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SUPPLEMENTARY MATERIAL FOR DIMERS AND IMAGINARY GEOMETRY

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A. Background Materials. Here we recall some background materials mentioned in Section 2 of the main file [2].

A.1. *Schramm–Loewner evolution.* SLE, or Schramm–Loewner evolution, is a family of conformally invariant random curves in the unit disc $\mathbb{D}$ which are supposed to describe several aspects of statistical physics models. In this paper we are concerned with the radial version of SLE, see [5] for more details. SLE$_\kappa$ with $\kappa \leq 4$ in the unit disc $\mathbb{D}$ starting from 1 and targeted at 0 is the curve γ (parametrised by $[0, \infty)$) described via the (unique) family of conformal maps $g_t : \mathbb{D} \to \mathbb{D} \setminus \gamma[0, t]$ with $g_t(0) = 0$ and $g_t'(0) > 0$, where g_t satisfies the following differential equation (called the radial Loewner equation) for each $z \in \mathbb{D} \setminus \gamma[0, t]$:

$$\frac{\partial g_t(z)}{\partial t} = g_t(z) \frac{e^{i\sqrt{\kappa} B_t} + g_t(z)}{e^{i\sqrt{\kappa} B_t} - g_t(z)}; \quad g_0(z) = z. \tag{A.1}$$

SLE$_\kappa$ enjoys conformal invariance, hence one can obtain radial SLE$_\kappa$ curves in other domains and/or for other starting and target points simply by applying conformal maps to the curve described by (A.1). It is worthwhile to note here that radial SLE$_\kappa$ in $\mathbb{D}$ starting at 1 and targeted at 0 is symmetric under conjugation in distribution.

A.2. *Gaussian Free Field (GFF).* In this section, we recall the definition of the GFF mostly to fix normalisation. See [14] and [1] for more thorough definitions. Let D be a domain in $\mathbb{C}$. Let G_D be the Green function in D (with Dirichlet boundary conditions), i.e., $G_D(x, y) = \pi \int_0^\infty p_t^D(x, y) dt$ where p_t^D is the transition kernel for a Brownian motion killed upon exiting D. The Gaussian free field, viewed as a stochastic process indexed by test functions $f \in C^\infty(\bar{D})$, is the centered Gaussian process such that (h, f) is

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a normal random variable and covariance between (h, f) and (h, g) given by $\iint G_D(x, y) f(x) g(y) dx dy$. Alternatively, if f_n is an orthonormal basis for the Sobolev space $H_0^1(D)$ induced by the Dirichlet inner product

$$(f, g)_\nabla = \frac{1}{2\pi} \int_D \nabla f \cdot \nabla g, \tag{A.2}$$

then $h = \sum_n X_n f_n$ where X_n are i.i.d. standard real Gaussian random variables. As a consequence of Weyl's law, it can be seen that this sum converges almost surely in the Sobolev space $H^{-\eta}(D)$ for any $\eta > 0$. Note that even though the GFF is not defined pointwise we will sometimes freely abuse notations and write $\mathbb{E}(h(x_1) \dots h(x_k))$ for its k-point moments (which are unambiguously defined if the x_i are distinct).

Finally, for any function u on the boundary of D, a GFF in D with boundary condition given by u is defined to be a Dirichlet GFF in D plus the harmonic extension of u in D. (In fact this also makes sense when u is rougher than a function, provided that it can be integrated against harmonic measure).

A.3. *Wilson's algorithm.* Recall from the introduction of [2] that given a finite weighted, oriented graph G with a marked "root" vertex (and such that there exists a path from any point to the root), a *spanning tree* of G is an oriented subgraph T containing all vertices of G, exactly one outgoing edge for each non-root vertex, and such that if we forget about the orientations T is connected and has no cycle. Note that starting from any point x, by following the outgoing path one obtains a self-avoiding path from x to the root which will be called the branch of the tree starting at (or emanating from) x which we will often write as γ_x. The measure defined in picking a tree with a probability proportional to the edges is called (with a small abuse of language) the Uniform Spanning Tree measure or UST for short.

A crucial tool for studying the UST is Wilson's celebrated algorithm, which we recall here for convenience (since we need it in the context of directed and weighted graphs). This relies on the notion of loop-erased random walk, which we briefly explain now. Consider a simple random walk $(X_s)_{0 \le s \le T}$ run until some (possibly random) time T. The loop-erasure Y of $(X_s)_{0 \le s \le T}$ is obtained from X by chronologically erasing the loops. By chronologically erasing the loops, we mean that we keep track of the current loop-erasure Y^t which we update as follows. Given Y^t and the next position of the walk X_{t+1}, if $X_{t+1} \notin Y^t$ we set $Y^{t+1} = Y^t \cup X_{t+1}$, and if $X_{t+1} \in Y^t$ we set Y^{t+1} as the part of Y^t up to and including X_{t+1}. The latter case is what we call erasing a loop. See [6] for general background on loop-erased random walks. We also observe that the law of the path when loops are erased in a chronological order, and the law of the path when loops are erased in a reverse chronological order coincide, even if the graph is oriented (i.e., the random walk is nonreversible). See Lemma 7.2.1 in [6] for a proof.

We now describe Wilson's algorithm for sampling a UST. In the first step, we fix a vertex, the root. To sample a uniform spanning tree rooted at that vertex, we now pick any other vertex and perform a loop-erased random walk until the walk hits the root. In the next step we update the root by adding this loop-erased path to the old root, and iterate this procedure, (each time stopping when we hit the updated root), until there are no vertices remaining. It is easy to see that such a process in the end produces a tree, where the outgoing edge is taken to be towards the initial root or equivalently the first step of the loop-erased walk out of this vertex. This tree has the law of the uniform spanning tree ([8]). Furthermore, and crucially, the law of this resulting tree does not depend on the choice of the ordering of vertices from which we draw loop-erased paths.

It follows that from every vertex, one can generate branches of the uniform spanning tree from that vertex to the root by sampling loop erased random walk from those vertices to the root. As mentioned before, often we will be interested in the **wired** uniform spanning tree, which is the UST on the graph where we have identified together a given set of "boundary" vertices to be the root.

A.4. *Continuum Uniform spanning tree.* The Continuum (Wired) Uniform Spanning Tree, in a simply connected domain D, is the scaling limit of the discrete UST in an approximation $D^{\#\delta}$ of D where we have wired all vertices of $\partial D^{\#\delta}$. The convergence is in the following underlying topology. For any metric space X, let $\mathcal{H}(X)$ denote the space of subsets of X equipped with the Hausdorff metric. Let $\mathcal{P}(z, w, D)$ be the space of all continuous paths in $\overline{D}$ from a point $z \in \overline{D}$ to $w \in \bar{D}$. We consider the Schramm space $D \times D \times \cup_{z,w \in \overline{D}} \mathcal{P}(z, w)$ which we view as a subset of $\mathcal{H}(\overline{D} \times \overline{D} \times \mathcal{H}(\overline{D}))$ equipped with the induced distance. A discrete wired UST is embedded into this space by considering $\{(x, y, \gamma_{xy}) | x, y \in D^{\#\delta}\}$ where γ_{xy} is the unique path connecting x and y in the tree, possibly including a part of the boundary if x and y belong to different connected components. We view the Schramm space as a subset of the compact space $\mathcal{H}(\overline{D} \times \overline{D} \times \mathcal{H}(\overline{D}))$ equipped with its metric. This is the **Schramm topology**.

All we will need to recall for now is the following fact. Let $z_1, z_2, \ldots, z_k$ be distinct points in D. A landmark result of Lawler, Schramm and Werner [7] shows that loop erased random walk in a domain $D^{\#\delta}$ of $\delta\mathbb{Z}^2$ approximating

D from the vertex closest to z_i converges as $\delta \to 0$ to a radial SLE_2 curve (denote it by γ_{z_i}). This SLE_2 curve starts from a point on the boundary picked according to harmonic measure from z_i, and is targeted at z_i. This convergence is in the Hausdorff sense. From this it was deduced (cf. [12], Theorem 11.3 or Corollary 1.2 in [7]) that the UST converges in the Schramm sense to an object which we call the *continuum UST*. We call the curve γ_{z_i} obtained as the scaling limit of the discrete UST branch the *branch from* z_i of the continuum UST (note that this is a typical observable for the Schramm topology). It is shown in Theorem 1.5 of [12] that the branch from a point in a domain is almost surely unique (i.e., there is a unique path connecting z to ∂D) and hence this branch is unique a.s. for Lebesgue-almost every point $z \in D$. (Actually Theorem 1.5 in [12] deals with UST in the sphere but the same argument works in a domain, see [12], Theorem 11.1.) It is therefore easy to deduce:

PROPOSITION A.1 (Wilson's algorithm in the continuum). *Let $D \subset \mathbb{C}$ be a simply connected domain and $z_1, \dots, z_k \in D$. We can sample the (a.s. unique) branches of the continuum wired UST in a domain D from $z_1, \dots, z_k$ as follows. Given the branches η_i from z_i for $1 \le i < j$, we inductively sample the branch from z_j as follows. We pick a point p from the boundary of $D' := D \setminus \cup_{1 \le i \le j} \eta_i$ according to harmonic measure from z_j and draw an SLE_2 curve in D' from p to z_j. The joint law of the branches does not depend on the order in which we sample the branches.*

B. Winding of curves. In this section, we provide detailed proofs of the results (Lemmas 2.1 to 2.4) in Section 2.1 of [2], which concern basic deterministic facts about intrinsic and topological windings of a simple path.

B.1. *Intrinsic and topological winding.*

LEMMA B.1 (Lemma 2.1 in [2]). *Let $\gamma : [0,1] \mapsto \mathbb{C}$ be a smooth self avoiding curve with $\gamma'(s) \neq 0$ for all s. We have*

$$W_{\mathrm{int}}(\gamma) = W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)). \tag{B.1}$$

PROOF. Let $x\ [2\pi]$ denote x modulo 2π for any real number x. Let $\mathrm{Arg} \in (-\pi, \pi]$ be the principal branch of argument with branch cut $(-\infty, 0]$. Note that

$$\lim_{u \to t, u < t} \mathrm{Arg}(\gamma(u) - \gamma(t)) \equiv \pi + \mathrm{Arg}(\gamma'(t))[2\pi],$$
$$\lim_{u \to s, u > s} \mathrm{Arg}(\gamma(u) - \gamma(s)) \equiv \mathrm{Arg}(\gamma'(s))\ [2\pi].$$

Therefore for any t,

$$\begin{aligned} W(\gamma[0,t],\gamma(t)) + W(\gamma[0,t],\gamma(0)) &\equiv \big(\pi + \mathrm{Arg}(\gamma'(t)) - \mathrm{Arg}(\gamma(0)-\gamma(t))\big) \\ &\quad + \big(\mathrm{Arg}(\gamma(t)-\gamma(0)) - \mathrm{Arg}(\gamma'(0))\big)\ [2\pi] \\ &\equiv \mathrm{Arg}(\gamma'(t)) - \mathrm{Arg}(\gamma'(0))\ [2\pi] \\ &\equiv W_{\mathrm{int}}(\gamma[0,t])\ [2\pi] \end{aligned}$$

where notice that $\pi + \mathrm{Arg}(\gamma(t)-\gamma(0))$ cancels with $\mathrm{Arg}(\gamma(0)-\gamma(t))$. Equivalently we can write

$$W(\gamma[0,t],\gamma(t)) + W(\gamma[0,t],\gamma(0)) = W_{\mathrm{int}}(\gamma[0,t]) + 2\pi k_t \tag{B.2}$$

for some $k_t \in \mathbb{Z}$. However, as t goes to 0, both $W(\gamma[0,t],\gamma(t))+W(\gamma[0,t],\gamma(0))$ and $W_{\mathrm{int}}(\gamma[0,t])$ go to 0 which implies $k_0 = 0$. Since γ is smooth and self avoiding, it is easy to check that both the winding terms are continuous in t. But this implies k_t is continuous in t and hence we conclude $k_t = 0$ for all t. This completes the proof. □

B.2. *Distortion estimates.* In this section we record some distortion estimates required in order to give a proof of Lemma 2.4 of [2]. This will be stated and proved in Corollary B.7.

Lemma B.2 (Distortion estimates). *Let D be a domain containing 0 and let $R = R(0,D)$. Let g be a conformal map defined on D mapping 0 to 0. Then for any $z \in B(0,R/8)$,*

$$|g(z)| \le 4|zg'(0)| \quad ; \quad |g(z) - g'(0)z| < 6\frac{|z|^2}{R}|g'(0)|.$$

In particular, the image of a straight line joining 0 to a point at a distance $\varepsilon < R/8$ under g lies within a cone of angle $\arctan(6\frac{\varepsilon}{R})$.

Proof. The first statement follows from applying the Growth theorem (see Theorem 3.23 in [5]) to the function $g(Rz/4)/(Rg'(0)/4)$, which is defined on the unit disc by Koebe's $1/4$ theorem. The second statement follows from applying Proposition 3.26 in [5] with $r = 1/2$ to the same function. The final assertion is immediate since the distance of $g(z)$ from $g'(0)z$ is at most $|g'(0)z|6\varepsilon/R$ for all z with $|z| \le \varepsilon$. □

Lemma B.3 (distortion of argument). *Let K be a closed subset of $\bar{\mathbb{D}}$ such that $H = \mathbb{D} \setminus K$ is simply connected (i.e. K is a hull). Further assume that the diameter of K is smaller than some $\delta < 1/2$ and $1 \notin B(K, \delta^{1/2})$. Let*

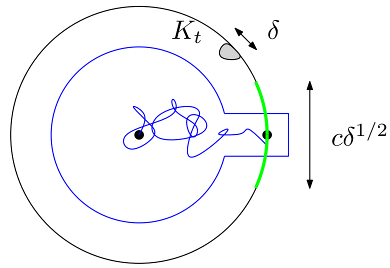

Fig 1: Sketch of proof of Lemma B.3. The probability that the Brownian motion exits through the green arc all the while staying within the region bounded by blue arcs is $c\delta^{1/2}$. This is a lower bound for $\mathrm{Harm}_{H_T}(0, I^{\pm})$.

$\tilde{g}$ denote the conformal map sending H to $\mathbb{D}$ with $\tilde{g}(0) = 0$ and $\tilde{g}(1) = 1$. Then

$$|\arg_{\tilde{g}'(H)}(\tilde{g}'(0)) - \arg_{\tilde{g}'(H)}(\tilde{g}'(1))| < C\delta^{1/2},$$

where C is a universal constant. Here $\arg_{\tilde{g}'(H)}(\cdot)$ is the argument in $\tilde{g}'(H)$ (which does not contain 0), defined up to a global unimportant additive constant.

Proof. Let T be the capacity of K seen from 0 and let $(K_t)_{t\le T}$ be a growing sequence of hulls of capacity t such that $K_T = K$. Let g_t denote the Loewner maps associated to the family $(K_t)_{t\le T}$, with the usual convention $g_t(0) = 0$ and $g_t'(0) \in \mathbb{R}^+$ and note that $\tilde{g} = g_T/g_T(1)$. Let $W : [0, \infty) \mapsto \mathbb{R}$ be the driving function for the radial Loewner differential equation (A.1) for the maps $(g_t)_{t\ge 0}$ and $H_t = \mathbb{D} \setminus K_t$.

