

Cycle structure of the interchange process and representation theory

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Abstract

Consider the process of random transpositions on the complete graph K_n . We use representation theory to give an exact, simple formula for the expected number of cycles of size k at time t , in terms of an incomplete Beta function. Using this we show that the expected number of cycles of size k jumps from 0 to its equilibrium value, $1/k$, at the time where the giant component of the associated random graph first exceeds k . Consequently we deduce a new and simple proof of Schramm’s theorem on random transpositions, that giant cycles emerge at the same time as the giant component in the random graph. We also calculate the “window” for this transition and find that it is quite thin. Finally, we give a new proof of a result by the first author and Durrett that the random transposition process exhibits a certain slowdown transition. The proof makes use of a recent formula for the character decomposition of the number of cycles of a given size in a permutation, and the Frobenius formula for the character ratios.

1 Introduction and main results

Consider the complete graph K_n on n vertices, and let σ_t be the random walk on S_n that results when considering the interchange process on K_n . That is, σ_t is the usual random transposition process (see e.g. [14]) sped up by a factor $\binom{n}{2}$.

Our first result gives an exact and surprisingly simple formula for the expected number of cycles of size k at time t .

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Theorem 1. Fix any $1 \leq k \leq n$ and let $s_k(t)$ be the number of cycles of size k at time t in σ_t . Then

$$\mathbb{E}(s_k(t)) = \binom{n}{k} \left[\frac{1}{k} x \phi(x) + \int_x^1 \phi(y) dy \right],$$

where $\phi(y) = y^{n-k}(1-y)^{k-1}$ and $x = e^{-tk}$.

This integral is known as the incomplete beta function (the integral from 0 to 1 is the regular beta function). The proof of Theorem 1 is based on representation theory. The key argument is a formula of Gil Alon and one of us [1] for the character decomposition of the number of cycles of a permutation, as well as Frobenius' formula for the values of the character ratios.

Our second result uses the above formula to show that in the limit as $n \rightarrow \infty$, the quantity $\mathbb{E}(s_k(t))$ exhibits a sharp transition from the value 0 to $1/k$ at a time $t_{n,k}$ which is essentially $(-1/k) \log(1 - k/n)$. We note that this time is an order magnitude smaller than the mixing time for σ_t which (with this parametrization) is $(\log n)/n$ (see [7] or [4]). The width of this transition is shown to be order $1/n^{3/2}$ when k is of order n (which corresponds to a width of order \sqrt{n} in the traditional scaling of random transpositions). This is reminiscent of the cutoff phenomenon for the mixing time, see [7], [12] or [11, §18] for the cutoff phenomenon.

Theorem 2. Let $1 \leq k < n$ and let $t_{n,k}$ be the unique t such that $e^{-kt} = (n-k)/(n-1)$. Then

$$\left| \mathbb{E}(s_k(t)) - \frac{1}{k} \mathbf{1}_{\{t > t_{n,k}\}} \right| \leq Cq \exp \left\{ -c(n-k) \min\{|t - t_{n,k}|^2 k^2, 1\} \right\}$$

where $q = q_n(k)$ is a polynomial factor, $q = n^{3/2} k^{-3/2} (n-k)^{-1/2}$.

(Here and below c and C pertain to unspecified positive universal constants, possibly different from one place to another).

At first sight it might seem as if Theorem 2 cannot possibly hold. After all, a large cycle of σ_t (say of size $\frac{1}{3}n$) is necessarily contained in the giant component of a corresponding Erdős-Rényi graph $G(n, t)$ with edge density t (see e.g. [14]). It is easy to check that $t_{n,k}$ is the first time that the giant component has a relative size which exceeds $1/3$, so it is clear that $\mathbb{E}(s_k(t))$ must be close to 0 before $t_{n,k}$. What Theorem 2 says is that $\mathbb{E}(s_k(t))$ suddenly reaches its equilibrium value precisely at that time, and does not change afterward. Note however that as t increases above $t_{n,k}$, the cluster continues to grow. (See Figure 1). So how come the probability for

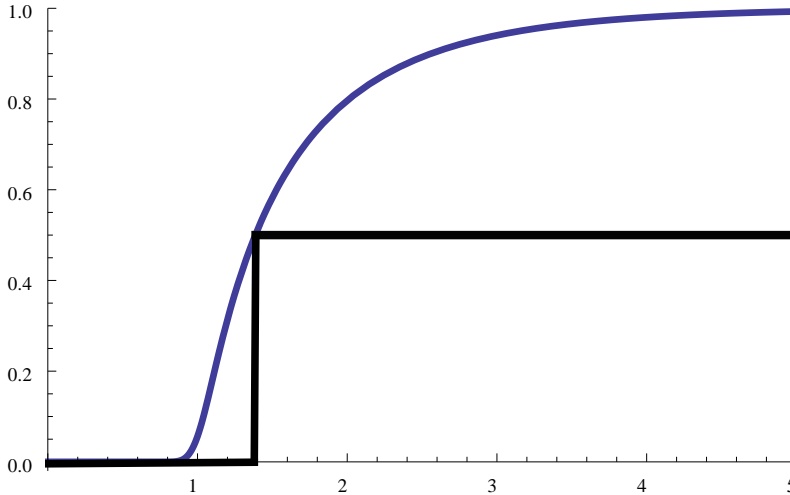


Figure 1: Approximate plots of $(n^2/k)\mathbb{E}(s_k(t/n))$, together with the relative size of the giant component, for $n = 200$ and $k = 100$. The scaling factor in front is chosen so that if $k/n \rightarrow \alpha$ then the limiting step function takes the values 0 and α .

