Equivalence of Liouville measure and Gaussian free field

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Abstract

Given an instance h of the Gaussian free field on a planar domain D and a constant $\gamma \in (0, 2)$, one can use various regularization procedures to make sense of the *Liouville quantum gravity measure* $\mu := e^{\gamma h(z)} dz$. It is known that the field h a.s. determines the measure μ_h. We show that the converse is true: namely, h is measurably determined by μ_h. More generally, given a random closed fractal subset X endowed with a Frostman measure σ_X whose support is X (independent of h), we construct a quantum measure μ_X and ask the following: how much information does μ_X contain about the free field? We conjecture that μ_X always determines h restricted to X, in the sense that it determines its harmonic extension off X. We prove the conjecture in the case where X is an independent SLE_κ curve equipped with its quantum natural time, and in the case where X is Liouville Brownian motion (that is, standard Brownian motion equipped with its quantum clock). The proof in the latter case relies on properties of nonintersecting planar Brownian motion, including the value of some nonintersection exponents.

1 Introduction

1.1 Background

Let $\gamma \in (0, 2)$ and let h be a Gaussian free field (GFF) defined in some domain $D \subset \mathbb{R}^2$ together with some boundary conditions. Consider the (formal) Riemannian metric tensor

$$e^{\gamma h(x)} dx^2. \tag{1}$$

The tensor (1) gives rise to a random geometry known in physics as (critical) Liouville quantum gravity (LQG); see [Dav88, DK89, Nak04, DS11, Gar13, MS13b, DMS14, Ber13, GRV13] for a series of works both within the physics and mathematics literature on the subject. A rigorous construction of the metric space associated to (1) is still an open problem (except in the case $\gamma = \sqrt{8/3}$, where this has very recently been announced in [MS13b, MS15]), but one can make rigorous sense of (1) in other ways, for example as a measure on a space with a conformal structure. Using these interpretations, (1) has been conjectured to represent (in some sense) the scaling limit of certain decorated random planar maps. There are various ways to formulate this constructure (in terms of metric space structure, conformal structure, loop structure, etc.) and the loop structure formulation has been recently proved [GMS15, GS15a, GS15b, GM15]. The parameter γ in (1) is related to the weighting of the planar maps by a given statistical physics model. See for example the surveys [Gar13, Ber15b].

The Liouville measure μ_h, which is the natural volume form of this metric (e.g., the conjectured limit of the uniform distribution on the vertices of the planar map) was defined in [DS11] as well as by Rhodes and Vargas in [RV11], building on work of Høegh-Krohn, Polyakov, and Kahane [HK71, Pol81, Kah85]. Another natural object called Liouville Brownian motion (LBM), which is the canonical diffusion in the geometry of LQG, was introduced in [Ber13] as well as by Garban, Rhodes and Vargas in [GRV13].

1.2 Aim of the paper

The main purpose of this paper is to analyse the following question: how much of the geometry of Liouville Quantum Gravity is encoded by the Liouville measure μ_h? As we will see, the answer turns out to be *everything*, in a precise sense. Beyond the intrinsic appeal of this question, we will see that this result has applications to the question of uniqueness in the main theorem of [DMS14].

More generally, much of the emerging theory of LQG concerns certain random fractals X, coupled in a certain way with the underlying Gaussian free field h. They typically come equipped with a 'natural' quantum measure supported on X. In this paper we will pay particular attention to the case where X is an independent SLE_κ curve equipped with its so-called quantum natural parameterisation, or the case where X is the range of a Liouville Brownian motion equipped with its quantum clock, but there are many other examples. It is natural to wonder how much information these measures contain about the underlying field h. We conjecture that, as soon as X is harmonically nontrivial, such measures encode *everything* about the restriction of h to X (that is, the harmonic extension of h off X), and nothing more. We prove this result in the two cases mentioned above. At a technical level, the SLE case follows from conformal invariance and the estimates used to prove the 'full domain' result (meaning Theorem 1.1), while the Liouville Brownian motion case requires very different ideas, and in particular relies on properties of nonintersecting planar Brownian motion (including the value of nonintersecting exponents derived by Lawler, Schramm and Werner [LSW01a, LSW01b]).

1.3 First results

Let D be a domain of $\mathbb{R}^2$ and let h be a Gaussian free field with zero boundary conditions on D. We can use a regularization procedure to define an area measure on D:

$$\mu = \mu_h := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz, \tag{2}$$

where dz is Lebesgue measure on D, $h_\varepsilon(z)$ is the mean value of h on the circle $\partial B(z, \epsilon)$ and the limit represents weak convergence in the space of measures on D. The limit exists almost surely, at least if ε is restricted to powers of two [DS11], and an alternative definition is provided in [RV11, RV10] using Kahane's theory of Gaussian multiplicative chaos. Note however that Kahane's theory only provides convergence in distribution and hence with that approach it is not immediately clear whether h determines μ_h. This problem was however resolved recently by Shamov [Sha], so that the methods of [DS11, RV11, RV10] all give equivalent ways of constructing μ_h from h. For a more recent, self-contained and elementary proof, the reader can also consult [Ber15a].

Before taking the limit, it is clear that h_ϵ and μ_{h_ϵ} determine each other. After taking the limit, μ_h is clearly determined by h, as noted above. But from [DS11] we know that μ_h will almost surely assign full measure to the set of so-called γ-thick points of h (see [She07]). The (Euclidean) Hausdorff dimension of thick points has been computed in [HMP10], and is shown to be equal to $2 - \frac{\gamma^2}{2}$, almost surely. So one may wonder whether μ_h still determines h, thus determining the quantum surface. The worry is that μ_h might only retain information about points which are in some sense exceptional for the field h. Fortunately, these points are sufficiently dense and together they contain enough information that we shall be able to determine the field h from the measure μ_h. Our first main result below states this in a very general way.

Note that if $h = h_0 + g$, where h_0 is a Gaussian Free Field on D, and g is a possibly random continuous function, the Liouville quantum gravity measure μ_h associated to h is well defined, and is simply the measure having density $e^{\gamma g}$ with respect to μ_{h_0}.

Theorem 1.1. *Let $h = h_0 + g$ where h_0 is a zero boundary* GFF *on a simply connected domain $D \subset \mathbb{C}$ and g is a random continuous function. Denote by μ_h its Liouville quantum measure with parameter $\gamma \in (0, 2)$. Then h is determined by μ_h almost surely. That is, h is measurable with respect to the σ-algebra generated by $\{\mu_h(A) : A$ open in $D\}$.*

Remark 1.2. Theorem 1.1 actually covers various types of GFFs (Dirichlet boundary conditions, Neumann boundary conditions, mixed boundary conditions, the whole plane GFF, etc.) via the domain Markov property. Likewise, by absolute continuity, Theorem 1.1 also covers the quantum surfaces defined in [DMS14] including quantum cones, wedges, spheres and disks.

Remark 1.3. If $g \in H_0^1$ is deterministic then $h + g$ is absolutely continuous with respect to h so the theorem is trivially implied by the case $g \equiv 0$. But here we only assume that g is continuous, so g can be much rougher, and moreover g may depend on h.