Since the conformal radius $R(0, H_t)$ is at least the inradius which is at least $1-\delta$, we obtain that the capacity t of K_t seen from 0 is always smaller than $-\log(1-\delta) \le C\delta$ for $t \le T$, in particular $T \le C\delta$. Let $A_t = \partial\mathbb{D} \setminus g_t(\bar{\mathbb{D}} \setminus K_t)$, i.e., A_t is the part of $\partial\mathbb{D}$ where g_t maps the boundary of K_t. By definition, $e^{iW_t} \in A_t$ for any t. Let $\mathrm{Harm}_D(z, S)$ denote the harmonic measure seen from z of a set S in the domain D. Since $\mathrm{dist}(1, K) \ge \delta^{1/2}$, we can find $\theta_- < 0 < \theta_+$ such that

$$\mathrm{Harm}_H(0, I^{\pm}) \ge c\delta^{1/2},\ I^{\pm} \cap K = \emptyset$$

where $I^{\pm}$ are the two arcs connecting $e^{i\theta_\pm}$ to 1 (the green arcs in Figure 1). Note that since $K_t \subset K_T = K$, the same holds for Harm_{H_t}. Applying conformal invariance of the harmonic measure, we get $\mathrm{Harm}_{\mathbb{D}}(g_t(I^{\pm})) \ge$

$c\delta^{1/2}$ and therefore $\mathrm{Diam}(g_t(I^{\pm})) \geq c\delta^{1/2}$. Finally since $I^{\pm}$ did not intersect K_T, $g_t(I_{\pm})$ does not intersect A_t and therefore $|e^{iW_t} - g_t(1)| \geq c\delta^{1/2}$.

Using this bound, the Loewner equation gives

$$|\partial_t g_t(1)| = |g_t(1)\frac{e^{iW_t} + g_t(1)}{e^{iW_t} - g_t(1)}| \leq C\delta^{-1/2}.$$

Integrating for a time $t \leq T \leq C\delta$ gives $|g_t(1)-1| \leq C\delta^{1/2}$. Recall Arg is the principal branch of argument in $(-\pi, \pi]$. This gives $|\operatorname{Arg}(g_T(1))| \leq C\delta^{1/2}$ and since by definition $g_t'(0) \in \mathbb{R}^+$,

$$|\operatorname{Arg} \tilde{g}_t'(0)| = |\operatorname{Arg} g_t'(0) - \operatorname{Arg} g_t(1)| \leq C\delta^{1/2}, \tag{B.3}$$

where $\tilde{g}_t = g_t/g_t(1)$ is the conformal map sending H_t to $\mathbb{D}$ such that $\tilde{g}(0) = 0$ and $\tilde{g}(1) = 1$. Note that $\operatorname{Arg} \tilde{g}_t'(0)$ is continuous in t since it is bounded by $C\delta^{1/2}$ (and so does not come into the region where Arg is discontinuous).

We deduce that the same bound as eq. (B.3) holds for $\arg_{\tilde{g}_T'(H_T)}$. Indeed note that one can find a curve $\lambda[0,1]$ connecting 1 to 0 which avoids K_T. By definition,

$$\arg_{\tilde{g}_t'(H_t)}(g_t'(0)) - \arg_{\tilde{g}_t'(H_t)}(g_t'(1)) = \Im\left(\int_{\tilde{g}_t'(\lambda)} \frac{dz}{z}\right) = \int_0^1 \Im\left(\frac{\tilde{g}_t''(\lambda(s))\lambda'(s)}{\tilde{g}_t'(\lambda(s))}\right) ds.$$

Thus the left hand side is continuous in t since $\tilde{g}_t$ has a conformal extension in a neighbourhood of λ which implies that all its derivatives are continuous in t in this neighbourhood. Also for $t = 0$ the difference between the left hand side and $\operatorname{Arg} \tilde{g}_t'(0)$ is 0 trivially. This concludes the proof. ☐

B.3. *Change in winding under conformal maps.* We are now able to return to the proof of Lemma 2.4 in [2] which will be stated and proved in Corollary B.7. We will use a deformation argument and the main step is to prove the following intermediate lemma.

LEMMA B.4. *Let D, D' be domains with locally connected boundary. Let ψ be a conformal map sending D to D'. Let $\gamma : [0,1] \mapsto \bar{D}$ be a curve (not necessarily simple) in $\bar{D}$. Assume the following*

(i) The endpoints of γ are smooth and simple.
(ii) $\arg_{\psi'(D)} \psi'$ extends continuously to a neighbourhood in D of $\gamma(0)$ and $\gamma(1)$.
(iii) There exists a continuous curve $\tilde{\gamma} \subset \overline{\mathbb{D}}$ such that $\varphi(\tilde{\gamma}) = \gamma$, where $\varphi : \bar{\mathbb{D}} \mapsto \bar{D}$ is the extension of a conformal map in $\mathbb{D}$. (Such an extension exists by our assumption that ∂D is locally connected.) See Remark B.5 below for an equivalent reformulation of this condition.

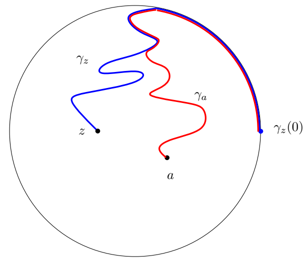

Fig 2: A typical application of Lemma B.4. Here $\gamma = \gamma_z$ (the curve going from 1 to z in a continuum UST) and ψ is the Loewner map removing γ_a. Note that $\psi(\gamma_z)$ is well-defined as a continuous curve in $\bar{\mathbb{D}}$. so condition (iii) is satisfied.

Then,

$$\begin{aligned}&W\left(\psi(\gamma), \psi(\gamma(1))\right) + W\left(\psi(\gamma), \psi(\gamma(0))\right)\\&\quad = W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)) + \arg_{\psi'(D)}(\psi'(\gamma(1))) - \arg_{\psi'(D)}(\psi'(\gamma(0))\end{aligned}$$

where $\arg_{\psi'(D)}$ *here is any determination of the argument on* $\psi'(D)$*.*

Proof. We write arg for $\arg_{\psi'(D)}$ in this proof for notational convenience. First, we assume that D is the unit disc $\mathbb{D}$ and $\tilde{\gamma} = \gamma$. We define a family of conformal maps,

$$\psi_t(z) = \psi(tz)/t \text{ for } t \in (0,1], \qquad \psi_0(z) = \psi'(0)z.$$

The map ψ_0 is just a rotation and scaling so the topological winding does not change and $\arg(\psi_0'(\gamma(1))) = \arg(\psi_0'(\gamma(0)) = \arg(\psi'(0))$. Thus we have:

$$\begin{aligned}&W\left(\psi_0(\gamma), \psi_0(\gamma(1))\right) + W\left(\psi_0(\gamma), \psi_0(\gamma(0))\right)\\&\qquad = W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)) + \arg(\psi_0'(\gamma(1))) - \arg(\psi_0'(\gamma(0)).\end{aligned}$$

On the other hand, since locally at $\gamma(0)$ and $\gamma(1)$, ψ acts only by rotation and scaling we see that for all t,

$$\begin{aligned}&W\left(\psi_t(\gamma), \psi_t(\gamma(1))\right) + W\left(\psi_t(\gamma), \psi_t(\gamma(0))\right)\\&\qquad \equiv W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)) + \arg(\psi_t'(\gamma(1))) - \arg(\psi_t'(\gamma(0))[2\pi].\end{aligned} \tag{B.4}$$

Indeed, let Arg denote the argument in $(-\pi, \pi]$ and note that

$$\begin{aligned}&W\left(\psi_t(\gamma), \psi_t(\gamma(1))\right)\\&\equiv \lim_{s \to 1} \operatorname{Arg}(\psi_t(\gamma(s)) - \psi_t(\gamma(1))) - \operatorname{Arg}(\psi_t(\gamma(0)) - \psi_t(\gamma(1)))\,[2\pi]\\&\equiv \operatorname{Arg}(\psi_t'(\gamma(1))) + \operatorname{Arg}(\gamma'(1)) + \pi - \operatorname{Arg}(\psi_t(\gamma(0)) - \psi_t(\gamma(1)))\,[2\pi]\end{aligned}$$

and

$$\begin{aligned} & W\left(\psi_t(\gamma), \psi_t(\gamma(0))\right) \\ \equiv & \operatorname{Arg}(\psi_t(\gamma(1)) - \psi_t(\gamma(0))) - \lim_{s \to 0} \operatorname{Arg}(\psi_t(\gamma(s)) - \psi_t(\gamma(0)))\,[2\pi] \\ \equiv & \operatorname{Arg}(\psi_t(\gamma(1)) - \psi_t(\gamma(0))) - \operatorname{Arg}(\psi_t'(\gamma(0))) - \operatorname{Arg}(\gamma'(0))\,[2\pi] \end{aligned}$$

and also

$$W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)) \equiv \operatorname{Arg}(\gamma'(1)) - \operatorname{Arg}(\gamma'(0))\,[2\pi].$$

Both sides of eq. (B.4) match when $t = 0$ and the right hand side is clearly continuous in t since ψ' extends continuously to $\gamma(0)$ and $\gamma(1)$, so we only have to argue that the left hand side is continuous. First, note that the curves $\psi_t(\gamma)$ are continuous in t in the Hausdorff sense. Indeed the continuity is obvious for $t < 1$ and at $t = 1$ continuity follows from the fact that the map ψ extends continuously to the boundary of $\mathbb{D}$. Now we argue that the topological winding of the curves are continuous in t, which will follow from the continuity of $\arg(\psi')$ up to the boundary. Let us argue for the winding around $\gamma(0)$ at a fixed time t_0. Since γ is smooth in a neighbourhood of $\gamma(0)$, we can find $\varepsilon > 0$ such that $|\arg \gamma'(s) - \arg \gamma'(0)| \leq \varepsilon$ for all $s \leq \varepsilon$. We can further assume by continuity of $\arg \psi'$ that ε is such that $|\arg(\psi_t'(\gamma(s)) - \arg(\psi_{t_0}'(\gamma(0))| \leq \varepsilon$ for $s \leq \varepsilon$ and $|t - t_0| \leq \varepsilon$. This shows that $\psi_t(\gamma[0, \varepsilon])$ is a smooth curve whose tangent at any point is always within 2ε of $\arg(\psi_{t_0}'(\gamma(0)) + \arg \gamma'(0)$. It is easy to see that such a curve cannot exit a cone of direction $\arg(\psi_{t_0}'(\gamma(0)) + \arg \gamma'(0)$ and of angle 2ε and therefore satisfies $W(\psi_t(\gamma[0, \varepsilon]), \psi_t(\gamma(0))) \leq 2\varepsilon$. On the other hand $\psi_t(\gamma[\varepsilon, 1])$ stays uniformly away from $\psi_t(\gamma(0))$ so $W(\psi_t(\gamma[\varepsilon, 1]), \psi_t(\gamma(0)))$ is continuous in t. Overall $W(\psi_t(\gamma), \psi_t(\gamma(0)))$ is continuous in t. The argument for the winding around $\gamma(1)$ is identical and we are done in the case $D = \mathbb{D}$.

For the general case, take the conformal map $\psi \circ \varphi : \mathbb{D} \to D'$. Using our previous argument for $\varphi : \mathbb{D} \to D$ and $\psi \circ \varphi$ gives two equations connecting the winding of $\tilde{\gamma}$ with the winding of γ and $\psi(\gamma)$. Combining the two and noting that the equation do not depend on the particular realisation of arg, we conclude. □

Remark B.5. The condition (iii) in Lemma B.4 is easier to understand when appealing to the notion of *conformal boundary* (see [3]). To explain what that is, we fix a conformal map $\varphi : D \to \mathbb{D}$ and equip D with the distance induced by φ, i.e., we set for $z, z' \in D$, $d_\varphi(z, z') = |\varphi(z) - \varphi(z')|$. The conformal closure $\mathrm{cl}(D)$ of D is defined as the completion of D with respect to this distance. (The conformal boundary can then be defined to be $\mathrm{cl}(D) \setminus D$; this notion is then equivalent to that of Poisson and Martin boundaries induced by Brownian motion on D, as well as prime ends in [11].) Note that $\mathrm{cl}(D)$ is then identified with the closed disc $\bar{\mathbb{D}}$ and so a point in $\mathrm{cl}(D)$ projects to a unique point on $\bar{D}$ since by local connectedness of ∂D, φ^{-1} extends to the closed disc, see Theorem 2.6 in [11].

With these definitions, assumption (iii) simply says that γ is a continuous curve in $\mathrm{cl}(D)$, i.e., γ is the projection of a continuous curve in $\mathrm{cl}(D)$.

Putting Lemma B.4 together with Lemma 2.2 in [2], we obtain the following.

LEMMA B.6. *Let D, D' be domains with locally connected boundaries and let ψ be a conformal map sending D to D'. Let $\gamma : [0,1] \to \bar{D}$ be a curve in $\bar{D}$ and assume that it is smooth and simple at $\gamma(1)$. Assume further that $x = \gamma(0) \in \partial D$ and that $\arg_{\psi'(D)}(\psi')$ extends continuously to $\gamma(0)$ and $\gamma(1)$. Then, letting $x' = \psi(x)$,*

$$\begin{aligned} &W\left(\psi(\gamma), \psi(\gamma(1))\right) - W(\gamma, \gamma(1)) \\ &\qquad = \arg_{\psi'(D)}(\psi'(\gamma(1))) + \arg_{D;x}(\gamma(1)) - \arg_{D';x'}(\psi(\gamma(1))). \end{aligned}$$

where we choose *the global constants defining the arguments so that the chain rule holds at $x = \gamma(0)$:*

$$\arg_{D';x'}((\psi \circ \gamma)'(0)) = \arg_{D,x}(\gamma'(0)) + \arg_{\psi'(D)}(\psi'(x)). \tag{B.5}$$

Furthermore if $\arg_{\psi'(D)}(\psi')$ does not extend continuously to $\gamma(0)$, the formula still holds up to a global constant in $\mathbb{R}$ not depending on γ and depending only on the choice of the constants in the definition of the arguments.

PROOF. First assume that $\gamma(0) = x$ is a smooth simple point of γ, then the results follows directly from applying Lemma B.4 together with Lemma 2.2 in [2]. Now to get rid of this assumption, note that if we modify γ in a small enough neighbourhood of x, we cannot change the value of the left hand side. We can therefore always change γ into a curve satisfying our assumptions so we are done. For the case where $\arg(\psi')$ does not extend to $\gamma(0)$, note that as in the proof of Lemma B.4, we can approximate ψ by a map ψ_t that has derivative in $\gamma(0)$. Applying the result for the smooth case modulo a global constant and then taking the approximation to 0 (i.e., $t \to 1$) yields the result. □

Finally in the case where we want to compute winding with respect to a point different from the endpoint – which in practice is what we will do to truncate the winding of SLE – the distortion lemma gives us a version of the Lemma B.6 with an error term.

Corollary B.7 (Lemma 2.4 in [2]). *Let D, D' be bounded domains with locally connected boundary and let ψ be conformal map sending D to D'. Let $\gamma : [0,1] \mapsto \bar{D}$ be a curve in $\bar{D}$. Assume further that $\arg_{\psi'(D)}(\psi')$ extends continuously to $\gamma(0)$ and $\gamma(1)$. Let z be a point in $D \setminus \gamma[0,1]$ and let $R = R(z, D)$ be its conformal radius and assume that $|z - \gamma(1)| \leq R/8$. Then, letting $x = \gamma(0)$ and $x' = \psi(x)$,*

$$\begin{aligned} \text{(B.6)} \quad & W(\psi(\gamma), \psi(z)) - W(\gamma, z) \\ & = \arg_{\psi'(D)}(\psi'(z)) + \arg_{D;x}(z) - \arg_{D';x'}(\psi(z)) + O(|z - \gamma(1)|/R), \end{aligned}$$

where the implicit constant in $O(|z - \gamma(1)|/R)$ is universal. The constants in the arguments are defined as in Lemma B.6. Furthermore if $\arg_{\psi'(D)}(\psi')$ does not extend to $\gamma(0)$, the formula still holds up to a global constant in $\mathbb{R}$ depending only on the choice of the constants for the arguments and not on γ.

