction.

Let us give two examples, both related to models we have already encountered, and which illustrate this idea.

Example 9.2. *Consider a Gaussian free field on the upper half plane $\mathbb{H}$, with boundary conditions $-\lambda$ on $(-\infty, 0)$, $+\lambda$ on $(0, 1)$ and $\lambda+\alpha$ on $(1, \infty)$, where $\lambda = \sqrt{\pi/8}$ and $\alpha > 0$. In other words, instead of the two values $\pm\lambda$ on either side of zero which form the boundary conditions in Theorem 8.16, there is a third boundary value, $\lambda+\alpha$, along the interval $(1, \infty)$. We may still expect the existence of a curve γ starting from 0 which keeps $-\lambda$ and $+\lambda$ on its left and right, but we should not expect it to be conformally invariant unless we specify the position of the point (here $x = 1$) where the boundary condition changes from λ to $\lambda + \alpha$.*

Another example is provided by considering a Brownian excursion $(E_t)_{t\geqslant 0}$ in the upper half plane $\mathbb{H}$, and letting γ denote the *right* boundary of its associated filling. Suppose we discover a portion $\gamma[0, t]$ of this boundary, and seek to describe the future evolution of the curve. Then this is described by right boundary of a Brownian motion, starting from $\gamma(t)$, which may not touch the real line, or the right-hand side of $\gamma[0, t]$, but which may touch its left hand side. If we believe in conformal invariance and apply the conformal g_t which maps away $\gamma[0, t]$, then this becomes a Brownian motion which is reflected on $[O_t, \xi_t]$ (where O_t is the "left image" of 0, and ξ_t is the Loewner transform), and is not allowed to touch $\mathbb{R} \setminus [O_t, \xi_t]$, before touching ∞. Clearly, this description necessitates a third boundary point namely O_t) beyond the start and end point of the excursion, 0 and ∞.

In both these examples, we are looking for laws $\mu_{(D,a,b,o,\sigma)}$ which are indexed by a two-pointed domain (D, a, b) and a scale σ (so we look for a chord in D from a to b) and by an additional point $o \in \delta D$, the (Martin) boundary of D. It is only if we keep track of this additional marked point o that the corresponding laws may be expected to enjoy conformal invariance and domain Markov property.

Suppose that $\mu_{(D,a,b,\sigma,o)}$ is a family of such laws. We now explain that there are rather few possibilities for what these laws can be. Suppose we observe the corresponding hulls $(K_t)_{t\geqslant 0}$ in the upper half plane, parameterised by half-plane capacity, and let $o \in \mathbb{R}$ be the marked point. Let g_t denote the associated Loewner flow, ξ_t the Loewner transform, and set $O_t = g_t(o)$, for $t < \zeta(o)$. Let us try and describe the law for the evolution of the Loewner transform $(\xi)_{t\geqslant 0}$.

First, observe that Loewner's equation applies, so

$$\frac{d}{dt} O_t = \frac{2}{O_t - \xi_t}. \tag{79}$$

On the other hand, the domain Markov property assumption implies that

$$Z_t := \xi_t - O_t$$

is a continuous diffusion, i.e., the solution of an (autonomous) stochastic differential equation. The assumption of conformal invariance implies that Z_t enjoys in addition the Brownian scaling property. It is not hard to see that the only SDEs which enjoy Brownian scaling are in fact Bessel processes. This specifies a law for $Z = \xi - O$; from this one can recover the evolution of O via (79) and hence ξ itself, which defines a unique chordal evolution via Loewner's theorem.

Thus take $\kappa \geqslant 0$, and fix $\rho > -2$. *Define* Z_t (later to be equal to $\xi_t - O_t$) be the solution of the SDE

$$dZ_t = \sqrt{\kappa} dB_t + \frac{\rho + 2}{Z_t} dt. \tag{80}$$

After scaling by $\sqrt{\kappa}$, this is a Bessel process of dimension δ, where

$$\delta = 1 + \frac{2(\rho + 2)}{\kappa}.$$

The reason for the choice of the form of the constant $\rho + 2$ in (80) will become clear below (see (81)). For now, note that since $\rho > -2$, $\rho + 2 > 0$ and $\delta > 1$.

Note that Z_t is well defined for all $t \geqslant 0$, and *can* be equal to zero; indeed it can even start from 0 (in which case one must specify if it starts either from 0^+ or 0^-.) Either way, by definition, the sign of Z remains constant and is determined by its value (which is chosen to be nonnegative if starting from 0^+ and nonpositive if starting from 0^-).[14] Furthermore, note that $\int_0^t du/Z_u < \infty$, since in fact

$$\int_0^t \frac{du}{Z_u} = \frac{Z_t - \sqrt{\kappa} B_t - Z_0}{(\rho + 2)}.$$

[14]The degeneracy of the equation at $Z = 0$ means that the SDE falls outside the scope of classical SDE theory with Lipschitz or Hölder coefficients. By convention, to define a solution to (80) we start instead from the squared-Bessel equation, i.e., we take $X_t = Z_t^2$ which satisfies $dX_t = 2\sqrt{\kappa X_t} dB_t + \kappa \delta dt$; this is now an SDE with Hölder 1/2 coefficients for which strong (pathwise) uniqueness holds; the solutions are then a.s. nonnegative and we set $Z_t = \sqrt{X_t}$. When the dimension of the Bessel process δ satisfies $\delta > 1$, the integral $\int_0^t du/Z_u$ converges, and Z is a strong (i.e., adapted to the filtration of Brownian motion) solution of the equation (80) in the sense that $Z_t - Z_0 = \sqrt{\kappa} B_t + \int_0^t (\rho + 2) du/Z_u$, a.s. Thus, the fact that a strong solution to (80) exists which furthermore remains of constant sign is a consequence of the fact that $\delta > 1$.

