

Critical branching Brownian motion with absorption: particle configurations

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Abstract

We consider critical branching Brownian motion with absorption, in which there is initially a single particle at $x > 0$, particles move according to independent one-dimensional Brownian motions with the critical drift of $-\sqrt{2}$, and particles are absorbed when they reach zero. Here we obtain asymptotic results concerning the behavior of the process before the extinction time, as the position x of the initial particle tends to infinity. We estimate the number of particles in the system at a given time and the position of the right-most particle. We also obtain asymptotic results for the configuration of particles at a typical time.

1 Introduction

1.1 Main results

We consider branching Brownian motion with absorption. At time zero, there is a single particle at $x > 0$. Each particle moves independently according to one-dimensional Brownian motion with a drift of $-\mu$, and each particle independently splits into two at rate 1. Particles are absorbed when they reach the origin. This process was first studied in 1978 by Kesten [19], who showed that with positive probability there are particles alive at all times if $\mu < \sqrt{2}$, but all particles are eventually absorbed almost surely if $\mu \geq \sqrt{2}$.

In recent years, there has been a surge of renewed interest in this process. Some of this interest has been driven by connections between branching Brownian motion with absorption and the FKPP equation. See, for example, the work of Harris, Harris, and Kyprianou [16], who used branching Brownian motion with absorption to establish existence and uniqueness results for the FKPP traveling-wave equation. In other work, such as [5, 8, 9, 20], branching Brownian motion with absorption or a very similar process has been used to model a population undergoing selection. In this setting, particles represent individuals in a population, branching events correspond to births, the positions of the particles are the fitnesses of the individuals, and absorption at zero models the death of individuals whose fitness becomes too low.

In this paper, we consider branching Brownian motion with absorption in the critical case with $\mu = \sqrt{2}$. This process is known to die out with probability one, but we are able to use techniques developed in [5, 6] to obtain some new and rather precise results about the behavior

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of the process before the extinction time. We focus on asymptotic results about the number of particles, the position of the right-most particle, and the configuration of particles as the position x of the initial particle tends to infinity.

Let $N(s)$ be the number of particles at time s , and let $X_1(s) \geq X_2(s) \geq \dots \geq X_{N(s)}(s)$ denote the positions of the particles at time s . Let

$$Y(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)}. \quad (1)$$

Throughout the paper, we will use the constants

$$\tau = \frac{2\sqrt{2}}{3\pi^2}, \quad c = \tau^{-1/3} = \left(\frac{3\pi^2}{2\sqrt{2}} \right)^{1/3}. \quad (2)$$

Let $t = \tau x^3$, which is approximately the extinction time of the process when x is large. More precisely, it was shown in [6] that for all $\varepsilon > 0$, there is a positive constant β such that for sufficiently large x , the extinction time is between $t - \beta x^2$ and $t + \beta x^2$ with probability at least $1 - \varepsilon$.

Our first result shows how the number of particles evolves over time. For times s between Bx^2 and $(1 - \delta)t$, where B is a large constant and δ is a small constant, with high probability this result estimates the number of particles at time s to within a constant factor.

Theorem 1. *Fix $\varepsilon > 0$ and $\delta > 0$. Then there exists a positive constant B depending on ε and positive constants C_1 and C_2 depending on B , δ , and ε such that for sufficiently large x , we have*

$$\mathbb{P} \left(\frac{C_1}{x^3} e^{\sqrt{2}(1-s/t)^{1/3}x} \leq N(s) \leq \frac{C_2}{x^3} e^{\sqrt{2}(1-s/t)^{1/3}x} \right) > 1 - \varepsilon$$

for all $s \in [Bx^2, (1 - \delta)t]$.

For $0 \leq s \leq t$, define

$$L(s) = x \left(1 - \frac{s}{t} \right)^{1/3} = c(t - s)^{1/3}. \quad (3)$$

The next result shows that at time s , the right-most particle is usually slightly to the left of $L(s)$.

Theorem 2. *Suppose $0 < u < \tau$, and let $s = ux^3$. Let $\varepsilon > 0$. Then there exist $d_1 > 0$ and $d_2 > 0$, depending on u and ε , such that for sufficiently large x ,*

$$\mathbb{P} \left(L(s) - \frac{3}{\sqrt{2}} \log x - d_1 < X_1(s) < L(s) - \frac{3}{\sqrt{2}} \log x + d_2 \right) > 1 - \varepsilon.$$

We are also able to obtain results about the entire configuration of particles. The key idea is that at time s , the density of particles near $y \in (0, L(s))$ will be roughly proportional to

$$e^{-\sqrt{2}y} \sin \left(\frac{\pi y}{L(s)} \right). \quad (4)$$

Establishing a rigorous version of this statement requires proving two theorems. In Theorem 3, we consider the probability measure in which a mass of $1/N(s)$ is placed at the position of each

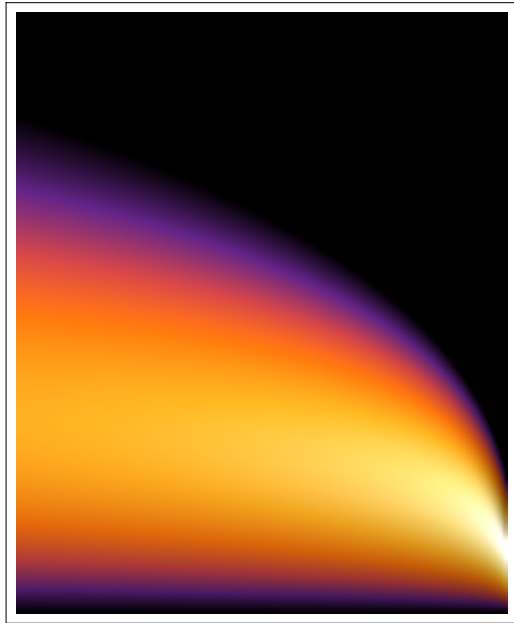


Figure 1: A plot of the asymptotic density of particles (4) started from a given position x . The horizontal axis is time and the vertical axis is space. Colors indicate the value of the density: black indicates no particles, and pale indicates a large number of particles. The curve delineating the black region is $L(s) = c(t-s)^{1/3}$, where t is the extinction time (at the bottom-right corner of the figure). Note that $x = L(0)$.

particle at time s . Because most particles are close to the origin and $\sin(\pi y/L(s)) \approx \pi y/L(s)$ for small y , in the limit this probability measure has a density proportional to $ye^{-\sqrt{2}y}$. In Theorem 4, we consider the probability measure in which a particle at position z is assigned a mass proportional to $e^{\sqrt{2}z}$. In this case, particles over the entire interval from 0 to $L(s)$ contribute significantly even in the limit, and the sinusoidal shape is observed.

For these results, we use \Rightarrow to denote convergence in distribution for random elements in the Polish space of locally finite measures on $(0, \infty)$, endowed with the vague topology. We also use δ_y to denote a unit mass at y .

Theorem 3. *Suppose $0 < u < \tau$, and let $s = ux^3$. Define the probability measure*

$$\chi(u) = \frac{1}{N(s)} \sum_{i=1}^{N(s)} \delta_{X_i(s)}.$$

Let μ be the probability measure on $(0, \infty)$ with density $g(y) = 2ye^{-\sqrt{2}y}$. Then $\chi(u) \Rightarrow \mu$ as $x \rightarrow \infty$.

Theorem 4. *Suppose $0 < u < \tau$, and let $s = ux^3$. Define the probability measure*

$$\eta(u) = \frac{1}{Y(s)} \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \delta_{X_i(s)/L(s)}.$$

Let ν be the probability measure on $(0, 1)$ with density $h(y) = \frac{\pi}{2} \sin(\pi y)$. Then $\eta(u) \Rightarrow \nu$ as $x \rightarrow \infty$.

1.2 Discussion and open problems

Beyond the general motivations recalled at the beginning of this introduction, we now explain some of the contexts in which these result might be applied, and some related open problems that are raised by them.

Yaglom limit laws. Let $x > 0$ be fixed, and consider a branching Brownian motion with critical drift started from one particle at x . Then with high probability, the process dies out in finite time. But conditional on survival up to a large time t , what does the process look like? More precisely, what is the empirical distribution of particles at times $0 \leq s \leq t$? This problem is known as the Yaglom conditional limit law, after a well known result of Yaglom [23] in the case of ordinary branching processes. We believe that the results in Theorems 3 and 4 may be relevant to this question as well.

An interesting related question is the following: conditional upon survival up to time t , what is the actual number of particles at that time? Note that the results of [6] give sharp estimates, up to constants, for the probability of survival up to time t .

Fleming-Viot processes. The process studied here (critical branching Brownian motion with absorption) shares several features with the Fleming-Viot process proposed by Burdzy et al. [10, 11], in the case where the underlying motion is simple random walk with negative drift and absorption at 0. For that process the main question concerns the limiting behaviour of the empirical distribution of particles, which under fairly general conditions is believed to be the minimal quasi-stationary distribution of the underlying motion. See [1] for a recent verification of this in the case where the underlying motion is that of a subcritical branching process. We point out that the function $e^{-\sqrt{2}x} \sin(\pi x/L)$ appearing in (4) (and before that in [5]) is precisely a Dirichlet eigenfunction of

$$\frac{1}{2} \frac{d^2}{dx^2} + \sqrt{2} \frac{d}{dx},$$

and corresponds to a quasi-stationary distribution of Brownian motion with drift $-\sqrt{2}$ in $(0, \infty)$. It is in fact the minimal such distribution in a sense similar to [1].

Extreme configurations. Theorem 2 gives us information about the position of the rightmost particle at time s , and localises it to within $O(1)$. A natural open problem is to get a convergence in distribution for the position of the rightmost particle. More generally, one can ask about the distribution of the particle configuration as seen from the rightmost particle, or from the median of the rightmost particle. We point out that these questions also make sense in the nearly-critical case studied in [5], and that the proof of Theorem 2 can be adapted to that setting. Getting information about the extremal configurations particles is of interest, among other things, because of the role of these particles in the spin glass interpretation of these branching diffusions. We note that the analogous question for a (free) branching Brownian motions has been settled in [2, 3].

1.3 Organization of the paper

The rest of this paper is organized as follows. In Section 2, we record some preliminary estimates about branching Brownian motion killed at the boundaries of a strip. We prove Theorems 1, 3, and 4 in Section 3. Theorem 2 is proved in Section 4. Throughout the paper, C , C' , and C'' will denote positive constants whose value may change from line to line. Numbered positive constants of the form C_k will not change their values from line to line.

2 Preliminary Estimates

In this section, we obtain or recall some preliminary estimates concerning branching Brownian motion in which particles are killed not only at the origin but also when they travel sufficiently far to the right. We will consider two cases. One is when the Brownian particles are killed at some level $L > 0$. The other is when particles are killed when they reach $L(s) = c(t - s)^{1/3}$ for some s .

As before, let $N(s)$ be the number of particles at time s , and denote the positions of the particles at time s by $X_1(s) \geq X_2(s) \geq \dots \geq X_{N(s)}(s)$. Define $Y(s)$ as in (1). Let $(\mathcal{F}_s, s \geq 0)$ denote the natural filtration associated with the branching Brownian motion. Let $q_s(x, y)$ denote the density of the branching Brownian motion, meaning that if initially there is a single particle at x and A is a Borel subset of $(0, \infty)$, then the expected number of particles in A at time s is

$$\int_A q_s(x, y) dy.$$

2.1 A constant right boundary

Let $L > 0$. We consider here the case in which particles are killed upon reaching either 0 or L . This case was studied in detail in [5]. The following result is Lemma 5 of [5].

Lemma 5. *For $s > 0$ and $x, y \in (0, L)$, let*

$$p_s(x, y) = \frac{2}{L} e^{-\pi^2 s / 2L^2} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L}\right),$$

and define $D_s(x, y)$ so that

$$q_s(x, y) = p_s(x, y)(1 + D_s(x, y)).$$

Then for all $x, y \in (0, L)$, we have

$$|D_s(x, y)| \leq \sum_{n=2}^{\infty} n^2 e^{-\pi^2(n^2-1)s/2L^2}. \quad (5)$$

Lemma 5 allows us to approximate $q_s(x, y)$ by $p_s(x, y)$ when s is sufficiently large. Lemma 6 below collects some further results about the density $q_s(x, y)$.

Lemma 6. *Fix a positive constant $b > 0$. There exists a constant C (depending on b) such that for all s such that $s \geq bL^2$, we have*

$$q_s(x, y) \leq Cp_s(x, y), \quad \forall x, y \in [0, L] \quad (6)$$

and for all s such that $s \leq bL^2$, we have

$$q_s(x, y) \leq \frac{CL^3}{s^{3/2}} p_s(x, y), \quad \forall x, y \in [0, L]. \quad (7)$$

The following inequalities hold in general (for all $s > 0$ and $x, y \in [0, L]$):

$$q_s(x, y) \leq \frac{Ce^{\sqrt{2}(x-y)}e^{-(x-y)^2/2s}}{s^{1/2}} \quad (8)$$

$$\int_0^L q_s(x, y) dy \leq e^s \quad (9)$$

$$\int_0^\infty q_s(x, y) ds \leq \frac{2e^{\sqrt{2}(x-y)}x(L-y)}{L} \quad (10)$$

$$\int_0^L e^{\sqrt{2}y} q_s(x, y) dy \leq e^{\sqrt{2}x} \min \left\{ 1, \frac{L-x}{s^{1/2}} \right\} \quad (11)$$

Proof. Equation (6) holds because the right-hand side of (5) is bounded by a constant when $s/L^2 \geq b$. The result (7) is established in the proof of Proposition 14 in [5] (see the argument between equations (53) and (54) of [5]) by breaking the sum on the right-hand side of (5) into blocks of size approximately L/\sqrt{s} . Equation (8) is equation (55) of [5] and is obtained by comparing $q_s(x, y)$ to the density of standard Brownian motion at time s . Equation (9) follows from the fact that the expected number of particles at time s is at most e^s because branching occurs at rate 1. Equation (10) follows from (28) and (51) of [5] and is proved using Green's function estimates for Brownian motion in a strip.

Finally, to prove (11), let $v_s(x, y)$ be the density of Brownian motion killed at 0 and L , meaning that if A is a Borel subset of $(0, L)$, then the probability that a Brownian motion started at x is in A at time s and has not hit 0 or L before time s is $\int_A v_s(x, y) dy$. By equation (28) of [5], we have

$$q_s(x, y) = e^{\sqrt{2}(x-y)} v_s(x, y). \quad (12)$$

Let $(B(t), t \geq 0)$ be standard Brownian motion with $B(0) = x$. Then, by the Reflection Principle,

$$\begin{aligned} \int_0^L v_s(x, y) dy &= \mathbb{P}(B(t) \in (0, L) \text{ for all } t \in [0, s]) \\ &\leq \mathbb{P} \left(\max_{0 \leq t \leq s} B(t) \leq L \right) \\ &= 2 \int_0^{L-x} \frac{1}{\sqrt{2\pi s}} e^{-y^2/2s} dy \\ &\leq \min \left\{ 1, \frac{L-x}{s^{1/2}} \right\}, \end{aligned} \quad (13)$$

and (11) follows from (12) and (13). \square

Let

$$Z(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \sin \left(\frac{\pi X_i(s)}{L} \right).$$

Lemma 7 below is part of Lemma 7 of [5], while Lemma 8 below follows immediately from Lemma 9 of [5].

Lemma 7. *For all initial configurations of particles at time zero, we have*

$$\mathbb{E}[Z(s)] = e^{-\pi^2 s/2L^2} Z(0) \quad (14)$$

and

$$\mathbb{E}[Y(s)] = \frac{4}{\pi} e^{-\pi^2 s / 2L^2} Z(0)(1 + D(s)), \quad (15)$$

where $|D(s)|$ is bounded above by the right-hand side of (5).