Proof. Let $\tilde{\gamma}$ be obtained by appending a straight line connecting $\gamma(1)$ to z. By Koebe's 1/4 theorem $\tilde{\gamma}$ is still a curve in D and it is obviously smooth at its last point so we can apply Lemma B.6 to it. On the other hand, by Lemma B.2, we see that the image of the straight segment has winding $O(|z - \gamma(t)|/R)$ since it stays in a cone of that angle. By additivity of the winding we are done. ☐

Note that the condition that the argument extends continuously near $\gamma(0)$ (condition (ii) in Lemma B.4, which also arises in Corollary B.7) is trivial if $\gamma(0)$ and $\gamma(1)$ are interior points but is non-trivial for boundary points. However Theorem 3.2 in [11] implies that if D, D' are *smooth* Jordan curves then this condition is satisfied. For completeness, we now give a slight generalisation of this fact, which gives us a simple geometric sufficient criterion for this condition to hold (essentially it suffices that the boundary is smooth locally near the point where we wish to extend the argument).

Lemma B.8. *Let ψ be a conformal map between two domains D and $\tilde{D}$ and let $x \in \partial D$ be fixed. Assume that D and $\tilde{D}$ have locally connected boundary and let λ be a parametrisation of ∂D coming from a map to the disc (i.e up to parametrisation, λ is the curve $(g(e^{it}))_{0 \leq t \leq 2\pi}$ with g conformal from $\mathbb{D}$ to D). If there exists an open interval I such that $x \in \lambda(I)$ and both $\lambda(I)$ and $\psi(\lambda(I))$ are smooth curves, then $\arg_{\psi'(D)} \psi'$ extends continuously* *to a neighbourhood of x. Here* arg *is any realisation of argument in the range of ψ'.*

PROOF. We consider first the case $D = \mathbb{D}$. Let us write $x' = \psi(x)$ and let I be an interval given as in the statement. Let $\tilde{\lambda}$ be a C^1 parametrisation of $\psi(\lambda(I))$ with non vanishing derivative and t_0 be such that $x' = \tilde{\lambda}(t_0)$. Since $\psi(\lambda(I))$ is smooth, we can assume by taking a smaller I if necessary that for all $t \in I$, $|\arg_{\psi'(D)} \tilde{\lambda}'(t) - \arg_{\psi'(D)} \tilde{\lambda}'(t_0)| \leq 1$. As in the proof of Lemma B.4, this implies that $\tilde{\lambda}$ stays in a cone of angle 2 around each of its points and in particular $\tilde{\lambda}$ is injective, or in other word it is a Jordan arc. It is then easy to see that we can find a sub-domain $E \subset D'$ such that E is bounded by a smooth Jordan curve and $\tilde{\lambda}(t) \in \partial E$ for t in a neighbourhood of t_0.

Let $K = \psi^{-1}(D' \setminus E)$ and let g be a map sending $\mathbb{D} \setminus K$ to $\mathbb{D}$ and x to x. Observe that $\arg(g')$ extends to x (e.g. by Schwarz reflection). Note also that the map $\varphi = \psi \circ g^{-1}$ is a conformal map sending $\mathbb{D}$ to E which is a smooth Jordan domain. By Theorem 3.2 in [11] $\arg(\varphi')$ extends continuously to the boundary. Finally by construction $\psi = \varphi \circ g$ in a neighbourhood of x so $\arg \psi'$ also extends to the boundary in a neighbourhood of x. We conclude with arbitrary D and D' by considering the maps from $\mathbb{D}$ to D and $\mathbb{D}$ to D' and by composition. □

B.4. *One-point winding lemma.* Schramm's theorem about one-point winding (Theorem 3.1 in [2]) deals with SLE curves towards 0 in the unit disc $\mathbb{D}$. We now provide an extension of this result for SLE curves towards an arbitrary point in $\mathbb{D}$.

LEMMA B.9 (Lemma 3.4 in [2]). *Let $z \in \mathbb{D}$ and let $\psi : \mathbb{D} \mapsto \mathbb{D}$ be the Möbius transformation mapping z to 0 and 1 to 1. If $\gamma_z(t) \in B(z, \varepsilon)$ where $\varepsilon \leq R(z, \mathbb{D})/8$, then we have:*

$$W(\psi(\gamma_z[-1,t]), 0) = W(\gamma_z[-1,t], z) + \pi - \arg_{\mathbb{D};1}(z) + \epsilon(t) \tag{B.7}$$

where the error term $|\epsilon(t)| \leq C\varepsilon/R(z, \mathbb{D})$ for some universal constant $C > 0$ and $\arg_{\mathbb{D},1}$ is the argument function with values in $(\pi/2, 3\pi/2)$ (so $\arg_{\mathbb{D};1}(0) = \pi$). Also for all s, t

$$\mathbb{P}(|\gamma_z(t) - z| > e^{-t+s} R(z, \mathbb{D})) \leq c e^{-c's} \tag{B.8}$$

where c, c' are independent of z.

PROOF. We are going to apply Lemma B.6 to the Möbius transform ψ of $\mathbb{D}$ mapping z to 0 and 1 to 1. Computing $\psi(w)$ for $w \in \mathbb{D}$:

$$\psi'(w) = \frac{1 - |z|^2}{(1 - w\bar{z})^2} \frac{1 - \bar{z}}{1 - z}. \tag{B.9}$$

Now we argue that $\arg_{\psi'(\mathbb{D})}(\cdot)$ can be taken to be $\mathrm{Arg} \in (-\pi, \pi]$ with branch cut $(-\infty, 0]$. Indeed let us describe $\psi'(\mathbb{D})$. First, $1 - \overline{z}\mathbb{D}$ is a disc of radius $|z|$ contained in the right half plane $\{z : \Re(z) > 0\}$. Therefore $1/(1 - \mathbb{D}\overline{z})$ is a subset of the right half plane since $z \mapsto 1/z$ preserves the right half plane. Thus $(1 - \mathbb{D}\overline{z})^{-2}$ does not intersect $(-\infty, 0]$. Multiplying by the positive real number $(1 - |z|^2)$ and rotating by $\theta = \mathrm{Arg}((1 - \overline{z})/(1 - z))$, we see that $\psi'(\mathbb{D})$ does not contain the half line w joining 0 and $e^{i(\theta-\pi)}$. Therefore $\arg_{\psi'(\mathbb{D})}$ coincides with the argument $\tilde{A}rg$ with branch cut w that can take the value 0 (since they differ by some multiple of 2π and agree at $\psi'(1)$ by definition). With our choice of θ, $\tilde{A}rg(\cdot)$ takes values in $(\theta - \pi, \theta + \pi]$. Therefore $\tilde{A}rg((1 - \bar{z})/(1 - z)) = \theta = \mathrm{Arg}((1 - \bar{z})/(1 - z))$ (since this is the only value congruent to θ mod 2π in the interval $(\theta - \pi, \theta + \pi]$) and hence,

$$\begin{aligned}\arg_{\psi'(\mathbb{D})}((1 - \overline{z})/(1 - z)) &= \tilde{A}rg(1 - \overline{z})/(1 - z)) = \theta \\ &= \mathrm{Arg}(1 - \overline{z})/(1 - z)) = 2\pi - 2\,\mathrm{Arg}(z - 1) = 2\pi - 2\arg_{\mathbb{D};1}(z)\end{aligned}$$

where we used the rules of addition for arguments when going across a branch cut. Plugging this in Lemma B.6 and making the cancellation, we have the desired expression (B.7).

Now without loss of generality, assume $t - s \geq 10$. Applying Koebe's 1/4 theorem (see Theorem 3.17 in [5]), we see that

$$B(0, e^{-t+s}/4) \subset \psi(B(z, e^{-t+s}R(z, \mathbb{D}))).$$

Hence applying Theorem 3.1 of [12] and conformal invariance we have (B.8). □

C. Flow lines of the GFF. In this section, we provide a proof of Theorem 2.8 in the main file which is restated below in Theorem C.1. The reader is advised to recall the definitions of intrinsic winding boundary conditions on a simply connected domain D with a marked point x (which we denote by $u_{D,x}$) from the main file.

THEOREM C.1 (Imaginary geometry coupling; Theorem 2.8 in [2]). *Let D be a simply connected domain with a marked point x on the boundary. Let $h = \frac{1}{\sqrt{2}} u_{(D,x)} + h^0_D$ where h^0_D is a GFF with Dirichlet boundary conditions in D. There exists a coupling between the continuum wired UST on D and h such that the following is true. Let $\{\gamma_i\}_{1\leq i\leq k}$ be the branches of the continuum wired UST from points $\{z_i\}_{1\leq i\leq k}$ in D and let $D' = D \setminus \cup_{1\leq i\leq k}\gamma_i$. Then the conditional law of h given $\{\gamma_i\}_{1\leq i\leq k}$ is the same as $\frac{1}{\sqrt{2}} u_{(D',x)} + h^0_{D'}$ where $h^0_{D'}$ is a GFF with Dirichlet boundary condition in D'. Furthermore, h is completely determined by the UST and vice-versa.*

Given a planar graph G with boundary ∂G (here the boundary is just a given subset of vertices of G), one can consider a uniform spanning tree in it with either wired (when all the boundary vertices are collapsed into a single vertex) or free boundary condition. Every spanning tree in a planar graph corresponds to a spanning tree in the dual graph. It is easy to see that the dual of a uniform spanning tree with free boundary conditions is a uniform spanning tree in the dual graph with a wired boundary condition. More generally, it will be convenient to consider trees with **mixed boundary conditions**, that is, partially free and partially wired. For example, we can divide the boundary $D^{\#\delta}$ into an union of arcs $\alpha^{\#\delta}$ and $\beta^{\#\delta}$ and consider the spanning tree with wired boundary on $\alpha^{\#\delta}$ and free boundary on $\beta^{\#\delta}$. Then the dual tree is a spanning tree of the dual $(D^{\#\delta})^\dagger$ with wired boundary conditions near $\beta^{\#\delta}$ and free boundary conditions near $\alpha^{\#\delta}$. See e.g. [7] for details.

With every spanning tree in a planar graph, one can associate a curve which forms the interface in between the spanning tree in the graph and the dual spanning tree in the dual graph. This interface visits every face, and is unique up to reparametrisation. This is called the **Peano curve** of the tree. In case of the mixed boundary spanning tree described above, the Peano curve joins the endpoints of the two arcs. The convergence of this Peano curve to chordal SLE_8 was also established in [7] using the following topology: for paths β, γ in $\bar{D}$, let $d_{\text{Peano}}(\beta, \gamma) = \inf \sup_t |\beta(t) - \gamma(t)|$ where the infimum is over all possible parametrisations of the curves β, γ. This is the **Peano curve topology**.

THEOREM C.2 ([7]). *Let ∂D be a C^1-smooth simple loop. Let $(D^\delta)^\dagger$ denote the dual graph of D^δ which is a subset of $(\mathbb{Z}+1/2)^2$.*

- *The uniform spanning tree with wired boundary condition as well as the uniform spanning tree with free boundary condition converge as $\delta \to 0$ in the Schramm topology. We call these limits the continuum wired uniform spanning tree and the continuum free uniform spanning tree respectively.*
- *Let α and β be smooth arcs in ∂D that are complementary: $\alpha \cup \beta = \partial D$ and let a and b be points in ∂D where α and β meet. Let $\alpha^{\#\delta}$ be an approximation of α in $\partial D^{\#\delta}$ and $\beta^{\#\delta}$ be an approximation of β in $\partial D^{\#\delta}$ in $(D^{\#\delta})^\dagger$ (the precise sense of this approximation can be found in Section 4 of Lawler, Schramm and Werner [7]). Consider the UST in $D^{\#\delta}$ with wired boundary condition on $\alpha^{\#\delta}$ and the UST in $(D^{\#\delta})^\dagger$ with wired boundary condition on $\beta^{\#\delta}$. Then the Peano curve in between these two trees converge in law (with the underlying*

topology being the Peano curve topology) to a chordal SLE_8 *curve from* a *to* b *in* D*.*

Now we describe a mapping which maps the Peano curve to an element in the Schramm space and essentially shows that the Peano curve topology is stronger than the Schramm topology. For this, we need to first go back to the discrete and describe the left and right boundary of the Peano curve.

In the graph $D^{\#\delta}$, we join together every dual vertex with the primal vertices adjacent to its face. These edges along with the primal and the dual trees form a triangulation. Every edge in the primal or the dual tree belongs to two triangles in this triangulation. Thus each triangle has an opposite triangle corresponding to the (primal or dual) edge where it belongs. The Peano curve visits each such triangle exactly once and we can fix the orientation of the Peano curve so that the primal tree edges always lie to its left. After drawing a certain number of edges in the Peano curve, the triangles visited by the Peano curve whose opposite triangle have not yet been visited form the boundary of the curve. The boundary triangles corresponding to the dual tree edges form the **left boundary** of the curve and the triangles corresponding to the primal tree edges form the **right boundary** of the curve.

It is known that a chordal SLE_8 is space filling almost surely (see Proposition 6.11 or more precisely Theorem 7.9 in [5]) which means that a.s. for all $z \in D$, the SLE_8 visits z. Since the curve is space filling, at any stopping time τ, the complement of the $\eta[0,\tau]$ consists of a single component. The boundary of $D \setminus \eta[0,t]$ can be naturally divided into two parts: the **left boundary** and the **right boundary**. By definition, the left boundary always contains a part of the wired boundary in the domain and the right boundary contains a part of the free boundary of the domain. Hence for $z, w \in \bar{D}$, we can grow the curve until the times τ_z, τ_w when it hits z and w respectively. One can think of the curve as oriented towards its target. Then we define a (mixed) continuum UST $\mathcal{T}$ (i.e., an element of the Schramm space) associated to η by taking the path connecting z and w to be simply given by the left boundary of $\eta(\min(\tau_z,\tau_w),\max(\tau_z,\tau_w))$. Furthermore, since almost surely, SLE_8 visits Lebesgue almost every point only once, it is easy to deduce that the above map is a.s. continuous. This implies that convergence in the Peano sense is stronger than in the Schramm sense for SLE_8.

One issue we have when applying Theorem C.2 is that the convergence of the Peano curve is written for spanning trees with mixed boundary condition only. Indeed, for fully wired boundary (or fully free boundary in the dual), the convergence is no longer to a chordal SLE_8 curve, but presumably to a certain space-filling loop. However, we could not locate a convergence result for this loop in the literature, so we bypass this problem using the following lemma.