However when the dimension δ satisfies instead $\delta \in [0, 1]$, corresponding to values of the parameter $\rho \leqslant -2$, the integral $\int_0^t du/Z_u$ a.s. does not converge absolutely. It can still be assigned a meaning e.g. via the notion of *principal value* (where we remove the contribution of the local time at zero to obtain a finite integral), see [17]. Alternatively, we may involve more randomness than just that of the driving Brownian motion and endow Z with a sign, chosen in an i.i.d. manner, each time it restarts from 0 (taken to be positive with probability $(1 + \beta)/2$ and negative with probability $(1 - \beta)/2$; here $\beta \in [-1, 1]$ is a skenwness parameter). This results in a signed version of a Bessel process called skew Bessel process, and symmetric in the particular case $\beta = 0$.

We may thus *define*

$$\begin{cases} O_t := o - 2\displaystyle\int_0^t \frac{du}{Z_u} \\ \xi_t := Z_t + O_t. \end{cases}$$

Definition 9.3. *The chordal $SLE_\kappa(\rho)$ in $\mathbb{H}$ from 0 to ∞, with initial marked point $o = O_0$, is the unique chordal evolution whose Lowener transform is $(\xi_t)_{t\geqslant 0}$.*

Note that ξ satisfies an equation

$$d\xi_t = dZ_t + dO_t = \sqrt{\kappa}dB_t + \frac{\rho}{Z_t}dt. \tag{81}$$

This explains the form of the constant $\rho + 2$ in (80); on the other hand, note that in comparison to (80), (81) is not an autonomous SDE. Yet this equation allows us to give some meaning to the parameter ρ: in comparison with regular SLE_κ, the driving function ξ is "attracted" to O_t if $\rho < 0$, and is "repelled" by it if $\rho > 0$. For $\rho = 0$ we recover ordinary SLE_κ.

Note that since Z_t may touch zero, it is possible that O_t touches ξ_t; equivalently the curve has absorbed the point o which was initially marked. Yet the evolution continues after, and in fact after the time $T = \zeta(o)$, the evolution is again that of a chordal $\mathrm{SLE}_\kappa(\rho)$ but with a new marked point which is immediately to the left $K_T \cap \mathbb{R}$ (assuming that $Z_0 \geqslant 0$, so that initially the marked point is to the left of the starting point of the curve). Thus, when the curve absorbs the marked point, we immediately move the marked point (potentially infinitesimally) in the direction which preserves the order between the starting point of the curve and the marked point.

Since Z is a Bessel process, it is not hard to find for which values of ρ the curve touches the real line:

Lemma 9.4. *Let $(K_t)_{t\geqslant 0}$ be chordal $SLE_\kappa(\rho)$ with some marked point o. Then $(K_t; o)_{t\geqslant 0}$ is scale-invariant. Furthermore, set $\rho_0 = \kappa/2 - 2 > -2$.*

- *If $\kappa \leqslant 4$ and $\rho \geqslant \rho_0$, then $K_\infty \cap \mathbb{R} = \{0\}$, a.s.*
- *If $\kappa \leqslant 4$ and $\rho < \rho_0$ and $o \leqslant 0$ (so initially the marked point is to the left of 0), then $K_\infty \cap \mathbb{R} = (-\infty, o] \cup \{0\}$, a.s.*

Thus for $\kappa \leqslant 4$ and $\rho < \rho_0$, the curve is sufficiently attracted to the marked point that it goes and hits it, even though $\kappa \leqslant 4$ so the curve would like to stay away from the real line. In that case the marked point is immediately moved to the left (infinitesimally) of its current value, and the process continues in the same manner. Eventually the curve covers all of $(-\infty, o]$ and its starting point 0, but nothing more.

Note also that so long as $\xi_t \neq O_t$, the curve is (by Girsanov's theorem) absolutely continuous with respect to SLE_κ. In particular, it will be simple, space-filling, or swallowing exactly in the same way that a regular chordal SLE_κ is.

There are interesting connections between a chordal $\mathrm{SLE}_{8/3}(\rho)$ for $\rho > -2$ and restriction measures. Recall that a sample K from a restriction measure of exponent $\alpha > 0$ is one such that

$$\mathbb{P}(K \cap A = \emptyset) = \phi_A'(0)^\alpha \tag{82}$$

for *all* compact hulls $A \in \mathcal{Q}$, i.e., which avoid zero.

Let γ be a chordal $\mathrm{SLE}_{8/3}(\rho)$ in $\mathbb{H}$ from 0 to ∞ with marked point $o = 0^-$, where $\rho > -2$. Let K denote the "left filling" associated to γ, obtained by filling in everything to the left of the curve. It is natural to ask if K satisfies a **one-sided restriction**, in which we require (82) to hold only for those A such that $\bar{A} \cap \mathbb{R} \subset (0, \infty)$. We let $\mathcal{Q}_+$ the set of hulls.

Using the same tools as Theorem 6.17, one can show:

Proposition 9.5. *Let K be as above. Then K satisfies a one-sided restriction with exponent*

$$\alpha = \frac{20 + 16\rho + 3\rho^2}{32} = \frac{(3\rho + 10)(2 + \rho)}{32}.$$

Note that when ρ spans $(-2, \infty)$ then α spans $(0, \infty)$. In particular, unlike in the two-sided case, one-sided restriction measures exist for *all* $\alpha > 0$.

A two-sided restriction measure always defines a one-sided restriction measures. We deduce:

Corollary 9.6. *Let $\alpha \geqslant 5/8$, and let K be a sample from a restriction measure. Then its right boundary is an $SLE_{8/3}(\rho)$ with*

$$\rho = \frac{-8 + 2\sqrt{24\alpha + 1}}{3}.$$

In particular, the right boundary of a Brownian excursion in the upper half plane is a chordal $\mathrm{SLE}_{8/3}(2/3)$ with marked point $o = 0^-$.

It is not hard to deduce as a corollary that two-sided restriction measures cannot exist for $\alpha < 5/8$.