Lemma 8. Fix a constant $b > 0$. Suppose initially there is a single particle at x . Then there exists a positive constant C , depending on b but not on L or x , such that for all $s \geq bL^2$,

$$\mathbb{E}[Z(s)^2] \leq \frac{C e^{\sqrt{2}x} e^{\sqrt{2}Ls}}{L^4}.$$

The next lemma is Lemma 8 in [5], where it is obtained as a straightforward application of results in [22]. It is also similar to Lemma 3.1 of [19].

Lemma 9. Suppose $f : (0, L) \rightarrow [0, \infty)$ is a bounded measurable function. Suppose initially there is a single particle at x . Then

$$\mathbb{E} \left[\sum_{i=1}^{N(s)} f(X_i(s)) \right] = \int_0^L f(y) q_s(x, y) dy$$

and

$$\mathbb{E} \left[\left(\sum_{i=1}^{N(s)} f(X_i(s)) \right)^2 \right] = \int_0^L f(y)^2 q_s(x, y) dy + 2 \int_0^s \int_0^L q_u(x, z) \left(\int_0^L f(y) q_{s-u}(z, y) dy \right)^2 dz du.$$

2.2 A curved right boundary

Fix any time $t > 0$. As in (3), for $s \in [0, t]$, let

$$L(s) = c(t - s)^{1/3},$$

where c was defined in (2). Consider branching Brownian motion with drift $-\sqrt{2}$ in which particles are killed if they reach zero, or if they reach $L(s)$ at time s . Note that all particles must be killed by time t because $L(t) = 0$. This right boundary was previously considered by Kesten [19]. We recall here some results that recently appeared in [6], where they were proved using techniques developed by Harris and Roberts [17].

Let

$$Z(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \sin \left(\frac{\pi X_i(s)}{L(s)} \right),$$

a quantity of crucial importance in what follows. The next result, which combines Proposition 10 and Corollary 11 of [6], provides a precise estimate of $\mathbb{E}[Z(s)]$.

Lemma 10. For $0 < r < s < t$, let

$$G_r(s) = \exp \left(- (3\pi^2)^{1/3} ((t - r)^{1/3} - (t - s)^{1/3}) \right) \left(\frac{t - s}{t - r} \right)^{1/6}. \quad (16)$$

There exist positive constants C_3 and C_4 such that if $0 < s < t$, then

$$Z(0)G_0(s) \exp(-C_3(t - s)^{-1/3}) \leq \mathbb{E}[Z(s)] \leq Z(0)G_0(s) \exp(C_4(t - s)^{-1/3})$$

and, more generally, if $0 < r < s < t$, then

$$Z(r)G_r(s) \exp(-C_3(t - s)^{-1/3}) \leq \mathbb{E}[Z(s)|\mathcal{F}_r] \leq Z(r)G_r(s) \exp(C_4(t - s)^{-1/3}).$$

The following result, which is the $r = 0$ case of Proposition 12 in [6], establishes bounds on the density up to a constant factor.

Lemma 11. *For $x, y > 0$ and $0 \leq s \leq t$, let*

$$\psi_s(x, y) = \frac{1}{L(s)} e^{-(3\pi^2)^{1/3}(t^{1/3} - (t-s)^{1/3})} \left(\frac{t-s}{t}\right)^{1/6} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right).$$

Fix a positive constant b . There exists a constant $A > 0$ and positive constants C' and C'' , with C'' depending on b , such that if $L(0)^2 \leq s \leq t - A$, then

$$q_s(x, y) \geq C' \psi_s(x, y)$$

and if $bL(0)^2 \leq s \leq t - A$, then

$$q_s(x, y) \leq C'' \psi_s(x, y).$$

We will also require estimates on the number of particles killed at the right boundary. The result below is the $s = 0$ case of Lemma 15 in [6].

Lemma 12. *Suppose there is initially a single particle at x , where $0 < x < L(0)$. Let R be the number of particles killed at $L(s)$ for some $s \in [0, t]$. Then there are positive constants C' and C'' such that*

$$C' h(x) \leq \mathbb{E}[R] \leq C'' (h(x) + j(x)),$$

where

$$h(x) = e^{\sqrt{2}x} \sin\left(\frac{\pi x}{ct^{1/3}}\right) t^{1/3} \exp(-(3\pi^2 t)^{1/3})$$

and

$$j(x) = x e^{\sqrt{2}x} t^{-1/3} \exp(-(3\pi^2 t)^{1/3}).$$

Finally, we will need the following bound on the second moment of $Z(s)$.

Proposition 13. *Fix $\kappa > 0$ and $\delta > 0$. Then there exists a positive constant C , depending on κ and δ but not on t , such that for all $t \geq 1$ and all s satisfying $\kappa t^{2/3} \leq s \leq (1 - \delta)t$,*

$$\text{Var}(Z(s)) \leq C \mathbb{E}[Z(s)]^2 \left(\frac{e^{\sqrt{2}L(0)}}{L(0)Z(0)} + \frac{e^{\sqrt{2}L(0)}Y(0)}{L(0)^2Z(0)^2} \right).$$

Proof. The proof is similar to the proof of Lemma 12 in [5]. Choose times $0 = s_0 < s_1 < \dots < s_K = s$ such that $\kappa t^{2/3} \leq s_{i+1} - s_i \leq 2\kappa t^{2/3}$ for $i = 0, 1, \dots, K - 1$. Note that $K \leq Ct^{1/3}$.

By Lemma 10, for $i = 0, 1, \dots, K - 1$,

$$\mathbb{E}[Z(s_{i+1}) | \mathcal{F}_{s_i}] = \exp\left(- (3\pi^2)^{1/3} ((t - s_i)^{1/3} - (t - s_{i+1})^{1/3})\right) \left(\frac{t - s_{i+1}}{t - s_i}\right)^{1/6} Z(s_i) D_i, \quad (17)$$

where

$$\exp(-C_3 \delta^{-1/3} t^{-1/3}) \leq D_i \leq \exp(C_4 \delta^{-1/3} t^{-1/3}). \quad (18)$$

Because the particles alive at time s_{i+1} are a subset of the particles that would be alive at time s_{i+1} if particles were killed at $L(s_i)$, rather than $L(s)$, for $s \in [s_i, s_{i+1}]$, and the right-hand side of (5) is bounded by a constant when $s \geq \kappa t^{2/3}$ and $L \leq ct^{1/3}$, it follows from (15) that

$$\mathbb{E}[Y(s_{i+1}) | \mathcal{F}_{s_i}] \leq CZ(s_i) \quad (19)$$

for $i = 0, 1, \dots, K - 1$. Let

$$Z'(s_{i+1}) = \sum_{i=1}^{N(s_{i+1})} e^{\sqrt{2}X_i(s_{i+1})} \sin\left(\frac{\pi X_i(s_{i+1})}{L(s_i)}\right),$$

which is the same as $Z(s_{i+1})$ except $L(s_i)$ rather than $L(s_{i+1})$ appears in the denominator. Because $\sin(\pi x/L(s_{i+1})) \leq C \sin(\pi x/L(s_i))$ for all $x \in [0, L(s_{i+1})]$, we have $Z(s_{i+1}) \leq CZ'(s_{i+1})$. By Lemma 8, if there is a single particle at x at time s_i , then

$$\text{Var}(Z(s_{i+1})|\mathcal{F}_{s_i}) \leq \mathbb{E}[Z(s_{i+1})^2|\mathcal{F}_{s_i}] \leq C \mathbb{E}[Z'(s_{i+1})^2|\mathcal{F}_{s_i}] \leq \frac{Ce^{\sqrt{2}x}e^{\sqrt{2}L(s_i)}(s_{i+1} - s_i)}{L(s_i)^4}.$$

Because particles move and branch independently, it follows by summing over the particles at time s_i that

$$\text{Var}(Z(s_{i+1})|\mathcal{F}_{s_i}) \leq \frac{CY(s_i)e^{\sqrt{2}L(s_i)}(s_{i+1} - s_i)}{L(s_i)^4} \leq Ct^{-2/3}Y(s_i)e^{\sqrt{2}L(s_i)}. \quad (20)$$

Using the conditional variance formula, equations (17) and (20), and the fact that $s < (1-\delta)t$,

$$\begin{aligned} \text{Var}(Z(s_{i+1})) &= \mathbb{E}[\text{Var}(Z(s_{i+1})|\mathcal{F}_{s_i})] + \text{Var}(\mathbb{E}[Z(s_{i+1})|\mathcal{F}_{s_i}]) \\ &\leq Ct^{-2/3}e^{\sqrt{2}L(s_i)} \mathbb{E}[Y(s_i)] + D_i^2 e^{-2(3\pi^2)^{1/3}((t-s_i)^{1/3} - (t-s_{i+1})^{1/3})} \left(\frac{t-s_{i+1}}{t-s_i}\right)^{1/3} \text{Var}(Z(s_i)). \\ &\leq Ct^{-2/3}e^{\sqrt{2}L(s_i)} \mathbb{E}[Y(s_i)] + D_i^2 e^{-2(3\pi^2)^{1/3}((t-s_i)^{1/3} - (t-s_{i+1})^{1/3})} \text{Var}(Z(s_i)). \end{aligned}$$

Therefore, by induction,

$$\begin{aligned} \text{Var}(Z(s)) &\leq Ct^{-2/3} \sum_{i=0}^{K-1} e^{\sqrt{2}L(s_i)} \left(\prod_{j=i+1}^{K-1} D_j^2 e^{-2(3\pi^2)^{1/3}((t-s_j)^{1/3} - (t-s_{j+1})^{1/3})} \right) \mathbb{E}[Y(s_i)] \\ &\leq Ct^{-2/3} \sum_{i=0}^{K-1} e^{\sqrt{2}L(s_i)} \left(\prod_{j=i+1}^{K-1} D_j^2 \right) e^{-2(3\pi^2)^{1/3}((t-s_{i+1})^{1/3} - (t-s)^{1/3})} \mathbb{E}[Y(s_i)]. \end{aligned}$$

By (19), for $i = 1, \dots, K - 1$, we have $\mathbb{E}[Y(s_i)] = \mathbb{E}[\mathbb{E}[Y(s_i)|\mathcal{F}_{s_{i-1}}]] \leq C \mathbb{E}[Z(s_{i-1})]$. By (18) and the fact that $K \leq Ct^{1/3}$, for $i = 0, 1, \dots, K - 1$ we have

$$\prod_{j=i+1}^{K-1} D_j^2 \leq \prod_{j=i+1}^{K-1} \exp(2C_4\delta^{-1/3}t^{-1/3}) \leq \exp\left(\sum_{j=1}^{K-1} 2C_4\delta^{-1/3}t^{-1/3}\right) \leq C.$$

It follows that

$$\begin{aligned} \text{Var}(Z(s)) &\leq Ct^{-2/3} \sum_{i=1}^{K-1} e^{\sqrt{2}L(s_i)} e^{-2(3\pi^2)^{1/3}((t-s_{i+1})^{1/3} - (t-s)^{1/3})} \mathbb{E}[Z(s_{i-1})] \\ &\quad + Ct^{-2/3} e^{\sqrt{2}L(0)} e^{-2(3\pi^2)^{1/3}((t-s_1)^{1/3} - (t-s)^{1/3})} Y(0). \end{aligned} \quad (21)$$

Denote the two terms on the right-hand side of (21) by T_1 and T_2 .

Because $[(t-s)/t]^{1/6}$ is bounded above and below by positive constants when $0 \leq s \leq (1-\delta)t$, it follows from Lemma 10 that there are constants C' and C'' , depending on δ , such that for $i = 0, 1, \dots, K$,

$$C'Z(0) \exp\left(- (3\pi^2)^{1/3}(t^{1/3} - (t-s_i)^{1/3})\right) \leq \mathbb{E}[Z(s_i)] \leq C''Z(0) \exp\left(- (3\pi^2)^{1/3}(t^{1/3} - (t-s_i)^{1/3})\right).$$

Therefore, using that $\sqrt{2}c = (3\pi^2)^{1/3}$,

$$\begin{aligned} T_1 &\leq Ct^{-2/3} \sum_{i=1}^{K-1} \exp\left(\sqrt{2}L(s_i) - 2(3\pi^2)^{1/3}((t-s_{i+1})^{1/3} - (t-s)^{1/3})\right. \\ &\quad \left. - (3\pi^2)^{1/3}(t^{1/3} - (t-s_{i-1})^{1/3})\right) Z(0) \\ &= Ct^{-2/3} \sum_{i=1}^{K-1} \exp\left((3\pi^2)^{1/3}(t-s_i)^{1/3} - 2(3\pi^2)^{1/3}((t-s_{i+1})^{1/3} - (t-s)^{1/3})\right. \\ &\quad \left. - (3\pi^2)^{1/3}(t^{1/3} - (t-s_{i-1})^{1/3})\right) Z(0) \\ &= Ct^{-2/3} \exp\left(2(3\pi^2)^{1/3}(t-s)^{1/3} - (3\pi^2)^{1/3}t^{1/3}\right) Z(0) \\ &\quad \times \sum_{i=1}^{K-1} \exp\left((3\pi^2)^{1/3}((t-s_i)^{1/3} - 2(t-s_{i+1})^{1/3} + (t-s_{i-1})^{1/3})\right). \end{aligned} \quad (22)$$

For $i = 0, 1, \dots, K-1$, we have $t-s_{i+1} \geq \delta t$, and so $(t-s_i)^{1/3} - (t-s_{i+1})^{1/3} \leq C$. Therefore, the sum on the right-hand side of (22) is bounded by $C(K-1) \leq Ct^{1/3}$. Thus, using $t > 1$ and Lemma 10 again,

$$\begin{aligned} T_1 &\leq Ct^{-1/3} \exp\left((3\pi^2)^{1/3}t^{1/3}\right) \exp\left(2(3\pi^2)^{1/3}((t-s)^{1/3} - t^{1/3})\right) \frac{Z(0)^2}{Z(0)} \\ &\leq Ct^{-1/3} \exp\left((3\pi^2)^{1/3}t^{1/3}\right) \frac{\mathbb{E}[Z(s)]^2}{Z(0)} \\ &\leq \frac{Ce^{\sqrt{2}L(0)} \mathbb{E}[Z(s)]^2}{L(0)Z(0)}. \end{aligned} \quad (23)$$

Also, using that $t^{1/3} - (t-s_1)^{1/3} \leq C$,

$$\begin{aligned} T_2 &= Ct^{-2/3} e^{\sqrt{2}L(0)} \exp\left(-2(3\pi^2)^{1/3}((t-s_1)^{1/3} - (t-s)^{1/3})\right) Y(0) \\ &\leq Ct^{-2/3} e^{\sqrt{2}L(0)} \exp\left(-2(3\pi^2)^{1/3}(t^{1/3} - (t-s)^{1/3})\right) Y(0) \\ &\leq \frac{Ce^{\sqrt{2}L(0)} Y(0) \mathbb{E}[Z(s)]^2}{L(0)^2 Z(0)^2}. \end{aligned} \quad (24)$$

The result now follows from (21), (23), and (24). \square

3 Number and configuration of particles

In this section, we return to the model presented in the introduction, in which there is initially a single particle at x and we are concerned with the asymptotic behavior of the process as $x \rightarrow \infty$.

3.1 The process before time κx^2

We first consider how the branching Brownian motion evolves during the initial period between time 0 and time κx^2 , where $\kappa > 0$ is an arbitrary positive constant. We will use the following result of Neveu [21].