LEMMA C.3. *Consider a sequence of arcs β_n and α_n in ∂D such that $\beta_n \cup \alpha_n = \partial D$ with the diameter of α_n being at most $1/n$. Let $\alpha_n^{\#\delta}$ and $\beta_n^{\#\delta}$ be a partition of $\partial D^{\#\delta}$ into connected segments which approximate α_n and β_n respectively as $\delta \to 0$ in the sense of [7]. Consider a mixed continuum tree $\mathcal{T}_n$ obtained as the scaling limit of the spanning tree $\mathcal{T}_n^{\#\delta}$ in $D_n^{\#\delta}$ with wired boundary condition in $\beta_n^{\#\delta}$ and free boundary condition in $\alpha_n^{\#\delta}$. (Such a scaling limit exist because the scaling limit of the Peano curve exists from [7] and the Peano curve topology is stronger than Schramm topology). For all $\varepsilon > 0$ there exists a coupling between the continuum mixed tree $\mathcal{T}_n$ and the continuum* wired *UST $\mathcal{T}_w$ in D (constructed in Section A.4) such that $\mathcal{T}_n \cap (D \setminus B(\alpha_n, \varepsilon)) = \mathcal{T}_w \cap (D \setminus B(\alpha_n, \varepsilon))$ for all $n > n_0(\varepsilon)$ with probability at least $1-\varepsilon$. In particular, $\mathcal{T}_n$ converges in probability to the continuum wired UST in D as $n \to \infty$, in the Schramm topology.*

PROOF OF LEMMA C.3. Let $\mathcal{T}^{\#\delta}$ denote the discrete tree with fully wired boundary condition. This lemma basically follows from Schramm's finiteness Theorem (Theorem 10.2 in [12], see also Lemma 4.18 in the main file [2]). For a fixed $\varepsilon > 0$, find a finite set V of vertices (corresponding to $\mathcal{Q}_j$ with $j = j_0(\varepsilon)$ in the notations of Lemma 4.18 of [2], for the domain $D^{\#\delta} \setminus B(\alpha_n^{\#\delta}, 2\varepsilon)$). Choose $n_0(\varepsilon)$ such that if $n \geq n_0(\varepsilon)$, the arc $\alpha_n^{\#\delta}$ is sufficiently small that for any $\delta < \delta_0(\varepsilon)$, no random walk starting from V hits ∂D on $\alpha_n^{\#\delta}$, with probability at least $1-\varepsilon$. (This is possible because a $\delta \to 0$, the random walks starting from V converge to a finite number of Brownian motions starting from a finite number of points in $D \setminus B(\alpha_n, 2\varepsilon)$).

On this event, which has probability at least $1-2\varepsilon$, the branches from $\mathcal{T}^{\#\delta}$ and $\mathcal{T}_n^{\#\delta}$ emanating from V can be taken to be identical thanks to Wilson's algorithm (which is valid both for $\mathcal{T}^{\#\delta}$ and $\mathcal{T}_n^{\#\delta}$, with the difference that in $\mathcal{T}_n^{\#\delta}$ the walks are reflected on $\alpha_n^{\#\delta}$ instead of being killed). Moreover, by Lemma 4.18 in [2], the choice of V is such that simple random walk started from any other vertex v will not exceed diameter ε before hitting one of the branches emanating from V, with probability at least $1-\varepsilon$, for any $\delta \leq \delta(\varepsilon)$.

On these two events, all the branches from V are identical in $\mathcal{T}_n^{\#\delta}$ and $\mathcal{T}^{\#\delta}$, and all branches from $D^{\#\delta} \setminus B(\alpha_n^{\#\delta}, \varepsilon)$ can be taken to be the same in $\mathcal{T}_n^{\#\delta}$ and $\mathcal{T}^{\#\delta}$. Hence we have found a coupling so that $\mathcal{T}_n^{\#\delta}$ and $\mathcal{T}^{\#\delta}$ agree in $D^{\#\delta} \setminus B(\alpha_n^{\#\delta}, \varepsilon)$ for any $\delta < \delta(\varepsilon)$. The result now follows by letting

$\delta \to 0$. □

Flow lines. The theory of imaginary geometry developed by Miller and Sheffield in [9, 10] following earlier work of Dubédat [4] and Schramm–Sheffield [13] gives us a way to couple SLE curve realised as flow lines of a Gaussian free field. Fix the constants:

$$\kappa = 2; \quad \kappa' = 8; \quad \lambda' = \frac{\pi}{\sqrt{8}}; \quad \lambda = \frac{\pi}{\sqrt{2}}; \quad \chi = \chi(\kappa) = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.$$

The notations are consistent with the notations in [9, 10]. We write $\chi = \chi(2)$, $\chi' = \chi(8) = -\chi(2)$.

Take a domain $D \subset \mathbb{C}$ with two marked points on the boundary a and b (possibly with a boundary which is rough). A canonical example is $(\mathbb{H}, \infty, 0)$ where $\mathbb{H}$ is the upper half plane. Fix $\alpha > 0$ which we refer to as the *angle* in what follows. Let $h_{\kappa'}$ be a GFF in $\mathbb{H}$ with boundary condition given by $-\lambda' + \alpha\chi'$ on the positive real line and $\lambda' + \alpha\chi'$ on the negative real line. Let h_κ be a GFF in $\mathbb{H}$ with boundary condition $\lambda + \alpha\chi$ on the positive real line and $-\lambda + \alpha\chi$ for the negative real line for κ. Let $\varphi_D : D \to \mathbb{H}$ be a conformal equivalence between $\mathbb{H}$ and D which maps b to 0 and a to ∞. Define a **mixed intrinsic winding boundary condition** GFF with angle α on (D, a, b) with parameter $x \in \{\kappa, \kappa'\}$ to be defined by

$$h_x \circ \varphi - \chi(x) \arg(\varphi') \tag{C.1}$$

When we do not specify the angle, we mean $\alpha = 0$ by default. So that the reader remains in context, this boundary condition will correspond to the mixed boundary uniform spanning tree (with a specific α) where the boundary type changes at a and b. Although φ is not unique, this definition defines a unique law via scale invariance.

It can be checked that the mixed intrinsic winding boundary condition $(\mathbb{D}, 1, e^{i\theta_b})$ can be defined as follows:

$$f_{1,b,\kappa'}(e^{i\theta}) = \begin{cases} -\lambda' + \chi'(\theta + \frac{\pi}{2}) \text{ if } 0 \le \theta \le \theta_b \\ \lambda' + \chi'(\theta + \frac{\pi}{2}) \text{ if } \theta_b < \theta < 2\pi \end{cases} \tag{C.2}$$

Replacing κ' by κ and λ' by $-\lambda$ we get a function $f_{1,b,\kappa}$. We extend the definition of $f_{1,b,\kappa}$ (resp. $f_{1,b,\kappa'}$) to $\theta_b = 2\pi$ by defining $f_{1,1,\kappa}(e^{i\theta}) = \lambda + \chi(\theta + \pi/2)$ (resp. $f_{1,1,\kappa'}(e^{i\theta}) = -\lambda' + \chi'(\theta + \pi/2)$). Hence $f_{1,1,\kappa}$ (resp. $f_{1,1,\kappa'}$) is the same as χ times (resp. χ' times) the intrinsic winding boundary condition plus a global shift. We extend this to all marked domains (D, a, b) via the coordinate change formula eq. (C.1); this defines functions $f_{a,b,\kappa}$ and $f_{a,b,\kappa'}$ on this doubly-marked domain, and with a slight abuse of notation we will still call $f_{a,b,x}$ the harmonic extension for $x \in \{\kappa, \kappa'\}$.

We are going to quote a special case of some theorems from [10].

THEOREM C.4. *Let (D, a, b) be a domain with two distinct marked points on its boundary. Let $\{z_n\}_{n\in\mathbb{N}}$ be a countable dense set of interior points in D. Let h be a zero boundary GFF in D and let $f_x = f_{a,b,x}$ be as above.*

Let $\eta'[0,\infty)$ be an $SLE_{\kappa'}$ curve in (D, a, b) from b to a parametrised by capacity. There exists a unique coupling between h, η' and a set of simple curves $\{\eta^{(i)}[0,\tau_i]\}_{i\in\mathbb{N}}$ in D starting at z_i for each $i \geq 1$ (defined up to monotone reparametrisation) where τ_i is the stopping time when the curves hit ∂D, such that the following is true.

- *i Let τ_i' be the stopping time when η' hits z_i ($\tau_i' < \infty$ almost surely since η' is space filling almost surely). Then the left boundary of $\eta'[0,\tau_i']$ is the same as η_i almost surely.*
- *ii For any stopping time τ' of η', the conditional law of $-h + f_{\kappa'}$ in $D \setminus \eta'[0,\tau']$ (notice that this is a connected domain since $SLE_{\kappa'}$ is space filling almost surely) is given by an independent GFF in $(D \setminus \eta'[0,\tau'], \eta'(\tau'), b)$ with mixed winding boundary condition with parameter κ'. Such a curve η' coupled with h is called a **counterflow line** in the terminology of of [10].*
- *iii For any $k \geq 1$, for all $1 \leq i, j \leq k$, the curves $\eta^{(i)}$ and $\eta^{(j)}$ have the coalescing property: if they intersect, they continue together until they hit ∂D. In particular, $D \setminus \cup_{i=1}^k \eta^{(i)}[0,\tau_i]$ is a connected domain. Furthermore, the conditional law of $h + f_\kappa$ given $\cup_{i=1}^k \eta^{(i)}[0,\tau_i]$ is given by an an independent GFF in $(D \setminus \cup_{i=1}^k \eta^{(i)}[0,\tau_i], a, b)$ with mixed intrinsic winding boundary condition with angle $\pi/2$. These curves are the **flow lines of h with angle $\pi/2$** in the terminology of [10]. The full set of curves $\eta^{(i)}$ is called a **tree of flow lines** starting from the countable set of points $\{z_i\}_{i\in\mathbb{N}}$ and angle $\pi/2$.*
- *iv The curve η' determines h and vice versa almost surely. The curves $(\eta^{(i)})_{i\in\mathbb{N}}$ determines h and vice versa almost surely.*

Furthermore in the case where $a = b$ we still get a coupling between h and a set of simple curves $\eta^{(i)}$ such that item iii above still holds.

PROOF. The coupling between η', h is described in Theorem 1.1 of [9] or Theorem 6.4 in [4], while the coupling between h and $\eta^{(i)}$ follows from Theorem 1.8 in [10]. Note that the latter still holds in the case $a = b$. The properties described in the coupling are special cases of Theorem 1.8 and

Theorem 1.13 of [10]. That the curve η' determines h and vice versa follows from Theorem 1.2 of [9, 10] and Theorem 1.16 of [10]. That the curves $\{\eta^{(i)}\}_{i\in\mathbb{N}}$ determine h follows from Theorem 1.10 of [10]. □

At this point, we know that in the mixed boundary case and $a \neq b$, the coupling above holds for h, the Peano path η' and the flow lines $\eta^{(i)}$. We know that for $\kappa = 2$, η' is a chordal SLE_8 and hence the scaling limit of the discrete Peano path of a mixed UST. Since convergence in the Peano sense is stronger than in the Schramm sense, we know that the left and right boundaries of the path η' are branches in the mixed continuum UST and its dual. By the above theorem these are also the flow lines associated to h with angle $\pm\pi/2$. We want to deduce by letting $b \to a$ that the tree of flow lines in the case $a = b$ described in the above theorem are the branches of the wired continuum UST.

Essentially, the argument goes as follows. By Lemma C.3 we know that as $b \to a$, the mixed continuum UST converges to the wired UST. The GFF determined by the branches of the mixed tree will be shown to converge. This requires an argument because the function which associates a field to a tree of flow lines is not known to be continuous, only measurable. However, the conditional law of the field with mixed boundary conditions, given a finite number of its flow lines, converges to that of a field with wired boundary conditions given these curves. Since these conditional distributions characterise entirely the joint distribution between all the branches and the field, the result follows.

LEMMA C.5. *Let (D, x) be a domain with a marked point. Let $\{z_i\}_{i\in\mathbb{N}}$ be a countable dense set of points in D. Then there exists a coupling between a GFF in (D, x) and the continuum wired UST in D such that flow line tree from $\{z_i\}_{i\in\mathbb{N}}$ with angle $\pi/2$ is equal to the branches of the continuum wired UST from $\{z_i\}_{i\in\mathbb{N}}$ almost surely.*

PROOF. Fix $k \in \mathbb{N}$ and any set of k points from $\{z_i\}_{i\in\mathbb{N}}$. Without loss of generality let this set be $\{z_1, \ldots, z_k\}$. It suffices to find a coupling between $(h, \mathcal{T})$ such that in this coupling h has χ times intrinsic winding boundary conditions and $\mathcal{T}$ is a wired UST and such that for each $k \geq 1$ the branches $\tilde{\eta}_1, \ldots, \tilde{\eta}_k$ emanating from $z_1, \ldots, z_k$ are equal to the flow lines $\eta_1, \ldots, \eta_k$ with angle $\pi/2$.

To find such a coupling, take a sequence $y_n \in \partial D$ such that $y_n \to x$ and call α_n the arc between y_n and x (which converges to $\{x\}$) and β_n the complementary arc (which converges to ∂D). Let $\mathcal{T}_n$ be a mixed boundary continuum uniform spanning tree with free boundary in the arc α_n and wired boundary in the rest. Let $\mathcal{T}_n$ be coupled with a continuum wired UST denoted by $\mathcal{T}$ in (D, x) as in Lemma C.3. The branches $\eta_{n,i}$ of $\mathcal{T}_n$ from $\{z_1, \dots, z_k\}$ (until they hit the wired boundary) in (D, x, y_n) are unique almost surely and are equal to $\tilde{\eta}_1, \dots, \tilde{\eta}_k$ with high probability as $n \to \infty$. Let h_n be the field determined by the branches of $\mathcal{T}_n$ viewed as flow lines (such an identification is possible when n is finite and the boundary conditions are mixed by Theorem C.4). This defines a coupling of $(h_n, \mathcal{T})$ such that the branches $\tilde{\eta}_1, \dots, \tilde{\eta}_k$ are equal with high probability to the flow lines of h_n with angle $\pi/2$. The marginal h_n in this coupling is a mixed intrinsic winding boundary GFF and so converges to a GFF with χ times intrinsic winding minus λ boundary conditions as $n \to \infty$. Hence $(h_n, \mathcal{T})$ is tight and we can consider a subsequential limit $(h, \mathcal{T})$. Notice that in such a coupling necessarily we have that $\tilde{\eta}_1, \dots, \tilde{\eta}_k$ are flow lines of h, hence we have the desired coupling (this follows easily from the fact that the law of $(\eta_{n,1}, \dots, \eta_{n,k})$ converges holds in total variation, see Lemma C.3). This completes the proof. $\square$

We summarise the findings of this section in the following theorem:

THEOREM C.6 (Coupling). *Let $D \subset \mathbb{C}$ be a domain which is not $\mathbb{C}$. Let $\mathcal{T}$ be a continuum spanning tree in D with wired condition and root $x \in \partial D$. There exists a coupling between the $\mathcal{T}$, a Gaussian free field h on (D, x) with intrinsic winding boundary condition such that $\mathcal{T}$ is the flow line tree of h with angle $\pi/2$.*

Consequently, given $z_1, \dots, z_k \in D$ and the set of branches $\mathcal{B}$ of $\mathcal{T}$ from these points to x, the conditional law of h in $D \setminus \mathcal{B}$ is given by a GFF in $D \setminus \mathcal{B}$ with intrinsic winding boundary condition in $(D \setminus \mathcal{B}, x)$. Also, $\mathcal{T}$ determines h and vice-versa.

The second paragraph of Theorem C.6 is exactly the statement of Theorem C.1, which is what we want.

D. Discrete estimates. In this section, we prove the technical random walk estimates that we need in Section 4 of the main file.

D.1. *Winding of random walk.*

LEMMA D.1 (Lemma 4.7 in [2]). *Fix $0 < r < R$. There exists $\alpha = \alpha(R/r) > 0$ and c such that for all $x \in \mathbb{C}$, $\delta \in (0, cr\delta_0)$, $v \in A(x, r + \frac{R-r}{3}, R - \frac{R-r}{3})^{\#\delta}$, for all u such that $\mathbb{P}_v(X_\tau = u) > 0$ where τ is the exit* *time of $A(x,r,R)$, and for all $n \geq 1$, we have:*

$$\mathbb{P}_v\left[\sup_{\mathcal{Y}\subset X[0,\tau]} |W(\mathcal{Y},x)| \geq n | X_\tau = u\right] \leq C(1-\alpha)^n.$$

where the supremum is over all continuous paths $\mathcal{Y}$ obtained by erasing portions from $X[0,\tau]$.