Corollary 9.7. *For any $\alpha < 5/8$, the two-sided restriction measure does not exist.*

Sketch of proof. Let γ be an $\mathrm{SLE}_{8/3}(\rho)$ curve and suppose $\rho < 0$. Let E be the event that i is to the right of the curve γ, i.e., i is not separated from 1 by γ. Then since $\rho < 0$ one has that $\mathbb{P}(E) > 1/2$ (which is what it would be for $\mathrm{SLE}_{8/3}(0)$)[15].

On the other hand, if K is a sample from two-sided restriction measure, then the probability that i is to the right of K is at most $1/2$ by symmetry (it can be strictly less if there is a positive probability that i is neither to the left nor to the right of K, e.g. if K is the filling of a Brownian excursion). However for $\alpha < 5/8$, the right boundary of a one-sided restriction measure with exponent α is an $\mathrm{SLE}_{8/3}(\rho)$ with $\rho < 0$, showing that this cannot come from a two-sided restriction measure. $\square$

[15] This requires an argument, which is sketched in [11].

9.3 Reversibility and duality

The definition of chordal SLE_κ as a chord in $\mathbb{H}$ from 0 to infinity is very *directional*; it is important that we think of the curve as growing from 0 to ∞ (and not the other way round) to define the Loewner transform. However if we think of an SLE curve as the scaling limit of an interface in a model of statistical mechanics (which is in many cases believed, and sometimes proved as we have argued) then there is no reason to consider the curve as oriented in one particular direction. For that reason Rohde and Schramm [18] conjectured that for $\kappa \leqslant 8$, SLE_κ is **reversible**. In other words, if γ is a chordal SLE_κ from 0 to ∞ in the upper half plane $\mathbb{H}$, and if $\psi(z) = -1/z$ is the Möbius inversion then $(\psi(\gamma(t)); t \geqslant 0)$ is, up to time parameterisation, also an SLE_κ from 0 to ∞ in $\mathbb{H}$. This conjectured has been proved (at least for $\kappa \leqslant 4$) by Dapeng Zhan [26].

Another remarkable property is the **Duplantier duality**, which, roughly speaking, states that the outer boundary η' of (a portion of) an SLE_κ curve η, should be, locally, (a version of) $\mathrm{SLE}_{\kappa'}$, where the parameters κ and κ' are related via the identity

$$\kappa\kappa' = 16. \tag{83}$$

Note that if $\kappa \geqslant 4$ then $\kappa' \leqslant 4$, and $\kappa = 4$ is self-dual. Thus if η is nonsimple ($\kappa > 4$), η' is a.s. simple.

For instance, when $\kappa = 8$ (in which case η corresponds to the scaling limit of the curve snaking around the Uniform Spanning Tree), then $\kappa' = 2$, corresponding to loop-erased random walk. Already at the discrete level, it is indeed already known that there are close connections between the Uniform Spanning Tree and Loop-Erased Random Walks, which are summarised through Wilson's celebrated algorithm for sampling a Uniform Spanning Tree using Loop-Erased Random Walk. A precise form of this conjecture was described by Dubédat in [6]. A form of this conjecture was proved rigorously by Dubédat in [7] and Zhan [25], who proved the precise form conjectured in [6]. This result takes the following form:

Theorem 9.8. *For $\kappa > 4$, the right boundary of the final hull of a chordal $SLE_\kappa(\kappa-4)$ from 0 to ∞ with marked point at 0^+ is a chordal $SLE_{\kappa'}(\frac{\kappa'-4}{2})$ from 0 to ∞ with marked point at 0^-.*

Note that in this result, the values of the ρ parameter, namely $\rho = \kappa - 4$ and $\rho' = (\kappa' - 4)/2$ both satisfy $\rho, \rho' > -2$.

9.4 Conformal Loop Ensemble

In many cases, in models of statistical mechanics one would like to keep track not just of a single interface but rather, in some sense, the whole collection of interfaces separating different clusters. The natural candidate is an object called *Conformal Loop Ensemble* (or CLE) for short and was first constructed in the case $\kappa = 6$ of percolation by Camia and Newman [4] under the name of "full scaling limit". A more general construction was

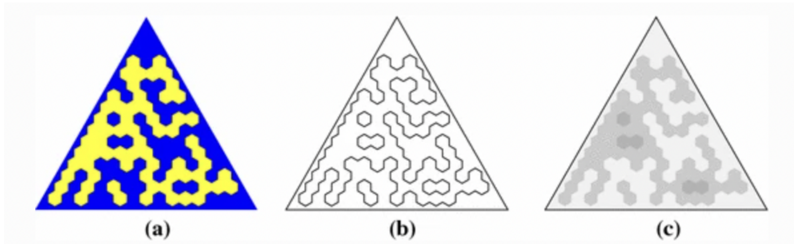

Figure 13: (a) Critical percolation configuration; (b) associated loop configuration; (c) shading corresponds to nesting. Image from [15].

proposed by Sheffield in [23] in the range of parameters $\kappa \in (8/3, 8]$ which turns out to be maximal; the resulting random collection (or ensemble) of loops is called CLE_κ. (We will say a few words about this construction in the case $\kappa > 4$ below). Locally, each loop in this ensemble is absolutely continuous with respect to SLE_κ. The loops will thus turn out to be simple and not touch the boundary or each other when $\kappa \leqslant 4$, but may touch the boundary, themselves or each other without crossing if $\kappa \in (4, 8]$. As $\kappa \downarrow 8/3$, the corresponding CLE_κ degenerates to the empty set, while as $\kappa \uparrow 8$ the ensemble becomes a unique space-filling loop which coincides with SLE_8. Thus the range of parameters $\kappa \in (8/3, 8]$ cannot be extended beyond these two bounds.

Subsequently an axiomatic characterisation in terms of conformal invariance and domain Markov property was proposed by Sheffield and Werner in [24]. In that same paper, a different construction in terms of the so-called Brownian loop soup was given for $\kappa \leqslant 4$. We summarise some of these results below.