a cycle of size exactly $\frac{1}{3}n$ does not grow with it? The answer lies in examining a completely random permutation. The expected number of cycles of length k in a random permutation of n elements is exactly $1/k$ and hence *does not depend on n* , of course, as long as $n \geq k$. This explains why the size of the giant component does not affect $\mathbb{E}(s_k)$, as long as it is bigger than k .

The proof of Theorem 2 follows by estimating the terms in Theorem 1 carefully. Let us just remark how the phase transition comes about. The integrand in Theorem 1 (i.e. ϕ) has a peak at exactly the critical value $t_{n,k}$. So in the subcritical case (i.e. t small so x large), $\int_x^1 \phi$ can be estimated simply by $\phi(x)$, which is small. In the supercritical case (i.e. x small), the integral is estimated by writing it as $\int_0^1 - \int_0^x$. The integral from 0 to 1 is easy to calculate, and gives the $1/k$ term, and the integral from 0 to x is the error term.

From this we can derive a new proof of the result, first proved by Schramm [14], that giant cycles (of macroscopic size) emerge at time $1/n$ with high probability asymptotically as $n \rightarrow \infty$.

Theorem 3. *Let $c > 1$ and let $t = c/n$. Let $C(t)$ denote the length of the largest cycle of σ_t . Then as $\varepsilon \rightarrow 0$,*

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(C(t) > n\varepsilon) = 1.$$

Schramm's results are stronger and also give information about the *joint* distribution of cycles, e.g. what is the probability that the largest cycle is in $[0.6n, 0.7n]$ and the second largest is in $[0.1n, 0.2n]$. We will not discuss the joint distribution in this paper.

Theorem 3 is the mean-field case of a well-known conjecture of Bálint Tóth [15] that the cycle structure of the interchange process on the graph \mathbb{Z}^d , $d \geq 3$, exhibits a phase-transition with infinite cycles appearing at some finite time (see e.g. [15] for definitions). This paper was initially motivated by the desire to see if it would be possible to use a representation-theoretic approach to this conjecture. Our proof of Theorem 3 can be seen as the mean-field case of this programme (see also Theorem 1 of [2] for a possible alternative route, which yields a slightly weaker conclusion). Let us remark that the recent resolution of Aldous' spectral gap conjecture [5] could aid one in an algebraic attack on the non-mean-field case.

Interestingly, the asymptotics leading to the transition in Theorems 2 and 3 do *not* come from the contribution of a single representation of S_n with all other contributions being negligible. Rather, it is the result of some very remarkable cancelations when summing up over all representations of S_n . These cancelations lead to the simple formula of Theorem 1. Here it is essential to have an exact computation that keeps track of *all* representation in order to establish the emergence of giant cycles of Theorem 3. Nevertheless, if one wants only weaker results, e.g. that whenever $t > 100t_{n,k}$ we already have $|\mathbb{E}(s_k(t)) - 1/k|$ exponentially small, then in this case a single representation (the trivial one, $[n]$) gives the main contribution, with the contribution of all the others combined negligible. In other words, ignoring the cancelation between representations will cause the information about the sharpness of the phase transition to be lost, but one may still show the existence of an "ordered phase".

Finally, we show how our formula allows us to recover a result of Berestycki and Durrett [3] concerning a slowdown transition for the interchange process. Let $d(t)$ denote the distance between σ_t and σ_0 , i.e., the minimal number of transpositions needed to write σ_t as a product of transpositions. (If one views σ_t as a random walk on the Cayley graph of S_n generated by the set of transpositions, then $d(t)$ is the graph distance between σ_t and σ_0).

Theorem 4. *Let $c > 0$ and let $t = c/n$. Then as $n \rightarrow \infty$,*

$$\frac{1}{n}\mathbb{E}(d(t)) \rightarrow u(c) := 1 - \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \frac{1}{c} (ce^{-c})^k. \quad (1)$$

The asymptotic behaviour (1) was first obtained in Theorem 3 of [3]. It is known that for $c < 1$, the function $u(c)$ simplifies to $u(c) = c/2$ while for $c > 1$, $u(c) < c/2$.

The function u is C^∞ everywhere except at $c = 1$, for which $u''(1) = -\infty$. This result is indicative of a slowdown transition, in the following sense: so long as $c < 1$, the random walk escapes from the origin at maximum speed (i.e., all but a negligible fraction of moves take it away from the origin), while for $c > 1$ the random walk starts decelerating brutally — indeed, the acceleration is $-\infty$ at $c = 1$. The methods of [3] relied heavily on the use of an associated Erdős-Renyi random graph process, and more precise results were obtained (e.g., a central limit theorem for $d(t)$). However we will see that the formula (1) is obtained with essentially no effort from our Theorem 1.