Before we present a more general setup in Section 1.4, we briefly explain an application of Theorem 1.1 to the peanosphere point of view on Liouville quantum gravity developed in [DMS14]. In the main result of that paper (Theorem 9.1), the authors consider a space-filling variant of SLE'_{κ}, $\kappa' = 16/\gamma^2$, on top of a γ-quantum cone, where the curve η' is parametrized by its quantum area (i.e., $\mu_h(\eta'([s,t])) = t - s$ for all $s \leq t \in \mathbb{R}$). We refer to [DMS14] and [She10] for the notion of quantum cone, while the space-filling variant of SLE was introduced in [MS13a]. The main theorem of [DMS14] is that the left and right boundary quantum length of the curve $\eta'([0,t])$, relative to time 0, evolve as a certain two dimensional Brownian motion $(L_t, R_t)_{t\in\mathbb{R}}$ whose covariance is given by $\cos(\pi\gamma^2/4)$. (In fact, this formula was only proved for $\gamma \in [\sqrt{2}, 2)$ in [DMS14], and the corresponding result for $\gamma \in [0, \sqrt{2})$ is being addressed in a work in progress [GHMS15].) In [DMS14, Chapter 10], the authors proved that this pair of Brownian motion in fact determines the quantum measure on the γ-quantum cone as well as the space-filling SLE almost surely, up to rotations, and used Theorem 1.1 of this paper to conclude that this in turn determines the free field h (up to rotations).

Corollary 1.4. *In the setting described above, $(L_t, R_t)_{t\in\mathbb{R}}$ determines the field h defining the γ-quantum cone almost surely (up to rotations). More precisely, h (modulo rotations) is measurable with respect to $(L_t, R_t)_{t\in\mathbb{R}}$.*

1.4 A more general setup

In this subsection we introduce a general conjecture that in some sense motivates the remainder of the paper. However, we stress that it is not necessary to read this subsection to follow the remainder of the paper.

Recall that a Borel measure on a domain D is *locally finite* if every point has a neighborhood of finite measure (or equivalently, if every compact set has finite measure). We will be interested in random pairs (σ, X), where σ is any (possibly random) locally finite measure on D and X is the (closed) support of σ. For example, X could be one of the random fractal sets that arise in SLE theory, and σ could be a 'natural' fractal measure associated to X. Let h be an instance of the GFF on D with some boundary conditions chosen independently from (σ, X).

We would now like to describe in some generality how to construct a "quantum" version $\mu_{X,h}$ of the measure σ. Fix $d \in (0, 2]$ and assume that σ has finite $(d-\epsilon)$-dimensional energy for all $\epsilon > 0$, i.e.,

$$\iint \frac{1}{|x-y|^{d-\varepsilon}} \sigma(dx)\sigma(dy) < \infty, \quad \text{for all } \varepsilon > 0. \tag{3}$$

(The reader may recall that, by Frostman's theorem, the Hausdorff dimension of a closed set X is the largest value of d for which there exists a non-trivial measure σ on X satisfying (3). In the discussion below, we will not require that d is the dimension of X, or that σ is in any sense an optimal measure on X. Once σ is fixed, choosing a smaller d than necessary for (3) will in some sense be equivalent to choosing a smaller γ.)

Now choose x so that $d = 2 - 2x$. If d happens to be the dimension of X (as will be the case in all of the examples treated in this paper), then x can be understood as the so-called (Euclidean) scaling exponent of X. Let Δ be related to x via the KPZ relation,

$$x = \frac{\gamma^2}{4}\Delta^2 + (1 - \frac{\gamma^2}{4})\Delta, \tag{4}$$

so Δ is the quantum scaling exponent associated to the Euclidean exponent x. Write $\hat{\gamma} = \gamma(1 - \Delta)$. Now, by Kahane's theory of multiplicative chaos (as explained, e.g., in Theorem 1.1 in [Ber15a]) there is a way to define a measure $\mu_{X,h}$ (which depends on h and σ) that can be formally written as follows:

$$\mu_{X,h}(dz) = \exp(\hat{\gamma}h(z) - \frac{\hat{\gamma}^2}{2}\mathbb{E}(h(z)^2))\sigma(dz).$$

We will not explain the details of this construction here. However, we do point out that $\hat{\gamma} < \sqrt{2d}$, which implies (by the theorem in [Ber15a]) that the measure $\mu_{X,h}$ is non-trivial and that its support is X.

We view $\mu_{X,h}$ as a natural quantum analogue of σ. An important feature of the definition of $\mu_{X,h}$ in [Ber15a] is that adding a constant C to h locally multiplies the measure by $e^{\hat{\gamma}C}$. By contrast, adding C to h locally multiplies the measure μ_h by $e^{\gamma C}$. In other words, if we rescale the overall μ_h volume by a factor of $A = e^{\gamma C}$, then we rescale the $\mu_{X,h}$ volume by a factor of

$$\hat{A} = e^{\hat{\gamma}C} = (e^{\gamma C})^{\hat{\gamma}/\gamma} = A^{1-\Delta}.$$

This is a way of saying that Δ is the natural scaling exponent associated to $\mu_{X,h}$.

Another important feature of the definition of $\mu_{X,h}$, also explained in [Ber15a], is that a typical point chosen from $\mu_{X,h}$ is the center of a log singularity of magnitude proportional to $\hat{\gamma}$. The reader may have wondered why we chose the particular value $\hat{\gamma} = \gamma(1 - \Delta)$ in the definition above. One reason is that (as explained above) it gives a scaling relation that matches the Δ predicted by KPZ theory. Another (essentially equivalent) reason is the idea (see e.g. (63) in [DS11]) that if one chooses a random small quantum ball conditioned to intersect a d-dimensional set, one expects to see a log singularity proportional to $\hat{\gamma}$ centered at that ball. In many instances, we like to think (heuristically) of σ as representing Euclidean measure restricted to X and $\mu_{X,h}$ as representing μ_h restricted to X, so it is natural (at least in these instances) to expect the log singularity at a typical point to be as described above.

We can now formulate the question we have in mind:

Question: To what extent does the measure $\mu_{X,h}$ determine the field h?

Clearly $\mu_{X,h}$ can only determine the field h 'restricted to X' in some sense. The issue of whether the restriction of h to a fractal subset X makes sense is itself not obvious. But if X is any 'local' set coupled with h (in particular, if X is any random set independent of h) then there is a natural way to define the harmonic extension (to the complement of X) of the values of h on X [SS13]. (If X is harmonically trivial, then this extension is just the *a priori* expectation of h.) We make the following conjecture:

Conjecture 1.5. *In the setting described above, the measure $\mu_{X,h}$ a.s. determines the harmonic extension of h off X.*

We will prove two particular cases of interest of this conjecture, dealing with an independent SLE_κ and Liouville Brownian motion respectively.

Note that the question makes sense and is interesting in even greater generality, assuming e.g. that h is a Gaussian log-correlated field in Euclidean space of some given dimension, and σ is some given locally finite measure with finite $(d-\varepsilon)$-dimensional energy for some d and for all $\varepsilon > 0$.