The setup is the following. Let D be a proper simply connected domain and let $\mathcal{C} = (\ell_j)_{j\geqslant 1}$ be a countable collection of random loops in D. It is convenient to allow the state space of loops to be arbitrary (i.e., continuous maps from the unit circle into D); nevertheless it will a.s. be the case that in $\mathcal{C}$, all loops are non-crossing, and cannot cross each other. We let μ_D the law of $\mathcal{C}$. Since we prefer to view $\mathcal{C}$ as unordered it is more appropriate to identify $\mathcal{C}$ with $\sum_{j=1}^\infty \delta_{\{\ell_j\}}$ which is a random point measure on the space $\mathcal{L}$ of loops. Equipped with the Prokhorov distance, this is a metric space and we let μ_D be the law of $\mathcal{C}$ on measures on $\mathcal{L}$. Associated to $\mathcal{C}$ is also a collection $\mathcal{C}^*$ of outermost loops: indeed, since the loops of $\mathcal{C}$ do not cross each other they have a nested structure: given $\ell, \ell' \in \mathcal{C}$ we have either ℓ' surrounds ℓ, or the other way around, or the two loops have disjoint interiors. A loop ℓ is called outermost if it is not surrounded by any loop; we then let Γ^* to be the set of outermost loops.

We will assume that the collection of laws μ_D are conformally invariant in the natural sense, and satisfy the following domain Markov property. Suppose $D' \subset D$. Let $U = D \backslash D'$. Let $\mathcal{C}$ be the collection of loops associated with μ_D. Suppose we discover all the loops of $\mathcal{C}$ which intersect U, say $\mathcal{C}^U = \{\ell \in \mathcal{C} : [\ell] \cap U \neq \emptyset\}$, where $[\ell]$ is the trace (or range) of the loop ℓ. These loops split D into a collection $(D_k)_{k\geqslant 1}$, i.e.,

$$D \setminus \bigcup_{\ell \in \mathcal{C}^U} ([\ell] \cap D) = \bigcup_{k=1}^{\infty} D_k$$

The domain Markov property is that conditionally given $\mathcal{C}^U$, the conditional law of $\mathcal{C}$ restricted to each D_k form independent ensembles of loops with respective law μ_{D_k}. Sheffield and Werner's theorem from [24] takes the following form.

Theorem 9.9. *For each $\kappa \in (8/3, 8]$, the CLE$_\kappa$ ensemble of loops satisfies the above conformal invariance and domain Markov property. Conversely, if μ_D satisfy conformal invariance and domain Markov property in the above sense, then there exists $\kappa \in (8/3, 8]$ such that μ_D is the law of CLE$_\kappa$ in D.*

This characterisation allowed Sheffield and Werner in [24] to provide a construction of the conformal loop ensemble CLE$_\kappa$ when $\kappa \leqslant 4$ in terms of the so-called Brownian loop soup. We now briefly explain what this means and how CLE is related to it.

A (rooted) loop is a continuous function $\gamma : [0,t] \to \mathbb{C}$ with $\gamma(0) = \gamma(t) = z$. The loop is then rooted at z. If $D \subset \mathbb{C}$ is a proper domain, we obtain a (σ-finite) measure on loops in D by setting

$$\mu^D_{\text{rooted}}(\cdot) = \int_{z\in D} \int_0^\infty \frac{1}{t} p_t^D(z,z) \mathbb{P}^D_{z\to z;t}(\cdot) dt \tag{84}$$

$$= \int_{z\in D} \int_0^\infty \frac{dt}{2\pi t^2} \mathbb{P}^{\mathbb{C}}_{z\to z;t}(\cdot; b[0,t] \subset D). \tag{85}$$

where $p_t^D(x,y)$ is the transition probability for Brownian motion killed outside of D, and $\mathbb{P}^D_{z\to z;t}$ is the law of a Brownian bridge of duration t from z to z, conditioned to stay in D, as in (7), whereas $\mathbb{P}^{\mathbb{C}}_{z\to z;t}(\cdot; b[0,t] \subset D)$ denotes the law of a Brownian bridge (in the full plane) from z to z, of duration t, but restricted to the event that the Brownian bridge remains in D. This is an (infinite, but σ-finite) measure on the space of rooted loops, equipped with the Borel topology inherited from the uniform metric after applying Brownian scaling so that both loops are defined on the interval $[0,1]$ (this turns the space of rooted loops into a complete metric space).

An *unrooted* loop is a continuous function $\gamma : \mathbb{U} \to \mathbb{C}$ (where $\mathbb{U} = \{z : |z| = 1\}$ is the unit circle), modulo the equivalence relation that two such functions γ_1, γ_2 are equivalent if there exists $\theta \in [0, 2\pi)$ such that $\gamma_1(z) = \gamma_2(ze^{i\theta})$ for all $z \in \mathbb{U}$. A rooted loop γ is naturally associated to an unrooted loop denoted by $\boldsymbol{\gamma}$. We obtain from the measure on rooted loops above an unrooted measure, denoted by μ^{loop}, in the obvious manner:

$$\mu^D_{\text{loop}}(A) = \mu^D_{\text{rooted}}(\{\gamma : \boldsymbol{\gamma} \in A\}).$$

This is an infinite but σ-finite measure on the space $\mathcal{L}$ of unrooted loops, equipped with the Borel $\sigma-$algebra inherited from the quotient metric of the above metric (which also turns the space of unrooted loops into a complete metric space).

The advantage of μ^D_{loop} over μ^D_{unrooted} is that it enjoys conformal invariance, in an obvious sense. Since the measure is infinite we cannot sample from this measure, but as it is σ-finite we may instead consider a Poisson point process with intensity μ^D_{loop}. More generally, given $\lambda \geqslant 0$, we may consider a Poisson point process

$$N_\lambda = \sum_{i \geqslant 1} \delta_{\{\gamma_i\}}$$

on the space of unrooted loops, with intensity[16] $(\lambda/2)\mu^D_{\text{loop}}$. This is the **Brownian loop soup with intensity** $\lambda/2$. By definition this is a random point measure, such that if for any $k \geqslant 1$, any pairwise disjoint $A_1, \ldots, A_k \subset \mathcal{L}$, then $(N_\lambda(A_1), \ldots, N_\lambda(A_k))$ are independent random variables with a Poisson distribution $(\lambda/2)\mu^D_{\text{loop}}(A_i)$, $1 \leqslant i \leqslant k$.