Lemma 14. *Consider branching Brownian motion with drift $-\sqrt{2}$ and no absorption, started with a single particle at the origin. For each $y \geq 0$, let $K(y)$ be the number of particles that reach $-y$ in a modified process in which particles are killed upon reaching $-y$. Then there exists a random variable W , with $P(0 < W < \infty) = 1$ and $E[W] = \infty$, such that*

$$\lim_{y \rightarrow \infty} ye^{-\sqrt{2}y} K(y) = W \quad a.s.$$

For our process which begins with a single particle at x , let $K(y)$ be the number of particles that would reach $x - y$, if particles were killed upon reaching $x - y$. Note that $K(y) < \infty$ almost surely because of Kesten's result [19] that critical branching Brownian motion with absorption dies out. If y is sufficiently large, then $ye^{-\sqrt{2}y} K(y)$ will have approximately the same distribution as the random variable W in Lemma 14. Our strategy for studying the branching Brownian motion between time 0 and time κx^2 will be to choose a sufficiently large constant y , wait for $K(y)$ particles to reach $x - y$, and then consider $K(y)$ independent branching Brownian motions started from $x - y$.

Let $\alpha \in \mathbb{R}$, and let

$$Z_\alpha = \sum_{i=1}^{N(\kappa x^2)} e^{\sqrt{2}X_i(\kappa x^2)} \sin\left(\frac{\pi X_i(\kappa x^2)}{x + \alpha}\right) \mathbf{1}_{\{X_i(\kappa x^2) \leq x + \alpha\}}. \quad (25)$$

The following result describes the behavior of the configuration of particles at time κx^2 .

Lemma 15. *For all $\varepsilon > 0$, there exists a positive constant C_5 , depending on κ and ε but not on x , such that for sufficiently large x ,*

$$\mathbb{P}(Y(\kappa x^2) \leq C_5 x^{-1} e^{\sqrt{2}x}) \geq 1 - \varepsilon. \quad (26)$$

Also, there exist positive constants C_6 and C_7 , depending on κ and ε but not on x or α , such that for sufficiently large x ,

$$\mathbb{P}(C_6 x^{-1} e^{\sqrt{2}x} \leq Z_\alpha \leq C_7 x^{-1} e^{\sqrt{2}x}) \geq 1 - \varepsilon. \quad (27)$$

Furthermore,

$$\lim_{x \rightarrow \infty} \mathbb{P}(X_1(\kappa x^2) \leq x + \alpha) = 1. \quad (28)$$

Proof. Choose $\eta > 0$ sufficiently small and $B > 0$ sufficiently large such that the random variable W in Lemma 14 satisfies $\mathbb{P}(W \leq 2\eta) < \varepsilon/8$ and $\mathbb{P}(W \geq B - \eta) < \varepsilon/8$. By Lemma 14, we can choose $y > 0$ large enough that, for some random variable W having the same distribution as the random variable W in Lemma 14,

$$\mathbb{P}(|ye^{-\sqrt{2}y} K(y) - W| \geq \eta) < \frac{\varepsilon}{8}.$$

These conditions imply that

$$\mathbb{P}(ye^{-\sqrt{2}y}K(y) \leq \eta) < \frac{\varepsilon}{4} \quad (29)$$

and

$$\mathbb{P}(ye^{-\sqrt{2}y}K(y) \geq B) < \frac{\varepsilon}{4}. \quad (30)$$

We can also choose y to be large enough that $y \geq 2|\alpha|$ and $Be^{-\sqrt{2}\alpha}/y < \varepsilon/8$.

For $1 \leq i \leq N(\kappa x^2)$ and $0 \leq s \leq \kappa x^2$, let $x_i(s)$ be the position of the particle at time s that is the ancestor of the particle at the location $X_i(\kappa x^2)$ at time κx^2 . Let $v_i = \inf\{s : x_i(s) = x - y\}$. Let $0 < u_1 < \dots < u_{K(y)}$ denote the times at which particles would hit $x - y$, if particles were killed upon reaching $x - y$. Note that $\{v_1, \dots, v_{N(\kappa x^2)}\} \subset \{u_1, \dots, u_{K(y)}\}$. Let \mathcal{G} denote the σ -field generated by the set of times $\{u_1, \dots, u_{K(y)}\}$. We can choose a positive number $\rho > 0$, depending on y but not on x , such that

$$\mathbb{P}(u_{K(y)} \leq \rho) > 1 - \frac{\varepsilon}{8}. \quad (31)$$

Throughout the proof, we will assume that x is large enough that $x \geq y$, so that particles are not killed at the origin before reaching $x - y$, and that $\kappa x^2/2 \geq \rho$, so that with high probability all particles will have reached $x - y$ well before time κx^2 .

Let

$$M(s) = \sum_{i=1}^{N(s)} X_i(s) e^{\sqrt{2}X_i(s)}. \quad (32)$$

It is well-known (see, for example, Lemma 2 of [15]) that the process $(M(s), s \geq 0)$ is a martingale. If there is initially a single particle at $x - y$, then by the Optional Sampling Theorem, the probability that some particle eventually reaches $x + \alpha$ is at most

$$\frac{(x - y)e^{\sqrt{2}(x-y)}}{(x + \alpha)e^{\sqrt{2}(x+\alpha)}}.$$

Therefore, conditional on \mathcal{G} , the probability that some descendant of a particle that reaches $x - y$ eventually reaches $x + \alpha$ is at most

$$\frac{K(y)(x - y)e^{\sqrt{2}(x-y)}}{(x + \alpha)e^{\sqrt{2}(x+\alpha)}} \leq \frac{e^{-\sqrt{2}\alpha}}{y} \cdot ye^{-\sqrt{2}y}K(y).$$

Thus, the unconditional probability that some descendant of a particle that reaches $x - y$ eventually reaches $x + \alpha$ is at most

$$\mathbb{P}(ye^{-\sqrt{2}y}K(y) > B) + \frac{Be^{-\sqrt{2}\alpha}}{y} < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{3\varepsilon}{8}.$$

In particular, $\mathbb{P}(X_1(\kappa x^2) > x + \alpha) \leq \mathbb{P}(u_{K(y)} > \rho) + 3\varepsilon/8 \leq \varepsilon/2$ for sufficiently large x , which by letting $\varepsilon \rightarrow 0$ implies (28).

Let $S(\alpha) = \{i : x_i(s) < x + \alpha \text{ for all } s \in [v_i, \kappa x^2]\}$. Then let

$$Y'_\alpha = \sum_{i=1}^{N(\kappa x^2)} e^{\sqrt{2}X_i(\kappa x^2)} \mathbf{1}_{\{i \in S(\alpha)\}}$$

and

$$Z'_\alpha = \sum_{i=1}^{N(\kappa x^2)} e^{\sqrt{2}X_i(\kappa x^2)} \sin\left(\frac{\pi X_i(\kappa x^2)}{x + \alpha}\right) \mathbf{1}_{\{i \in S(\alpha)\}}.$$

The argument in the previous paragraph implies that

$$\mathbb{P}(Y'_\alpha = Y(\kappa x^2) \text{ and } Z'_\alpha = Z_\alpha) \geq 1 - \frac{\varepsilon}{2}. \quad (33)$$

By the Strong Markov Property, the configuration of particles at time κx^2 has the same distribution as the configuration that we would get by starting with $K(y)$ particles at $x - y$ and stopping their descendants at the times $\kappa x^2 - u_i$. Furthermore, restricting to particles in $S(\alpha)$ is equivalent to killing particles when they reach $x + \alpha$. Therefore, the tools of Section 2, with $L = x + \alpha$, can be used to estimate the first and second moments of Y'_α and Z'_α .

We first apply (15) with $s = \kappa x^2 - u_i$, which when $u_{K(y)} \leq \rho$ is at least $\kappa x^2/2$. Because the right-hand side of (5) is bounded by a constant when s is of the order L^2 , it follows from (15) that there is a constant C , depending on κ , such that on the event $\{u_{K(y)} \leq \rho\}$,

$$\mathbb{E}[Y'_\alpha | \mathcal{G}] \leq CK(y)e^{\sqrt{2}(x-y)} \sin\left(\frac{\pi(x-y)}{x + \alpha}\right).$$

Using \sim to denote that the ratio of the two sides tends to one as $x \rightarrow \infty$, we have

$$\sin\left(\frac{\pi(x-y)}{x + \alpha}\right) \sim \frac{\pi(y + \alpha)}{x + \alpha} \sim \frac{\pi(y + \alpha)}{x}. \quad (34)$$

Because $y \geq 2|\alpha|$, it follows that there exists a constant C_8 such that on the event $\{u_{K(y)} \leq \rho\}$, for sufficiently large x ,

$$\mathbb{E}[Y'_\alpha | \mathcal{G}] \leq C_8 x^{-1} e^{\sqrt{2}x} \cdot y e^{-\sqrt{2}y} K(y).$$

Therefore, choosing $C_5 = 8C_8 B/\varepsilon$ and using (30), (31), and the conditional Markov's inequality,

$$\begin{aligned} \mathbb{P}(Y'_\alpha \geq C_5 x^{-1} e^{\sqrt{2}x}) &\leq \mathbb{P}(u_{K(y)} > \rho) + \mathbb{P}\left(y e^{-\sqrt{2}y} K(y) \geq B\right) + \mathbb{P}\left(Y'_\alpha \geq \frac{8\mathbb{E}[Y'_\alpha | \mathcal{G}]}{\varepsilon}\right) \\ &\leq \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{\varepsilon}{2}. \end{aligned} \quad (35)$$

The result (26) now follows from (35) and (33).

By (14), on the event $\{u_{K(y)} \leq \rho\}$, we have

$$\begin{aligned} e^{-\pi^2 \kappa x^2 / 2(x+\alpha)^2} K(y) e^{\sqrt{2}(x-y)} \sin\left(\frac{\pi(x-y)}{x + \alpha}\right) \\ \leq \mathbb{E}[Z_\alpha | \mathcal{G}] \leq e^{-\pi^2(\kappa x^2 - \rho) / 2(x+\alpha)^2} K(y) e^{\sqrt{2}(x-y)} \sin\left(\frac{\pi(x-y)}{x + \alpha}\right). \end{aligned}$$

Because (34) holds and $e^{-\pi^2 \kappa x^2 / 2(x+\alpha)^2} \sim e^{-\pi^2 \kappa / 2} \sim e^{-\pi^2(\kappa x^2 - \rho) / 2(x+\alpha)^2}$, there are constants C_9 and C_{10} , depending on κ , such that

$$C_9 x^{-1} e^{\sqrt{2}x} \cdot y e^{-\sqrt{2}y} K(y) \leq \mathbb{E}[Z'_\alpha | \mathcal{G}] \leq C_{10} x^{-1} e^{\sqrt{2}x} \cdot y e^{-\sqrt{2}y} K(y) \quad (36)$$

when $u_{K(y)} \leq \rho$ for sufficiently large x . Furthermore, by applying Lemma 8 to the configuration with a single particle at $x - y$ at time zero and then summing over the particles, we get

$$\text{Var}(Z'_\alpha | \mathcal{G}) \leq \frac{CK(y)e^{\sqrt{2}(x-y)}e^{\sqrt{2}(x+\alpha)}\kappa x^2}{2(x+\alpha)^4} \leq \frac{Ce^{\sqrt{2}\alpha} \cdot ye^{-\sqrt{2}y}K(y)}{y} (x^{-1}e^{\sqrt{2}x})^2$$

for sufficiently large x . By the conditional Chebyshev's Inequality, on the event $\{u_{K(y)} \leq \rho\}$,

$$\mathbb{P}\left(|Z'_\alpha - \mathbb{E}[Z'_\alpha | \mathcal{G}]| > \frac{1}{2} \mathbb{E}[Z'_\alpha | \mathcal{G}]\right) \leq \frac{4\text{Var}(Z'_\alpha | \mathcal{G})}{(\mathbb{E}[Z'_\alpha | \mathcal{G}])^2} \leq \frac{Ce^{\sqrt{2}\alpha}}{y \cdot ye^{-\sqrt{2}y}K(y)}.$$

In view of (29), it follows that for y large enough that $Ce^{\sqrt{2}\alpha}/(\eta y) < \varepsilon/8$ and sufficiently large x ,

$$\mathbb{P}\left(|Z'_\alpha - \mathbb{E}[Z'_\alpha | \mathcal{G}]| > \frac{1}{2} \mathbb{E}[Z'_\alpha | \mathcal{G}]\right) \leq \mathbb{P}(u_{K(y)} > \rho) + \mathbb{P}(ye^{-\sqrt{2}y}K(y) \leq \eta) + \frac{Ce^{\sqrt{2}\alpha}}{\eta y} < \frac{\varepsilon}{2}.$$

Combining this result with (36), we get that for sufficiently large x , the event

$$\frac{C_9}{2} \cdot x^{-1}e^{\sqrt{2}x} \cdot ye^{-\sqrt{2}y}K(y) \leq Z'_\alpha \leq \frac{3C_{10}}{2} \cdot x^{-1}e^{\sqrt{2}x} \cdot ye^{-\sqrt{2}y}K(y)$$

holds with probability at least $1 - \varepsilon/2$. Thus, using (29) and (30), for sufficiently large x we have

$$\frac{C_9\eta}{2} \cdot x^{-1}e^{\sqrt{2}x} \leq Z'_\alpha \leq \frac{3BC_{10}}{2} \cdot x^{-1}e^{\sqrt{2}x}$$

with probability at least $1 - \varepsilon$. The result (27) now follows by setting $C_6 = C_9\eta/2$ and $C_7 = 3BC_{10}/2$ and invoking (33). \square

3.2 A lower bound for $Z(s)$

Let $t = \tau x^3 = 2\sqrt{2}x^3/(3\pi^2)$. For $0 < s < t$, recall that

$$L(s) = x \left(1 - \frac{3\pi^2 s}{2\sqrt{2}x^3}\right)^{1/3} = c(t-s)^{1/3}$$

as in (3), and let

$$Z(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \sin\left(\frac{\pi X_i(s)}{L(s)}\right) \mathbf{1}_{\{X_i(s) \leq L(s)\}}.$$

Our goal in this subsection is to find a lower bound for $Z(s)$. Such a bound will be provided by Proposition 16 below.

To prove this result, we will consider the following new process, which will also be useful in later subsections. Fix $\alpha \in \mathbb{R}$, and let $t_\alpha = \tau(x + \alpha)^3$, so that $ct_\alpha^{1/3} = x + \alpha$, where c is defined in (2). For $0 \leq s \leq t_\alpha$, let $L_\alpha(s) = c(t_\alpha - s)^{1/3}$. Note that $L_0(s) = L(s)$. Now suppose that, in addition to being killed at the origin, particles to the right of $x + \alpha$ are killed at time κx^2 , and for $\kappa x^2 < s < t_\alpha + \kappa x^2$, particles are killed at time s if they reach $L_\alpha(s - \kappa x^2)$. Let $N_\alpha(s)$ be the number of particles alive at time s , and let $X_{1,\alpha}(s) \geq \dots \geq X_{N_\alpha(s),\alpha}(s)$ denote the positions of these particles at time s . Let

$$Z_\alpha(s) = \sum_{i=1}^{N_\alpha(s)} e^{\sqrt{2}X_{i,\alpha}(s)} \sin\left(\frac{\pi X_{i,\alpha}(s)}{L_\alpha(s - \kappa x^2)}\right).$$

Note that $Z_\alpha(\kappa x^2)$ is the same as Z_α defined in (25). Also, let

$$Y_\alpha(s) = \sum_{i=1}^{N_\alpha(s)} e^{\sqrt{2}X_{i,\alpha}(s)}. \quad (37)$$

Proposition 16. *For all $\varepsilon > 0$, there exists a constant $C > 0$ depending on κ , δ , and ε such that for sufficiently large x ,*

$$\mathbb{P}\left(Z(s) \geq Cx^{-1} \exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right) > 1 - \varepsilon$$

for all $s \in [2\kappa x^2, (1-\delta)t]$.