It is interesting to note that Theorem 4 is derived by applying Theorem 1 for constant k , unlike Theorems 2 and 3 for which the relevant regime is $k \asymp n$. It would be interesting to find applications of our formula in other regimes of k , for example $k = n - o(n)$.

2 Proofs

2.1 Preliminaries

For a permutation σ , let $\alpha_k(\sigma)$ denote the number of cycles of size k of σ , and $s_k(t) = \alpha_k(\sigma_t)$. For a representation $\rho : S_n \rightarrow GL(\mathbb{C}^{\dim \rho})$ of S_n , let χ_ρ be its character, i.e., $\chi_\rho(\sigma) = \text{Tr}(\rho(\sigma))$. Note that $\alpha_k(\sigma)$ is a class function. Hence, since characters form an orthonormal basis of class functions (see, e.g., Proposition 1.10.2 in [13]) there exists $a_\rho \in \mathbb{C}$ such that

$$\alpha_k(\sigma) = \sum_{\rho} a_\rho \chi_\rho(\sigma) \tag{2}$$

where the sum is over irreducible representations ρ . In [1] the value of a_ρ for all representations ρ was worked out. To describe this result we need to recall some standard background in the representation theory of the symmetric group. The irreducible representations of S_n are parametrized by Young diagrams of size n , that is, sequences $\lambda = [\lambda_1, \dots, \lambda_j]$ with $\lambda_1 \geq \dots \geq \lambda_j \geq 1$ and $\sum_i \lambda_i = n$, which may be thought of as a collection of boxes sitting on top of one another, with λ_i boxes in row i . See [13] for the details of this parameterization (we will not use it directly in this paper).

As it turns out, there are very few Young diagrams for which the corresponding coefficient a_ρ defined by (2) is nonzero. To state things as simply as possible, we restrict ourselves to the case where $k \leq n/2$ (the other case is similar and we come

back to this later). Consider the representation $\rho_i = [n-k, k-i, 1^i]$ where 1^i indicates that there are i rows with exactly one box. Then Theorem 3 of [1] states that a_ρ is zero unless $\rho = \rho_i$ for some $0 \leq i \leq k-1$, in which case a_ρ is simply equal to $(-1)^i/k$, or $\rho = [n]$ the trivial representation, in which case $a_\rho = 1/k$.

Lemma 1. *For a representation ρ , let $r(\rho) = \chi_\rho(\tau)/d_\rho$ be the character ratio of ρ , i.e., when τ is any transposition, $\chi_\rho(\tau) = \text{Tr}(\rho(\tau))$ is the character of ρ at τ , and d_ρ is the dimension of ρ . Then*

$$\mathbb{E}(s_k(t)) = \sum_{\rho} a_\rho d_\rho \exp \left\{ \binom{n}{2} (r(\rho) - 1)t \right\}. \quad (3)$$

Proof. To compute $s_k(t)$, let Q be the infinitesimal generator of the random walk. That is, viewing Q as an element of the group algebra $\mathbb{C}[S_n]$, $Q = \sum_{i < j} [(i, j) - \text{id}]$ (here, (i, j) denotes the transposition of i and j and id is the identity permutation). Thus, $\mathbb{P}(\sigma_t = s) = e^{tQ}(o, s) = f(s)$, say. Then

$$\begin{aligned} \mathbb{E}(s_k(t)) &= \sum_{s \in S_n} \alpha_k(s) \mathbb{P}(\sigma_t = s) \\ &= \sum_{\rho} \sum_{s \in S_n} a_\rho \chi_\rho(s) f(s) \\ &= \sum_{\rho} a_\rho \text{Tr} \left(\sum_{s \in S_n} \rho(s) f(s) \right) \\ &= \sum_{\rho} a_\rho \text{Tr} \widehat{f}(\rho), \end{aligned} \quad (4)$$

where $\widehat{f}(\rho)$ denotes the Fourier transform of f at ρ . Note that $f(s) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k Q^k(s)$. Since convolutions become (matrix) products after taking Fourier transform, we get

$$\widehat{f}(\rho) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \widehat{Q}(\rho)^k = \exp(t\widehat{Q}(\rho))$$

where the second \exp is a matrix exponential. Now note that $Q(s)$ is constant on conjugacy classes, i.e., is a class function (here is where we use that we do the interchange process on the complete graph, and not on any other graph). It easily follows from Schur's lemma that $\widehat{Q}(\rho) = \lambda_\rho I$ for some $\lambda_\rho \in \mathbb{R}$, and thus we conclude that

$$\widehat{f}(\rho) = e^{t\lambda_\rho} I. \quad (5)$$

To obtain the constant λ_ρ we proceed as follows: on the one hand we have $\text{Tr } \widehat{Q}(\rho) = d_\rho \lambda_\rho$, so $\lambda_\rho = (1/d_\rho) \text{Tr } \widehat{Q}(\rho)$. On the other hand, $Q = \sum_{i < j} [(ij) - \text{id}]$ so by linearity of the trace,