The Brownian loop soup is conformally invariant since μ^D_{loop} is conformally invariant. Furthermore, the Brownian loop soup enjoys a natural **restriction property**: given two proper simply connected domains D, D' with $D' \subset D$, and given $N_\lambda = \sum_{i \geqslant 1} \delta_{\{\gamma_i\}}$ a Brownian loop soup in D, if we consider

$$N'_\lambda = \sum_{i \geqslant 1} \delta_{\{\gamma_i\}} 1_{\{[\gamma_i] \subset D'\}} \tag{86}$$

where $[\gamma_i]$ denotes the trace (i.e. the range or image set) of the unrooted loop γ_i, then N'_λ is a Brownian loop soup in D'. That is, the point process consisting of those loops remaining entirely in D' is the Brownian loop soup in D'. This is immediate from the definition of a Poisson point process and the definition of the loop measure coming from the second expression in (85).

We now explain how to associate a conformal loop ensemble to a Brownian loop soup. Let N_λ be a realisation of the Brownian loop soup in D with intensity $\lambda/2$. We can consider **clusters of loops** associated to N_λ, where two loops γ, γ' are considered connected if there is a finite chain $\gamma_0, \ldots, \gamma_k$, such that $\gamma_0 = \gamma$ and $\gamma_k = \gamma'$, and $[\gamma_{i-1}] \cap [\gamma_i] \neq \emptyset$ for $1 \leqslant i \leqslant k$. It is not hard to see that as λ increases, the clusters of loops increase monotonically, making it plausible that there exists a phase transition at some critical parameter λ_c such that the loop clusters are nontrivial for $\lambda < \lambda_c$, while they cover (almost) all of D for $\lambda > \lambda_c$, in the sense that there is a unique cluster. For $\lambda < \lambda_c$, we may fill in each cluster; the *outer-boundary* of the resulting compact set will be a simple loop. The set of all outer boundaries of filled-in Brownian loop soup clusters are therefore a plausible candidate for the set of outermost loops in a CLE.

Theorem 9.10 ([24])**.** *Let N_λ denote a Brownian loop soup in D with intensity $\lambda/2$, and let $\mathcal{C}_\lambda$ denote the collection of outer boundaries of filled in loop clusters in N_λ.*

(a) If $\lambda > 1$, then there is a.s. a unique cluster of loops in N_λ.

[16] The factor $1/2$ in the intensity has to do with the fact that the loops we consider here are oriented; if we were to consider unoriented loops, which is natural both in the context of percolation and from the point of view of the geometric description of the outermost cluster boundaries, then this factor $1/2$ would not need to be there.

(b) If $\lambda \leqslant 1$, then a.s. $\mathcal{C}_\lambda$ consists of a countable collection of disjoint simple loops. These satisfy the axioms of Theorem 9.9 and are therefore a CLE$_\kappa$ for some value of $\kappa \in (8/3, 4]$. In fact, κ is determined from λ via the relation

$$\lambda = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}. \tag{87}$$

Let us make a few observations about this theorem:

- First, this confirms that the percolation phenomenon which was discussed heuristically above takes place, and identifies the critical intensity λ_c to be $\lambda_c = 1$.

- We point out that, assuming that the loop clusters are nontrivial (say $\lambda < \lambda_c$) then the fact that the collection $\mathcal{C}_\lambda$ satisfies the axioms of Theorem 9.9 is relatively easy to see (conformal invariance follows directly from the conformal invariance of the Brownian loop soup, and the domain Markov property comes from the restriction property of the Brownian loop soup (86) and elementary properties of Poisson point processes).

- The relation (87) between the intensity of the loop soup and the parameter κ describing the geometry of the outer boundary is the same, up to a sign factor, as the one used to construct restriction measures from SLE$_\kappa$. Indeed, recall that by exercise 3 on Sheet 8, given $\kappa \leqslant 8/3$, we obtain a restriction measure of exponent $\alpha = (6-\kappa)/(2\kappa)$ by adding to a chordal SLE$_\kappa$ path γ the collection of loops encountered by γ from a Brownian loop soup with intensity

$$\lambda = \frac{(8 - 3\kappa)(6 - \kappa)}{2\kappa}. \tag{88}$$

  Note that the relation (88) is almost the same as (87), except for a sign change. Furthermore $\kappa \leqslant 8/3$ in (88), while in (87), κ is by assumption greater than 8/3. Thus the intensity of the loop soup is always nonnegative, as required.

- Given the relationship between κ and λ, it is reasonable to expect that $\lambda_c = 1$ (which corresponds to $\kappa = 4$). Indeed, at $\kappa = 4$ the CLE loops are very close to touching the boundary or one another. Increasing the value of λ just a little therefore results in a unique percolation cluster.

A Beurling's projection theorem

We prove a result of Beurling which concerns the probability that complex Brownian motion $(B_t)_{t\geqslant 0}$ starting from 0 hits a relatively closed subset $A \subseteq \mathbb{D}$ before leaving $\mathbb{D}$. It states that the probability does not increase if we replace A by its *radial projection* $A^* = \{|z| : z \in A\}$. For $A = [\varepsilon, 1)$ we can compute the hitting probability exactly. This provides general source of lower bounds for harmonic measure. We also prove a symmetry estimate, in the case where A is a simple path, for the probability that Brownian motion hits a given side of A. Finally, we prove a maximal inequality for H^1 functions of Brownian motion.

Write T_A for the hitting time of A given by

$$T_A = \inf\{t \geqslant 0 : B_t \in A\}.$$

Theorem A.1. *Let A be a relatively closed subset of $\mathbb{D}$. Then*

$$\mathbb{P}_0(T_{A^*} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A < T(\mathbb{D})).$$

The proof relies on the following folding inequality[17]. Define the folding map ϕ on $\mathbb{C}$ by $\phi(x + iy) = x + i|y|$.