1.5 Result in the case of SLE

Let h be a Gaussian free field on $\mathbb{H}$ with free boundary conditions (alternatively, the reader can also think of the case where $(\mathbb{H}, h)$ is a γ-quantum wedge if familiar with this notion). Let η be an independent SLE_κ curve with $\kappa = \gamma^2$ (we emphasise that this is the standard non-space-filling curve, and in fact here $\kappa < 4$ so the curve η is simple). We let X be the range of η, i.e. $X = \eta([0,\infty))$, and equip X with the so-called quantum natural time defined by Theorem 1.3 in [She10] (or Theorem 1.8 in the case of the wedge). That is, the measure $\mu_{X,h}(\eta[0,t]))$ is given by the quantum boundary length of $\eta([0,t])$ in either component of $\mathbb{H} \setminus X$ (by Theorem 1.3 of [She10], resp. Theorem 1.8, these measures are indeed a.s. equal). Equivalently, we map $\eta([0,t])$ away using the Loewner map g_t and measure the quantum length on $\mathbb{R}$ of $g_t(\eta([0,t]))$ on either side of 0, that is,

$$\mu_{X,h}(\eta([0,t])) := \nu_{h_t}([0^-, \xi_t]))$$

where ξ_t is the driving function of the Loewner equation, 0^- is the left-image of 0 by g_t, h_t is obtained from h by applying the change of coordinate rule of LQG:

$$h_t = h \circ g_t^{-1} + Q \log |(g_t^{-1})'|; \quad Q = \frac{\gamma}{2} + \frac{2}{\gamma}, \tag{5}$$

and if h is a field we denote by

$$\nu_h(dx) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/4} e^{\gamma h_\varepsilon(x)/2} dx; \quad x \in \mathbb{R} \tag{6}$$

the boundary length measure associated with h on $\partial\mathbb{H} = \mathbb{R}$. It is easy to check that $\mu_{X,h}$ indeed defines a nonnegative measure supported on X. Note that this definition is different from that in Conjecture 1.5, but we believe that the two notions coincide when σ is given by the so-called natural parametrisation of η defined in [LS11, LZ13, LR15], up to multiplication by a deterministic function related to the conformal radius.

Theorem 1.6. *In the above setup, $\mu_{X,h}$ determines the harmonic extension of h off X.*

1.6 Result in the case of Liouville Brownian motion

The second case of interest to us will be the case where X is the range of an independent Brownian motion $(B_t, t \le T_D)$, run until it leaves the domain D for the first time, and σ is the occupation measure of B (i.e., $\sigma(A) = \int_0^{T_D} \mathbf{1}_{\{B_s \in A\}} ds$ for all Borel set $A \subset D$). Then it is well known that the dimension of X is almost surely equal to 2 so $x = 0$ and $\Delta = 0$ as well. Hence, following the construction of Section 1.4, the measure $\mu_{X,h}$ is formally given by

$$\mu_{X,h}(A) = \int_0^t e^{\gamma h(B_s) - \frac{\gamma^2}{2}\mathbb{E}(h(B_s)^2)} \mathbf{1}_{\{B_s \in A\}} ds.$$

In other words, X is the range of an independent Liouville Brownian motion (as equivalently defined in [Ber13, GRV13]) and $\mu_{X,h}$ is the occupation measure of X induced by its *quantum clock*. This is the increasing process $(\phi(t), 0 \le t \le \tau_D)$ such that

$$\phi(t) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_0^t e^{\gamma h_\varepsilon(B_s)} ds.$$

Liouville Brownian motion is then defined as the process

$$Z_t := B(\phi^{-1}(t)); \quad t \le T_D = \phi(\tau_D).$$

Theorem 1.7. *Liouville Brownian motion determines the harmonic extension of h off its range. That is, the harmonic extension of h off $X = B[0, T_D]$ is measurable with respect to $(Z_t, t \le T_D)$ (or, equivalently, with respect to X and $(\phi(t), t \le \tau_D)$).*

The paper is organized as follows. In Section 2, we provide some relevant background on the Gaussian free field and prove a useful preliminary estimate. In Section 3 we give the proof of Theorem 1.1. Section 4 then gives the proof of Theorem 1.6 which covers the SLE case. Finally, Section 5 contains the proof of Theorem 1.7. This is the most technical part of the paper.

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2 Preliminaries

In this section we recall some background on the Gaussian free field (GFF). We focus on the zero boundary GFF since our technical proofs are mainly in the setting of zero boundary GFF.

Let D be a domain in $\mathbb{C}$ with harmonically nontrivial boundary (i.e. the harmonic measure of ∂D is positive as seen from any point in D). We denote by $H_0(D)$ the Hilbert-space closure of $C_0^\infty(D)$ [the space of compactly supported smooth functions in D], equipped with the Dirichlet inner product

$$(f,g)_\nabla = \frac{1}{2\pi}\int_D \nabla f(z)\cdot\nabla g(z)\,dz. \tag{7}$$

A zero boundary *Gaussian free field* on D is given by the formal sum

$$h = \sum_{n=1}^{\infty} \alpha_n f_n, \quad (\alpha_n) \quad \text{i.i.d} \quad N(0,1) \tag{8}$$

where $\{f_n\}$ is an orthonormal basis for $H_0(D)$. Although this expansion of h does not converge in $H(D)$, it can be shown that convergence holds almost surely in the space of distributions. See [She07, Ber15b] for more details.

If $V, V^\perp \subset H_0(D)$ are complementary orthogonal subspaces, then h can be decomposed as the sum of its projections onto V and $V^\perp$. In particular, for a domain $U \subset D$, we can take $V = H_0(U)$ and $V^\perp$ the set of functions in $H_0(D)$ which are harmonic in U. This allows us to decompose h as the sum of a zero boundary Gaussian free field on U and a random distribution which is harmonic on U, with both terms independent. We call the former field the *projection of h onto U*.

We record a lemma which will be used frequently.

Lemma 2.1. *For a simply connected domain D, $z \in D$, let h be the zero boundary Gaussian free field on D and h^{har} be the projection of h onto the space of functions in $H_0(D)$ that are harmonic inside $B_r(z)$, where $r \le \operatorname{dist}(z, D)$. For $\varepsilon < r/4$, let*

$$\Delta_\varepsilon = \max_{x\in B_\varepsilon(z)} h^{\text{har}}(x) - \min_{x\in B_\varepsilon(z)} h^{\text{har}}(x).$$

Then $\mathbb{E}[\Delta_\varepsilon^2] \le C(\varepsilon/r)^2$ where C is an universal constant independent of ε, r, z, D.

Proof. By translation and scaling, we only need to prove the case when $z = 0$ and $r = 1$.