PROOF. The proof is divided in two steps. First we construct a set U' close to u and macroscopic such that if the random walk is started in U' it has a good chance to exit the annulus through u. In the second step we use the conditional crossing estimate (Lemma 4.4 in [2]) to control the winding: every time the walk crosses the positive real line, the chance that it will wind once more before exiting at u is small because there is a good chance to go hit the macroscopic set U'.

Step 1. Let us assume first that u is a point on the outer boundary of the annulus. Up to a rotation and a translation of the graph, we can assume that $x = 0$ and u is on the negative real axis. We also simplify notations by writing $A = A(x,r,R)$. For a continuous path γ we write $\gamma^{\#\delta}$ for a discrete path staying at distance $2\delta/\delta_0$ of γ (such a path can be constructed thanks to our Russo–Seymour–Welsh crossing assumption).

We start as in the proof of Lemma 4.4 in [2] by controlling the function $h(v) = \mathbb{P}_v[X_\tau = u]$. More precisely we claim that there exists $c > 0$ such that if $\rho = \frac{R-r}{10}$, for all δ small enough, for all $a \in U := \big(A \cap \partial B(u,\rho)\big)^{\#\delta}$ and $b \in U' := \big(\{z | \arg(z-u) \in [-\frac{\pi}{4}, \frac{\pi}{4}]\} \cap \partial B(u,\rho)\big)^{\#\delta}$,

$$h(b) \geq ch(a). \tag{D.1}$$

Indeed fix $b \in U'$ and $a \in U$. Since h is harmonic there exists a path γ from a to u along which h is non-decreasing and we can assume that γ does not intersect U outside of a (otherwise change a to be the last intersection point). Let τ_γ be the first hitting time of $\gamma \cup \partial A$ by the simple random walk. By harmonicity we have

$$h(b) = \mathbb{E}_b[h(X_{\tau_\gamma})].$$

On the other hand, by the crossing estimate, the random walk has a positive probability c independent of ρ to hit γ irrespective of the relative positions of a, b, u. (For example this is at least the probability to surround $\partial B(u,\rho) \cap A$ in the clockwise and anticlockwise directions, staying in the annulus $A(u, 99\rho/100, 101\rho/100)$.) Hence using this event to lower bound the expectation, we find $h(b) = \mathbb{E}_b[h(X_{\tau_\gamma})] \geq h(a)c$. This proves the claim (D.1).

Step 2. It will be easier to bound crossings of the real line. Let ℓ_+ be a path connecting two boundary pieces of A and approximating the interval $[r, R]$. Here approximating means the path stays within $O(\delta/\delta_0)$ from ℓ_+ and the existence of such a path is guaranteed by the crossing estimate. Similarly, define ℓ_- approximating $[-R, -r]$.

Let $\tau_{+,1}$ be the first hitting time of ℓ_+ and by induction let $\tau_{-,i}$ the first hitting time of ℓ_- after $\tau_{+,i}$ and $\tau_{+,i}$ the hitting time of ℓ_+ after $\tau_{-,i-1}$. Let I_+ the number of τ_+ before τ, i.e $I_+ = |\{i|\tau_{+,i} \leq \tau\}|$.

Note that there exists $\alpha \in (0,1)$ depending only on R/r such that for any $w \in \ell_+$,

$$\mathbb{P}_w(\tau_{-,1} < \tau) \leq 1 - \alpha.$$

Hence applying the strong Markov property n times,

$$\mathbb{P}_v(\tau_{+,n} \leq \tau; X_\tau = u) \leq (1-\alpha)^n \sup_{w \in \ell_+} \mathbb{P}_w(X_\tau = u). \tag{D.2}$$

However, we claim that for any $w \in \ell_+$, $\mathbb{P}_w(X_\tau = u) = h(w) \leq h(v)/\alpha$ where α depends only on R/r, whenever $v \in A(0, r + (R-r)/3, R - (R-r)/3)$. Indeed, by the maximum principle, $h(w) \leq \sup_U h \leq (1/c) \inf_{U'} h$ by (D.1). On the other hand, $h(v) \geq \alpha \inf_U h$ since there is a uniformly positive chance α of hitting U before τ whenever v is in the allowed region. Hence $h(v) \geq \alpha h(w)$ for any $w \in \ell_+$ and v in the allowed region. Plugging into (D.2), we deduce

$$\mathbb{P}_v(\tau_{+,n} \leq \tau | X_\tau = u) \leq (1/\alpha)(1-\alpha)^n.$$

However notice that deterministically,

$$\sup_{\mathcal{Y}} |W(\mathcal{Y}, x)| \leq 2\pi(I_+ + 1)$$

and hence the result follows.

Finally it is clear that the above proof extends to the case where u is a point in the inner boundary instead of being in the outer boundary. □

The next lemma is Lemma 4.8 in [2] which, we recall, is an analogue of Lemma D.1 for the largest scale.

Lemma D.2 (Lemma 4.8 in [2]). *Fix $r < R$ There exists $\alpha = \alpha(R/r) > 0$ and c, C such that for all $\delta \leq cr\delta_0$, for all $x \in \mathbb{C}$, for all domains D such* *that* $\mathrm{dist}(x, \partial D^{\#\delta}) = R$ *(in particular* $D \supset B(x,R)$*), for all* $v \in A(x, r + \frac{R-r}{3}, R - \frac{R-r}{3})^{\#\delta}$*, writing* τ *for the exit time of* $D \setminus B(x,r)$*,*

$$\text{(D.3)} \qquad \forall n \geq 1, \qquad \mathbb{P}_v\Big(\sup_{\mathcal{Y} \subset X[0,\tau]} |W(\mathcal{Y}, x)| \geq n \Big| X_\tau \in \partial D^{\#\delta}\Big) \leq C(1-\alpha)^n,$$

and for all $u \in B(x,r)$ *such that* $\mathbb{P}(X_\tau = u) > 0$*,*

$$\text{(D.4)} \qquad \forall n \geq 1, \qquad \mathbb{P}_v\Big(\sup_{\mathcal{Y} \subset X[0,\tau]} |W(\mathcal{Y}, x)| \geq n \Big| X_\tau = u\Big) \leq C(1-\alpha)^n.$$

In both cases, the supremum is over all continuous paths $\mathcal{Y}$ *obtained by erasing portions from* $X[0,\tau]$*.*

Before we start the proof, we remind the reader that the above result (D.3) could be wrong if we conditioned on the precise exit point $X_\tau \in \partial D^{\#\delta}$, so it is important to work with the above formulation.

Proof. We start with the proof of (D.3). Note that doing a full turn outside $B(x,R)$ implies necessarily that the walk has left $D^{\#\delta}$ since $\mathrm{dist}(x, \partial D^{\#\delta}) = R$. This readily implies $\mathbb{P}(X_\tau \in \partial D^{\#\delta}) \geq 1 - \alpha(R/r)$; hence in order to show (D.3) it suffices to prove the corresponding unconditional statement:

$$\text{(D.5)} \qquad \forall n \geq 1, \qquad \mathbb{P}_v\Big(\sup_{\mathcal{Y} \subset X[0,\tau]} |W(\mathcal{Y}, x)| \geq n\Big) \leq C(1-\alpha)^n.$$

This is easy to see: indeed, as in the proof of Lemma D.1, let $\ell_+ = [r, \infty)^{\#\delta}$ and and $\ell_- = (-\infty, -r]^{\#\delta}$ be the *infinite* half lines starting from the inner radius to infinity. Every time the walk hits ℓ_+ there is a probability at least $\alpha = \alpha(R/r) > 0$ that the walk will leave D without touching ℓ_- and then ℓ_+ again (and hence without making another turn): let $\tau_{+,i}$ and $\tau_{-,i}$ be defined as before. It is clear from the uniform crossing estimate that there exists $\alpha = \alpha(R/r) > 0$ such that

$$\text{(D.6)} \qquad \forall v \in \ell_+, \quad \mathbb{P}_v[X[0, \tau_{+,1}] \text{ does a full turn outside } B(x,R)] \geq \alpha.$$

(Note that this holds even if v is quite far away from $B(x,R)$ by taking rectangles of sufficiently large scale in the crossing assumption). Iterating and applying the Markov property immediately implies (D.5).

We now turn to (D.4). This is essentially the same as Lemma D.1, with a few modifications. We take ℓ_-, ℓ_+ to be the infinite half-lines as above, and note that

$$\mathbb{P}(\tau_{+,n} \leq \tau) \leq (1-\alpha)^n$$

by (D.6). We conclude as in the proof of Lemma D.1, noting that the maximum principle applies in any domain, and that if $v \in A(x, r+(R-r)/3, R-(R-r)/3)$ the probability that the walk will hit the set U' is still uniformly bounded below by $\alpha(R/r)$. □

LEMMA D.3 (Lemma 4.9 in [2]). *There exists $\alpha \in (0,1)$ and $c, C > 0$ depending only on the constants in the uniform crossing conditions such that the following holds. For all $\delta < c\delta_0 R$,*

$$\forall n \geq 1, \qquad \mathbb{P}_v\big(\big|W(\gamma_v^{\#\delta}[-1,0],v) - \mathbb{E}W(\gamma_v^{\#\delta}[-1,0],v)\big| \geq n\big) \leq C(1-\alpha)^n.$$

PROOF. We drop the superscript $\cdot^{\#\delta}$ for convenience and use a replica technique: let $\tilde{\gamma}_v$ be an independent realisation of γ_v. Clearly it suffices to show that

$$\forall n \geq 1, \qquad \mathbb{P}_v\big(|W(\gamma_v[-1,0],v) - W(\tilde{\gamma}_v([-1,0],v)| \geq n\big) \leq C(1-\alpha)^n.$$

Let λ be a continuous parametrisation of ∂D (as usual, this exists because of our assumption that ∂D is locally connected). Then note that if we choose S, S' such that $\lambda(S) = \gamma_v(0)$ and $\lambda(S') = \tilde{\gamma}_v(0)$ then by additivity of winding

$$|W(\gamma_v[-1,0],v) - W(\tilde{\gamma}_v([-1,0],v)| = |W(\lambda([S,S']),v)|$$

Fix a domain $D^{\#\delta}$ and a starting point v, let $R = \text{dist}(v, \partial D^{\#\delta})$ and let $r = R/2$, we let the constant c, C, α be chosen so that Lemma D.2 applies. Take two independent random walks starting from v run until they leave D, and consider the times at which they each exit $B(v, 3R/4)$ for the first time after the last exit from $B(v, R/2)$. Let X, X' be the continuation of the two walks from that point onwards and let w, w' be their starting points respectively on $\partial B(v, 3R/4)^{\#\delta}$. Observe that both X, X' are independent random walks, starting from $w, w' \in A(v, r+\frac{R-r}{3}, R-\frac{R-r}{3})^{\#\delta}$ and conditioned to exit $D \setminus B(v,r)$ through $\partial D^{\#\delta}$. We now make use of the following topological fact:

$$|W(\lambda[S,S'])| \leq 4\pi + \sup_{\mathcal{Y} \subset X} |W(\mathcal{Y},v)| + \sup_{\mathcal{Y}' \subset X'} |W(\mathcal{Y}',v)|.$$

which immediately proves the lemma using Lemma D.2. To see the above inequality, we simply point out that the winding of any loop around any point is bounded by 2π (note that this is true even if the loop is nonsimple, so long as it is noncrossing, and that λ is such a curve). Also note that if the walks X, X' do not intersect each other, then we can form a simple loop by joining w, w' by a circular arc, the loop erasures $\mathcal{Y}$ and $\mathcal{Y}'$ of X and X' after the last time they touch this circular arc, and $\lambda([S,S'])$. If the walks do intersect, then we can do the same but looking at loop erasures of the walks starting from their last intersection point. This completes the proof. □

D.2. *Proof of exponential tail of loop-erased winding (Proposition 4.12).* We now provide proofs of the sub-lemmas stated in the main file for the proof of Proposition 4.12 or, more precisely, Lemma 4.13.

LEMMA D.4 (Lemma 4.14 in [2]). *Recall $\mathcal{Y}, B_0$ from Lemma 4.13 (and above) in [2]. Then*

$$\mathcal{Y} \cap B_0^c \subset \bigcup_{0 \le i \le I} \bigcup_{k \in \mathcal{G}_i} X[\tau_k, \tau_{k+1}].$$

Furthermore, one can write $\mathcal{Y} \cap B_0^c = \cup_{i \le I} \cup_{k \in \cup \mathcal{G}_i} \mathcal{Y}_k$ where $\mathcal{Y}_k$ are disjoint intervals of the loop erased random walk of the form $\mathcal{Y}_k = (Y_{j_k}, Y_{j_k+1}, \dots, Y_{j_k+i_k})$ and $\mathcal{Y}_k \subset X[\tau_k, \tau_{k+1}]$.

PROOF. We consider the chronological erasure of loops so at each time T we have a loop-erased path $Y^{(T)}$ obtained by erasing loops from the random walk $(X_t)_{t \le T}$.

Notice that by construction, after time $\tau_{k_{\max}}$ the random walk does not return to $B(0, r_1)$ so $\mathcal{Y} \subset Y^{(\tau_{k_{\max}})}$ (it might be smaller because more loops can occur, erasing further the final set $\mathcal{Y}$, but no more points can be added to $\mathcal{Y}$ since we do not return to the annulus). This justifies looking only at $k \le k_{\max}$ above.

Observe that if X does a full turn between times τ_k and τ_{k+1}, then $Y^{(\tau_{k+1})} \subset B(v, r_{i(k)+1})$. Indeed if $Y^{(\tau_k)} \subset B(v, r_{i(k)+1})$ the statement is trivial. Otherwise $Y^{(\tau_k)}$ includes a path from $C_{i(k)-1}$ to $C_{i(k)+1}$ which has to be crossed by the full turn at some time T and this erases everything outside of $B(v, r_{i(k)+1})$.

In particular, we see that $Y^{\tau_{\kappa_{-1}+1}} \subset B(v, r_0)$. This implies that $\mathcal{Y} \cap B_0^c \subset X[\tau_{\kappa_{-1}+1}, \tau_{k_{\max}}]$ since all the rest of X either was erased or contributes to parts of Y outside of $\mathcal{Y} \cap B_0^c$. Using the same argument, we can erase the parts of the walk before τ_{κ_0} that lie outside of C_1. But we need to keep all the visits to C_0 (that is all the elementary pieces of random walk in $\mathcal{G}_0$); so we get

$$\mathcal{Y} \cap B_0^c \subset \bigcup_{k \in \mathcal{G}_0} X[\tau_k, \tau_{k+1}] \cup X[\tau_{\kappa_0+1}, \tau_{k_{\max}}].$$

We now proceed by induction to complete the proof of the first part of the statement.