Proof. We consider the process defined above. Recall that $(\mathcal{F}_u)_{u \geq 0}$ is the natural filtration associated with the branching Brownian motion. By Lemma 10 and the Markov property, there exist positive constants C' and C'' , depending on κ and δ , such that for all $s \in [2\kappa x^2, (1-\delta/2)t_\alpha]$,

$$C'Z_\alpha G_0(s - \kappa x^2) \leq \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}] \leq C''Z_\alpha G_0(s - \kappa x^2).$$

Because $(t_\alpha - (s - \kappa x^2))^{1/3} - (t_\alpha - s)^{1/3}$ is bounded by a constant, it follows from (16) that

$$\begin{aligned} C'Z_\alpha \exp\left(- (3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s)^{1/3})\right) &\leq \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}] \\ &\leq C''Z_\alpha \exp\left(- (3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s)^{1/3})\right). \end{aligned} \quad (38)$$

Likewise, by Proposition 13,

$$\begin{aligned} \text{Var}(Z_\alpha(s)|\mathcal{F}_{\kappa x^2}) &\leq C \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]^2 \left(\frac{e^{\sqrt{2}L_\alpha(0)}}{L_\alpha(0)Z_\alpha} + \frac{e^{\sqrt{2}L_\alpha(0)}Y(\kappa x^2)}{L_\alpha(0)^2 Z_\alpha^2} \right) \\ &= C \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]^2 \left(\frac{e^{\sqrt{2}x}e^{\sqrt{2}\alpha}}{(x+\alpha)Z_\alpha} + \frac{e^{\sqrt{2}x}e^{\sqrt{2}\alpha}Y(\kappa x^2)}{(x+\alpha)^2 Z_\alpha^2} \right). \end{aligned}$$

Let A be the event that $Y(\kappa x^2) \leq C_5 x^{-1} e^{\sqrt{2}x}$ and $Z_\alpha \geq C_6 x^{-1} e^{\sqrt{2}x}$, where C_5 and C_6 are the constants from Lemma 15 applied with $\varepsilon/8$ in place of ε . Lemma 15 then gives $\mathbb{P}(A) > 1 - \varepsilon/4$ for sufficiently large x . On A , we have

$$\text{Var}(Z_\alpha(s)|\mathcal{F}_{\kappa x^2}) \leq C \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]^2 e^{\sqrt{2}\alpha} \left(\frac{x}{C_6(x+\alpha)} + \frac{C_5 x}{C_6^2(x+\alpha)^2} \right).$$

Therefore, if α is chosen to be a large enough negative number that $Ce^{\sqrt{2}\alpha}/C_6 < \varepsilon/8$, then $\text{Var}(Z_\alpha(s)|\mathcal{F}_{\kappa x^2}) \leq (\varepsilon/8) \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]^2$ on A for sufficiently large x . It follows from the conditional Chebyshev's Inequality that for sufficiently large x ,

$$\mathbb{P}\left(Z_\alpha(s) < \frac{1}{2} \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]\right) \leq \mathbb{P}(A^c) + \frac{4\varepsilon}{8} < \frac{3\varepsilon}{4}. \quad (39)$$

By (38), on A we have

$$\mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}] \geq Cx^{-1} \exp\left(\sqrt{2}x - (3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s)^{1/3})\right).$$

Thus, using (39) and the fact that $\mathbb{P}(A^c) < \varepsilon/4$, there is a positive constant C such that for all $s \in [2\kappa x^2, (1 - \delta/2)t_\alpha]$,

$$\mathbb{P}\left(Z_\alpha(s) \geq Cx^{-1} \exp(\sqrt{2}x - (3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s)^{1/3}))\right) \geq 1 - \varepsilon$$

for sufficiently large x . Note that $|t_\alpha^{1/3} - t^{1/3}|$ is bounded by a constant which depends on α , and thus on ε . Likewise,

$$\sup_{\kappa x^2 \leq s \leq (1-\delta/2)t_\alpha} |(t_\alpha - s)^{1/3} - (t - s)^{1/3}| \quad (40)$$

is bounded by a constant which depends on α and δ . Furthermore, we have $\sqrt{2}x = (3\pi^2)^{1/3}t^{1/3}$. Because $(1 - \delta/2)t_\alpha \geq (1 - \delta)t$ for sufficiently large x , we obtain the result of the proposition with $Z_\alpha(s)$ in place of $Z(s)$, provided that α is a sufficiently large negative number.

To complete the proof, recall that $L(s) = c(t - s)^{1/3}$ and $L_\alpha(s - \kappa x^2) = c(t_\alpha - s + \kappa x^2)^{1/3}$, where $t = \tau x^3$ and $t_\alpha = \tau(x + \alpha)^3$. Therefore, there is a constant $\alpha_0 < 0$ such that if $\alpha < \alpha_0$, then $L_\alpha(s - \kappa x^2) < L(s)$ for sufficiently large x . Also, $L(s)/2 < L_\alpha(s - \kappa x^2)$ for sufficiently large x . Thus, if $\alpha < \alpha_0$, there exists a constant C such that for sufficiently large x ,

$$\sin\left(\frac{\pi z}{L_\alpha(s - \kappa x^2)}\right) \leq C \sin\left(\frac{\pi z}{L(s)}\right)$$

for all $z \in [0, L_\alpha(s - \kappa x^2)]$. Because killing particles at a right boundary can only reduce the number of particles in the system, it follows that if $\alpha < \alpha_0$, then $Z_\alpha(s) \leq CZ(s)$ for sufficiently large x . The result follows. \square

3.3 Upper bounds for $Z(s)$ and $Y(s)$

Recall that $t = \tau x^3$. The next lemma shows that it is unlikely for any particle ever to get far to the right of $L(s)$ for $s \in [2\kappa x^2, (1 - \delta)t]$.

Lemma 17. *Let $\varepsilon > 0$. For all $\alpha > 0$, let $t_\alpha = \tau(x + \alpha)^3$, and let $L_\alpha(s) = c(t_\alpha - s)^{1/3}$ for $0 \leq s \leq t_\alpha$. Then there exists a positive constant C_{11} , depending on κ , δ , and ε but not on α or x , such that for sufficiently large x ,*

$$\mathbb{P}(X_1(s) \leq L_\alpha(s - \kappa x^2) \text{ for all } s \in [\kappa x^2, (1 - \delta)t]) \geq 1 - \varepsilon - C_{11}e^{-\sqrt{2}\alpha}.$$

Proof. Suppose there is a particle at the location $z \leq ct_\alpha^{1/3} = x + \alpha$ at time κx^2 . By Lemma 12 with $t = t_\alpha$, the probability that a descendant of this particle reaches $L_\alpha(s - \kappa x^2)$ for some $s \in [\kappa x^2, (1 - \delta/2)t_\alpha]$ is at most

$$Ce^{-(3\pi^2 t_\alpha)^{1/3}} \left(e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L_\alpha(0)}\right) t_\alpha^{1/3} + ze^{\sqrt{2}z} t_\alpha^{-1/3} \right).$$

Therefore, using the bound $zt_\alpha^{-1/3} \leq c$ and applying the Markov property, we get that the conditional probability, given $\mathcal{F}_{\kappa x^2}$, on the event $X_1(\kappa x^2) < x + \alpha$, that a particle reaches $L_\alpha(s - \kappa x^2)$ for $\kappa x^2 \leq s \leq (1 - \delta/2)t_\alpha$ is at most

$$Ce^{-\sqrt{2}(x+\alpha)} \left(t_\alpha^{1/3} Z_\alpha(\kappa x^2) + Y(\kappa x^2) \right). \quad (41)$$

Let A be the event that $X_1(\kappa x^2) < x + \alpha$, $Y(\kappa x^2) \leq C_5 x^{-1} e^{\sqrt{2}x}$ and $Z_\alpha(\kappa x^2) \leq C_7 x^{-1} e^{\sqrt{2}x}$, where C_5 and C_7 are the constants from Lemma 15 with $\varepsilon/3$ in place of ε . On A , for sufficiently large x , the expression in (41) is at most

$$C t_\alpha^{1/3} x^{-1} e^{-\sqrt{2}\alpha} + C x^{-1} e^{-\sqrt{2}\alpha} \leq C_{11} e^{-\sqrt{2}\alpha}.$$

Because $\mathbb{P}(A) > 1 - \varepsilon$ for sufficiently large x by Lemma 15 and the fact that $(1 - \delta/2)t_\alpha \geq (1 - \delta)t$ for sufficiently large x , the result follows. \square

The next lemma shows that at any fixed time $s \in [2\kappa x^2, (1 - \delta)t]$, it is unlikely that there is any particle near or to the right of $L(s)$.

Lemma 18. *Let $a > 0$ be a positive constant. Let $\varepsilon > 0$. Then for sufficiently large x , we have*

$$\mathbb{P}(X_1(s) > L(s) - a) < \varepsilon$$

for all $s \in [2\kappa x^2, (1 - \delta)t]$.

Proof. We consider the process defined at the beginning of subsection 3.2 in which at time κx^2 , particles to the right of $x + \alpha$ are killed, and for $\kappa x^2 < s < t_\alpha + \kappa x^2$, particles are killed at time s if they reach $L_\alpha(s - \kappa x^2)$. By (28), for sufficiently large x , the probability that some particle is killed at time κx^2 is at most $\varepsilon/4$. By applying Lemma 17 with $\varepsilon/4$ in place of ε and choosing $\alpha > 0$ large enough that $C_{11} e^{-\sqrt{2}\alpha} < \varepsilon/4$, we get that the probability that a particle is killed between times κx^2 and $(1 - \delta)t$ is at most $\varepsilon/2$. Thus, with probability at least $1 - 3\varepsilon/4$, no particle is killed until at least time $(1 - \delta)t$.

Suppose $s \in [2\kappa x^2, (1 - \delta)t]$. Let $K_\alpha(s)$ be the number of particles at time s between $L(s) - a$ and $L_\alpha(s - \kappa x^2)$. By Lemma 11 with t_α in place of t , we have

$$\begin{aligned} \mathbb{E}[K_\alpha(s) | \mathcal{F}_{\kappa x^2}] &\leq C t_\alpha^{-1/3} e^{-(3\pi^2)^{1/3} (t_\alpha^{1/3} - (t_\alpha - s + \kappa x^2)^{1/3})} Z_\alpha \\ &\quad \times \int_{L(s)-a}^{L_\alpha(s-\kappa x^2)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L_\alpha(s-\kappa x^2)}\right) dy. \end{aligned}$$

For sufficiently large x , the expression $L_\alpha(s - \kappa x^2) - (L(s) - a) = c(t_\alpha - s + \kappa x^2)^{1/3} - c(t - s)^{1/3} + a$ is bounded above by a constant depending on α and a , and thus

$$\int_{L(s)-a}^{L_\alpha(s-\kappa x^2)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L_\alpha(s-\kappa x^2)}\right) dy \leq \frac{C e^{-\sqrt{2}L_\alpha(s-\kappa x^2)}}{L_\alpha(s-\kappa x^2)} \leq C t_\alpha^{-1/3} e^{-\sqrt{2}L_\alpha(s-\kappa x^2)}.$$

Therefore, on the event that $Z_\alpha \leq C_7 x^{-1} e^{\sqrt{2}x}$, where C_7 is the constant from Lemma 15 with $\varepsilon/8$ in place of ε , for sufficiently large x ,

$$\begin{aligned} \mathbb{E}[K_\alpha(s) | \mathcal{F}_{\kappa x^2}] &\leq C t_\alpha^{-2/3} x^{-1} \exp(\sqrt{2}x - (3\pi^2)^{1/3} (t_\alpha^{1/3} - (t_\alpha - s + \kappa x^2)^{1/3}) - \sqrt{2}L_\alpha(s - \kappa x^2)) \\ &\leq C x^{-3} \exp(\sqrt{2}x - \sqrt{2}(x + \alpha) + (3\pi^2)^{1/3} (t_\alpha - s + \kappa x^2)^{1/3} - (3\pi^2)^{1/3} (t_\alpha - s + \kappa x^2)^{1/3}) \\ &\leq C x^{-3} \end{aligned}$$

because the exponential is a constant which depends on α . Therefore, by the conditional Markov's Inequality and Lemma 15, for sufficiently large x ,

$$\mathbb{P}(K_\alpha(s) > 0) \leq \mathbb{P}(Z_\alpha > C_7 x^{-1} e^{\sqrt{2}x}) + Cx^{-3} < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}.$$

Because with probability at least $1 - 3\varepsilon/4$, no particle is killed until at least time $(1 - \delta)t$, it follows that for sufficiently large x , we have $\mathbb{P}(X_1(s) > L(s) - a) < \varepsilon$ for all $s \in [2\kappa x^2, (1 - \delta)t]$. \square

Proposition 19. *For all $\varepsilon > 0$, there exists a constant $C > 0$ depending on κ , δ , and ε such that for sufficiently large x ,*

$$\mathbb{P}\left(Z(s) \leq Cx^{-1} \exp((3\pi^2)^{1/3}(t-s)^{1/3})\right) > 1 - \varepsilon$$

for all $s \in [2\kappa x^2, (1 - \delta)t]$.

Proof. We again work with the process defined at the beginning of subsection 3.2. By (38) and the conditional Markov's Inequality, there is a constant C depending on κ , δ , and ε such that for all $s \in [2\kappa x^2, (1 - \delta)t]$,

$$\mathbb{P}\left(Z_\alpha(s) \leq CZ_\alpha \exp(- (3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s)^{1/3}))\right) > 1 - \frac{\varepsilon}{4}.$$

Therefore, by (27), for all $s \in [2\kappa x^2, (1 - \delta)t]$,

$$\mathbb{P}\left(Z_\alpha(s) \leq Cx^{-1} \exp(\sqrt{2}x - (3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s)^{1/3}))\right) > 1 - \frac{\varepsilon}{2}$$

for sufficiently large x . Because $|t_\alpha^{1/3} - t^{1/3}|$ and the expression in (40) are bounded by constants depending on α and $\sqrt{2}x = (3\pi^2 t)^{1/3}$, it follows that

$$\mathbb{P}\left(Z_\alpha(s) \leq Cx^{-1} \exp((3\pi^2)^{1/3}(t-s)^{1/3})\right) > 1 - \frac{\varepsilon}{2} \quad (42)$$

for sufficiently large x .

From Lemma 17 with $\varepsilon/8$ in place of ε , we see that with probability at least $1 - \varepsilon/8 - C_{11}e^{-\sqrt{2}\alpha}$, no particles are killed between times κx^2 and $(1 - \delta)t$. Therefore, if α is chosen large enough that $C_{11}e^{-\sqrt{2}\alpha} < \varepsilon/8$, then with probability at least $1 - \varepsilon/4$, we have $N_\alpha(s) = N(s)$ and $X_i(s) = X_{i,\alpha}(s)$ for $i = 1, \dots, N(s)$. Furthermore, provided α is also large enough that $L_\alpha(s - \kappa x^2) \geq L(s)$, for sufficiently large x it holds that for $0 \leq x \leq L(s)$, we have

$$\sin\left(\frac{\pi x}{L_\alpha(s - \kappa x^2)}\right) \geq C \sin\left(\frac{\pi x}{L(s)}\right)$$

for some positive constant C . By Lemma 18, for sufficiently large x the probability that $X_1(s) > L(s)$ is less than $\varepsilon/4$. It follows that for sufficiently large x , we have $Z_\alpha(s) \geq CZ(s)$ with probability at least $1 - \varepsilon/2$. Combining this observation with (42) yields the result. \square

Proposition 20. For all $\varepsilon > 0$, there exists a constant $C > 0$ depending on κ , δ , and ε such that for sufficiently large x ,

$$\mathbb{P}\left(Y(s) \leq Cx^{-1} \exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right) > 1 - \varepsilon$$

for all $s \in [2\kappa x^2, (1-\delta)t]$.