$$\text{Tr } \widehat{Q}(\rho) = \sum_{i < j} \chi_\rho((ij)) - \chi_\rho(\text{id}) = \binom{n}{2} (\chi_\rho(\tau) - d_\rho),$$

where $\chi_\rho(\tau)$ denotes the character of τ evaluated at an arbitrary transposition (again, it is not important which one, since characters are class functions). Hence, dividing by d_ρ , we get

$$\lambda_\rho = \binom{n}{2} (r(\rho) - 1) \tag{6}$$

where $r(\rho) = \chi_\rho(\tau)/d_\rho$ is the character ratio at a transposition. Combining (4), (5) and (6), we have arrived at

$$\mathbb{E}(s_k(t)) = \sum_{\rho} a_\rho d_\rho \exp \left\{ \binom{n}{2} (r(\rho) - 1)t \right\}$$

as needed. □

2.2 Proof of Theorem 1, case $k \leq n/2$.

We now explain how to compute the various terms in the sum involved in (3) in order to get to the exact formula of Theorem 1. We assume for now $k \leq n/2$.

Proof. We start by recalling Frobenius's formula, which gives the values of the character ratios for any representation. Suppose ρ is parametrized by the Young tableau (ρ_1, \dots, ρ_j) , then (see [6, Equation $(D - 2)$, p. 40] or [7, Lemma 7]),

$$r(\rho) = \frac{1}{n(n-1)} \sum_{i=1}^j \rho_i^2 - (2i-1)\rho_i. \tag{7}$$

Thus if $\rho = \rho_i$ is the representation given by $\rho_i = [n-k, k-i, 1^i]$, $0 \leq i \leq k-1$, then

$$\begin{aligned} n(n-1)r(\rho_i) &= (n-k)^2 - (n-k) + (k-i)^2 - 3(k-i) \\ &\quad + 1 - 5 + \dots + 1 - (2(i+2) - 1) \\ &= (n-k)^2 - (n-k) + k^2 - 2ik + i^2 - 3k + 3i - (i^2 + 3i) \end{aligned}$$

$\overbrace{\hspace{10em}}^{n-k}$						
$n-k+i+1$	$n-k$	\cdots	$n-2k+i+2$	$n-2k+i$	\cdots	1
k	$k-i-1$	\cdots	1			
i	$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} i$					
\cdots						
1						

Figure 2: Sizes of the hooks for the diagram $[n-k, k-i, 1^i]$.

$$= (n-k)^2 - n - 2k + k^2 - 2ik. \quad (8)$$

Notice the cancelation of the terms in i^2 , which is crucial for the following calculations.

By the hook-length formula (see e.g. (4.12) in [10]), it is also easy to compute d_{ρ_i} , the dimension of the representation ρ_i (see Figure 2):

$$\dim(\rho_i) = \frac{n!(n-2k+i+1)}{i!k(n-k)!(k-i-1)!(n-k+i+1)}. \quad (9)$$

We now plug these various expressions into (3). Recall that $a_\rho = (-1)^i/k$ when $\rho = \rho_i$, and $a_\rho = 1/k$ when ρ is the trivial representation $[n]$. We obtain the following expression, after rearranging the terms (mostly to isolate the i -independent terms and take them out of the sum). Let $m = k-1$, and $x = \exp(-kt)$. Then

$$\mathbb{E}(s_k(t)) = \frac{1}{k} + C_{n,k} \sum_{i=0}^m \frac{m!}{i!(m-i)!} \frac{n-2k+i+1}{n-k+i+1} (-1)^i x^i \quad (10)$$

where

$$\begin{aligned} C_{n,k} &= \frac{1}{k} \frac{n!}{k!(n-k)!} \exp \left\{ t \binom{n}{2} \left(\frac{(n-k)^2 - n - 2k + k^2}{n(n-1)} - 1 \right) \right\} \\ &= \frac{1}{k} \binom{n}{k} \exp \left\{ \frac{t}{2} (n^2 - 2nk + k^2 - n - 2k + k^2) - t \frac{n^2 - n}{2} \right\} \\ &= \frac{1}{k} \binom{n}{k} \exp \{-tk(n+1-k)\} = \frac{1}{k} \binom{n}{k} x^{n-m}. \end{aligned} \quad (11)$$

To compute the right-hand side of (10), note that $\sum_{i=0}^m \frac{m!}{i!(m-i)!} (-1)^i x^i = (1-x)^m$, hence

$$\begin{aligned}
\mathbb{E}(s_k(t)) &= \frac{1}{k} + C_{n,k} \sum_{i=0}^m \frac{m!}{i!(m-i)!} \frac{n-2k+i+1}{n-k+i+1} (-1)^i x^i \\
&= \frac{1}{k} + C_{n,k} \sum_{i=0}^m \binom{m}{i} (-x)^i \left(1 - \frac{k}{n-k+i+1}\right) \\
&= \frac{1}{k} + C_{n,k} (1-x)^m - C_{n,k} k \sum_{i=0}^m \binom{m}{i} (-x)^i \frac{1}{n-m+i} \\
&= \frac{1}{k} + A_1 - A_2,
\end{aligned}$$

say.