For the last sentence, note that the edges of Y are created by the random walk in order from v to the boundary. Therefore the set of edges created in an interval of time by the random walk has to form an interval in the

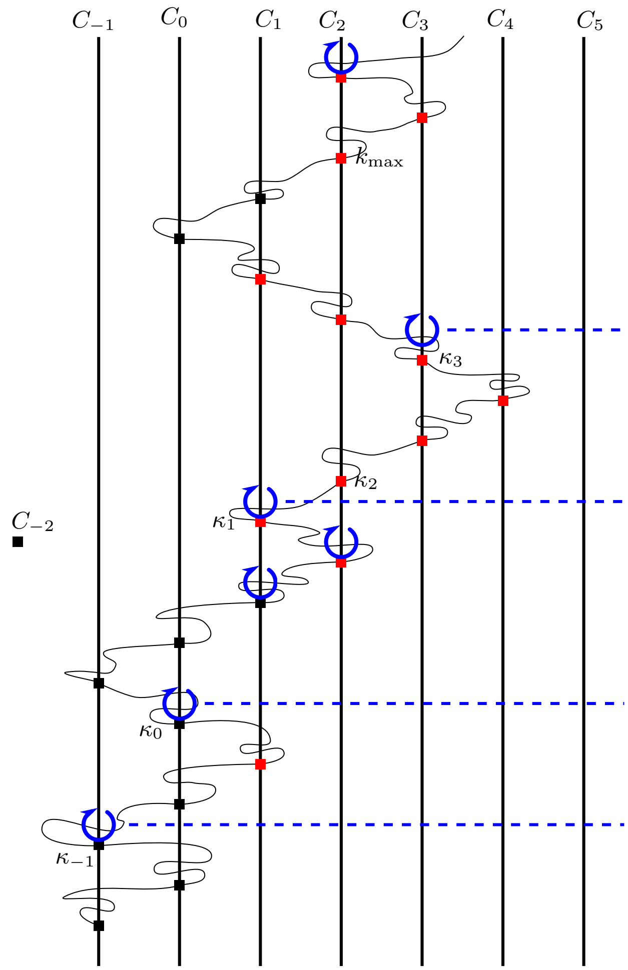

Fig 3: Schematic illustration of the proofs in Section D.2. Time runs upward. The times τ_k are indicated by the boxes, with a blue circular arrows above time τ_k indicating when $X[\tau_k, \tau_{k+1}]$ does a full turn in an annulus. The horizontal blue half lines indicate that everything below these lines is erased and ignored in the decomposition of Lemma D.4. In this example, there is no full turn at scale 2 after κ_1 so we have $\kappa_2 = \kappa_1 + 1$ as indicated. There is no crossing of C_4 after κ_3 so $\kappa_4 = \infty$ and $I = 3$, and the event pictured is exactly how we see that I has exponential tail. The red boxes denote the set $\mathcal{S}^1$ in the proof of Lemma D.5.

loop-erased walk Y. We emphasise that here we only look at the times when edges are added to Y, the full loop-erased walk, and not the loop-erased path $Y^{(t)}$ at any intermediate point of time. □

The next delicate lemma in the proof of Lemma 4.13 in [2] is the following:

LEMMA D.5 (Lemma 4.15 in [2]). *There exists $C, c, c' > 0$, such that, for all $\delta \leq ce^{-t}\operatorname{dist}(v, \partial D^{\#\delta})\delta_0$, for all $n > 0$,*

$$\mathbb{P}(\sum_{0\leq j\leq I} |\mathcal{G}_j| \geq n) \leq C\exp(-c'n).$$

PROOF. Fix a $\delta < ce^{-t}\delta_0$ with c small enough that all the estimates below are valid for such a δ. First we observe that there exist $c_1 > 0$ such that

$$\mathbb{P}(I \geq n) \leq e^{-c_1 n}. \tag{D.7}$$

Indeed, condition on the sequence of crossing positions $\mathcal{S} = (X_{\tau_k})_{k\geq 0}$. Recall that by construction $i(k_{\max} - 1) = 1$, therefore for every $i \geq 2$, just after the last crossing τ_k of C_i before $k_{\max}$ the random walk goes to C_{i-1}. In particular if there is a full turn within the annulus $A(v, r_i, r_{i+1})$ during $[\tau_k, \tau_{k+1}]$, then by definition $\kappa_{i+1} = +\infty$. By Lemma 4.4 and Corollary 4.5 in [2], the conditional probability of making such a full turn has a uniformly positive probability to occur for each i so we are done.

To proceed, we have to work on the law of $\mathcal{S}$ so it is more tricky. Note that by Corollary 4.5 in [2] again, we immediately see that the number of crossings of C_i after τ_{κ_i} has geometric tail. Indeed, conditionally on $\mathcal{S}$, the event that there are n crossings of C_i after τ_{κ_i} is the event that the last n crossings of C_i – which are measurable with respect to $\mathcal{S}$ – were not followed by a full turn in the annulus $A(v, r_i, r_{i+1})$. But a full turn in this annulus has uniformly positive probability by Corollary 4.5 in [2] at each visit, conditionally on $\mathcal{S}$. So this conditional probability has geometric tail, and so does the unconditional probability.

However $|\mathcal{G}_i|$ is the number of visits to C_i after κ_{i-1}, so we also need to exclude the possibility that the random walk alternates many times between C_i and C_{i+1} without crossing C_{i-1}. We are actually going to prove that the number of crossings of C_i between two successive crossings of C_{i-1} has an exponential tail which is uniform over i. The difficulty comes from the fact that we are looking close to random times which are not stopping times, so we don't know the law of the walk. To get around this issue we will need to be more careful and discover the set $\mathcal{S}$ in steps.

For any $j \geq 0$, let us define τ^j, i^j and $\mathcal{S}^j$ as before but using C_j as the reference smallest circle rather than C_{-1} as above. More precisely, we define $\tau_0^j = 0$, τ_1^j to be the first crossing of C_j. Then by induction, whenever $i^j(k) = j$ we define τ_{k+1}^j to be the first crossing of C_{j+1} after τ_k^j and otherwise if $i^j(k) > j$ then we set τ_{k+1}^j to be the first crossing of $C_{i^j(k)\pm 1}$ after τ_k^j. Finally we set $\mathcal{S}^j = (X(\tau_k^j))_k$. If needed, let us modify this a bit and define a sequence $\tilde{\mathcal{S}}^j$ by inserting in the sequence $\mathcal{S}^j$ the point $X(\tau_{k_{\max}-1})$ where the last crossing of C_1 happens. Let $\{\tilde{\tau}_k^j\}_{k\geq 0}$ be the times corresponding to the sequence $\tilde{\mathcal{S}}^j$.

As before, conditionally on $\tilde{\mathcal{S}}^j$ the pieces of random walk $X[\tilde{\tau}_k^j, \tilde{\tau}_{k+1}^j]$ are independent. For any k such that $i^j(k) = j$, the piece of random walk $X[\tilde{\tau}_k^j, \tilde{\tau}_{k+1}^j]$ is distributed as a random walk starting at some point on C_j and conditioned on its exit point of $B(v, r_{j+1})$. By Corollary 4.6 in [2], this random walk has a strictly positive probability to intersect C_{j-1} and this probability is independent of everything else. Therefore in $\tilde{\mathcal{S}}^j$, looking at the successive visits to C_j, the gaps between visits where the walk also crossed C_{j-1} have an uniform exponential tail, even if we condition on $\tilde{\mathcal{S}}_j$. Let $\Xi_1^j, \Xi_2^j, \dots$ be the number of such visits to C_j in between visits to C_{j-1}. Since conditioned on $\tilde{\mathcal{S}}^j$, the walks are independent, we conclude from the above discussion that $\Xi_1^j, \Xi_2^j, \dots$ are independent and have an exponential tail which is uniform and independent of everything else.

Now let $\mathcal{I}_{j-1} = \{k \in \mathcal{V}_{j-1} : k \geq \kappa_{j-1}\}$ (which has uniform geometric tail by (D.7)). Thus

$$|\mathcal{G}_j| \leq \sum_{1 \leq i \leq |\mathcal{I}_{j-1}|} \Xi_i$$

We conclude that $|\mathcal{G}_j|$ has an exponential tail which is uniform over everything else. In other words, there exist constants $c_2, C_2 > 0$ such that for all $j \geq 1$

$$\mathbb{E}(e^{c_2|\mathcal{G}_j|} | \tilde{\mathcal{S}}^j) \leq C_2 \tag{D.8}$$

Now we need to sum $|\mathcal{G}_j|$ over $1 \leq j \leq I$. Note that $\mathcal{G}_j$ are not independent so we cannot directly obtain the desired exponential tails (although stretched exponential tails follow automatically). However, there is enough independence so that we can reveal information step by step and bound the probabilities. Note that $\mathcal{S} \supseteq \tilde{\mathcal{S}}^0 \supseteq \tilde{\mathcal{S}}^1 \supseteq \dots$ and $\mathcal{G}_j$ is measurable with respect to $\tilde{\mathcal{S}}^{j-1}$. Therefore for any $m \geq 1$, discovering the $\tilde{\mathcal{S}}^j$ successively with decreasing j,

$$\mathbb{E}(e^{c_2 \sum_{1\leq j\leq m} |\mathcal{G}_j|}) = \mathbb{E}(e^{c_2 \sum_{2\leq j\leq m} |\mathcal{G}_j|} \mathbb{E}(e^{c_2|\mathcal{G}_1|}|\tilde{\mathcal{S}}^1))$$
$$\leq C_2 \mathbb{E}(e^{c_2 \sum_{2\leq j\leq m} |\mathcal{G}_j|}) \leq C_2^m$$

using (D.8) for the first inequality and the final inequality is done by iterating. Therefore

$$\mathbb{P}(\sum_{0\leq j\leq I} |\mathcal{G}_j| > n) \leq \mathbb{P}(\sum_{0\leq j\leq \varepsilon n} |\mathcal{G}_j| > n) + \mathbb{P}(I \geq \varepsilon n)$$
$$\leq (C_2)^{\varepsilon n} e^{-n} + e^{-c_1 \varepsilon n}$$

The result follows by choosing ε small enough depending only on C_2. □

E. Comparisons of capacity. In this section we prove Lemma E.6 which is Lemma 4.25 from the main file. Recall the setup: we have two domains D and $\tilde{D}$, $z_0 \in D^{\#\delta} \cap \tilde{D}^{\#\delta}$ and we couple the brances γ and $\tilde{\gamma}$ starting from z_0 for the wired USTs in $D^{\#\delta}$ and $\tilde{D}^{\#\delta}$ respectively.

Firstly, we drop the δ in this section for notational convenience. Since we are only interested in a single point, it is also convenient to drop the linear shift in the parametrisation of $\gamma, \tilde{\gamma}$ by $\log R(z_0, D), \log R(z_0, \tilde{D})$ respectively, hence we do it in this Section. As explained in the main file (and also in the following paragraph), the idea is to compare the rate of *change* of capacity in the two domains, so this global shift in parametrisation is immaterial in the argument.

For $z \in \gamma \cap \tilde{\gamma}$, call respectively T and $\tilde{T}$ the corresponding capacity in γ and $\tilde{\gamma}$ seen from their respective starting point up to z. The idea will be that T and $\tilde{T}$ are so close as a function of z that, if we sample the capacity at which we cut randomly (using a Poisson point process in practice), the law of corresponding random points along γ and $\tilde{\gamma}$ are very close in total variation and can be made to agree exactly. The technical difficulty with that strategy is that a uniform bound on $T - \tilde{T}$ is not sufficient, we need to show that "the derivative of $T - \tilde{T}$ with respect to z" is small.

We first need some technical estimates on the comparison of the two capacities. In this section D will be a fixed domain. We denote by γ and $\tilde{\gamma}$ two simple paths from a point $z_0 \in D$ to respectively ∂D and infinity and we will always assume that there exists I such that $\gamma \cap B(z_0, e^{-I}) = \tilde{\gamma} \cap B(z_0, e^{-I})$. For points $z, z' \in \gamma$ we will write $\gamma(z, z')$ for the part of the path between z and z', $\gamma(z, \partial D)$ for the path between z and the boundary and similarly for $\tilde{\gamma}$. We denote by T and $\tilde{T}$ the capacity in D or $\mathbb{C}$, i.e $T(z) = -\log R(z_0, D \setminus \gamma(z, \partial D))$ and $\tilde{T}(z) = -\log R(z_0, \mathbb{C} \setminus \tilde{\gamma}(z, \infty))$, where recall that $R(z, D)$ denotes the conformal radius of z in D.

The following lemma is elementary and well known.

LEMMA E.1. *Let D be a simply connected domain and z_0 in D. Let B_t be a Brownian motion started in z_0 and τ the exit time of D. We have*

$$\log R(z_0, D) = \mathbb{E}(\log|B_\tau - z_0|).$$

PROOF. Let g be a map sending D to $\mathbb{D}$ and z_0 to 0. Let $\phi(z) = \frac{g(z)}{z-z_0}$ for $z \neq z_0$ and $\phi(z_0) = g'(z_0)$. Then ϕ is holomorphic in D therefore $\log|\phi| = \log|g| - \log|z-z_0|$ is harmonic. Furthermore by definition $R(z_0, D) = 1/|\phi(z_0)|$ and for $z \in \partial D$, $|g(z)| = 1$ hence $\log|\phi|(z) = -\log|z - z_0|$ on ∂D.

Applying the optional stopping theorem, we deduce that

$$-\log R(z_0, D) = \mathbb{E}(-\log|B_{t\wedge\tau} - z_0|)$$

Note that τ always has polynomial tail (even if D is unbounded) and hence $\sup_{t\leq\tau} \log|B_{t\wedge\tau} - z_0|$ has exponential tail and hence is integrable. So the use of the dominated convergence theorem is justified and hence the result follows. □

LEMMA E.2. *There exist positive constants C, c such that for any pair of paths γ and $\tilde{\gamma}$ and any $z \in \gamma \cap B(z_0, e^{-I})$, we have*

$$|T(z) - \tilde{T}(z)| \leq C(T(z) + \tilde{T}(z))e^{-c(T(z)-I)}.$$

In particular, if $T - I$ is large enough, then

$$|T(z) - \tilde{T}(z)| \leq C'T(z)e^{-c(T(z)-I)}.$$

PROOF. Let B_t be a Brownian motion starting from z_0 and let τ and $\tilde{\tau}$ denote respectively the hitting time of $\partial D \cup \gamma(z, \partial D)$ and $\tilde{\gamma}(z, \infty)$. Let τ_I denote the exit time of $B(z_0, e^{-I})$.

By Lemma E.1, $T(z) - \tilde{T}(z) = \mathbb{E}_{z_0}[-\log|B_\tau - z_0| + \log|B_{\tilde{\tau}} - z_0|]$ but the right hand side is 0 whenever $\tau \leq \tau_I$. Furthermore, $d(z_0, \gamma(z, \partial D)) \leq e^{-T(z)}$ by Schwarz's lemma. Hence let z' be such that $z' \in \gamma(z, \partial D)$ and such that $|z' - z_0| \leq e^{-T(z)}$. Then by the Beurling estimate (see [5, Theorem 3.76]), $\mathbb{P}(\tau_I \geq \tau) \leq C\sqrt{|z' - z_0|/e^{-I}} \leq Ce^{-(1/2)(T-I)}$. Also by Beurling estimate, $\log|(B_\tau - z_0)|$ has an exponential tail and therefore its second moment is comparable to the square of its first moment. Hence we obtain, applying Cauchy–Schwarz,

$$\begin{aligned} &|\mathbb{E}\log|(B_\tau - z_0)/(B_{\tilde{\tau}} - z_0)|\mathbb{1}_{\tau\geq\tau_I}| \\ &\leq \sqrt{\mathbb{E}\log^2|(B_\tau - z_0)/(B_{\tilde{\tau}} - z_0)|\mathbb{P}(\tau \geq \tau_I)} \leq C(T(z) + \tilde{T}(z))e^{-c'(T(z)-I)}. \end{aligned}$$

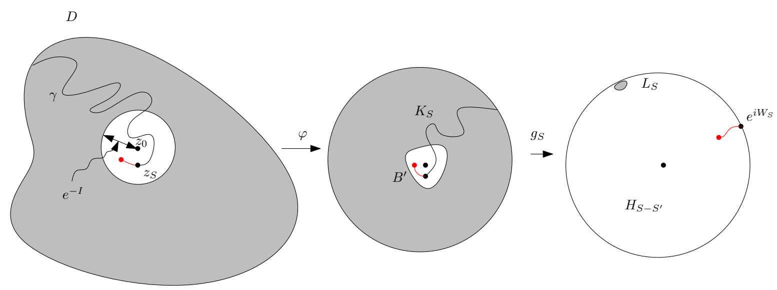

Fig 4: Sketch of the maps in the proof of Lemma E.3

This completes the proof as the second inequality is easily derivable from the first. □

Now we will compare the growth rate of the capacities of γ and $\tilde{\gamma}$. We will also need to introduce $\bar{T}(z) = -\log R(z_0; B(z_0, e^{-I})\backslash\gamma(z, \partial D))$. In other words this is the capacity of $\gamma(z, \partial D)$ within $B(z_0, e^{-I})$. The next lemma depends on the geometry of the curves so it is convenient to go to the unit disc. Let $\varphi : D \to \mathbb{D}$ be the conformal map to the unit disc sending z_0 to 0 and such that $\varphi'(z_0) > 0$.