Lemma A.2. *Let A be a relatively closed subset of $\mathbb{D}$. Then*

$$\mathbb{P}_0(T_{\phi(A)} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A < T(\mathbb{D})).$$

Proof. We exclude the case where $0 \in A$ for which the inequality is clear. Consider the set $\rho(A) = A \cup \{\bar{z} : z \in A\}$, symmetrized by reflection. Set $R_0 = 0$ and define, recursively for $k \geqslant 1$,

$$S_k = \inf\{t \geqslant R_{k-1} : B_t \in \rho(A) \text{ or } B_t \notin \mathbb{D}\}, \quad R_k = \inf\{t \geqslant S_k : B_t \in \mathbb{R}\}.$$

Then S_k and R_k are stopping times and $R_{k-1} \leqslant S_k \leqslant R_k < \infty$ for all k, almost surely.

Set $K = \inf\{k \geqslant 1 : S_k = R_k\}$, where we take $\inf \emptyset = \infty$ as usual. On the event $\{K = \infty\}$, we have $R_{k-1} < S_k < R_k < T_{\mathbb{R}\setminus\mathbb{D}} < \infty$ for all k, so the sequences $(S_k : k \geqslant 1)$ and $(R_k : k \geqslant 1)$ have a common accumulation point $T^* \leqslant T_{\mathbb{R}\setminus\mathbb{D}}$. We can write $(\mathbb{D} \setminus \rho(A)) \cap \mathbb{R}$ as a countable union of disjoint open intervals $\cup_n I_n$. By a straightforward harmonic measure estimate, there is a constant $C < \infty$ such that $\mathbb{P}_0(B_t \in I_n \text{ for some } t < T_{\mathbb{R}\setminus\mathbb{D}}) \leqslant C\,\mathrm{Leb}(I_n)$ for all n so, by Borel-Cantelli, almost surely, B visits only finitely many of the intervals I_n before $T_{\mathbb{R}\setminus\mathbb{D}}$. Hence, on $\{K = \infty\}$, almost surely $B_{T^*} = \lim_k B_{S_k} = \lim_k B_{R_k}$ is an endpoint of one of the intervals I_n. But, almost surely, B does not hit any of these endpoints. Hence $K < \infty$ almost surely.

Set $A^+ = \phi(A) \cap A$ and $A^- = \phi(A) \setminus A$. Take a sequence of independent random variables $(\varepsilon_k)_{k\geqslant 1}$, independent of B, with $\mathbb{P}(\varepsilon_k = \pm 1) = 1/2$ for all k. Set $\hat{\varepsilon}_k = \delta_k \varepsilon_k$, where

[17] Our proof of the folding inequality is new, though based on ideas from an argument of Oksendal. Whereas Oksendal cuts up the events whose probabilities are to be compared into pieces where symmetry can be invoked to make the comparison, we obtain the inequality from a global inclusion of events, using stochastic calculus to obtain the needed symmetry.

$\delta_k = \pm 1$ according as $B_{S_k} \in \rho(A^{\pm})$. Then $(\hat{\varepsilon}_k)_{k\geqslant 1}$ has the same distribution as $(\varepsilon_k)_{k\geqslant 1}$ and is also independent of B. Write $B_t = X_t + iY_t$ and define new processes $\tilde{B}$, $\tilde{Y}$ and $\hat{B}$ by setting

$$\tilde{B}_t = X_t + i\tilde{Y}_t = X_t + i\varepsilon_k Y_t, \quad \hat{B}_t = X_t + i\hat{\varepsilon}_k Y_t, \quad \text{for} \quad R_{k-1} \leqslant t < R_k, \quad k = 1, \dots K$$

and $\tilde{B}_t = X_t + i\tilde{Y}_t = \hat{B}_t = B_t$ for $t \geqslant R_K$. Then we have

$$\tilde{Y}_t = \int_0^t \left(\sum_{k=1}^K \varepsilon_k 1_{\{R_{k-1}\leqslant s<R_k\}} + 1_{\{s\geqslant R_K\}} \right) dY_s$$

almost surely, where the right hand side is understood as an Itô integral in the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ given by

$$\mathcal{F}_t = \sigma(\varepsilon_k, B_s : k \geqslant 1, s \leqslant t).$$

Thus $\tilde{Y}$ is a continuous $(\mathcal{F}_t)_{t\geqslant 0}$-local martingale with quadratic variation $[\tilde{Y}]_t = t$. Morever we have $[X, \tilde{Y}] = 0$. Hence $\tilde{B}$ is a Brownian motion by Lévy's characterization, Similarly $\hat{B}$ is also a Brownian motion (in a different filtration). Note that, with obvious notation, for all k,

$$\tilde{T}(\mathbb{D}) = \hat{T}(\mathbb{D}) = T(\mathbb{D}), \quad \tilde{S}_k = S_k, \quad \tilde{R}_k = R_k.$$

Suppose that $\tilde{B}$ hits $\phi(A)$ before $T(\mathbb{D})$. Note that $\tilde{B}$ cannot hit $\phi(A)$ before S_1 and, if it does not hit $\phi(A)$ at S_k, then it cannot do so until S_{k+1}. Also, if $R_K < T(\mathbb{D})$, then $\tilde{B}_{R_K} \in \rho(A) \cap \mathbb{R} \subseteq \phi(A)$. Hence the only possible values for $T_{\phi(A)}$ are $S_1, \dots, S_K$ and R_K. Now, if $T_{\phi(A)} = S_k$ for some $k \leqslant K$, then either $\tilde{B}_{S_k} \in A^+$ so $\hat{B}_{S_k} = \tilde{B}_{S_k} \in A$, or $\tilde{B}_{S_k} \in A^-$ so $\hat{B}_{S_k} = \bar{\tilde{B}}_{S_k} \in A$. On the other hand, if $T_{\phi(A)} = R_K$, then $\hat{B}_{R_K} = \tilde{B}_{R_K} \in \rho(A) \cap \mathbb{R} \subseteq A$. In all cases $\hat{B}$ hits A before $T(\mathbb{D})$. Hence $\{\tilde{T}_{\phi(A)} < T(\mathbb{D})\} \subseteq \{\hat{T}_A < T(\mathbb{D})\}$ and the folding inequality follows on taking probabilities. $\square$