This can be simplified a little bit. Observe indeed that

$$\begin{aligned}
A_2 &= k C_{n,k} \sum_{i=0}^m \binom{m}{i} (-x)^i \frac{1}{n-m+i} \\
&= k C_{n,k} x^{-(n-m)} \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{x^{n-m+i}}{n-m+i} \\
&= \binom{n}{k} \sum_{i=0}^m \binom{m}{i} (-1)^i \int_0^x y^{n-m+i-1} dy \\
&= \binom{n}{k} \int_0^x y^{n-k} (1-y)^m dy
\end{aligned}$$

On the other hand, $A_1 = \frac{1}{k} \binom{n}{k} (1-x)^m x^{n-m}$, so we are led to the following exact expression for $\mathbb{E}(s_k(t))$: with $x = e^{-tk}$,

$$\mathbb{E}(s_k(t)) = \frac{1}{k} + \binom{n}{k} \left[\frac{1}{k} x^{n-m} (1-x)^m - \int_0^x y^{n-k} (1-y)^m dy \right].$$

Recalling the notation ϕ from the statement of the theorem, we can rewrite this shortly as

$$\mathbb{E}(s_k(t)) = \frac{1}{k} + \binom{n}{k} \left[\frac{1}{k} x \phi(x) - \int_0^x \phi(y) dy \right]. \quad (12)$$

The theorem is basically finished: we only need the following simple transformation:

Lemma 2.

$$\binom{n}{k} \int_0^x \phi(y) dy = \frac{1}{k} - \binom{n}{k} \int_x^1 \phi(y) dy$$

Proof. Denote

$$I(x) = \int_x^1 \phi(y) dy = \int_0^1 \phi(y) dy - \int_0^x \phi(y) dy.$$

The integral from 0 to 1 is just the Beta integral, so

$$\int_0^1 y^{n-k}(1-y)^{k-1} dy = \frac{\Gamma(n-k+1)\Gamma(k)}{\Gamma(n-k+1+k)}.$$

For completeness, here is a short proof: Fubini's theorem shows that for every $x, y > 0$, $\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty e^{-u-v} u^{x-1} v^{y-1} du dv$, hence after a change of variables $z = u+v$, $t = u/z$, so that $du dv = z dz dt$, we get

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-z} z^{x+y-1} dz \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Hence $\int_0^1 t^{x-1} (1-t)^{y-1} dt = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, as required.

Multiplying by $\binom{n}{k}$ gives the $\frac{1}{k}$, showing the lemma. \square

Thus the $\frac{1}{k}$ from lemma 2 cancels with the $\frac{1}{k}$ in (12) and leads us to

$$\mathbb{E}(s_k(t)) = \binom{n}{k} \left[\frac{1}{k} x^{n-m} (1-x)^{k-1} + I(x) \right] \quad (13)$$

(with I as in the proof of lemma 2). This finishes the proof of Theorem 1 in the case $k \leq n/2$. \square

2.3 Proof of Theorem 1, case $k > n/2$.

The calculation in this case is very similar to the one in the previous case. In fact, we find the similarity eerie and needing of explanation. What we will show is that (10) holds, which means that all the calculation from (10) on holds as is, including the conclusion of the theorem. So we only need to show (10).

Recall from [1] that if $k > n/2$, the representations that arise in the character decomposition of α_k are of two types: for $0 \leq i \leq 2k - n - 2$, we have $\rho_i = [k - i - 1, n - k + 1, 1^i]$, for which $a_i = (-1)^{i+1}/k$, while for $2k - n \leq i \leq k - 1$, we have $\rho_i = [n - k, k - i, 1^i]$ coming with a coefficient $a_i = (-1)^i/k$. Note in

$\overbrace{\hspace{10em}}^{k-i-1}$						
k	$k-i-1$	\cdots	$2k-n-i$	$2k-n-i-2$	\cdots	1
$n-k+i+1$	$n-k$	\cdots	1			
i	$\left. \begin{array}{l} \\ \\ \end{array} \right\} i$					
\cdots						
1						

Figure 3: Sizes of the hooks for the diagram $[k-i-1, n-k+1, 1^i]$

particular, that for $2k-n \leq i \leq k-1$, the representation and the coefficient is formally identical to those used in the case $k \leq n/2$, and hence the corresponding dimension and character ratio are still given (formally) by the same expressions (8) and (9). Note also that there is no representation associated with $i = 2k-n-1$.