Proof. We again work with the process defined at the beginning of subsection 3.2. Recall the definition of $Y_\alpha(s)$ from (37). By Lemma 17, we can choose $\alpha > 0$ sufficiently large that with probability at least $1 - \varepsilon/2$, we have $X_1(s) \leq c(t_\alpha - s + \kappa x^2)^{1/3}$ for all $s \in [\kappa x^2, (1-\delta)t_\alpha]$. Therefore, for all $s \in [2\kappa x^2, (1-\delta)t]$, we have $\mathbb{P}(Y_\alpha(s) = Y(s)) > 1 - \varepsilon/2$.

By Lemma 11 with t_α in place of t , for all $s \in [2\kappa x^2, (1-\delta)t]$,

$$\begin{aligned} \mathbb{E}[Y_\alpha(s) | \mathcal{F}_{\kappa x^2}] &\leq \frac{C}{L_\alpha(s - \kappa x^2)} e^{-(3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s + \kappa x^2)^{1/3})} Z_\alpha \int_0^{L_\alpha(s - \kappa x^2)} \sin\left(\frac{\pi y}{L_\alpha(s - \kappa x^2)}\right) dy \\ &\leq C e^{-(3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s + \kappa x^2)^{1/3})} Z_\alpha. \end{aligned}$$

By combining this result with the conditional Markov's inequality and (27), we get that there is a constant C such that for sufficiently large x ,

$$\mathbb{P}\left(Y_\alpha(s) \leq Cx^{-1} e^{\sqrt{2}x} e^{-(3\pi^2)^{1/3}(t_\alpha^{1/3} - (t_\alpha - s + \kappa x^2)^{1/3})}\right) > 1 - \frac{\varepsilon}{2}$$

for all $s \in [2\kappa x^2, (1-\delta)t]$. Because $|(t_\alpha - s - \kappa x^2)^{1/3} - (t - s)^{1/3}|$ is bounded by a constant depending on α and $(3\pi^2)^{1/3}t_\alpha^{1/3} = \sqrt{2}(x - \alpha)$, there is a constant C depending on α such that

$$\mathbb{P}\left(Y_\alpha(s) \leq Cx^{-1} \exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right) > 1 - \frac{\varepsilon}{2}$$

for all $s \in [2\kappa x^2, (1-\delta)t]$. The result follows because $\mathbb{P}(Y_\alpha(s) = Y(s)) > 1 - \varepsilon/2$. \square

3.4 Moments of functions of branching Brownian motion

Suppose $\kappa > 0$ and $\delta > 0$. Let $\varepsilon > 0$. Choose a constant $B > 0$ sufficiently large that if $s = BL^2$, the right-hand side of (5) is at most ε . Now fix a time s such that

$$(B + 3\kappa)x^2 \leq s \leq (1 - \delta)t.$$

Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $\phi : [0, 1] \rightarrow \mathbb{R}$ be bounded continuous functions. Let $\|f\| = \sup_{x \geq 0} |f(x)|$ and $\|\phi\| = \sup_{0 \leq x \leq 1} |\phi(x)|$. We are interested here in the quantities

$$\sum_{i=1}^{N(s)} f(X_i(s)). \tag{43}$$

and

$$\sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \phi\left(\frac{X_i(s)}{L(s)}\right) \mathbf{1}_{\{X_i(s) < L(s)\}}. \tag{44}$$

Let $r = s - Bx^2$. Let A be the event that $X_1(u) \leq L(s)$ for all $u \in [r, s]$. By Proposition 20, there is a positive constant C such that

$$\mathbb{P}\left(Y(r) \leq Cx^{-1} \exp((3\pi^2)^{1/3}(t-r)^{1/3})\right) > 1 - \varepsilon \quad (45)$$

for sufficiently large x . Because $L(r) - L(s)$ is bounded above by a constant, Lemma 18 implies that

$$\mathbb{P}(X_1(r) \leq L(s)) > 1 - \varepsilon \quad (46)$$

for sufficiently large x . Because $M(r)$, as defined in (32), is bounded by $X_1(r)Y(r)$, we have

$$M(r) \leq CL(s)x^{-1} \exp((3\pi^2)^{1/3}(t-r)^{1/3})$$

when the events in (45) and (46) both occur. By the Optional Sampling Theorem, the probability, conditional on \mathcal{F}_r , that some particle reaches $L(s)$ between times r and s is at most $M(r)/(L(s)e^{\sqrt{2}L(s)})$. Therefore,

$$\mathbb{P}(A^c) \leq 2\varepsilon + Cx^{-1} \exp((3\pi^2)^{1/3}(t-r)^{1/3} - \sqrt{2}L(s)). \quad (47)$$

Because $\sqrt{2}L(s) = (3\pi^2)^{1/3}(t-s)^{1/3}$, the exponential on the right-hand side of (47) is bounded by a constant. Therefore, the second term on the right-hand side of (47) tends to zero as $x \rightarrow \infty$, and thus $\mathbb{P}(A^c) < 3\varepsilon$ for sufficiently large x .

Let S be the set of all $i \in \{1, \dots, N(s)\}$ such that for all $u \in [r, s]$, the particle at time u that is the ancestor of the particle at $X_i(s)$ at time s is positioned to the left of $L(s)$. We will work with the quantities

$$X(f) = \sum_{i=1}^{N(s)} f(X_i(s)) \mathbf{1}_{\{i \in S\}}$$

and

$$X'(\phi) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \phi\left(\frac{X_i(s)}{L(s)}\right) \mathbf{1}_{\{i \in S\}}.$$

Note that $X(f)$ and $X'(\phi)$ equal the sums in (43) and (44) respectively on the event A , so we have the following result.

Lemma 21. *Suppose ε , B , r , and s are as defined above. Then for sufficiently large x , with probability greater than $1 - 3\varepsilon$, the quantity $X(f)$ equals the sum in (43) and $X'(\phi)$ equals the sum in (44) for all bounded continuous functions $f : [0, \infty) \rightarrow \mathbb{R}$ and $\phi : [0, 1] \rightarrow \mathbb{R}$.*

Because $X(f)$ and $X'(\phi)$ are the sums that would be obtained if particles were killed at $L(s)$ between times r and s , we can compute conditional moments of $X(f)$ and $X'(\phi)$ by applying Lemma 9 with Bx^2 in place of s and $L(s)$ in place of L . For the rest of this subsection, we define $q_u(x, y)$ as in Lemma 5 with $L(s)$ in place of L .

Define

$$\hat{Z} = \sum_{i=1}^{N(r)} e^{\sqrt{2}X_i(r)} \sin\left(\frac{\pi X_i(r)}{L(s)}\right) \mathbf{1}_{\{X_i(r) \leq L(s)\}}. \quad (48)$$

Note that \hat{Z} is defined in the same way as $Z(r)$, except that $L(s)$ is used instead of $L(r)$ in the denominator of the sine function and in the indicator. Lemma 18 implies that with probability

tending to one as $x \rightarrow \infty$, we have $X_1(r) \leq L(r) - 2(L(r) - L(s))$. Therefore, there are positive constants C' and C'' such that for sufficiently large x ,

$$\mathbb{P}(C'Z(r) \leq \hat{Z} \leq C''Z(r)) > 1 - \varepsilon. \quad (49)$$

Lemma 22. *For sufficiently large x , we have*

$$\left| \mathbb{E}[X(f)|\mathcal{F}_r] - \hat{Z} \frac{\pi}{L(s)^2} e^{-\pi^2 Bx^2/2L(s)^2} \int_0^\infty f(y)g(y) dy \right| < \frac{2\pi\|f\|\varepsilon}{L(s)^2} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z},$$

where $g(y) = 2ye^{-\sqrt{2}y}$ as in Theorem 3.

Proof. Because the right-hand side of (5) is at most ε when $s = Bx^2$, it follows from Lemma 9 and Lemma 5 that

$$\begin{aligned} \mathbb{E}[X(f)|\mathcal{F}_r] &= \sum_{i=1}^{N(r)} \int_0^{L(s)} f(y)q_{Bx^2}(X_i(r), y) dy \\ &= \frac{2(1+D)}{L(s)} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z} \int_0^{L(s)} f(y)e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) dy, \end{aligned}$$

where $|D| < \varepsilon$. Note that

$$\lim_{x \rightarrow \infty} L(s) \int_0^{L(s)} f(y)e^{-\sqrt{2}y} \left| \frac{\pi y}{L(s)} - \sin\left(\frac{\pi y}{L(s)}\right) \right| dy = 0$$

and

$$\lim_{x \rightarrow \infty} \int_{L(s)}^\infty f(y)e^{-\sqrt{2}y} \cdot \pi y dy = 0.$$

It follows that

$$L(s) \int_0^{L(s)} f(y)e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) dy = \int_0^\infty f(y)e^{-\sqrt{2}y} \cdot \pi y dy + \gamma(x),$$

where $\gamma(x) \rightarrow 0$ as $x \rightarrow \infty$. Therefore,

$$\mathbb{E}[X(f)|\mathcal{F}_r] = \hat{Z} \frac{\pi(1+D)}{L(s)^2} e^{-\pi^2 Bx^2/2L(s)^2} \left(\int_0^\infty f(y)g(y) dy + \frac{2\gamma(x)}{\pi} \right). \quad (50)$$

To obtain the result from (50), first note that the error term involving $\gamma(x)$ is bounded by $2(1+\varepsilon)L(s)^{-2}e^{-\pi^2 Bx^2/2L(s)^2}\hat{Z}\gamma(x)$, and then bound the remaining error term involving D by $\pi\varepsilon L(s)^{-2}e^{-\pi^2 Bx^2/2L(s)^2}\|f\|\hat{Z}$. \square

Lemma 23. *There is a constant C such that for sufficiently large x ,*

$$\text{Var}(X(f)|\mathcal{F}_r) \leq \frac{CY(r)e^{\sqrt{2}L(s)}}{x^{11/2}}.$$

Proof. By summing over the contributions of the particles at time r and applying Lemma 9, we get

$$\begin{aligned} \text{Var}(X(f)|\mathcal{F}_r) &\leq \sum_{i=1}^{N(r)} \int_0^{L(s)} f(y)^2 q_{Bx^2}(X_i(r), y) dy \\ &\quad + 2 \sum_{i=1}^{N(r)} \int_0^{Bx^2} \int_0^{L(s)} q_u(X_i(r), z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z, y) dy \right)^2 dz du. \end{aligned} \quad (51)$$

The first term on the right-hand side of (51) is bounded by $\|f\|^2 \mathbb{E}[X(1)|\mathcal{F}_r]$, where $X(1)$ denotes the value of $X(f)$ when $f(x) = 1$ for all x . Consequently, by Lemma 22, this term is bounded above by $C\hat{Z}x^{-2} \leq CY(r)x^{-2} \leq CY(r)e^{\sqrt{2}L(s)}/x^{11/2}$.

It remains to bound the second term. The strategy is very similar to that used in the proof of Proposition 14 in [5] and involves splitting the outer integral into four pieces. Suppose $0 < w < L(s)$. Using Lemma 5 and equations (6) and (10),

$$\begin{aligned} &\int_0^{Bx^2/2} \int_0^{L(s)} q_u(w, z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z, y) dy \right)^2 dz du \\ &\leq \int_0^{Bx^2/2} \int_0^{L(s)} q_u(w, z) \left(\int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) dy \right)^2 dz du \\ &\leq \frac{C}{L(s)^2} \int_0^{Bx^2/2} \int_0^{L(s)} q_u(w, z) e^{2\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right)^2 \left(\int_0^{L(s)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) dy \right)^2 dz du \\ &\leq \frac{C}{L(s)^4} \int_0^{L(s)} e^{2\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right)^2 \left(\int_0^{Bx^2/2} q_u(w, z) du \right) dz \\ &\leq \frac{Ce^{\sqrt{2}w}}{L(s)^4} \int_0^{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right)^2 \frac{w(L(s) - z)}{L(s)} dz \\ &\leq \frac{Ce^{\sqrt{2}w} e^{\sqrt{2}L(s)}}{L(s)^6}. \end{aligned} \quad (52)$$

Using Lemma 5 and (7),

$$\begin{aligned}
& \int_{Bx^2/2}^{Bx^2-L(s)^{7/4}} \int_0^{L(s)} q_u(w, z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z, y) dy \right)^2 dz du \\
& \leq \int_{Bx^2/2}^{Bx^2-L(s)^{7/4}} \int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) e^{-\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) \\
& \quad \times \left(\int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) \cdot \frac{CL(s)^3}{(Bx^2-u)^{3/2}} dy \right)^2 dz du \\
& \leq CL(s)^3 e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) \left(\int_{Bx^2/2}^{Bx^2-L(s)^{7/4}} \frac{1}{(Bx^2-u)^3} du \right) \\
& \quad \times \left(\int_0^{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right)^3 dz \right) \left(\int_0^{L(s)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) dy \right)^2 \\
& \leq CL(s)^3 e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) \cdot \frac{1}{L(s)^{7/2}} \cdot \frac{e^{\sqrt{2}L(s)}}{L(s)^3} \cdot \frac{1}{L(s)^2} \\
& = \frac{Ce^{\sqrt{2}w} e^{\sqrt{2}L(s)}}{L(s)^{11/2}} \sin\left(\frac{\pi w}{L(s)}\right). \tag{53}
\end{aligned}$$

Using (8), we get

$$\begin{aligned}
& \int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \int_0^{2L(s)/3} q_u(w, z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z, y) dy \right)^2 dz du \\
& \leq \int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \int_0^{2L(s)/3} \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) \\
& \quad \times e^{-\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) \left(\int_0^{L(s)} \frac{Ce^{\sqrt{2}(z-y)}}{(Bx^2-u)^{1/2}} dy \right)^2 dz du \\
& \leq \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) \left(\int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \frac{1}{Bx^2-u} du \right) \left(\int_0^{2L(s)/3} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) dz \right) \\
& \leq \frac{Ce^{\sqrt{2}w} e^{2\sqrt{2}L(s)/3} \log L(s)}{L(s)} \sin\left(\frac{\pi w}{L(s)}\right) \tag{54}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \int_{2L(s)/3}^{L(s)} q_u(w, z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z, y) dy \right)^2 dz du \\
& \leq \int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \int_{2L(s)/3}^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) \\
& \quad \times e^{-\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) \left(\int_0^{L(s)} \frac{C e^{\sqrt{2}(z-y)} e^{-(z-y)^2/2(Bx^2-u)}}{(Bx^2-u)^{1/2}} dy \right)^2 dz du \\
& \leq \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) \left(\int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \frac{1}{Bx^2-u} du \right) \\
& \quad \times \int_{2L(s)/3}^{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) \left(\int_0^{L(s)} e^{-\sqrt{2}y} e^{-(z-y)^2/2L(s)^{7/4}} dy \right)^2 dz \\
& \leq \frac{C \log L(s)}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) \\
& \quad \times \int_{2L(s)/3}^{L(s)} e^{\sqrt{2}z} \left(\int_0^{L(s)/3} e^{-\sqrt{2}y} e^{-(L(s)/3)^2/2L(s)^{7/4}} dy + \int_{L(s)/3}^{L(s)} e^{-\sqrt{2}y} dy \right)^2 dz \\
& \leq \frac{C \log L(s)}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) \left(\int_{2L(s)/3}^{L(s)} e^{\sqrt{2}z} dz \right) \left(e^{-L(s)^{1/4}/18} + e^{-\sqrt{2}L(s)/3} \right)^2 \\
& \leq \frac{C \log L(s)}{L(s)} e^{\sqrt{2}w} e^{\sqrt{2}L(s)} e^{-L(s)^{1/4}/9} \sin\left(\frac{\pi w}{L(s)}\right). \tag{55}
\end{aligned}$$

Finally, using (9),

$$\begin{aligned}
& \int_{Bx^2-1}^{Bx^2} \int_0^{L(s)} q_u(w, z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z, y) dy \right)^2 dz du \\
& \leq \int_{Bx^2-1}^{Bx^2} \int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) e^{-\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) (\|f\|e)^2 dz du \\
& \leq \frac{C e^{\sqrt{2}w}}{L(s)^2} \sin\left(\frac{\pi w}{L(s)}\right). \tag{56}
\end{aligned}$$

The expressions in (52), (53), (54), (55), and (56) are all bounded by $C e^{\sqrt{2}w} e^{\sqrt{2}L(s)} / L(s)^{11/2}$. Because $L(s)$ and x are the same to within a constant factor, we get after summing over the positions of the particles at time r that the second term on the right-hand side of (51) is bounded by $CY(r) e^{\sqrt{2}L(s)} / x^{11/2}$. The result follows. \square

Lemma 24. *For sufficiently large x , we have*

$$\left| \mathbb{E}[X'(\phi) | \mathcal{F}_r] - \frac{4\hat{Z}}{\pi} e^{-\pi^2 Bx^2/2L(s)^2} \int_0^1 \phi(y) h(y) dy \right| < \frac{4\|\phi\|\varepsilon}{\pi} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z},$$

where $h(y) = \frac{\pi}{2} \sin(\pi y)$ as in Theorem 4.