We parametrise the curve γ by capacity: for $T \geq T_0 = -\log R(z_0, D)$, let z_T be the point on γ such that $T(z_T) = T$ and define $g_T : D\backslash\gamma(z_T, \partial D) \to \mathbb{D}$. Let $K_T = \varphi(\gamma(z_T, \partial D))$ for $T \geq T_0$. Then $(K_T)_{T\geq T_0}$ is a growing family of hulls in the unit disc. Let $(g_T)_{T\geq T_0}$ be the (radial) Loewner flow in $\mathbb{D}$ and $(W_T)_{T\geq T_0}$ be the driving function (with $W_T \in \mathbb{R}$) corresponding to $(K_T)_{T\geq T_0}$. Also set $B' = \varphi(B(z_0, e^{-I}))$, $H_T = \mathbb{D} \setminus K_T$ and finally set $L_T = g_T(H_T \setminus B')$.

Lemma E.3. *There exist positive constants A, a, ε_0 such that the following holds. Let S and S' be fixed with $0 < S' - S < \varepsilon_0$. Let*

$$d = \inf_{S\leq T\leq S'} d(L_T, e^{iW_T}).$$

Let $w = z_S$ and $w' = z_{S'}$. If $S - I$ is large enough and if d is sufficiently large so that both $d \geq Ae^{-\frac{a}{4}(S-I)}$ and $d^2 \geq A(S' - S)$ hold, then

$$1 - Ae^{-a(S-I)}/d^2 < \frac{\bar{T}(w') - \bar{T}(w)}{S' - S} < 1 + Ae^{-a(S-I)}/d^2.$$

Proof. Let ρ be the image of $\gamma(w', w)$ under $g_S \circ \varphi$, i.e, the red curve in the rightmost image of Figure 4. First we note that by definition

$$\bar{T}(w) - S = -\log R(0, \mathbb{D} \setminus L_S), \tag{E.1}$$

$$\bar{T}(w') - S = -\log R(0, \mathbb{D} \setminus (L_S \cup \rho)), \tag{E.2}$$

$$S' - S = -\log R(0, \mathbb{D} \setminus \rho). \tag{E.3}$$

Let τ_ρ, τ_L and τ_∂ denote the hitting time of respectively ρ, L_S and $\partial\mathbb{D}$ by a Brownian motion B starting from zero. Combining (E.2), (E.1), (E.3) and Lemma E.1, we have

$$\bar{T}(w') - \bar{T}(w) - (S' - S) = \mathbb{E}^0\left[-\log|B_{\tau_\partial \wedge \tau_L \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_L}|\right]. \tag{E.4}$$

Hence it suffices to show that

$$|\mathbb{E}^0\left[-\log|B_{\tau_\partial \wedge \tau_L \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_L}|\right]| \leq \frac{Ce^{-c(S-I)}}{d^2}(S' - S). \tag{E.5}$$

Notice that if $\tau_\partial < \tau_L \wedge \tau_\rho$ then the random variable in the right hand side of (E.4) is zero. We will consider separately the two cases where τ_L is smallest and also τ_ρ is smallest below in Steps 2 and 3 respectively, but in step 1, we establish some geometric estimates on L_S and ρ.

Step 1. First we prove that the distance between L_S and ρ is at least $d/10$ for small enough $S' - S$. From the choice of d, we can draw an arc I_ℓ to the left of the leftmost point in $L_S \cap \partial D$ and I_r to the right of the rightmost point in $L_S \cap \partial D$ both of length $d/4$ (and hence they do not intersect e^{iW_S}). Let $b \in L_S \cap \partial\mathbb{D}$ and let a', c be the two extremities of the arcs I_ℓ, I_r which are farthest from L_S. Using the radial Loewner equation applied to $h_t = g_{S+t} \circ g_S^{-1}$, at $z = b$ for now, we see that

$$|\partial_t h_t(b)| = |h_t(b)\frac{e^{iW_{S+t}} + h_t(b)}{e^{iW_{S+t}} - h_t(b)}| \leq \frac{2}{d} \tag{E.6}$$

for all $t \leq S' - S$ (this is because over this interval of time, we must have $d(h_t(b), e^{iW_{t+S}}) \geq d$ and hence the denominator in the right hand side is greater than d). Applying the same Loewner equation at $z = a'$ and $z = c$, so long as $d(h_t(a'), e^{iW_{t+S}}) \geq d/2$, and so long as that $t \leq S' - S$, we get $|\partial_t h_t(a')| \leq 4/d$. Hence $|\partial_t h_t(a') - \partial_t h_t(b)| \leq 6/d$ on that interval. Combining with the fact that $h_t(b)$ is at least d away from $e^{iW_{t+S}}$ and eq. (E.6), we deduce that the time it would take for $h_t(a')$ to be less than $d/2$ away from $e^{iW_{t+S}}$ is at least $d^2/24$. Hence if $S' - S \leq d^2/C$ for some sufficiently large constant $C > 24$, then the condition $d(h_t(a'), e^{iW_{t+S}}) \geq d/2$ is in fact always fulfilled throughout $t \in [0, S' - S]$. Observe that as $S' - S \to 0$, $d \to$

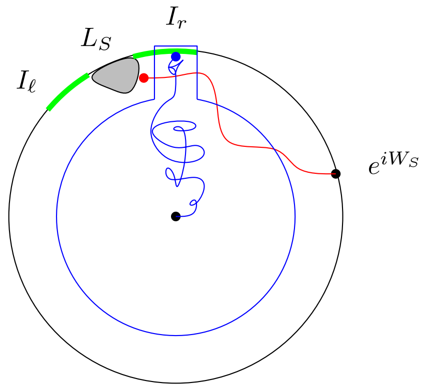

Fig 5: Proof of step 1.

$d(L_S, e^{iW_S}) > 0$ and hence we can always make such a choice. Consequently, we deduce that a bound similar to eq. (E.6) holds at $z = a'$ and $z = c$ with a different constant: for $t \le S' - S$,

$$|\partial_t h_t(z)| = |h_t(z)\frac{e^{iW_{S+t}} + h_t(z)}{e^{iW_{S+t}} - h_t(z)}| \le \frac{4}{d}, \quad z = a', c. \tag{E.7}$$

Applying the above bound for the extremities of both the arcs I_ℓ, I_r and integrating this bound over $[0, S'-S]$ we see that $h_{S'-S}(I_\ell)$ and $h_{S'-S}(I_r)$ are arcs whose length is $(d/4)(1+O(1/C))$. Hence this can be made arbitrarily close to $d/4$. Note that the lengths of $h_{S'-S}(I_\ell)$ and $h_{S'-S}(I_r)$ are nothing else but the harmonic measures seen from 0 in $\mathbb{D} \setminus \rho$ of I_ℓ, I_r respectively.

From this we can conclude that ρ does not come within distance $d/10$ of L_S. Indeed, if it did, the harmonic measures of I_r or I_ℓ would be small. More precisely, there is a universal constant $c > 0$ such that conditionally on $B_{\tau_\partial} \in I_r$ say, B crosses any set coming within distance $d/10$ of L_S with probability at least c: for instance this necessarily happens if B stays in a certain deterministic set consisting of a union of a rectangle of dimensions $d \times (d/20)$ and a disc of radius $1 - d/2$ (see blue set in the accompanying Figure 5). This would imply that the harmonic measure of I_ℓ or I_r in $\mathbb{D} \setminus \rho$ is at most $d/4 - c$ which is a contradiction to the fact that they can be made arbitrarily close to $d/4$ by choosing $S' - S$ small enough. Note also, for future reference, that $h_{S'-S}(I_\ell)$ and $h_{S'-S}(I_r)$ are arcs of length at least $d/8$ and which don't intersect $h(\rho)$ (since ρ does not hit $\partial\mathbb{D}$).

Step 2. Now we estimate the right hand side of (E.4), which was

$$\mathbb{E}^0\left[-\log|B_{\tau_\partial \wedge \tau_L \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_L}|\right],$$

in the case where the Brownian motion hits ρ before L or $\partial\mathbb{D}$. In this case the first and second terms in the random variable above cancel each other and we need to estimate $\mathbb{E}^0(1_{\{\tau_\rho<\tau_\partial\wedge\tau_L\}}\mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial\wedge\tau_L}|))$. Let $\mathcal{C}$ denote the circle of radius $d/10$ centered at B_{τ_ρ}. Notice that by Step 1, since ρ does not come within distance $d/10$ of L_S,

$$|\mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial\wedge\tau_L}|))| \le |\log(1-\mathrm{Diam}(L_S))|\mathbb{P}^{B_{\tau_\rho}}(\tau_{\mathcal{C}}<\tau_\partial) \tag{E.8}$$

where $\tau_{\mathcal{C}}$ is the stopping time when the Brownian motion hits $\mathcal{C}$. Now, observe that

$$\mathrm{Diam}(L_S) \le C\ \mathrm{Harm}_{\mathbb{D}\setminus L_S}(0;L_S) \le Ce^{-C'(S-I)} \tag{E.9}$$

by Beurling's estimate, where $\mathrm{Harm}_D(z,\cdot)$ denotes harmonic measure in D seen from z. Hence

$$\mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial\wedge\tau_L}|)) \le Ce^{-c'(S-I)}\mathbb{P}^{B_{\tau_\rho}}(\tau_{\mathcal{C}}<\tau_\partial)$$

using $|\log(1-x)|=O(x)$. Now it remains to bound $\mathbb{P}^{B_{\tau_\rho}}(\tau_{\mathcal{C}}<\tau_\partial)$ from above. Let $z=B_{\tau_\rho}$. Then it is elementary to check that we have $\mathbb{P}^z(\tau_{\mathcal{C}}<\tau_\partial)\le c(1-|z|)/d$ for some constant $c>0$ (this is the probability of reaching distance $d/10$ within a half-plane from $i(1-|z|)$: this can be seen by applying the map $z\mapsto z^2$ and Beurling's estimate). Plugging this back into (E.8), we conclude:

$$|\mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial\wedge\tau_L}|))| \le Ce^{-c'(S-I)}\frac{1-|B_{\tau_\rho}|}{d} \le \frac{Ce^{-c'(S-I)}}{d}|\log|B_{\tau_\rho}||.$$

Taking expectations with respect to $\mathbb{E}^0$ on the event $\tau_\rho<\tau_\partial\wedge\tau_L$, we deduce that

$$\begin{aligned}|\mathbb{E}^0[1_{\tau_\rho<\tau_\partial\wedge\tau_L}\mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial\wedge\tau_L}|))]| &\le \frac{Ce^{-c'(S-I)}}{d}\mathbb{E}^0|\log|B_{\tau_\rho}|| \\ &= \frac{Ce^{-c'(S-I)}}{d}(S'-S),\end{aligned}$$

which is slightly better than the bound $(Ce^{-c'(S-I)}/d^2)(S'-S)$ we are aiming for.

Step 3. Finally we consider the case where $\tau_L<\tau_\partial\wedge\tau_\rho$, which is the most delicate. If the Brownian motion hits L first then the first and third terms cancel each other and we need to estimate $\mathbb{E}^0(\mathbb{E}^{B_{\tau_L}}(\log|B_{\tau_\partial\wedge\tau_\rho}|)1_{\tau_L<\tau_\partial\wedge\tau_\rho})$ using the strong Markov property. We now claim that, almost surely,

$$\Big|\frac{\mathbb{E}^{B_{\tau_L}}(\log|B_{\tau_\partial\wedge\tau_\rho}|)}{\mathbb{E}^0(\log|B_{\tau_\partial\wedge\tau_\rho}|)}\Big| \le \frac{C}{d^2}e^{-c'(S-I)} \tag{E.10}$$

To do this we are going use conformal invariance of harmonic measure and map out ρ via $h_{S'-S}$. Then we can use an explicit bound on the Poisson kernel on the disc which allows us to compare the harmonic measure of a set seen from 0 and from a point in L_S. By conformal invariance of harmonic measure, letting $h = h_{S'-S}$

$$\begin{aligned}\mathbb{E}^{B_{\tau_L}}(-\log|B_{\tau_\partial \wedge \tau_\rho}|) &= -\int_\rho \log|z| \;\operatorname{Harm}_{\mathbb{D}\setminus\rho}(B_{\tau_L}, dz)\\ &= -\int_{h(\rho)} \log|h^{-1}(z)| \operatorname{Harm}_{\mathbb{D}}(h(B_{\tau_L}), dz)\\ &= -\int_{h(\rho)} \log|h^{-1}(z)| \frac{d\operatorname{Harm}_{\mathbb{D}}(h(B_{\tau_L}), dz)}{d\operatorname{Harm}_{\mathbb{D}}(0, dz)} \operatorname{Harm}_{\mathbb{D}}(0, dz)\end{aligned}$$

Observe that if we write $h(B_{\tau_L}) = re^{i\theta} \in L_{S'}$, then the Radon-Nikodym derivative above is simply the Poisson kernel and at the point $z = e^{it}$ it is thus equal to $P_r(\theta - t)$ where

$$P_r(\theta) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

Note that when $e^{it} \in h(\rho)$ then $|\theta - t| \geq d/100$ by Step 1. Indeed, we know that $h(I_\ell)$ and $h(I_r)$ are arcs of length at least $d/8$ which do not intersect $h(\rho)$ (see the end remark of Step 1) and we know that the diameter of $L_{S'}$ is at most $Ce^{-C'(S-I)} \leq d/10^3$ by assumption for a choice of constants $A > C, a < C'$ and large enough $S - I$. Hence the difference in arguments for a point in $L_{S'}$ and a point in $h(\rho)$ is at least $d/100$. Thus we have $\cos(\theta - t) \leq 1 - d^2/10^5$. Hence

$$P_r(\theta - t) \leq \frac{1 - r^2}{(1-r)^2 + 2rd^2/10^5} \leq \frac{2Ce^{-C'(S-I)}}{d^2}$$

by our assumptions on d, the above choice of constants a, A and the fact that

$$1 - r \leq \operatorname{Diam}(L_{S'}) \leq Ce^{-C'(S'-I)} \leq Ce^{-C'(S-I)}$$

by eq. (E.9). Since $\mathbb{E}^0(\log|B_{\tau_\partial \wedge \tau_\rho}|) = S' - S$, putting everything together we obtain Lemma E.3. □

Taking the increment $S' - S$ to 0 we get the following corollary:

LEMMA E.4. *Let a, A be as in Lemma E.3. Let*

$$d = \inf_{S \leq T \leq S+1/10} d(L_T, e^{iW_T}).$$

Let w and w' be the two points on γ corresponding respectively to S and $S+1/10$. Let μ and $\bar{\mu}$ be the measures on γ obtained by capacity in D and $B(z_0, e^{-I})$ respectively: that is, if $z = z_T$ and $z' = z_{T'}$ then $\mu(\gamma(z, z')) = |T' - T|$, and likewise for $\bar{\mu}$. If $S - I$ is large enough and if $d \geq Ae^{-\frac{a}{4}(S-I)}$, then, uniformly over $\gamma(w, w')$, we have

$$1 - Ae^{-a(S-I)}/d^2 \leq \frac{d\bar{\mu}}{d\mu} \leq 1 + Ae^{-a(S-I)}/d^2.$$

Now we check that the assumption on d in the previous lemma holds when the curve is SLE_2 with high probability.