For $0 \leq i \leq 2k-n-2$, consider the representation $\rho_i = [k-i-1, n-k+1, 1^i]$ for which the coefficient is $a_{\rho_i} = (-1)^{i+1}/k$. Then a calculation using Frobenius' character formula shows (still keeping the notation $m = k-1$),

$$\begin{aligned}
n(n-1)r(\rho_i) &= (m-i)^2 - (m-i) + (n-m)^2 - 3(n-m) \\
&\quad + 1 - 5 + \dots + 1 - (2(i+2) - 1) \\
&= (m-i)^2 - (m-i) + (n-m)^2 - 3(n-m) - (i^2 + 3i) \\
&= -2ik + (n-k)^2 + k^2 - 2k - n
\end{aligned}$$

Comparing with (8), observe that this is exactly the same expression.

It remains to compute $d(\rho_i)$. In order to do so we use again the hook length formula, which gives us the following (see Figure 3):

$$\dim(\rho_i) = \frac{n!(2k-n-i-1)}{i!k(n-k)!(k-i-1)!(n-k+i+1)}.$$

Formally, this is once again exactly the same result as in case $k \leq n/2$ but with a minus sign. But since in this case $a_i = (-1)^{i+1}/k$ instead of $(-1)^i/k$, the two minus signs cancel out and the formula which computes $\mathbb{E}(s_k(t))$ is *exactly the same*, except for the fact that there is no term corresponding to $i = 2k-n-1$. However, observe that the term in (10) corresponding to $i = 2k-n-1$ is zero, so it does nothing to the sum when we keep it.

This shows that (10) holds also when $k > n/2$ and we conclude that, even when $k > n/2$,

$$\mathbb{E}(s_k(t)) = \binom{n}{k} \left[\frac{1}{k} x^{n-m} (1-x)^{k-1} + I(x) \right].$$

This finishes the proof of Theorem 1. \square

2.4 Proof of Theorem 2

We need to estimate the terms appearing in Theorem 1. For this we denote $B_1 = \frac{1}{k} \binom{n}{k} x^{n-k} (1-x)^{k-1}$ and $B_2 = \binom{n}{k} I(x)$ where I is as in the previous section. With these definitions $\mathbb{E}(s_k(t)) = xB_1 + B_2$ (the reason for taking one x out of B_1 is technical — it allows us to use the same notations for the estimate of B_1 and of B_2 , nothing more). Another useful notation would be

$$\psi_{n,k}(y) = (n-k) \log(y) + (k-1) \log(1-y),$$

so that the x -dependent term in B_1 is $e^{\psi(x)}$ and the integrand in $I(x)$ is $e^{\psi(y)}$. Note that $\psi_{n,k}$ has a unique maximum over $[0, 1]$, attained at $y_{n,k} = (n-k)/(n-1)$. The key lemma is the following estimate:

Lemma 3. *There exists $C > 0$ such that for all $k \leq n$, and for all $x \in [0, 1]$ we have:*

$$B_1 \leq \frac{Cq}{k} \exp \left\{ -\frac{\min\{|\varepsilon|, 1/4\}^2 (n-k)}{4} \right\} \quad \varepsilon = \frac{x - y_{n,k}}{y_{n,k}}, \quad (14)$$

where $q = n^{3/2} k^{-3/2} (n-k)^{-1/2}$, as in the statement of theorem 2.

Proof. We start by defining

$$\Phi_{n,k}(\varepsilon) := \exp(\psi_{n,k}(y_{n,k}(1+\varepsilon))) \quad (15)$$

and then

$$\begin{aligned} \Phi_{n,k}(\varepsilon) &= \exp \left\{ (n-k) \log \left(\frac{n-k}{n-1} (1+\varepsilon) \right) + (k-1) \log \left(1 - \frac{n-k}{n-1} (1+\varepsilon) \right) \right\} \\ &= \exp \{ (\psi_{n,k}(y_{n,k}) + \delta) \}. \end{aligned}$$

To bound δ we first assume that $|\varepsilon| \leq 1/4$ and get

$$\delta = (n-k) \log(1+\varepsilon) + (k-1) \log \left(1 - \varepsilon \frac{n-k}{k-1} \right) \leq$$

$$\begin{aligned}
&\leq (n-k)\varepsilon - (n-k)\frac{\varepsilon^2}{4} + (k-1)(-\varepsilon)\frac{n-k}{k-1} \\
&\leq -\frac{(n-k)\varepsilon^2}{4},
\end{aligned} \tag{16}$$

where we have used for the first log the fact that for $|\varepsilon| \leq 1/4$, $\log(1+\varepsilon) \leq \varepsilon - \varepsilon^2/4$, and for the second log simply $\log(1+a) \leq a$. To remove the restriction on ε , note that $\psi_{n,k}$ is monotone increasing over $[0, y_{n,k}]$ and monotone decreasing over $[y_{n,k}, 1]$. So we can write $\delta \leq -\frac{1}{4}(n-k) \min\{|\varepsilon|, 1/4\}^2$, for all ε .