Proof of Theorem A.1. The map ϕ folds $\mathbb{C}$ along $\mathbb{R}$ and fixes the point i. Note that ϕ preserves the class of relatively closed subsets of $\mathbb{D}$. Set $\phi_0 = \phi$ and consider for $n \geqslant 1$ the map ϕ_n which folds $\mathbb{C}$ along $\exp(2^{-n}\pi i)\mathbb{R}$ and fixes 1. Set $\psi_n = \phi_n \circ \dots \circ \phi_0$. For all $n \geqslant 0$, by the folding inequality and rotation invariance,

$$\mathbb{P}_0(T_{\phi_n(A)} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A < T(\mathbb{D}))$$

and so by induction

$$\mathbb{P}_0(T_{\psi_n(A)} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A < T(\mathbb{D})).$$

Consider the set

$$A(n) = \{ze^{i\theta} : z \in A,\ |\theta| \leqslant 2^{-n}\pi\}.$$

Then $A(n)$ is relatively closed and

$$A^* = A(n)^* \subseteq \{xe^{i\theta} : x \in A^*,\ \theta \in [0, 2^{-n}\pi]\} = \psi_n(A(n))$$

so

$$\mathbb{P}_0(T_{A^*} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_{\psi_n(A(n))} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_{A(n)} < T(\mathbb{D})).$$

On letting $n \to \infty$ we have $T_{A(n)} \uparrow T_A$ almost surely, so we obtain

$$\mathbb{P}_0(T_{A^*} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A \leqslant T(\mathbb{D})).$$

By scaling, the same inequality holds when $\mathbb{D}$ is replaced by $s\mathbb{D}$ for any $s \in (0,1)$ and the result follows on taking the limit $s \to 1$. □

Theorem A.3 (Beurling's estimate)**.** *Let A be a relatively closed subset of $\mathbb{D}$ and let $\varepsilon \in (0,1)$. Suppose that A contains a continuous path from the circle $\{|z| = \varepsilon\}$ to the boundary $\partial\mathbb{D}$. Then*

$$\mathbb{P}_0(T_A \geqslant T(\mathbb{D})) \leqslant 2\sqrt{\varepsilon}.$$

Proof. By the intermediate value theorem, we must have $[\varepsilon, 1) \subseteq A^*$. Then, by Beurling's projection theorem, it will suffice to consider the case where $A = [\varepsilon, 1)$. Consider the conformal map $\mathbb{D} \setminus A \to \mathbb{H}$ given by $\phi = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$, where

$$\phi_1(z) = i\frac{1-z}{1+z}, \quad \phi_2(z) = \frac{1+\varepsilon}{1-\varepsilon}z, \quad \phi_3(z) = \sqrt{z^2+1}, \quad \phi_4(z) = az, \quad a = \frac{1-\varepsilon}{2\sqrt{\varepsilon}}.$$

Then $\phi(0) = i$ and the left and right sides of A are mapped to the interval $(-a, a)$. Then, by conformal invariance of Brownian motion,

$$\mathbb{P}_0(T_A \geqslant T(\mathbb{D})) = \mathbb{P}_i(|B_{T(\mathbb{H})}| > a) = \frac{2}{\pi}\cot^{-1} a$$

and the claimed estimate follows using the bound $\sin x \geqslant 2x/\pi$ for $x \in [0, \pi/2]$. □

B Smirnov's theorem

We now discuss Smirnov's proof of Cardy's formula for percolation on the triangular lattice. Consider the lattice of edge length δ. Sites of the lattice are coloured black or white independently with probability $1/2$. Take any Jordan domain D with three distinct boundary points $a(1), a(\tau), a(\tau^2)$, ordered positively, where $\tau = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Write Φ for the unique conformal isomorphism from D to the triangle Δ with corresponding boundary points $1, \tau, \tau^2$. For $z \in D$ and $\alpha \in \{1, \tau, \tau^2\}$, write $Q_\alpha(z)$ for the event that z is separated from the boundary segment $a(\tau\alpha)a(\tau^2\alpha)$ by a simple black path from $a(\alpha)a(\tau\alpha)$ to $a(\tau^2\alpha)a(\alpha)$. Set $H_\alpha(z) = H_\alpha^\delta(z) = \mathbb{P}(Q_\alpha(z))$. By a black path we mean any path in the lattice which visits only black points. The functions $H_\alpha(z)$ are constant in the interior of lattice triangles with discontinuities at the edges. Let f_α denote the unique affine function on Δ with $f_\alpha(\alpha) = 1$ and $f_\alpha(\tau\alpha) = f_\alpha(\tau^2\alpha) = 0$, and set $h_\alpha = f_\alpha \circ \Phi$.

Theorem B.1 (Smirnov)**.** *For $\alpha = 1, \tau, \tau^2$, H_α^δ converges uniformly on D to h_α as $\delta \to 0$.*

It follows, in particular, by taking $z \in \partial D$, that the asymptotic crossing probabilities for this percolation model are indeed conformally invariant and are given by Cardy's formula.

Before sketching the proof, we will describe a variant of the Cauchy–Riemann equations and of conjugate harmonic functions, associated with the angle $2\pi/3$. For $\alpha = 1, \tau, \tau^2$, and f analytic, set

$$f_\alpha = \mathrm{Re}(f/\alpha).$$

Then f_α is harmonic and we can recover f by

$$\alpha f = f_\alpha + \frac{i}{\sqrt{3}}(f_{\alpha\tau} - f_{\alpha\tau^2}).$$

Also, for any $\eta \in \mathbb{C}$, the directional derivatives satisfy

$$\nabla_\eta f_\alpha(z) = \frac{\partial}{\partial \varepsilon}\,|_{\varepsilon=0}\, \mathrm{Re}\left(\frac{f(z+\varepsilon\eta)}{\alpha}\right) = Re\left(\frac{f'(z)\eta}{\alpha}\right) = \nabla_{\tau\eta} f_{\tau\alpha}(z).$$

These are the $2\pi/3$-*Cauchy–Riemann equations*, and $(f_1, f_\tau, f_{\tau^2})$ is the *harmonic triple* of f.