Proof. Because the right-hand side of (5) is at most ε when $s = Bx^2$, it follows from Lemma 9 and Lemma 5 that

$$\begin{aligned}\mathbb{E}[X'(\phi)|\mathcal{F}_r] &= \sum_{i=1}^{N(r)} \int_0^{L(s)} e^{\sqrt{2}y} \phi\left(\frac{y}{L(s)}\right) q_{Bx^2}(X_i(r), y) dy \\ &= \frac{2(1+D)}{L(s)} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z} \int_0^{L(s)} \phi\left(\frac{y}{L(s)}\right) \sin\left(\frac{\pi y}{L(s)}\right) dy \\ &= \frac{4(1+D)}{\pi} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z} \int_0^1 \phi(y) h(y) dy\end{aligned}$$

where $|D| < \varepsilon$. Because h is a probability density, the error term involving D is bounded by $(4\|\phi\|\varepsilon/\pi)e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z}$, as claimed. \square

Lemma 25. *There is a constant C such that for sufficiently large x ,*

$$\text{Var}(X'(\phi)|\mathcal{F}_r) \leq \frac{CY(r)e^{\sqrt{2}L(s)} \log x}{x^2}.$$

Proof. By summing over the contributions of the particles at time r and applying Lemma 9, we get

$$\begin{aligned}\text{Var}(X'(\phi)|\mathcal{F}_r) &\leq \sum_{i=1}^{N(r)} \int_0^{L(s)} e^{2\sqrt{2}y} \phi\left(\frac{y}{L(s)}\right)^2 q_{Bx^2}(X_i(r), y) dy \\ &\quad + 2 \sum_{i=1}^{N(r)} \int_0^{Bx^2} q_u(X_i(r), z) \left(\int_0^{L(s)} e^{\sqrt{2}y} \phi\left(\frac{y}{L(s)}\right) q_{Bx^2-u}(z, y) dy \right)^2 dz du\end{aligned}\tag{57}$$

To bound the first term on the right-hand side of (57), note that if $0 < w < L(s)$, then, by Lemma 5 and (6),

$$\begin{aligned}\int_0^{L(s)} e^{2\sqrt{2}y} \phi\left(\frac{y}{L(s)}\right)^2 q_{Bx^2}(w, y) dy &\leq \frac{Ce^{\sqrt{2}w}}{L(s)} \sin\left(\frac{\pi w}{L(s)}\right) \int_0^{L(s)} e^{\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) dw \\ &\leq \frac{Ce^{\sqrt{2}w} e^{\sqrt{2}L(s)}}{L(s)^2} \sin\left(\frac{\pi w}{L(s)}\right).\end{aligned}\tag{58}$$

We bound the second term on the right-hand side of (57) by breaking the outer integral into two

pieces. Using (10), if $0 < w < L(s)$, then

$$\begin{aligned}
& \int_0^{Bx^2/2} \int_0^{L(s)} q_u(w, z) \left(\int_0^{L(s)} e^{\sqrt{2}y} \phi\left(\frac{y}{L(s)}\right) q_{Bx^2-u}(z, y) dy \right)^2 dz du \\
& \leq \int_0^{Bx^2/2} \int_0^{L(s)} q_u(w, z) \left(\int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) \sin\left(\frac{\pi y}{L(s)}\right) dy \right)^2 dz du \\
& \leq C \int_0^{Bx^2/2} \int_0^{L(s)} q_u(w, z) e^{2\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right)^2 dz du \\
& \leq C \int_0^{L(s)} e^{\sqrt{2}w} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right)^2 \frac{w(L(s) - z)}{L(s)} dz \\
& \leq \frac{C e^{\sqrt{2}w} e^{\sqrt{2}L(s)}}{L(s)^2}.
\end{aligned} \tag{59}$$

Furthermore, by using (11) in the third line, making the substitution $v = Bx^2 - u$ in the fourth line, and breaking the inner integral into the piece from 0 to 1 and the piece from 1 to $Bx^2/2$ in the fifth line, we get

$$\begin{aligned}
& \int_{Bx^2/2}^{Bx^2} \int_0^{L(s)} q_u(w, z) \left(\int_0^{L(s)} e^{\sqrt{2}y} \phi\left(\frac{y}{L(s)}\right) q_{Bx^2-u}(z, y) dy \right)^2 dz du \\
& \leq \int_{Bx^2/2}^{Bx^2} \int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\frac{\pi w}{L(s)}\right) e^{-\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) \left(\int_0^{L(s)} e^{\sqrt{2}y} q_{Bx^2-u}(z, y) dy \right)^2 dz du \\
& \leq \frac{C e^{\sqrt{2}w}}{L(s)} \sin\left(\frac{\pi w}{L(s)}\right) \int_{Bx^2/2}^{Bx^2} \int_0^{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) \min\left\{1, \frac{(L(s) - z)^2}{Bx^2 - u}\right\} dz du \\
& \leq \frac{C e^{\sqrt{2}w}}{L(s)} \sin\left(\frac{\pi w}{L(s)}\right) \int_0^{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) \left(\int_0^{Bx^2/2} \min\left\{1, \frac{(L(s) - z)^2}{v}\right\} dv \right) dz \\
& \leq \frac{C e^{\sqrt{2}w}}{L(s)} \sin\left(\frac{\pi w}{L(s)}\right) \int_0^{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) (1 + (L(s) - z)^2 \log x) dz \\
& \leq \frac{C e^{\sqrt{2}w} e^{\sqrt{2}L(s)} \log x}{L(s)^2} \sin\left(\frac{\pi w}{L(s)}\right).
\end{aligned} \tag{60}$$

The expressions in (58), (59), and (60) are all bounded by $(C e^{\sqrt{2}w} e^{\sqrt{2}L(s)} \log x)/L(s)^2$. By summing over the positions of the particles at time r , we get that the right-hand side of (57) is bounded by $(CY(r) e^{\sqrt{2}L(s)} \log x)/x^2$, which implies the result. \square

3.5 Proofs of Theorems 1, 3, and 4

In this subsection, we use the results of subsection 3.4 to prove Theorems 1, 3, and 4.

Proof of Theorem 1. Let $\kappa = 1$. Choose B as at the beginning of subsection 3.4. Choose $s \in [(B + 3\kappa)x^2, (1 - \delta)t]$, and let $r = s - Bx^2$ as in subsection 3.4. Throughout the proof, the constants C , C' , and C'' will be allowed to depend on B , δ and ε . Recall that $X(1)$ denotes the value of $X(f)$ when $f(x) = 1$ for all x . By Lemma 21,

$$\mathbb{P}(X(1) = N(s)) > 1 - 3\varepsilon \tag{61}$$

for sufficiently large x . By Lemma 22,

$$(1 - 2\varepsilon)\hat{Z}\frac{\pi}{L(s)^2}e^{-\pi^2 Bx^2/2L(s)^2} \leq \mathbb{E}[X(1)|\mathcal{F}_r] \leq (1 + 2\varepsilon)\hat{Z}\frac{\pi}{L(s)^2}e^{-\pi^2 Bx^2/2L(s)^2}. \quad (62)$$

Using Lemma 23 and the conditional Chebyshev's Inequality,

$$\mathbb{P}\left(|X(1) - \mathbb{E}[X(1)|\mathcal{F}_r]| > \frac{1}{2}\mathbb{E}[X(1)|\mathcal{F}_r] \middle| \mathcal{F}_r\right) \leq \frac{CY(r)e^{\sqrt{2}L(s)}}{x^{11/2}\mathbb{E}[X(1)|\mathcal{F}_r]^2} \leq \frac{CY(r)e^{\sqrt{2}L(s)}}{x^{3/2}\hat{Z}^2}. \quad (63)$$

By (49), Proposition 16, Proposition 19, and Proposition 20, there are constants C , C' and C'' such that with probability at least $1 - 4\varepsilon$, we have

$$C'x^{-1}\exp((3\pi^2)^{1/3}(t-r)^{1/3}) \leq \hat{Z} \leq C''x^{-1}\exp((3\pi^2)^{1/3}(t-r)^{1/3}) \quad (64)$$

and

$$Y(r) \leq Cx^{-1}\exp((3\pi^2)^{1/3}(t-r)^{1/3}). \quad (65)$$

Thus, on an event of probability at least $1 - 4\varepsilon$, the quantity on the right-hand side of (63) is bounded above by

$$Cx^{-1/2}\exp(\sqrt{2}L(s) - (3\pi^2)^{1/3}(t-r)^{1/3}) = Cx^{-1/2}\exp((3\pi^2)^{1/3}(t-s)^{1/3} - (3\pi^2)^{1/3}(t-r)^{1/3}),$$

which tends to zero as $x \rightarrow \infty$ because the exponential term is bounded by a constant. By (62), on this same event of probability $1 - 4\varepsilon$, there are constants C' and C'' such that

$$C'x^{-3}\exp((3\pi^2)^{1/3}(t-s)^{1/3}) \leq \frac{1}{2}\mathbb{E}[X(1)|\mathcal{F}_r] \leq \frac{3}{2}\mathbb{E}[X(1)|\mathcal{F}_r] \leq C''x^{-3}\exp((3\pi^2)^{1/3}(t-s)^{1/3}).$$

Combining these results with (61), we get

$$\mathbb{P}\left(C'x^{-3}\exp((3\pi^2)^{1/3}(t-s)^{1/3}) \leq N(s) \leq C''x^{-3}\exp((3\pi^2)^{1/3}(t-s)^{1/3})\right) > 1 - 7\varepsilon$$

for sufficiently large x . Because the constants C' and C'' do not depend on s and

$$(3\pi^2)^{1/3}(t-s)^{1/3} = \sqrt{2}\left(1 - \frac{s}{\tau x^3}\right)^{1/3}x,$$

the result follows. \square

We will now prove slightly more general versions of Theorems 3 and 4. The generalizations will be useful later for the proof of Theorem 2. Instead of setting $s = ux^3$ and letting $x \rightarrow \infty$, we consider a sequence $(x_n)_{n=1}^\infty$ tending to infinity. We define $X_i(s)$, $N(s)$, $X(f)$, and $X'(\phi)$ as before, but with the initial particle located at x_n rather than at x . Note that $X_i(s)$, $N(s)$, $X(f)$, and $X'(\phi)$ depend on n , even though this dependence is not recorded in the notation. The following proposition implies Theorem 3. Here \sim means that the ratio of the two sides tends to one as $n \rightarrow \infty$.

Proposition 26. *Suppose $0 < u < \tau$. Consider a sequence of times $(s_n)_{n=1}^\infty$ such that $s_n \sim ux_n^3$. Let*

$$\chi_n(u) = \frac{1}{N(s_n)} \sum_{i=1}^{N(s_n)} \delta_{X_i(s_n)}.$$

Define μ as in Theorem 3. Then $\chi_n(u) \Rightarrow \mu$ as $n \rightarrow \infty$.

Proof. To show that $\chi_n(u) \Rightarrow \mu$ as $n \rightarrow \infty$, it suffices to show (see, for example, Theorem 16.16 of [18]) that for all bounded continuous functions $f : [0, \infty) \rightarrow \mathbb{R}$, we have

$$\frac{1}{N(s_n)} \sum_{i=1}^{N(s_n)} f(X_i(s_n)) \rightarrow_p \int_0^\infty g(y)f(y) dy, \quad (66)$$

where $g(y) = 2ye^{-\sqrt{2}y}$ for $y \geq 0$ and \rightarrow_p denotes convergence in probability as $n \rightarrow \infty$.