On the other hand, from Stirling's formula it is easy to see that there exists a universal constant C such that for all n and all $k \leq (n/2)$,

$$\binom{n}{k} \leq C \sqrt{\frac{n}{k(n-k)}} \frac{n^n}{k^k(n-k)^{n-k}}. \tag{17}$$

Combining (17) and (16) we get

$$\begin{aligned}
\binom{n}{k} \Phi_{n,k}(\varepsilon) &\leq C \sqrt{\frac{n}{k(n-k)}} \exp \left\{ k \log \left(\frac{n}{k} \right) + (n-k) \log \left(\frac{n}{n-k} \right) \right. \\
&\quad \left. + (n-k) \log \left(\frac{n-k}{n-1} \right) + (k-1) \log \left(\frac{k-1}{n-1} \right) + \delta \right\} \\
&= C \sqrt{\frac{n}{k(n-k)}} \cdot \frac{n-1}{k-1} \exp \left\{ n \log \left(\frac{n}{n-1} \right) + k \log \left(\frac{k-1}{k} \right) + \delta \right\} \\
&\leq \frac{Cn^{3/2}}{k^{3/2}(n-k)^{1/2}} \exp \left\{ -\frac{\min\{|\varepsilon|, 1/4\}^2(n-k)}{4} \right\}
\end{aligned} \tag{18}$$

Multiplying further by $1/k$, we get the claimed bound on B_1 in Lemma 3. \square

Lemma 4. *When $x > y_{n,k}(1+\varepsilon)$ and $\varepsilon > 0$*

$$B_2 \leq Cq \exp \left\{ -\frac{\min\{|\varepsilon|, 1/4\}^2(n-k)}{4} \right\} \tag{19}$$

while if $x < y_{n,k}(1+\varepsilon)$ and $\varepsilon < 0$,

$$\left| B_2 - \frac{1}{k} \right| \leq Cq \exp \left\{ -\frac{\min\{|\varepsilon|, 1/4\}^2(n-k)}{4} \right\}. \tag{20}$$

where $q = q_n(k)$ is as in Lemma 3 and Theorem 2.

Proof. We analyze these two cases separately:

Case 1: $x > y_{n,k}(1 + \varepsilon)$ with $\varepsilon > 0$. Then, since $\psi_{n,k}$ is monotone decreasing after $y_{n,k}$, we have

$$I(x) = \int_x^1 e^{\psi_{n,k}(y)} dy \leq e^{\psi_{n,k}(y_{n,k}(1+\varepsilon))},$$

and hence

$$B_2 \leq \binom{n}{k} e^{\psi_{n,k}(y_{n,k}(1+\varepsilon))} = \binom{n}{k} \Phi_{n,k}(\varepsilon). \quad (21)$$

Case 2: $x < y_{n,k}(1 + \varepsilon)$ with $\varepsilon < 0$. As explained in the introduction, here we will Lemma 2, which gives us

$$B_2 = \binom{n}{k} I(x) = \frac{1}{k} - \binom{n}{k} \int_0^x e^{\psi_{n,k}(y)} dy$$

and this time, since $\psi_{n,k}$ is monotone increasing over $[0, y_{n,k}(1 + \varepsilon)]$ we conclude that

$$\int_0^x e^{\psi_{n,k}(y)} dy \leq e^{\psi_{n,k}(y_{n,k}(1+\varepsilon))}.$$

so that

$$|B_2 - 1/k| \leq \binom{n}{k} \Phi_{n,k}(\varepsilon) \quad (22)$$

Hence in both cases (21) and (22), we conclude by (18). \square

With these two lemmas, we are now able to finish the proof of Theorem 2.

Proof. Suppose $t = t_{n,k} + \varepsilon$, with $\varepsilon \in \mathbb{R}$. Then

$$x = e^{-tk} = e^{-kt_{n,k}} e^{-k\varepsilon} = y_{n,k} e^{-k\varepsilon}$$

so

$$\frac{|x - y_{n,k}|}{y_{n,k}} = |e^{-k\varepsilon} - 1| > c \min\{|k\varepsilon|, 1\} = c \min\{k|t - t_{n,k}|, 1\}. \quad (23)$$

Thus

$$\begin{aligned} \left| \mathbb{E}(s_k(t)) - \frac{1}{k} \mathbf{1}_{\{t > t_{n,k}\}} \right| &= \left| xB_1 + B_2 - \frac{1}{k} \mathbf{1}_{\{t > t_{n,k}\}} \right| \\ \text{By lemmas 3 and 4} &\leq Cq \exp\left(-\frac{\min\{|x - y_{n,k}|/y_{n,k}, 1/4\}^2(n-k)}{4}\right) \\ \text{By (23)} &\leq Cq \exp\left(-c \min\{k|t - t_{n,k}|, 1\}^2(n-k)\right). \end{aligned}$$

As needed. \square

2.5 Proof of Theorem 3

We now turn to the proof of Theorem 3. Let $t = c/n$ and let $X_n(\varepsilon) = \frac{1}{n} \sum_{k \geq n\varepsilon} ks_k(t)$ be the relative mass of cycles greater than $n\varepsilon$. Then we have the following lemma:

Lemma 5. *Assume $c > 1$ and let $\theta(c) = \max\{z > 0 : 1 - z > e^{-zc}\}$. Then $\theta(c) > 0$ and*

$$\liminf_{n \rightarrow \infty} \mathbb{E}(X_n(\varepsilon)) \geq \theta(c) - \varepsilon.$$