It suffices to control the gradient of h^{har} in $B_{1/2}(0)$. In the proof of [DS11, Lemma 4.5], the authors show that the minimum of h^{har} in $B_{1/2}(0)$ has super exponential tail which is independent of the domain D containing the unit disk. (In fact, we point out that a simpler proof of that lemma can be obtained using the Borell–Tsirelson inequality for Gaussian processes). The same is true for the maximum of h^{har}. In particular, the second moment of $\|h^{\text{har}}\|_{\infty,B_{1/2}(0)}$ is bounded by a universal constant C. By a standard gradient estimate of harmonic functions, $\|\nabla h^{\text{har}}\|_{\infty,B_{\frac{1}{4}}(0)} \le C\|h^{\text{har}}\|_{\infty,B_{1/2}(0)}$ where C is another universal constant. So for $\varepsilon < 1/4$, we have

$$\mathbb{E}[\Delta_\varepsilon^2] \le C\varepsilon^2\mathbb{E}[\|\nabla h^{\text{har}}\|^2_{\infty,B_{1/4}(0)}] \le C\varepsilon^2.$$

□

3 Proof of Theorem 1.1: full domain case

We will prove Theorem 1.1 by making sense of the statement that $e^{\gamma h}$ is the Radon-Nikodym derivative of μ_h with respect to Lebesgue measure. Pick a positive radially symmetric smooth function η which has integral 1 and is supported on the unit disk. Let $\eta^\epsilon(x) = \frac{1}{\epsilon^2}\eta(\frac{x}{\epsilon})$. We define h^ϵ by letting

$$e^{\gamma h^\epsilon(x)} = \int_D \eta^\epsilon(x-z)d\mu_h(z). \tag{9}$$

Then $\mu_{h^\epsilon} = e^{\gamma h^\epsilon}dz$ is the convolution of μ_h with η^ϵ, which is an approximation to μ_h. Roughly speaking, we will show that $h^\epsilon - \mathbb{E}[h^\varepsilon]$ converges to h in probability as $\varepsilon \to 0$. Since h^ϵ is determined by μ_h, h is determined by μ_h. We will achieve this via the following two lemmas.

Lemma 3.1 (Variance estimate). *Suppose D is a simply connected domain and $D' = \{x \in D | \operatorname{dist}(x, \partial D) > \varepsilon_0\}$, where ε_0 is a fixed constant. h is the zero boundary GFF on D and h^ϵ and h_ϵ are defined as above. For all $z \in D', 0 < \epsilon < \frac{\epsilon_0}{4}$, let $f_\epsilon(z) = h^\varepsilon(z) - h_\varepsilon(z)$. Then we have*

$$\operatorname{Var}[f_\varepsilon(z)] \le C\log(\varepsilon_0/\varepsilon),$$

where C is a universal constant independent with D, ε_0 and z.

Lemma 3.2 (Covariance estimate). *Let h be the zero boundary* GFF *on $\mathbb{D}$ and f_ε be defined as in Lemma 3.1 for $D = \mathbb{D}$. Then for $x_1, x_2 \in r\mathbb{D}$ and $\varepsilon < |x_1 - x_2|/100$,*

$$\operatorname{Cov}[f_\varepsilon(x_1), f_\varepsilon(x_2)] \le C_r \frac{\varepsilon}{|x_2 - x_1|} \log^{1/2} \frac{|x_1 - x_2|}{\varepsilon}$$

where C_r only depends on $r \in (0,1)$.

Given Lemma 3.1 and Lemma 3.2, we can get Theorem 1.1 in the case that h is the zero boundary GFF on $\mathbb{D}$.

Proposition 3.3. *If h is a zero boundary* GFF *on $\mathbb{D}$, then μ_h determines h almost surely.*

Proof. Suppose ρ is a smooth function supported on $r\mathbb{D}$ where $r < 1$. It is sufficient to show that (h, ρ) is measurable with respect to μ_h.

$$\operatorname{Var}[(f_\varepsilon, \rho)] = \int_{\mathbb{D}\times\mathbb{D}} dxdy \operatorname{Cov}[f_\varepsilon(x), f_\varepsilon(y)]\rho(x)\rho(y)$$
$$= \int_{\{|x-y|>\varepsilon^{1/2}\}} dxdy \operatorname{Cov}[f_\varepsilon(x), f_\varepsilon(y)]\rho(x)\rho(y) + \int_{\{|x-y|<\varepsilon^{1/2}\}} dxdy \operatorname{Cov}[f_\varepsilon(x), f_\varepsilon(y)]\rho(x)\rho(y).$$

By Lemma 3.2

$$1_{\{|x-y|>\varepsilon^{1/2}\}} \operatorname{Cov}[f_\varepsilon(x), f_\varepsilon(y)] \le C_r \varepsilon^{1/2} \log^{1/2}(\varepsilon^{-1}).$$

Therefore
$$\lim_{\varepsilon\to 0}\int_{\mathbb{D}\times\mathbb{D}} dxdy1_{\{|x-y|>\varepsilon^{1/2}\}}\operatorname{Cov}[f_\varepsilon(x),f_\varepsilon(y)]\rho(x)\rho(y)=0.$$
On the other hand
$$\int_{\mathbb{D}\times\mathbb{D}} dxdy1_{\{|x-y|<\varepsilon^{1/2}\}}\operatorname{Cov}[f_\varepsilon(x),f_\varepsilon(y)]\rho(x)\rho(y)\le C\varepsilon\log\frac{r}{\varepsilon}.$$
Therefore $\lim\limits_{\varepsilon\to 0}\operatorname{Var}[(f_\varepsilon,\rho)]=0$. In other words (recalling that $\mathbb{E}((h_\varepsilon,\rho))=0$), $(h^\varepsilon,\rho)-\mathbb{E}[(h^\varepsilon,\rho)]-(h_\varepsilon,\rho)$ tends to 0 in L^2.

In addition, $(h_\varepsilon-h,\rho)$ also tends to 0 in L^2. So $(h^\varepsilon,\rho)-\mathbb{E}[(h^\varepsilon,\rho)]$ tends to (h,ρ) in L^2. This implies that the random variable (h,ρ) is measurable with respect to μ_h.

So far, we have proved that for all smooth function supported on $\mathbb{D}$, (h,ρ) is measurable with respect to μ_h, which means h is almost surely determined by μ_h. □

With this in hand it is not hard to get a proof of Theorem 1.1 (we only need to add to h a part which is a continuous function.)

Proof of Theorem 1.1. We first assume $D=\mathbb{D}$, h_0 is an instance of a zero boundary GFF on D and $g=h-h_0$ is the random continuous function in Theorem 1.1. Let μ_{h_0} be the Liouville quantum measure of h_0. Defined h^ε and h_0^ε by
$$e^{\gamma h^\epsilon}=\int_D \eta^\epsilon(x-z)d\mu_h(z),$$
$$e^{\gamma h_0^\epsilon}=\int_D \eta^\epsilon(x-z)d\mu_{h_0}(z),$$
where η is the same function used in (9).

By the intermediate value theorem, for all x, there is a ξ_x such that
$$\int_D \eta^\epsilon(x-z)d\mu_h(z)=\exp\{\gamma g(\xi_x)\}\int_D \eta^\epsilon(x-z)d\mu_{h_0}(z),\quad |\xi_x-x|\le\varepsilon. \tag{10}$$
Let $g^\varepsilon(x)=g(\xi_x)$. By taking log on both sides of (10), we have $h^\epsilon=h_0^\epsilon+g^\varepsilon$. Denote $h_\varepsilon,h_{0,\varepsilon},g_\varepsilon$ to be the circle average process of h,h_0,g respectively. Then $\forall\rho\in C_c^\infty(D)$.
$$(h^\varepsilon,\rho)-(h_\varepsilon,\rho)=(h_0^\varepsilon,\rho)-(h_{0,\varepsilon},\rho)+(g^\varepsilon-g_\varepsilon,\rho). \tag{11}$$
By the argument in Proposition 3.3, $(h_0^\varepsilon,\rho)-(h_{0,\varepsilon},\rho)-\mathbb{E}[(h_0^\varepsilon,\rho)]$ tends to 0 in L^2 as ε tends to 0.