Fix a bounded continuous function $f : [0, \infty) \rightarrow \mathbb{R}$. Let $\varepsilon > 0$, and choose B as at the beginning of subsection 3.4. Let $r_n = s_n - Bx^2$. By Lemma 21, for sufficiently large n ,

$$\mathbb{P}\left(\frac{1}{N(s_n)} \sum_{i=1}^{N(s_n)} f(X_i(s_n)) = \frac{X(f)}{X(1)}\right) > 1 - 3\varepsilon. \quad (67)$$

By Lemma 23 and the conditional Chebyshev's Inequality,

$$\begin{aligned} \mathbb{P}\left(\left|X(f) - \mathbb{E}[X(f)|\mathcal{F}_{r_n}]\right| > x_n^{-19/6}e^{\sqrt{2}L(s_n)} \middle| \mathcal{F}_{r_n}\right) &\leq \frac{CY(r_n)e^{\sqrt{2}L(s_n)}}{x_n^{11/2}} \cdot \frac{x_n^{19/3}}{e^{2\sqrt{2}L(s_n)}} \\ &\leq \frac{CY(r_n)x_n^{5/6}}{e^{\sqrt{2}L(s_n)}}. \end{aligned} \quad (68)$$

Both (64) and (65) hold, with r_n in place of r , with probability at least $1 - 4\varepsilon$ for sufficiently large n . Because $(t - r_n)^{1/3} - (t - s_n)^{1/3}$ is bounded by a constant, the expression obtained by replacing $Y(r_n)$ on the right-hand side of (68) by the upper bound from (65) tends to zero as $n \rightarrow \infty$, and thus is less than ε for sufficiently large n . The same convergence holds with $X(1)$ in place of $X(f)$ on the left-hand side of (68). Thus, for sufficiently large n , on an event of probability at least $1 - 5\varepsilon$, we have

$$\frac{\mathbb{E}[X(f)|\mathcal{F}_{r_n}] - x_n^{-19/6}e^{\sqrt{2}L(s_n)}}{\mathbb{E}[X(1)|\mathcal{F}_{r_n}] + x_n^{-19/6}e^{\sqrt{2}L(s_n)}} \leq \frac{X(f)}{X(1)} \leq \frac{\mathbb{E}[X(f)|\mathcal{F}_{r_n}] + x_n^{-19/6}e^{\sqrt{2}L(s_n)}}{\mathbb{E}[X(1)|\mathcal{F}_{r_n}] - x_n^{-19/6}e^{\sqrt{2}L(s_n)}}.$$

This inequality, when combined with Lemma 22, becomes

$$\begin{aligned} &\frac{\hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/2L(s_n)^2} \left(\int_0^\infty f(y)g(y) dy - 2\|f\|\varepsilon\right) - x_n^{-19/6}e^{\sqrt{2}L(s_n)}}{\hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/2L(s_n)^2} (1 + 2\varepsilon) + x_n^{-19/6}e^{\sqrt{2}L(s_n)}} \\ &\leq \frac{X(f)}{X(1)} \leq \frac{\hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/2L(s_n)^2} \left(\int_0^\infty f(y)g(y) dy + 2\|f\|\varepsilon\right) + x_n^{-19/6}e^{\sqrt{2}L(s_n)}}{\hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/2L(s_n)^2} (1 - 2\varepsilon) - x_n^{-19/6}e^{\sqrt{2}L(s_n)}}. \end{aligned}$$

When (64) holds, we have $x_n^{-3}e^{\sqrt{2}L(s_n)} \leq C\hat{Z}L(s_n)^{-2}$, and thus for sufficiently large n ,

$$x_n^{-19/6}e^{\sqrt{2}L(s_n)} \leq \hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/2L(s_n)^2} \varepsilon.$$

Therefore, for sufficiently large n ,

$$\frac{1}{1 + 3\varepsilon} \left(\int_0^\infty f(y)g(y) dy - 2\|f\|\varepsilon - \varepsilon \right) \leq \frac{X(f)}{X(1)} \leq \frac{1}{1 - 3\varepsilon} \left(\int_0^\infty f(y)g(y) dy + 2\|f\|\varepsilon + \varepsilon \right)$$

with probability at least $1 - 5\varepsilon$. In view of (67), we can let $\varepsilon \rightarrow 0$ to obtain (66). \square

The following proposition implies Theorem 4.

Proposition 27. *Suppose $0 < u < \tau$. Consider a sequence of times $(s_n)_{n=1}^\infty$ such that $s_n \sim ux_n^3$ as $n \rightarrow \infty$. Let*

$$\eta_n(u) = \frac{1}{Y(s_n)} \sum_{i=1}^{N(s_n)} e^{\sqrt{2}X_i(s_n)} \delta_{X_i(s_n)/L(s_n)}.$$

Let ν be defined as in Theorem 4. Then $\eta_n(u) \Rightarrow \nu$ as $n \rightarrow \infty$.

Proof. The proof is very similar to the proof of Theorem 3. It suffices to show that we have $\mathbb{P}(X_1(s_n) < L(s_n)) \rightarrow 1$ as $n \rightarrow \infty$, and that for all bounded continuous functions $\phi : [0, 1] \rightarrow \mathbb{R}$,

$$\frac{1}{Y(s_n)} \sum_{i=1}^{N(s_n)} e^{\sqrt{2}X_i(s_n)} \phi\left(\frac{X_i(s_n)}{L(s_n)}\right) \rightarrow_p \int_0^1 \phi(y)h(y) dy. \quad (69)$$

That $\mathbb{P}(X_1(s_n) < L(s_n)) \rightarrow 1$ as $n \rightarrow \infty$ follows immediately from Lemma 18 with $a = 0$.

Fix a bounded continuous function $\phi : [0, 1] \rightarrow \mathbb{R}$. Let $\varepsilon > 0$, and choose B as at the beginning of subsection 3.4. Let $r_n = s_n - Bx_n^2$. Let $X'(1)$ denote the value of $X'(\phi)$ when $\phi(x) = 1$ for all $x \in [0, 1]$. By Lemma 21, for sufficiently large x ,

$$\mathbb{P}\left(\frac{1}{Y(s_n)} \sum_{i=1}^{N(s_n)} e^{\sqrt{2}X_i(s_n)} \phi\left(\frac{X_i(s_n)}{L(s_n)}\right) = \frac{X'(\phi)}{X'(1)}\right) > 1 - 3\varepsilon. \quad (70)$$

By Lemma 25 and the conditional Chebyshev's Inequality,

$$\begin{aligned} \mathbb{P}\left(\left|X'(\phi) - \mathbb{E}[X'(\phi)|\mathcal{F}_{r_n}]\right| > x_n^{-4/3} e^{\sqrt{2}L(s_n)} \Big| \mathcal{F}_{r_n}\right) &\leq \frac{CY(r_n)e^{\sqrt{2}L(s_n)} \log x_n}{x_n^2} \cdot \frac{x_n^{8/3}}{e^{2\sqrt{2}L(s_n)}} \\ &\leq \frac{CY(r_n)x_n^{2/3} \log x_n}{e^{\sqrt{2}L(s_n)}}. \end{aligned} \quad (71)$$

Recall that (64) and (65) both hold with probability at least $1 - 4\varepsilon$ for sufficiently large n . The expression obtained by replacing $Y(r_n)$ with the right-hand side of (65) on the right-hand side of (71) tends to zero as $x_n \rightarrow \infty$, and the same result holds when $X'(\phi)$ is replaced by $X'(1)$ on the left-hand side. Thus, for sufficiently large n , on an event of probability at least $1 - 5\varepsilon$, we have

$$\frac{\mathbb{E}[X'(\phi)|\mathcal{F}_{r_n}] - x_n^{-4/3} e^{\sqrt{2}L(s_n)}}{\mathbb{E}[X'(1)|\mathcal{F}_{r_n}] + x_n^{-4/3} e^{\sqrt{2}L(s_n)}} \leq \frac{X'(\phi)}{X'(1)} \leq \frac{\mathbb{E}[X'(\phi)|\mathcal{F}_{r_n}] + x_n^{-4/3} e^{\sqrt{2}L(s_n)}}{\mathbb{E}[X'(1)|\mathcal{F}_{r_n}] - x_n^{-4/3} e^{\sqrt{2}L(s_n)}}.$$

Combining this inequality with Lemma 24 gives

$$\begin{aligned} &\frac{4\pi^{-1} \hat{Z} e^{-\pi^2 Bx_n^2/2L(s_n)^2} \left(\int_0^1 \phi(y)h(y) dy - \|\phi\|\varepsilon\right) - x_n^{-4/3} e^{\sqrt{2}L(s_n)}}{4\pi^{-1} \hat{Z} e^{-\pi^2 Bx_n^2/2L(s_n)^2} (1 + \varepsilon) + x_n^{-4/3} e^{\sqrt{2}L(s_n)}} \\ &\leq \frac{X'(\phi)}{X'(1)} \leq \frac{4\pi^{-1} \hat{Z} e^{-\pi^2 Bx_n^2/2L(s_n)^2} \left(\int_0^1 \phi(y)h(y) dy + \|\phi\|\varepsilon\right) + x_n^{-4/3} e^{\sqrt{2}L(s_n)}}{4\pi^{-1} \hat{Z} e^{-\pi^2 Bx_n^2/2L(s_n)^2} (1 - \varepsilon) - x_n^{-4/3} e^{\sqrt{2}L(s_n)}}. \end{aligned}$$

Because $x_n^{-1} e^{\sqrt{2}L(s_n)} \leq C\hat{Z}$ when (64) holds, we have $x_n^{-4/3} e^{\sqrt{2}L(s_n)} \leq 4\pi^{-1} \hat{Z} e^{-\pi^2 Bx_n^2/2L(s_n)^2} \varepsilon$ for sufficiently large n when (64) holds. Therefore, for sufficiently large n ,

$$\frac{1}{1 + 2\varepsilon} \left(\int_0^1 \phi(y)h(y) dy - \|\phi\|\varepsilon - \varepsilon\right) \leq \frac{X'(\phi)}{X'(1)} \leq \frac{1}{1 - 2\varepsilon} \left(\int_0^1 \phi(y)h(y) dy + \|\phi\|\varepsilon + \varepsilon\right)$$

with probability at least $1 - 5\varepsilon$. In view of (70), we can let $\varepsilon \rightarrow 0$ to obtain (69). \square

4 Position of the right-most particle

In this section, we prove Theorem 2. Consider branching Brownian motion without killing and with a drift of $-\sqrt{2}$. Let $u(t, w)$ be the probability that if at time zero there is a single particle at the origin, then the position of the right-most particle at time t will be greater than or equal to w . Define $m(t) = \inf\{w : u(t, w) \geq 1/2\}$. By Proposition 8.2 on page 127 of [7], applied with $y_0 = -1$, and by Corollary 1 on page 130 of [7], applied with $\alpha(r, t) = -1$, there exist positive constants T, C', C'' , and C_{12} such that if $t \geq T$, then

$$u(t, w) \leq C'' e^t \int_{-1}^0 \frac{e^{-(w+\sqrt{2}t-z)^2/2t}}{\sqrt{2\pi t}} (1 - e^{-2(z+1)(w-m(t))/t}) dz \quad (72)$$

and

$$u(t, w) \geq C' e^t \int_{-1}^0 \frac{e^{-(w+\sqrt{2}t-z)^2/2t}}{\sqrt{2\pi t}} (1 - e^{-2(z+1)(w-m(t))/t}) dz \quad (73)$$

for all $w \geq m(t) + 1$, where

$$\left| m(t) + \frac{3}{2\sqrt{2}} \log t \right| \leq C_{12}. \quad (74)$$

See (8.4) and (8.18) of [7] for the bounds on $m(t)$, and observe that $m(t)$ here corresponds to $m_{1/2}(t) - \sqrt{2}t$ in the notation of [7].

Lemma 28. *Suppose $0 < \gamma \leq 1$. Suppose that $t = \gamma x^2$ and that $w = -(3/2\sqrt{2}) \log t + y$, where $1 + C_{12} \leq y \leq C_{13}x$ for some positive constant C_{13} . Then there exists $x_0 > 0$, depending on γ , such that for $x \geq x_0$,*

$$C' y e^{-\sqrt{2}y} e^{-y^2/2t} \leq u(t, w) \leq C'' y e^{-\sqrt{2}y} e^{-y^2/2t},$$

where C' and C'' are positive constants that do not depend on γ .

Remark 29. *We note that similar bounds on u may be obtained directly by PDE methods, and these have in fact been used in [13] to reprove Bramson's logarithmic correction result of [7] and to extend it to the setup of periodic branching rates (see [14]).*

Proof. We may assume that x is large enough that $t \geq \max\{1, T\}$. If $-1 \leq z \leq 0$, then using (74),

$$\frac{2(z+1)(w-m(t))}{t} \leq \frac{2(y - (3/2\sqrt{2}) \log t - m(t))}{t} \leq \frac{2(y + C_{12})}{t} \leq \frac{4y}{t}. \quad (75)$$

It follows that

$$1 - e^{-2(z+1)(w-m(t))/t} \leq \frac{4y}{t}. \quad (76)$$

Because $y \leq C_{13}x$ and $t = \gamma x^2$, the expression in (75) tends to zero as $x \rightarrow \infty$. Therefore, if $-1/2 \leq z \leq 0$, we have, for sufficiently large x ,

$$1 - e^{-2(z+1)(w-m(t))/t} \geq \frac{1}{2} \cdot \frac{2(z+1)(w-m(t))}{t} \geq \frac{y - (3/2\sqrt{2}) \log t - m(t)}{2t} \geq \frac{y - C_{12}}{2t} \geq \frac{Cy}{t}. \quad (77)$$

Next, observe that

$$e^{-(w+\sqrt{2}t-z)^2/2t} = e^{-(y-z)^2/2t} e^{(3/2\sqrt{2})(y-z)(\log t)/t} e^{-9(\log t)^2/16t} e^{-\sqrt{2}(y-z)} e^{-t^{3/2}}.$$

If $-1 \leq z \leq 0$, then $e^{-\sqrt{2}z} \leq e^{\sqrt{2}z} \leq 1$. Also, $e^{-9(\log t)^2/16t}$ tends to one as $x \rightarrow \infty$. Furthermore, because $t = \gamma x^2$ and $y \leq C_{13}x$, we have $e^{(3/2\sqrt{2})(y-z)(\log t)/t} \rightarrow 1$ and $e^{-(y-z)^2/2t}/e^{-y^2/2t} \rightarrow 1$ as $x \rightarrow \infty$. It follows that there exists $x_0 > 0$, depending on γ , and positive constants C' and C'' such that if $x \geq x_0$, then

$$C' e^{-y^2/2t} e^{-\sqrt{2}y} e^{-t} t^{3/2} \leq e^{-(w+\sqrt{2}t-z)^2/2t} \leq C'' e^{-y^2/2t} e^{-\sqrt{2}y} e^{-t} t^{3/2}. \quad (78)$$

Combining (72), (76), and (78), we get that for sufficiently large x ,

$$\begin{aligned} u(t, w) &\leq C e^t \int_{-1}^0 \frac{e^{-(w+\sqrt{2}t-z)^2/2t}}{\sqrt{2\pi t}} (1 - e^{-2(z+1)(w-m(t))/t}) dz \\ &\leq C e^t \int_{-1}^0 \frac{e^{-y^2/2t} e^{-\sqrt{2}y} e^{-t} t^{3/2}}{\sqrt{2\pi t}} \cdot \frac{y}{t} dz \\ &\leq C y e^{-\sqrt{2}y} e^{-y^2/2t}. \end{aligned} \quad (79)$$

By similar reasoning using (73), (77), and (78), we get that for sufficiently large x ,

$$\begin{aligned} u(t, w) &\geq C e^t \int_{-1/2}^0 \frac{e^{-(w+\sqrt{2}t-z)^2/2t}}{\sqrt{2\pi t}} (1 - e^{-2(z+1)(w-m(t))/t}) dz \\ &\geq C e^t \int_{-1/2}^0 \frac{e^{-y^2/2t} e^{-\sqrt{2}y} e^{-t} t^{3/2}}{\sqrt{2\pi t}} \cdot \frac{y}{t} dz \\ &\geq C y e^{-\sqrt{2}y} e^{-y^2/2t}. \end{aligned} \quad (80)$$

The result follows from (79) and (80). \square

Lemma 30. *Suppose $0 < \gamma \leq 1$. Suppose $t \leq \gamma x^2$ and $w \geq C_{14}x$ for some positive constant C_{14} . Then there exists $x_0 > 0$, depending on γ , such that for $x \geq x_0$,*

$$u(t, w) \leq C \gamma^{-3/2} x^{-3} w e^{-\sqrt{2}w} e^{-C_{15}/\gamma} \quad (81)$$

for some positive constants C and C_{15} that do not depend on γ .