Conversely, if we are given C^1 functions f_1, f_τ, f_{τ^2} such that, for $\alpha \in \{1, \tau, \tau^2\}$, for all η,

$$\nabla_\eta f_\alpha(z) = \nabla_{\tau\eta} f_{\tau\alpha}(z),$$

then f, defined by

$$f = f_1 + \frac{i}{\sqrt{3}}(f_\tau - f_{\tau^2}),$$

is holomorphic and $f_\alpha = \mathrm{Re}(f/\alpha)$ for all α.

Sketch proof of Theorem B.1. For z the centre of a lattice triangle in D and η a vector from z to one of the three neighbouring triangle centres, for $\alpha \in \{1, \tau, \tau^2\}$, the events $Q = Q_\alpha(z+\eta) \setminus Q_\alpha(z)$ and $\tilde{Q} = Q_{\tau\alpha}(z+\tau\eta) \setminus Q_{\tau\alpha}(z)$ have the same probability. To see this, label the vertices of the triangle at z by X, Y, Z, where X is opposite to η and we move anticlockwise around the triangle. Note that Q is the event that there exist disjoint black paths from Y to $a(\alpha\tau^2)a(\alpha)$ and from Z to $a(\alpha)a(\tau\alpha)$ and also a white path from X to $a(\alpha\tau)a(\alpha\tau^2)$. On the other hand, $\tilde{Q}$ is a similar event but where the path from Y must be white, and that from X must be black. To see that $\mathbb{P}(Q) = \mathbb{P}(\tilde{Q})$, explore the lattice from $a(\alpha)$ just as far as is needed to find suitable black paths (for Q) from Y and Z. Supposing this done, the conditional probability of the required white path from X is the same as if we required it to be black (and disjoint from the other paths). Hence Q and $\tilde{Q}$ both have the same probability as the event of three disjoint black paths to the required boundary segments.

Set $P_\alpha(z, \eta) = \mathbb{P}(Q)$. We have shown that

$$P_\alpha(z, \eta) = P_{\tau\alpha}(z, \tau\alpha). \tag{89}$$

This is a discrete version of the $2\pi/3$-Cauchy–Riemann equations for the triple $(H_1, H_\tau, H_{\tau^2})$. The rest of the proof is analytic. We accept here without proof the following results

Lemma B.2 (Hölder estimate)**.** *There are constants $\varepsilon > 0$ and $C < \infty$, depending only on $(D, a(1), a(\tau), a(\tau^2))$, such that*

$$|H_\alpha(z) - H_\alpha(z')| \leqslant C(|z - z'| \wedge \delta)^\varepsilon.$$

Also, $H_\alpha(a(\alpha)) \to 1$ as $\delta \to 0$.

The proof uses a a classical method for regularity estimates in percolation due to Russo, Seymour and Welsh.

Lemma B.3. *For any equilateral triangular contour Γ, of side length ℓ, interpolating neighbouring centres of lattice triangles, define the discrete contour integral*

$$\int_\Gamma^\delta H(z)dz = \delta \sum_{z \in A_1} H(z) + \delta\tau \sum_{z \in A_\tau} H(z) + \delta\tau^2 \sum_{z \in A_{\tau^2}} H(z),$$

where A_α is the set of centres along the side parallel to α. (Make some convention at the corners.) Then

$$\int_\Gamma^\delta H_\alpha(z)dz = \frac{1}{\tau} \int_\Gamma^\delta H_{\alpha\tau}(z)dz + O(\ell\delta^\varepsilon).$$

The proof is an elementary, if complicated, resummation argument, using the identity

$$H_\alpha(z+\eta) - H_\alpha(z) = P_\alpha(z, \eta) - P_\alpha(z+\eta, -\eta)$$

and, from the preceding lemma, the estimate $P_\alpha(z, \eta) \leqslant C\delta^\varepsilon$ for some stray terms.

The Hölder estimate implies that every sequence $\delta_n \downarrow 0$ contains a subsequence δ_{n_k} such that $H_\alpha^{\delta_{n_k}}$ converges uniformly on D for all α, and any such subsequential limits, h_α say, must have boundary values $h_\alpha(a(\alpha)) = 1$ and $h_\alpha(z) = 0$ on $a(\alpha\tau)a(\alpha\tau^2)$. Moreover, by Lemma B.3, we must have

$$\int_\Gamma h_\alpha(z)dz = \frac{1}{\tau} \int_\Gamma h_{\alpha\tau}(z)dz.$$

Set $h = h_1 + (i/\sqrt{3})(h_\tau - h_{\tau^2})$, then

$$\int_\Gamma h(z)dz = 0$$

for all Γ, so h is holomorphic by Morera's theorem, and $h_\alpha = \mathrm{Re}(h/\alpha)$ is harmonic for all α. Hence we obtain

$$\nabla_\eta h_\alpha = \nabla_{\tau\alpha} h_{\tau\alpha}.$$

(This can be considered as the limiting form of the key observation on the discrete model (89), but the limit has not been justified directly.) This relation implies that the directional derivatives of h_1 on $a(\tau^2)a(1)$ and $a(\tau)a(1)$ at an angle τ to the tangent are zero. Thus we have a (conformally-invariant) Dirichlet-Neumann problem for h_1. In the case $D = \Delta$, the affine function f_1 is obviously a solution, and moreover it is the only solution. Hence the functions $H_1^\delta, H_\tau^\delta, H_{\tau^2}^\delta$ each have exactly one uniform limit point as $\delta \to 0$, given by h_1, h_τ, h_{τ^2} respectively, as required. □

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