Let ω be its modulus of continuity:
$$\omega_g(x,\varepsilon)=\max\{|g(x)-g(y)|:|y-x|\le\varepsilon\},\quad \forall x\in D,\varepsilon<\operatorname{dist}(x,\partial D). \tag{12}$$

Since g is a continuous function,

$$\lim_{\varepsilon\to 0}(\omega_g(x,\varepsilon),\rho(x)) = 0 \; a.s. \quad \forall \rho \in C_c^\infty(D). \tag{13}$$

By (13), $(g^\varepsilon - g_\varepsilon, \rho)$ tends to 0 a.s. as ε tends to 0. Hence

$$\lim_{\varepsilon\to 0}(h^\varepsilon,\rho) - \mathbb{E}[(h_0^\varepsilon,\rho)] = (h,\rho) \text{ in probability.}$$

As the same argument at the end of the proof of Proposition 3.3, we conclude the proof of Theorem 1.1 for $D = \mathbb{D}$.

For a general domain D, Theorem 1.1 is obtained by first conformally mapping D to $\mathbb{D}$ and then applying the coordinate change formula. □

3.1 The variance estimate

Proof of Lemma 3.1. Consider the disk $B_{\frac{\varepsilon_0}{2}}(z)$. We have the decomposition $h = h_0^{\text{supp}} + h_0^{\text{har}}$, where h_0^{supp} is the projection of h onto $B_{\frac{\varepsilon_0}{2}}(z)$ and h_0^{har} is the expectation of h conditional on h outside $B_{\frac{\varepsilon_0}{2}}(z)$. It is harmonic in $B_{\frac{\varepsilon_0}{2}}(z)$ and coincides with h on $B^c_{\frac{\varepsilon_0}{2}}(z)$.

Let ξ_0 be a point in $B_\varepsilon(z)$ such that

$$\int_{B_\varepsilon(z)} \eta^\varepsilon(z-y)d\mu_h(y) = e^{\gamma h_0^{\text{har}}(\xi_0)} \int_{B_\varepsilon(z)} \eta^\varepsilon(z-y)d\mu_{h_0^{\text{supp}}}(y).$$

Thus

$$f_\varepsilon(z) = [h_0^{\text{har}}(\xi_0) - h_0^{\text{har}}(z)] + [\frac{1}{\gamma}\log\int_{B_\varepsilon(z)} \eta^\varepsilon(z-y)d\mu_{h_0^{\text{supp}}}(y) - h_{0,\varepsilon}^{\text{supp}}(z)]. \tag{14}$$

Now,

$$\begin{aligned}\text{Var}[h_0^{\text{har}}(\xi_0) - h_0^{\text{har}}(z)] &\le \mathbb{E}[(h_0^{\text{har}}(\xi_0) - h_0^{\text{har}}(z))^2] \\ &\le \mathbb{E}(\Delta_\varepsilon(z)^2) \\ &\le C(\varepsilon/r)^2\end{aligned}$$

by Lemma 2.1. Thus set

$$I = \frac{1}{\gamma}\log\int_{B_\varepsilon(0)} \eta^\varepsilon(x)d\mu_h(x) - h_\varepsilon(0),$$

where h is the zero boundary GFF on $\mathbb{D}$. Hence, from (14), the Cauchy–Schwarz inequality, and applying the scaling $x \mapsto \frac{x-z}{\varepsilon_0}$ we deduce that in order to prove Lemma 3.1, it suffices to prove

$$\text{Var}(I) \le C|\log\varepsilon|. \tag{15}$$

Furthermore, we can assume without loss of generality $\varepsilon = 2^{-n}$ for some integer n.

We define the following quantities. Let h^1 be the projection of h to $B_{1/2}(0)$, $h_1^{\mathrm{har}} = h - h^1$ and ξ_1 be a point in $B_\varepsilon(z)$ such that

$$\int_{B_\varepsilon(0)} \eta^\varepsilon d\mu_h = e^{\gamma h_1^{\mathrm{har}}(\xi_1)} \int_{B_\varepsilon(0)} \eta^\varepsilon d\mu_{h^1}.$$

Then by induction, for $k < n-1$, let h^{k+1} be the projection of h^k to $B_{2^{-k-1}}(0)$, $h_{k+1}^{\mathrm{har}} = h^k - h^{k+1}$, which is harmonic in $B_{2^{-k-1}}(0)$. Hence $h = \sum_{i=1}^{n-1} h_i^{\mathrm{har}} + h^{n-1}$. Also, by induction we can choose ξ_{k+1} to be a point in $B_\varepsilon(z)$ such that

$$\int_{B_\varepsilon(0)} \eta^\varepsilon d\mu_{h^k} = e^{\gamma h_{k+1}^{\mathrm{har}}(\xi_{k+1})} \int_{B_\varepsilon(0)} \eta^\varepsilon d\mu_{h^{k+1}}.$$

Then since $h_\varepsilon(0) = \sum_{i=1}^{n-1} h_i^{\mathrm{har}}(0) + h_\varepsilon^{n-1}(0)$ by the mean value property of harmonic functions, we can write:

$$I = \sum_{k=1}^{n-1} X_k + R, \tag{16}$$

where

$$X_k = h_k^{\mathrm{har}}(\xi_k) - h_k^{\mathrm{har}}(0); \text{ and } R = \frac{1}{\gamma} \log \int_{B_\varepsilon(0)} \eta^\varepsilon(x) d\mu_{h^{n-1}}(x) - h_\varepsilon^{n-1}(0).$$

To bound $\mathrm{Var}[R]$, we map $B_{2^{-n+1}}(0) = B_{2\varepsilon}(0)$ to $\mathbb{D}$ by scaling. By the rule of changing coordinates for μ_h,

$$R \overset{\mathrm{d}}{=} \frac{1}{\gamma} \log \int_{B_{\frac{1}{2}}(0)} \eta(2x) d\mu_h(x) - h_{\frac{1}{2}}(0) - \frac{Q}{\gamma} \log 2\varepsilon, \tag{17}$$

where h is the zero boundary GFF on $\mathbb{D}$, and $h_{\frac{1}{2}}(0)$ refers to the circle average process at distance $1/2$ evaluated at $z = 0$. Thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathrm{Var}[R] = \mathrm{Var}[\frac{1}{\gamma} \log \int_{B_{\frac{1}{2}}(0)} \eta(2x) d\mu_h(x) - h_{\frac{1}{2}}(0)] \\
&\le \mathrm{Var}[h_{\frac{1}{2}}(0)] + 2\sqrt{\mathrm{Var}[h_{\frac{1}{2}}(0)]\,\mathrm{Var}[\frac{1}{\gamma} \log \int_{B_{\frac{1}{2}}(0)} \eta(2x) d\mu_h(x)]} + \mathrm{Var}[\frac{1}{\gamma} \log \int_{B_{\frac{1}{2}}(0)} \eta(2x) d\mu_h(x)].
\end{aligned}$$

Observe that $\mathrm{Var}[h_{\frac{1}{2}}(0)] \le \mathbb{E}[h_{\frac{1}{2}}(0)^2] < \infty$. Also,

$$\log \int_{B_{1/2}(0)} \eta(2x) d\mu_h(x) \le \log \max_{x \in B_1(0)} \{\eta(x)\} + \log \mu_h(B_{\frac{1}{2}}(0)) \tag{18}$$

and

$$\log \int_{B_{1/2}(0)} \eta(2x) d\mu_h(x) \geq \log \min_{x \in B_{1/2}(0)} \{\eta(x)\} + \log \mu_h(B_{\frac{1}{4}}(0)), \tag{19}$$

Both right hand sides in (18) and (19) have finite second moment. Hence

$$\mathbb{E}\left[\left(\log \int_{B_{1/2}(0)} \eta(2x) d\mu_h(x)\right)^2\right] < \infty.$$

We deduce that

$$\mathrm{Var}[R] \leq C \tag{20}$$

where C does not depend on ε.