LEMMA E.5. *Suppose γ is a radial SLE_2 process independent of I. Define d for γ as in Lemma E.4. For any $a, A > 0$ we have*

$$d \geq Ae^{-a(S-I)} \text{ with probability } \geq 1 - c\exp(-c'(S-I)). \tag{E.11}$$

for some $c, c' > 0$ depending only on A, a. Now suppose γ is a loop-erased random walk in $D^{\#\delta}$ from z_0 to ∂D and I as defined in Theorem 4.21 in the main file (applied for a single point). We emphasise that I might not be independent of γ. Let d be as above. Then there exists $\delta = \delta(S)$ such that conditioned on I, if $S - I$ is large enough and $\delta \leq \delta(S)$, for any $a, A > 0$, we have

$$d \geq Ae^{-a(S-I)} \text{ with probability } \geq 1 - \frac{c}{2}\exp(-c'(S-I)). \tag{E.12}$$

PROOF. Let T be the smallest t such that $\gamma(z(t)) \in B(z_0, e^{-I})$, and let $e^{i\theta_t}$ be one of the two images $g_t \circ \varphi(z)$ for $T \leq t \leq S$ where $z = z(T)$. Then note that $e^{i\theta_S} \in \partial\mathbb{D} \cap L_S$. Moreover we claim that with high probability, if $S - I$ is large, then $e^{i\theta_S}$ is far away from e^{iW_S}.

We can follow the evolution of $e^{i\theta_t}$ under the Loewner flow g_t for $T \leq t \leq S$ as follows. If $Y_t = \theta_t - W_t$ then Y solves a stochastic differential equation:

$$dY_t = \cot(Y_t/2) - dW_t \; ; \; Y_T = 0.$$

(see (6.13) in [5]). By comparing with a Bessel process (of dimension 5 in the case $\kappa = 2$), we can see that the probability for Y_t to hit $[0, \delta]$ in any particular interval of length $O(1)$ is polynomial in δ. Thus for any given S such that $S - I$ is large enough, then for any c, C,

$$\mathbb{P}\left(\min_{S \leq t \leq S+1/10} Y_t \leq 10e^{-c(S-I)}\right) \leq C' \exp(-c'(S-I))$$

is exponentially small in $S-I$. Now, since $\mathrm{Diam}(L_S) \leq Ce^{-c(S-I)}$ by Beurling's estimate (see eq. (E.9)) we see that

$$d \geq Ce^{-C(S-I)} \text{ with probability } \geq 1 - C' \exp(-c'(S-I)).$$

This proves eq. (E.11).

To deduce eq. (E.12), we recall that loop-erased random walk converges to SLE_2 under our assumptions by [15]. Then use Remark 4.24 in the main file to note that conditionally on $I = i$, the law of γ after the first time it hits $B(z_0, e^{-I})$ is absolutely continuous with respect to the unconditional law $(\gamma_t, t \geq T_i)$ (where T_i is the first time the path enters $B(z_0, e^{-i})$). Also the derivative of the conditional law with respect to the unconditional law is bounded by $C > 0$. Since the distance d is a continuous function (in the mesh size δ) of γ the estimate eq. (E.11) applies with $\kappa = 2$ for all δ sufficiently small and $S - I$ large enough. □

Now we can finally prove our coupling estimate. Recall the full coupling $(\gamma, \tilde{\gamma})$ (Theorem 4.21 with a single point), where γ is a loop-erased random walk in $D^{\#\delta}$ and $\tilde{\gamma}$ is a loop-erased random walk in $\tilde{D}^{\#\delta}$ starting from a vertex v where $\tilde{D}$ is arbitrary. It is recommended to think of $\tilde{D}$ as the full plane.

Lemma E.6. *There exists a universal constant c such that the following holds. For any $t > 0$ there exists $\delta = \delta(t)$ such that for any $\delta \in (0, \delta(t))$, we can find a pair of random variables $(X, \tilde{X})$ such that individually, X and $\tilde{X}$ are each independent of $(\gamma, \tilde{\gamma})$ and*

$$\mathbb{P}[\gamma(t + X) = \tilde{\gamma}(t + \tilde{X})] \geq 1 - Ce^{-ct}.$$

Furthermore, both X and $\tilde{X}$ are random variables which are bounded (by $1/20$).

In this proof, we will need to choose constants depending on each other so we number them.

Proof. Let μ be the measure on γ defined as in Lemma E.4 and let $\tilde{\mu}$ be the equivalent measure for the curve $\tilde{\gamma}$. Note that Lemma E.4 allows us to compare μ to $\bar{\mu}$ and also $\tilde{\mu}$ to $\bar{\mu}$ since γ and $\tilde{\gamma}$ are assumed to coincide within $B(z_0, e^{-I})$. Consequently, we will have a way of comparing μ and $\tilde{\mu}$ and show that the Radon-Nikodym derivative of one with respect to the other is very close to one, from which the result will follow.

Here are the details. Let $z = z(t)$ be the point of capacity t seen from v in D for γ. Let $\tilde{T}(z)$ be the capacity of $\tilde{\gamma}$ at z seen from v in $\tilde{D}$. Let $\mathcal{A}$ be the event that $I \leq t/2$, $|\tilde{T}(z(t)) - t| \leq e^{-c_1 t}$ and both $\gamma, \tilde{\gamma}$ do not exit $B(z_0, e^{-t/2})$ after capacity t. Recall that I has exponential tail (Theorem 4.21 in the main file) and for small enough $\delta = \delta(t)$, we see from Lemma E.2 and Schramm's estimate (Theorem 3.1 in the main file) that both $\gamma, \tilde{\gamma}$ do not exit $B(z_0, e^{-I})$ after capacity t with exponentially high probability for large enough t. Thus overall, the probability of $\mathcal{A}$ is at least $1 - e^{-c_2 t}$ for some $c_2 > 0$ for small enough $\delta(t)$.

Let d, a, A be as in Lemma E.4 for γ, D and let $\tilde{d}, \tilde{a}, \tilde{A}$ be the equivalent quantities for $\tilde{\gamma}, \tilde{D}$. By eq. (E.12), we observe that the event $\mathcal{B} := \{d > Ae^{-a/4t}, \tilde{d} > \tilde{A}e^{-\tilde{a}/4t}\}$ has probability at least $1 - e^{-c_3 t}$ for some $c_3 > 0$.

Applying Lemma E.4, we find that for t sufficiently large on $\mathcal{A} \cap \mathcal{B}$,

$$1 - e^{-c_4 t} \leq \frac{d\mu}{d\tilde{\mu}} \leq 1 + e^{-c_4 t} \tag{E.13}$$

on a subset of the path including $\Gamma := \tilde{\gamma}(t + e^{-c_1 t}, t - e^{-c_1 t} + \frac{1}{20})$ for some small enough $\delta(t)$. We let $\nu = (1 - e^{-c_4 t})1_\Gamma \tilde{\mu}$. Now we choose $0 < c_5 < c_4$ and we define three independent Poisson processes on γ, say P_1, P_2, P_3, with densities respectively $e^{c_5 t}\nu$, $e^{c_5 t}(\tilde{\mu} - \nu)$ and $e^{c_5 t}(\mu - \nu)$. Note also that the processes P_1 and P_2 depend only on $\tilde{\gamma}$.

Let $\tilde{z}$ be the first point in $P_1 \cup P_2$ after capacity t in $\tilde{D}$ and let z be the first point in $P_1 \cup P_3$ after capacity t in D. Let $X = T(z) - t$ and $\tilde{X} = T(\tilde{z}) - t$; if there are no points in $P_1 \cup P_3$ or $P_1 \cup P_2$ we abort the coupling (this has probability smaller than $e^{-\exp(c_5 t/30)}$). Note that $X, \tilde{X} \leq 1/20$ by construction and they are each independent of $(\gamma, \tilde{\gamma})$ (since the conditional marginal law of both X and $\tilde{X}$ is exponential; however, note that the pair $(X, \tilde{X})$ is *not* independent from $(\gamma, \tilde{\gamma})$). Now observe that with very high probability both $z, \tilde{z} \in P_1$ and therefore are equal. Indeed, $\mathbb{P}(\tilde{z} \notin P_1) \leq e^{(c_5 - c_4)t}$ and similarly $\mathbb{P}(z \notin P_1) \leq e^{(c_5 - c_4)t}$ using eq. (E.13) which concludes the proof. □

F. General domains. In this section we state the following theorem which is the main result of Section 3.5 of the main file [2]. Recall that our definition of $u_{(D,x)}$ from the main file.

THEOREM F.1. *Let D be as above. Let f be any bounded Borel test function defined on $\bar{D}$. Let $h = h^0_{\mathrm{GFF}} + \chi u_{(D,x)}$ be the GFF coupled to the UST according to the imaginary geometry coupling of Theorem C.1 and*

$u_{(D,x)}$ is as in the main file. Then (h_t^D, f) converges to (h_{GFF}^D, f) in $L^2(\mathbb{P})$ and in probability as $t \to \infty$, where $h_{\text{GFF}}^D = \chi^{-1} h + \pi/2$.

PROOF. Let h_t^D be the winding field in D as defined as in (3.1) or (3.2) from the main file as appropriate. Let $\mathcal{T}^D$ denote the continuum wired UST in D. Fix a conformal map $\varphi : \mathbb{D} \to D$ sending the marked point 1 to the marked point $x \in \partial D$ (note that since ∂D is locally connected, any conformal map from $\mathbb{D}$ to D extends continuously to ∂D, see [11]). Let $h^{\mathbb{D}}$ denote the intrinsic winding field associated to the continuum spanning tree $\mathcal{T} = \varphi^{-1}(\mathcal{T}^D)$.

Let $\tilde{\gamma}_w(t)$ denote the branch of the UST towards w in $\mathcal{T}^D$. Let $\mathcal{A}^D(t,w)$ be the event that $|\tilde{\gamma}_w(t) - w| < e^{-t/2} R(w, D)$. Applying Koebe's 1/4 theorem, we conclude using Theorem 2.11 from the main file that $\mathbb{P}(\mathcal{A}^D(t,w)) \geq 1 - e^{-ct}$.

First note that in the case of a smooth domain, using Lemma 2.4 from the main file (we can apply this since we are dealing with conformal images of continuous curves in $\mathbb{D}$ and we can also extend $\arg \varphi'$ to 1 using Lemma B.8), we conclude that (writing $w = \varphi(z)$),

$$h_t^D \circ \varphi(z) 1_{\mathcal{A}^D(t,w)} = \Big(h_t^{\mathbb{D}}(z) + \arg_{\varphi'(\mathbb{D})}(\varphi'(z)) + \epsilon(z) \Big) 1_{\mathcal{A}^D(t,w)} \tag{F.1}$$

where $|\epsilon(z)| < e^{-ct}$. In the case of general boundary, using Lemma 2.4 from the main file, we get the same equation (F.1) up to a global constant. Using this and (3.11) from the main file [2], we immediately conclude that

$$\mathbb{E}((h_t^D \circ \varphi(z) - \arg_{\varphi'(\mathbb{D})}(\varphi'(z)))^2 1_{\mathcal{A}^D(t,w)}) \leq C(1+t).$$

We now claim that the same estimate also holds on the complement of $\mathcal{A}^D(t,w)$, which will follow from Lemma 2.10 of the main file. Indeed, note that even when $\mathcal{A}^D(t,w)$ does not hold, the curve $\tilde{\gamma}_w[-1,t]$ must come within distance $e^{-t/2} R(w,D)$ of w. So by Koebe's 1/4 theorem there is a (random) time t' satisfying $t/2 - \log 4 < t' < t/2$ so that $|\tilde{\gamma}_w(t') - w| < e^{-t/2} R(w,D)$. We then simply apply Lemma 2.4 of the main file at t' instead of t. More precisely, we write

$$h_t^D \circ \varphi(z) = h_{t'}^D \circ \varphi(z) + W(\tilde{\gamma}_w[t',t]; w). \tag{F.2}$$

Using eq. (F.1) and applying Lemma 2.10 of [2] to $h_{t'}^{\mathbb{D}}$ in $\mathbb{D}$ we see that the second moment of $h_{t'}^D \circ \varphi(z) - \arg_{\varphi'(\mathbb{D})}(\varphi'(z))$ is bounded by $C(1+t)^2$. Moreover the second moment of $W(\tilde{\gamma}_w[t',t]; w)$ is bounded by $C(1+t-t')^2$ using Lemma 2.10 of [2] again, but this time in D.

Thus overall, we get

$$\mathbb{E}\Big((h_t^D \circ \varphi(z) - \arg_{\varphi'(\mathbb{D})}(\varphi'(z)))^2\Big) \le C(1+t) \tag{F.3}$$

Also in the general case, (F.3) is valid up to a global constant (meaning that for every choice of constant in the left hand side, the inequality holds for an appropriate choice of C in the right hand side). Now note that

$$\mathbb{E}(h_t^D\circ\varphi - h_{\mathrm{GFF}}^D\circ\varphi, f\circ\varphi)^2 \le 2\mathbb{E}(h_t^D\circ\varphi - h_t^{\mathbb{D}} - \arg_{\varphi'(\mathbb{D})}\varphi', f\circ\varphi)^2 + 2\mathbb{E}(h_t^{\mathbb{D}} - h_{\mathrm{GFF}}^{\mathbb{D}}, f\circ\varphi)^2$$

The second term on the right hand side above converges to 0 via Theorem 3.13 of the main file. The first term can be written as

$$\mathbb{E}(h_t^D \circ \varphi - h_t^{\mathbb{D}} - \arg_{\varphi'(\mathbb{D})}\varphi', f\circ\varphi)^2 = \int_{\mathbb{D}^2} dzdy f\circ\varphi(z) f\circ\varphi(y)$$

$$\times\, \mathbb{E}(h_t^D\circ\varphi(z) - h_t^{\mathbb{D}}(z) - \arg_{\varphi'(\mathbb{D})}\varphi'(z))(h_t^D\circ\varphi(y) - h_t^{\mathbb{D}}(y) - \arg_{\varphi'(\mathbb{D})}\varphi'(y))$$

$$\le \int_{\mathbb{D}^2} dzdy f\circ\varphi(z) f\circ\varphi(y)\big[\mathbb{E}(\epsilon(z)\epsilon(y)1_{\mathcal{A}^D(t,\varphi(z))\cap\mathcal{A}^D(t,\varphi(y))}) + c(1+t)e^{-ct}\big]$$

where $|\epsilon(z)| \le e^{-ct}$ by (F.1). The result follows. □

REMARK F.2. We point out that the integral of $\arg_{\varphi'(\mathbb{D})}(\varphi'(\cdot))$ might be infinite even in the smooth case. Also for the joint convergence of the moments, we do not need the domain to be bounded.

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