Proof. Fix $\alpha < \theta(c)$ and let $\varepsilon n \leq k \leq \alpha n$. It follows that

$$t_{n,k} < \frac{c - \delta}{n} \quad \text{for some } \delta > 0. \quad (24)$$

Indeed, $k/n \leq \alpha < \theta(c)$ gives that $1 - k/n > e^{-(c-\delta)(k/n)}$ and with

$$e^{-kt_{n,k}} \stackrel{\text{def}}{=} \frac{n - k}{n - 1} > 1 - \frac{k}{n} > e^{-(c-\delta)(k/n)}$$

we get (24). Hence by Theorem 2, noting that $q_n(k) \leq C$ for any k in this range,

$$\mathbb{E}(s_k(t)) > \frac{1}{k} - C \exp(-\delta'n)$$

for some $C, \delta' > 0$ depending on α and c . Summing up between $k = \lceil \varepsilon n \rceil$ and $k = \lfloor \alpha n \rfloor$, we get

$$\mathbb{E}(X_n(\varepsilon)) > (\alpha - \varepsilon) - Cn^2 \exp(-\delta'n)$$

The lemma follows immediately by letting $n \rightarrow \infty$, since α was arbitrarily close to $\theta(c)$. \square

Lemma 6. *For any $\varepsilon > 0$, on an event of probability tending to 1 as $n \rightarrow \infty$,*

$$X_n(\varepsilon) \leq \theta(c) + \varepsilon.$$

Proof. Let $G(t)$ be the (random) graph where there is an edge (i, j) if and only if the transposition (i, j) has occurred prior to t . Then $G(t)$ is a realization of $G(n, p)$ with $p = 1 - \exp(-c/n) \sim c/n$. Furthermore, it is well-known and easy to see that any cycle of σ_t is a subset of some connected component of $G(t)$. By the Erdős-Renyi theorem (see, e.g., [8, 9]), it is known that on an event of high probability, $G(t)$ has only one component greater than εn (the giant component) and thus $X_n(\varepsilon) \leq Y_n$, where Y_n is the rescaled size of the largest component. Furthermore, the same papers show

$$\frac{Y_n}{n} \rightarrow \xi(c),$$

in probability, where $\xi(c)$ is the survival probability of a Poisson (c) Galton-Watson process. Hence $\xi(c) = \theta(c)$ and the lemma follows. \square

Proof of Theorem 3. Recall that the theorem states that a cycle $> \varepsilon n$ exists with high probability. Consider the random variable $E_n = \theta(c) + \varepsilon - X_n(\varepsilon)$. Then E_n is bounded by 2, nonnegative with high probability (from the Galton-Watson argument), and has expectation bounded by 2ε (from Lemma 5). Hence by Markov's inequality $\mathbb{P}(E_n > \sqrt{\varepsilon}) \leq 3\sqrt{\varepsilon}$ for all n sufficiently large. On the complement event $X_n(\varepsilon) > \theta(c) - \sqrt{\varepsilon}$ and hence $X_n(\varepsilon) > 0$, in which case the length of the longest cycle is $> n\varepsilon$. Since ε was arbitrary, Theorem 3 follows. \square

2.6 Proof of Theorem 4

Recall that we wish to estimate $d(t)$, the graph distance to $\sigma(t)$. Let $N(t)$ denote the number of cycles of σ_t . It is well known and easy to check that $d(t) = n - N(t)$. Let $\varepsilon > 0$ and fix $K \geq 1/\varepsilon$. Then observe that, since there can never be more than $n\varepsilon$ cycles greater than K ,

$$\frac{1}{n}\mathbb{E}(N(t)) = \frac{1}{n} \sum_{k=1}^K \mathbb{E}(s_k(t)) + O(\varepsilon).$$

Thus Theorem 4 follows from the claim: for all fixed $k \geq 1$, and all $c > 0$, if $t = c/n$,

$$\frac{1}{n}\mathbb{E}(s_k(t)) \rightarrow \frac{k^{k-2}}{k!} \frac{1}{c} (ce^{-c})^k. \quad (25)$$

For this we apply Theorem 1 with these values of k and t , and note that $x = e^{-kt}$ satisfies $x \rightarrow 1$, $1 - x \sim kc/n$ and $x^{n-k} \rightarrow e^{-ck}$ (the notation $a \sim b$ is short for $a = b(1 + o(1))$, here and below). Since $\binom{n}{k} \sim n^k/k!$, we deduce immediately that (recall that $\phi(x) = x^{n-k}(1-x)^{k-1}$),

$$\frac{1}{k} \binom{n}{k} x \phi(x) \sim n \frac{k^{k-2}}{k!} \frac{1}{c} (ce^{-c})^k.$$

Moreover, it is plain that $\frac{1}{n} \binom{n}{k} \phi(y)$ is uniformly bounded for $x \leq y \leq 1$, hence

$$\frac{1}{n} \binom{n}{k} \int_x^1 \phi(y) dy \leq C(1-x) \rightarrow 0.$$

(25) follows immediately, hence so does Theorem 4.

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