By Lemma 2.1 and the definition of X_k, $\mathbb{E}[X_k^2] \leq C 4^{k-n}$ where C is independent of ε. Thus by the Cauchy–Schwarz inequality,

$$\mathbb{E}[(\sum_{k=1}^{n-1} X_k)^2] \leq n\mathbb{E}[\sum_{k=1}^{n-1} X_k^2] \leq C \log \varepsilon^{-1}$$

Together with (16) and (20) and using the Cauchy–Schwarz inequality again, this proves (15) and hence completes the proof of Lemma 3.1. □

3.2 The covariance estimate

Proof of Lemma 3.2. For fixed x_1 and x_2 in $\mathbb{D}$, let L be the segment dividing $\mathbb{D}$ into two components and bisecting $\overline{x_1 x_2}$. Let U_i be the connected component of $\mathbb{D} \setminus L$ containing x_i, $i = 1, 2$. Let h^i be the projection of h onto U_i and $h^{\mathrm{har}} = h - h^1 - h^2$ which is harmonic on U_1 and U_2. Then for $\varepsilon < |x_1 - x_2|/100$,

$$f_\varepsilon(x_i) = \frac{1}{\gamma} \log \int_{B_\varepsilon(x_i)} \eta^\varepsilon(x_i - z) e^{\gamma h^{\mathrm{har}}(z)} d\mu_{h^i}(z) - h^{\mathrm{har}}(x_i) - h^i_\varepsilon(x_i).$$

As in the proof of Lemma 3.1, let ξ_i be a point in $B_\varepsilon(x_i)$ such that

$$\int_{B_\varepsilon(x_i)} \eta^\varepsilon(x_i - z) e^{\gamma h^{\mathrm{har}}(z)} d\mu_{h^i} = e^{\gamma h^{\mathrm{har}}(\xi_i)} \int_{B_\varepsilon(x_i)} \eta^\varepsilon(x_i - z) d\mu_{h^i}.$$

Let

$$\tilde{f}_\varepsilon(x_i) = \frac{1}{\gamma} \log \int_{B_\varepsilon(x_i)} \eta^\varepsilon(x_i - z) d\mu_{h^i} - h^i_\varepsilon(x_i)$$

and $\Delta_i = h^{\mathrm{har}}(\xi_i) - h^{\mathrm{har}}(x_i)$. Then $f_\varepsilon(x_i) = \tilde{f}_\varepsilon(x_i) + \Delta_i$. Therefore

$$\mathrm{Cov}[f_\varepsilon(x_1), f_\varepsilon(x_2)] = \mathrm{Cov}[\tilde{f}_\varepsilon(x_1), \tilde{f}_\varepsilon(x_2)] + \mathrm{Cov}[\tilde{f}_\varepsilon(x_1), \Delta_2] + \mathrm{Cov}[\tilde{f}_\varepsilon(x_2), \Delta_1] + \mathrm{Cov}[\Delta_1, \Delta_2].$$

By Lemma 3.1, $\mathrm{Var}[\tilde f_\varepsilon(x_i)] \le C_r \log \frac{|x_1-x_2|}{\varepsilon}$. By Lemma 2.1, $\mathrm{Var}[\Delta_i] \le C \frac{\varepsilon^2}{|x_2-x_1|^2}$. $\tilde f_\varepsilon(x_1)$ and $\tilde f_\varepsilon(x_2)$ are independent, which means $\mathrm{Cov}[\tilde f_\varepsilon(x_1), \tilde f_\varepsilon(x_2)] = 0$. Therefore, by the Cauchy–Schwarz inequality again,

$$\begin{aligned}\mathrm{Cov}[f_\varepsilon(x_1), f_\varepsilon(x_2)] &\le (\mathrm{Var}[\tilde f_\varepsilon(x_1)]\,\mathrm{Var}[\Delta_2])^{1/2} + (\mathrm{Var}[\tilde f_\varepsilon(x_2)]\,\mathrm{Var}[\Delta_1])^{1/2} + (\mathrm{Var}[\Delta_1]\,\mathrm{Var}[\Delta_2])^{1/2}\\ &\le C_r \frac{\varepsilon}{|x_2-x_1|} \log^{1/2} \frac{|x_1-x_2|}{\varepsilon}.\end{aligned}$$

This conclude the proof of Lemma 3.2, and with it the proof of Theorem 1.1. □

4 Proof of Theorem 1.6: SLE case

The proof of Theorem 1.6 is fairly simple from what we have done above and conformal invariance. Indeed we essentially need the same theorem as above but in the boundary case, which is what we do now.

4.1 Boundary measure case

Given a free boundary GFF with a certain normalization, recall the definition of the quantum boundary length in (6). In this section we explain that quantum boundary length also determines the GFF restricted to the boundary. Since since most of the argument will be just a variant the area case, we will only point out the difference.

For the sake of concreteness, we focus the following setup. Let $\mathbb{D}_+ = \mathbb{D} \cap \mathbb{H}$ be the upper unit disk. Let h be the mixed GFF on $\mathbb{D}_+$ which has zero boundary condition on $\partial\mathbb{D}_+ \setminus \mathbb{R}$ and free boundary condition on $[-1,1]$. Let ν_h be the boundary measure induced by h on $[-1,1]$: recall that, as per (6), this is simply

$$\nu_h(dx) = \varepsilon^{\gamma^2/4} e^{\gamma h_\varepsilon(x)/2} dx.$$

It is not hard to see that h gives rise to a nontrivial distribution on $[-1,1]$ which will be denoted by $h|_{[-1,1]}$. (This distribution is itself a log-correlated field in one-dimension, but this will not be relevant here). The harmonic extension of h off $[-1,1]$ is then trivially measurable with respect to $h|_{[-1,1]}$.

Then we have:

Theorem 4.1. *ν_h determines $h|_{[-1,1]}$. In another word, $\forall \rho \in C_0^\infty(-1,1)$, (h,ρ) is measurable with respect to ν_h. The same is also true if we replace h by $h+g$ where g is a random continuous function on $\overline{\mathbb{D}_+}$*

Proof. For all $x \in (-1,1)$, it is still possible to define $h^\varepsilon(x)$ by

$$e^{\frac{\gamma}{2} h^\varepsilon(x)} = \int_{\mathbb{R}} \eta^\varepsilon(x-z) d\nu_h(z).$$

where η^ε is obtained by scaling a one dimensional bump function. Thus we can define $f_\varepsilon(x)$ by

$$f_\varepsilon(x) = h^\varepsilon(x) - h_\varepsilon(x)$$

where $h_\varepsilon(x)$ is the upper semi-circle average of h for $x \in (-1, 1)$. Note the key fact that the free boundary GFF enjoys the following Markov property: for any $U \subset \mathbb{D}_+$, we can write $h = h^{\text{supp}} + h^{\text{harm}}$ where h^{supp} is a zero boundary GFF in U, h^{harm} is harmonic in U, and crucially, if $\partial U \cap [-1, 1] \neq \emptyset$, it has Neumann boundary condition on $\partial U \cap [-1, 1]$. In particular, by reflection we obtain a harmonic function across $U \cup \bar{U} \cup (\partial U \cap [-, 1, 1])$. Thus the estimate of Lemma 2.1 is valid even if $z \in \partial U$, with $r = \text{dist}(z, (\partial U) \setminus [-, 1, 1])$.