Proof. If $-1 \leq z \leq 0$, then

$$1 - e^{-2(z+1)(w-m(t))/t} \leq \frac{2(z+1)(w-m(t))}{t} \leq \frac{Cw}{t}. \quad (82)$$

Also, for sufficiently large x ,

$$e^{-(w+\sqrt{2}t-z)^2/2t} = e^{-t} e^{-\sqrt{2}(w-z)} e^{-(w-z)^2/2t} \leq C e^{-t} e^{-\sqrt{2}w} e^{-C_{14}^2 x^2/t}. \quad (83)$$

By (72), (82), and (83), we get that when $T \leq t \leq \gamma x^2$,

$$u(t, w) \leq C w e^{-\sqrt{2}w} t^{-3/2} e^{-C_{14}^2 x^2/t}.$$

The function $t \mapsto t^{-3/2} e^{-C_{14}^2 x^2/t}$ is increasing when $t \leq (2C_{14}^2 x^2)/3$ which means that for $\gamma \leq 2C_{14}^2/3$, we have

$$u(t, w) \leq C \gamma^{-3/2} x^{-3} w e^{-\sqrt{2}w} e^{-C_{14}^2/2\gamma}$$

whenever $T \leq t \leq \gamma x^2$. This is enough to imply (81) except in the case when $t < T$. However, when $t < T$, by the Many-to-One Lemma and Markov's Inequality, $u(t, w)$ is bounded above by e^t times the probability that an individual Brownian particle started at the origin is to the right of w by time t . For the purpose of obtaining an upper bound on $u(t, w)$, we may ignore the drift of $-\sqrt{2}$. Therefore, using that

$$\int_z^\infty e^{-x^2/2} dx \leq z^{-1} e^{-z^2/2},$$

we have

$$u(t, w) \leq e^t \int_{w/\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \leq \frac{e^t \sqrt{t}}{\sqrt{2\pi w}} e^{-w^2/2t} \leq \frac{e^T T}{\sqrt{2\pi w}} e^{-w^2/2T}.$$

Because $w \geq C_{14}x$, this expression is bounded above by the right-hand side of (81) for $x \geq x_0$, where x_0 depends on γ . \square

We now return to the setting of Theorem 2, in which there is initially a particle at x and particles are killed when they reach the origin.

Lemma 31. *Let $\varepsilon > 0$. Let $0 < u < \tau$, and let $s = ux^3$. Let $\gamma > 0$. Let D be the number of particles that are killed at the origin between times $s - \gamma x^2$ and s . Then there exists a positive constant C , depending on u and ε but not on γ , such that for sufficiently large x ,*

$$\mathbb{P}\left(D > C\gamma x^{-1} \exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right) \leq 6\varepsilon.$$

Proof. Let $A = 2\gamma$, and let $r = s - Ax^2$. For $u \in [s - \gamma x^2, s]$ define $X_u(1)$ in the same way as $X(1)$, but with u playing the role of s . That is, $X_u(1)$ consists of the number of particles at time u whose ancestor was positioned to the left of $L(u)$ at time v for all $v \in [r, u]$. By the argument leading to Lemma 21,

$$\mathbb{P}(N(u) = X_u(1) \text{ for all } u \in [s - \gamma x^2, s]) > 1 - 3\varepsilon \tag{84}$$

for sufficiently large x . By Lemma 22, there is a positive constant C such that $\mathbb{E}[X_u(1)|\mathcal{F}_r] \leq Cx^{-2}\hat{Z}$ for sufficiently large x , where \hat{Z} is defined as in (48) but with u in place of s . The argument leading to (49) implies that on an event with probability greater than $1 - \varepsilon$, we have $\mathbb{E}[X_u(1)|\mathcal{F}_r] \leq Cx^{-2}Z(r)$ for all $u \in [s - \gamma x^2, s]$ for sufficiently large x , where C is some other positive constant.

Define times $s - \gamma x^2 = u_0 < u_1 < \dots < u_j = s$, where the u_i are chosen such that $1/2 \leq u_i - u_{i-1} \leq 1$ for $i = 1, 2, \dots, j$. For $i = 0, 1, \dots, j-1$, let D_i be the number of particles that are killed at the origin between times u_i and u_{i+1} . Let D'_i be the number of such particles that are descended from particles at time u_i that are counted in $X_{u_i}(1)$, meaning that their ancestor was positioned to the left of $L(u_i)$ throughout the time period $[r, u_i]$. Even in the absence of killing between times u_i and u_{i+1} , the expected number of descendants at time u_{i+1} produced by a given particle at time u_i is at most $e^{u_{i+1}-u_i} \leq e$. It follows that for sufficiently large x ,

$$\mathbb{E}[D'_i|\mathcal{F}_r] \leq e \mathbb{E}[X_{u_i}(1)|\mathcal{F}_r] \leq Cx^{-2}Z(r)$$

for all i on an event of probability at least $1 - \varepsilon$, and therefore,

$$\mathbb{E}\left[\sum_{i=0}^{j-1} D'_i \middle| \mathcal{F}_r\right] \leq C\gamma Z(r)$$

on an event of probability at least $1 - \varepsilon$. In view of Proposition 19, there is a positive constant C such that for sufficiently large x ,

$$\mathbb{E} \left[\sum_{i=0}^{j-1} D'_i \middle| \mathcal{F}_r \right] \leq C \gamma x^{-1} \exp \left((3\pi^2)^{1/3} (t-r)^{1/3} \right)$$

on an event of probability at least $1 - 2\varepsilon$. By Markov's Inequality, there is a positive constant C such that for sufficiently large x ,

$$\mathbb{P} \left(\sum_{i=0}^{j-1} D'_i > C \gamma x^{-1} \exp \left((3\pi^2)^{1/3} (t-r)^{1/3} \right) \right) \leq 3\varepsilon.$$

Because $\mathbb{P}(D = \sum_{i=0}^{j-1} D'_i) > 1 - 3\varepsilon$ by (84) and

$$\exp \left((3\pi^2)^{1/3} (t-r)^{1/3} \right) \leq C \exp \left((3\pi^2)^{1/3} (t-s)^{1/3} \right),$$

the result follows. \square

Proof of Theorem 2. Fix $d \in \mathbb{R}$. Let $\gamma \in (0, 1]$. Let $r = s - \gamma x^2$. Let

$$p_i = u \left(\gamma x^2, L(s) - \frac{3}{\sqrt{2}} \log x + d - X_i(r) \right).$$

Let $R(s)$ be the position of the right-most particle at time s for a modified process in which particles that reach the origin between times r and s are not killed. Then

$$\mathbb{P} \left(R(s) \geq L(s) - \frac{3}{\sqrt{2}} \log x + d \middle| \mathcal{F}_r \right) = 1 - \prod_{i=1}^{N(r)} (1 - p_i).$$

Therefore,

$$1 - \exp \left(- \sum_{i=1}^{N(r)} p_i \right) \leq \mathbb{P} \left(R(s) \geq L(s) - \frac{3}{\sqrt{2}} \log x + d \middle| \mathcal{F}_r \right) \leq \sum_{i=1}^{N(r)} p_i. \quad (85)$$

Consequently, the key to the proof will be obtaining a precise estimate of $\sum_{i=1}^{N(r)} p_i$.

Note that

$$p_i = u \left(\gamma x^2, L(s) - \frac{3}{2\sqrt{2}} \log \gamma x^2 + \frac{3}{2\sqrt{2}} \log \gamma + d - X_i(r) \right).$$

Because $L(r) - L(s)$ is bounded above by a constant depending on u , it follows from Lemma 18 that with probability tending to one as $x \rightarrow \infty$, we have

$$X_1(r) \leq L(s) + \frac{3}{2\sqrt{2}} \log \gamma + d - 1 - C_{12}, \quad (86)$$

where C_{12} is the constant from (74). By Lemma 28, on this event for sufficiently large x we have

$$C' R_i S_i T_i \leq p_i \leq C'' R_i S_i T_i \quad (87)$$

for all i , where

$$\begin{aligned} R_i &= L(s) + \frac{3}{2\sqrt{2}} \log \gamma + d - X_i(r), \\ S_i &= \exp \left(-\sqrt{2}(L(s) + (3/2\sqrt{2}) \log \gamma + d - X_i(r)) \right), \\ T_i &= \exp \left(-\frac{(L(s) + (3/2\sqrt{2}) \log \gamma + d - X_i(r))^2}{2\gamma x^2} \right). \end{aligned}$$

Let

$$a = L(s) - L(r) + \frac{3}{2\sqrt{2}} \log \gamma + d.$$

Then

$$R_i = L(r) \left(1 - \frac{X_i(r)}{L(r)} + \frac{a}{L(r)} \right). \quad (88)$$

Also,

$$S_i = \gamma^{-3/2} e^{-\sqrt{2}d} e^{-\sqrt{2}L(s)} e^{\sqrt{2}X_i(r)}. \quad (89)$$

Finally, because

$$\frac{L(s)^2}{2\gamma x^2} = \frac{c^2(t-s)^{2/3}}{2\gamma c^2 t^{2/3}} = \frac{1}{2\gamma} \left(1 - \frac{s}{t} \right)^{2/3} = \frac{1}{2\gamma} \left(1 - \frac{u}{\tau} \right)^{2/3},$$

we have

$$\begin{aligned} T_i &= \exp \left(-\frac{1}{2\gamma x^2} \left((L(r) - X_i(r))^2 + 2a(L(r) - X_i(r)) + a^2 \right) \right) \\ &= \exp \left(-\frac{L(s)^2 - (L(s)^2 - L(r)^2)}{2\gamma x^2} \left(1 - \frac{X_i(r)}{L(r)} \right)^2 - \frac{2a(L(r) - X_i(r)) + a^2}{2\gamma x^2} \right) \\ &= \exp \left(-\frac{1}{2\gamma} \left(1 - \frac{u}{\tau} \right)^{2/3} \left(1 - \frac{X_i(r)}{L(r)} \right)^2 \right) U_i, \end{aligned} \quad (90)$$

where $U_i \rightarrow 1$ as $x \rightarrow \infty$ uniformly in i because $a/x \rightarrow 0$ and $(L(s)^2 - L(r)^2)/x^2 \rightarrow 0$ as $x \rightarrow \infty$. Therefore, by (88), (89), and (90),

$$\begin{aligned} \sum_{i=1}^{N(r)} R_i S_i T_i &= \gamma^{-3/2} e^{-\sqrt{2}d} e^{-\sqrt{2}L(s)} L(r) \sum_{i=1}^{N(r)} U_i e^{\sqrt{2}X_i(r)} \left(1 - \frac{X_i(r)}{L(r)} + \frac{a}{L(r)} \right) \\ &\quad \times \exp \left(-\frac{1}{2\gamma} \left(1 - \frac{u}{\tau} \right)^{2/3} \left(1 - \frac{X_i(r)}{L(r)} \right)^2 \right). \end{aligned} \quad (91)$$

Consider the function $\phi : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\phi(z) = (1-z) \exp \left(-\frac{1}{2\gamma} \left(1 - \frac{u}{\tau} \right)^{2/3} (1-z)^2 \right).$$

By (69), applied with $s_n = ux_n^3 - \gamma x_n^2$, where $(x_n)_{n=1}^{\infty}$ is a sequence tending to infinity, we have,

$$\frac{1}{Y(r)} \sum_{i=1}^{N(r)} e^{\sqrt{2}X_i(r)} \phi \left(\frac{X_i(r)}{L(r)} \right) \rightarrow_p \frac{\pi}{2} \int_0^1 (1-z) \exp \left(-\frac{1}{2\gamma} \left(1 - \frac{u}{\tau} \right)^{2/3} (1-z)^2 \right) \sin(\pi z) dz. \quad (92)$$

Now let $\alpha = (2\gamma)^{-1/2}(1 - u/\tau)^{1/3}$ and make the substitution $y = \alpha(1 - z)$ to get that the right-hand side of (92) is

$$\frac{\pi}{2} \int_0^\alpha \frac{y}{\alpha} e^{-y^2} \sin\left(\frac{\pi y}{\alpha}\right) \cdot \frac{1}{\alpha} dy \asymp \frac{1}{\alpha^3} \asymp \gamma^{3/2}, \quad (93)$$

where \asymp means that the ratio of the two sides is bounded above and below by positive constants. Furthermore, $\sum_{i=1}^{N(r)} e^{\sqrt{2}X_i(r)} = Y(r)$ and $a/L(r)$ tends to zero as $x \rightarrow \infty$. It thus follows from (91), (92), and (93) that on the event (86), we have

$$\sum_{i=1}^{N(r)} R_i S_i T_i = e^{-\sqrt{2}d} e^{-\sqrt{2}L(s)} L(r) Y(r) H(u, x, \gamma), \quad (94)$$

where $H(u, x, \gamma)$ converges in probability as $x \rightarrow \infty$ to some number which is bounded between two positive constants that do not depend on γ . Note that $e^{-\sqrt{2}L(s)} = e^{-(3\pi^2)^{1/3}(t-s)^{1/3}}$. Therefore, because $Z(r) \leq Y(r)$, we can use Propositions 16 and 20 to conclude that with probability at least $1 - 2\varepsilon$, we have $C' \leq e^{-\sqrt{2}L(s)} L(r) Y(r) \leq C''$ for sufficiently large x . Combining this result with (87) and (94), we get that there are constants C_{16} and C_{17} , not depending on γ , such that for sufficiently large x ,

$$\mathbb{P}\left(C_{16}e^{-\sqrt{2}d} \leq \sum_{i=1}^{N(r)} p_i \leq C_{17}e^{-\sqrt{2}d}\right) > 1 - 3\varepsilon. \quad (95)$$

Now choose $d_2 > 0$ large enough that $C_{17}e^{-\sqrt{2}d_2} < \varepsilon$. By (85) and (95),

$$\begin{aligned} \mathbb{P}\left(X_1(s) \geq L(s) - \frac{3}{\sqrt{2}} \log x + d_2\right) &\leq \mathbb{P}\left(R(s) \geq L(s) - \frac{3}{\sqrt{2}} \log x + d_2\right) \\ &\leq C_{17}e^{-\sqrt{2}d_2} + 3\varepsilon \\ &\leq 4\varepsilon. \end{aligned} \quad (96)$$

Likewise, we can choose $d_1 > 0$ large enough that $\exp(-C_{16}e^{\sqrt{2}d_1}) \leq \varepsilon$. By (85) and (95),

$$\mathbb{P}\left(R(s) \leq L(s) - \frac{3}{\sqrt{2}} \log x - d_1\right) \leq \exp(-C_{16}e^{\sqrt{2}d_1}) + 3\varepsilon \leq 4\varepsilon. \quad (97)$$

It remains to bound the probability that $R(s) > L(s) - (3/\sqrt{2}) \log x - d_1$ but $X_1(s) \leq L(s) - (3/\sqrt{2}) \log x - d_1$. This could only happen if some particle reaches 0 between times r and s and then, for the modified process in which killing is suppressed during this time, some descendant particle is to the right of $L(s) - (3/\sqrt{2}) \log x - d_1$ at time s . However, by Lemma 31, with probability at least $1 - 6\varepsilon$, at most $C\gamma x^{-1} \exp((3\pi^2)^{1/3}(t-s)^{1/3}) = C\gamma x^{-1} e^{\sqrt{2}L(s)}$ particles reach the origin between times r and s . Conditional on this event, by Lemma 30, the expected number of these particles with a descendant to the right of $L(s) - (3/\sqrt{2}) \log x - y$ at time s is at most

$$C\gamma x^{-1} e^{\sqrt{2}L(s)} \cdot \gamma^{-3/2} x^{-3} L(s) e^{-\sqrt{2}(L(s) - (3/\sqrt{2}) \log x - d_1)} e^{-C_{15}/\gamma} \leq C_{18} \gamma^{-1/2} e^{\sqrt{2}d_1} e^{-C_{15}/\gamma}.$$

Combining this result with (97) and Markov's Inequality, and choosing γ small enough that $C_{18} \gamma^{-1/2} e^{\sqrt{2}d_1} e^{-C_{15}/\gamma} < \varepsilon$, we get, for sufficiently large x ,

$$\mathbb{P}\left(X_1(s) \leq L(s) - \frac{3}{\sqrt{2}} \log x - d_1\right) \leq 4\varepsilon + 6\varepsilon + C_{18} \gamma^{-1/2} e^{\sqrt{2}d_1} e^{-C_{15}/\gamma} \leq 11\varepsilon. \quad (98)$$

The result follows from (96) and (98). \square

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