This allows us to repeat *verbatim* the proof of Theorem 1.1, with some obvious changes: e.g., the integral I becomes

$$I = \frac{2}{\gamma} \log \int_{B_\varepsilon(0)} \eta^\varepsilon(x) d\nu_h(x) - h_\varepsilon(0).$$

Details are left to the reader. □

4.2 Proof of Theorem 1.6

Proof of Theorem 1.6. Let $h = h_0 + \varphi$, where h_0 is a free boundary GFF (normalised in some way – say zero mean on the semi-circle of radius 1 around the origin) and $\varphi(z) = -(2/\gamma) \log |z|$. Let η be an independent SLE_κ where $\kappa = \gamma^2$ and reparametrize η by its quantum length induced by h. Suppose T is the time when $\eta[0, T]$ has capacity 1 (which is a.s. finite and positive). If we can prove that $\mathcal{F} = \sigma(\{\eta(t) : 0 \leq t \leq T\})$ determines h restricted to $\eta[0, T]$, then by scaling and absolutely continuity, we also obtain the general case.

In this setup, define g_T, h_T as in (5): thus g_T is the Lowener map (which is really g_1 in the parametrisation by half-plane capacity) and $h_T = h \circ g_T^{-1} + Q \log |(g_T^{-1})'|$. Then by the properties of the (reverse) coupling between SLE and the Gaussian Free Field (Theorem 1.2 in [She10]), the marginal law of h_T is the same as that of h (namely, a free boundary GFF plus $(2/\gamma) \log(1/|\cdot|)$).

Now, note that the boundary measure ν_{h_T}, restricted to $g_T(\eta([0, T]))$, is measurable with respect to $\mathcal{F}$. Indeed, first observe that g_T is $\mathcal{F}$-measurable, and hence so is g_T^{-1}. Now, for $z \in [0^-, \xi_T]$, we can find $t = t_z$ such that $g_T(\eta(t)) = z$; hence by choice of our reparametrisation, $\nu_{h_T}([0^-, z]) = t$. Finally, t_z is clearly measurable with respect to $\mathcal{F}$ (as t_z is the first point on the curve which intersects $g_T^{-1}(z)$.)

Therefore by Theorem 4.1, the restriction of h_T to $g_T(\eta([0, T]))$ is measurable with respect to $\mathcal{F}$. (This is because Theorem 4.1 is an almost sure property, and the law of $h_0 + \varphi$ is absolutely continuous with respect to the law of h_0 on subsets bounded away from zero and infinity.)

Let h^{har} be the harmonic extension of h outside $\eta[0, T]$. Then $h^{\text{har}} \circ g_T^{-1} + Q \log |(g_T^{-1})'|$ is $\mathcal{F}$-measurable. Since g_T is $\mathcal{F}$-measurable, we have that h^{har} is $\mathcal{F}$-measurable too. The theorem follows by absolute continuity between h_0 and $h_0 + \varphi$ on sets bounded away from zero and infinity. $\square$

5 Proof of Theorem 1.7: case of LBM

In this section, let h be a zero boundary GFF on a simply connect domain D. Let B_t be an independent Brownian motion starting from $0 \in D$ and τ_D be the hitting time of ∂D. We recall the definition of the quantum clock

$$\phi(t) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_0^{t \wedge \tau_D} e^{\gamma h_\varepsilon(B_s)} ds.$$

Recall that $Z_t = B_{\phi^{-1}(t)}$ is, by definition, the Liouville Brownian motion on D stopped upon hitting the boundary.

To prove Theorem 1.7, we first fix D_0 to be a subdomain of D such that $0 \in D_0, \bar{D}_0 \subset D$ such that $\mathrm{dist}(D_0, \partial D) > 0$ and ∂D_0 is smooth. Let τ be the time when B_t hits ∂D_0. Observe that $\sigma(\{Z_t\}_{0 \leq t \leq \phi(\tau)}) = \sigma(\{B_t, \phi(t)\}_{0 \leq t \leq \tau})$. In fact, since $B_{\phi^{-1}(t)} = Z_t$, $\phi^{-1}(t)$ is the quadratic variation of Z_t. Therefore

$$\{\phi(t)\}_{0 \leq t \leq \tau} \in \sigma(\{Z_t\}_{0 \leq t \leq \phi(\tau)}), B_t = Z_{\phi(t)} \in \sigma(\{Z_t\}_{0 \leq t \leq \phi(\tau)}).$$

Let h^{har} be the projection of h onto the space of functions in $H_0^1(D)$ which are harmonic in $D \setminus B[0,\tau]$. h^{har} encodes the information of h restricted to $B[0,\tau]$.

At a conceptual level, the proof of this theorem follows similar lines as that of Theorem 1.1. (However the technical estimates will be quite a bit more complicated.) For $z \in D$, let $\omega(z, dx)$ be the harmonic measure of $D \setminus B[0,\tau]$ viewed from z. Let

$$\begin{aligned}
\tilde{h}_\varepsilon^{\mathrm{har}}(z) &= \int \omega(z, dx) \big[\frac{1}{\gamma} \log \int_0^\tau 1_{B_s \in B_\varepsilon(x)} d\phi(s) \big], \\
h_\varepsilon^{\mathrm{har}}(z) &= \int \omega(z, dx) h_\varepsilon(x), \\
g_\varepsilon &= \tilde{h}_\varepsilon^{\mathrm{har}} - h_\varepsilon^{\mathrm{har}}.
\end{aligned}$$

Here we modify the definition of $h_\varepsilon(x)$ a little bit. Since only inside D_0 matters, we assume that $h_\varepsilon(x)$ is the circle average for $x \in D_0$ and $h_\varepsilon(x) = 0$ for $x \in \partial D$. This can be achieved by multiplying the usual circle average field $h_\varepsilon(z)$ by a function in $C_0^\infty(D)$ which takes value 1 on a neighborhood of D_0.

In order to show that $\tilde{h}_\varepsilon^{\mathrm{har}}$ is a good estimator for h^{har}, we follow the strategy in Section 3. We first introduce the probability measure $\widetilde{\mathbb{P}}$, which is the conditional probability measure given $B[0,\tau]$. Let $\widetilde{\mathbb{E}}, \widetilde{\mathrm{Var}}$ and $\widetilde{\mathrm{Cov}}$ be the expectation, variance, and covariance under $\widetilde{\mathbb{P}}$.

Lemma 5.1. *Let X_n, X, Y_n be random variables measurable w.r.t. the σ-algebra $\mathcal{G}$ and $\mathcal{F} \subset \mathcal{G}$ be a sub σ-algebra. Let $\widetilde{\mathbb{E}}$ denote the conditional expectation w.r.t. $\mathcal{F}$. If*

$$\lim_{n\to\infty} \widetilde{\mathbb{E}}\left[(X_n - Y_n)^2\right] = 0, \lim_{n\to\infty} X_n = X$$

in probability, then Y_n converges to X in probability. In particular, if $Y_n \in \mathcal{F}$, then $X \in \mathcal{F}$.

Proof. It suffices to show that $X_n - Y_n$ converges to zero in probability. Let E be the event that $\{\widetilde{\mathbb{E}}\left[(X_n - Y_n)^2\right] > \varepsilon\}$. Then $E \in \mathcal{F}$. So

$$\mathbb{E}[(X_n - Y_n)^2 \mathbf{1}_{E^c}] = \mathbb{E}\left[\widetilde{\mathbb{E}}[(X_n - Y_n)^2]\mathbf{1}_{E^c}\right] \le \varepsilon.$$

By Markov inequality $\mathbb{P}\left[|X_n - Y_n|\mathbf{1}_{E^c} \ge \varepsilon^{1/4}\right] \le \varepsilon^{1/2}$. Therefore

$$\mathbb{P}[|X_n - Y_n| \ge \varepsilon^{1/4}] \le \mathbb{P}[E] + \mathbb{P}[E^c, |X_n - Y_n| \ge \varepsilon^{1/4}] \le o_\varepsilon(1) + \varepsilon^{1/2},$$

so the result follows. □

Lemma 5.2. *For all $\rho \in C_0^\infty(D)$, $\mathbb{E}[(h_\varepsilon^{\text{har}}, \rho)] = 0$ and $(h_\varepsilon^{\text{har}}, \rho)$ tends to (h^{har}, ρ) in probability.*

Proof. Without loss of generality, we can assume ρ is a probability density function. Then $(h^{\text{har}}, \rho) = (h, \hat{\rho})$ where $\hat{\rho}(x)dx$ represents the probability distribution obtained by first sampling z according to $\rho(z)dz$ and then sample a Brownian motion from z. Suppose X is the exit location of the domain $D \setminus B[0, \tau]$ for this Brownian motion. Then $\hat{\rho}(x)dx$ represents the distribution of X.

Let $\hat{\rho}_\varepsilon(x)dx$ be the probability measure obtained by first sample x according to $\hat{\rho}(x)dx$, then adding εU to x where U is a uniform unit vector. Using this construction $(h_\varepsilon^{\text{har}}, \rho)$ can be represented by $(h, \hat{\rho}_\varepsilon)$ (we emphasise that this is the usual L^2 inner product). By Lemma 5.1 we just need to show that under $\widetilde{\mathbb{P}}$, $(h, \hat{\rho}_\varepsilon)$ converges to $(h, \hat{\rho})$. In fact it suffice to prove that $\widetilde{\text{Var}}[(h, \hat{\rho}_\varepsilon - \hat{\rho})] \to 0$ almost surely. Suppose X and Y are two independent copies sampled from $\hat{\rho}(z)dz$, and U_1, U_2 are two independent uniform unit vector. Then it suffices to show that

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}}[G_D(\cdot, *)] = \widetilde{\mathbb{E}}[G_D(X, Y)] \tag{21}$$

where $\cdot$ represents either X or $X + \varepsilon U_1$, $*$ represents either Y or $Y + \varepsilon U_2$, and G_D is the Green function in D (this has to be shown in all four possible combinations of the terms).

We consider $G_D(X + \varepsilon U_1, Y + \varepsilon U_2)$, the others are similar or simpler. By the mean property of harmonic functions, and the fact G_D itself is harmonic off the diagonal, we have

$$\widetilde{\mathbb{E}}[G_D(X + \varepsilon U_1, Y + \varepsilon U_2)\mathbf{1}_{\{|X-Y|>2\varepsilon\}}] = \widetilde{\mathbb{E}}(G_D(X, Y)\mathbf{1}_{\{|X-Y|>2\varepsilon\}}).$$

Also,

$$\mathbb{E}(G_D(X + \varepsilon U_1, Y + \varepsilon U_2)\mathbf{1}_{\{|X-Y|\le 2\varepsilon\}}) \le C\log(1/\varepsilon)\widetilde{\mathbb{P}}(|X - Y| < 2\varepsilon).$$

But by a Beurling estimate ([Law05, Theorem 3.76]),

$$\widetilde{\mathbb{P}}[|X-Y|<\varepsilon]=o(|\log\varepsilon|^{-1}).$$

Hence (21) holds and the lemma is proved. □

We will prove the following analogues of Lemma 3.1 and Lemma 3.2.

Lemma 5.3. *Fix $\zeta>0$ and a compact set $D'\subset D_0$,*

$$\mathbb{E}\left[\widetilde{\mathrm{Var}}[g_\varepsilon(z)]\right]\le C|\log\varepsilon|^3 \tag{22}$$

where C is constant only depending ζ, D_0, D' but not z.

Lemma 5.4. *Fix $\zeta>0, k>0$ and a compact set $D'\subset D_0$,*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\widetilde{\mathrm{Cov}}[g_\varepsilon(z_1),g_\varepsilon(z_2)]\right]=0 \tag{23}$$

uniformly in $z_1,z_2\in D'$ such that $|z_1|,|z_2|>\zeta, |z_1-z_2|>|\log\epsilon|^{-k}$.

Assuming these lemmas, the proof of Theorem 1.7 follows. By the same argument in Proposition 3.3, Lemma 5.3 and Lemma 5.4 imply that for all $\zeta>0$ and $\rho\in C_0^\infty(D_0\setminus B_\zeta(o))$,

$$\lim_{\varepsilon\to 0}\widetilde{\mathrm{Var}}[(g_\varepsilon,\rho)]=\int\int\widetilde{\mathrm{Cov}}[g_\varepsilon(z_1),g_\varepsilon(z_2)]\rho(z_1)\rho(z_2)dz_1dz_2=0$$

in probability. Applying Lemma 5.2 and Lemma 5.1 to the case where $\mathcal{F}=\sigma(\{B_t,\phi(t):t\in[0,\tau]\})$, we have $(h^{\mathrm{har}},\rho)\in\sigma(\{B_t,\phi(t):t\in[0,\tau]\})$. By varying ζ and D_0, we obtain Theorem 1.7.

5.1 The variance estimate

Before proving Lemma 5.3, we start with a few preliminary estimates.

Lemma 5.5. *Let h be the zero boundary GFF on $2\mathbb{D}$, $x\in\mathbb{D}$, $r\in(0,1)$ and $y\in\partial B_r(x)$. Let B_t be a Brownian motion starting from x, τ^x be the hitting time of $\partial B_r(x)$. Then there exists a constant C independent of x,y such that*

$$\mathbb{E}\left[1/\phi(\tau_x)\Big|B_{\tau^x}=y\right]\le Cr^{-2-\gamma^2}. \tag{24}$$

Proof. By Kahane's convexity inequalities (see e.g. [GRV13, Appendix A]), and since $x\mapsto 1/x$ is convex, we can replace the Gaussian free field h_ε by a field X_ε which is exactly scale-invariant, translation invariant, and also rotationally invariant (see for instance Lemma 3.1 in

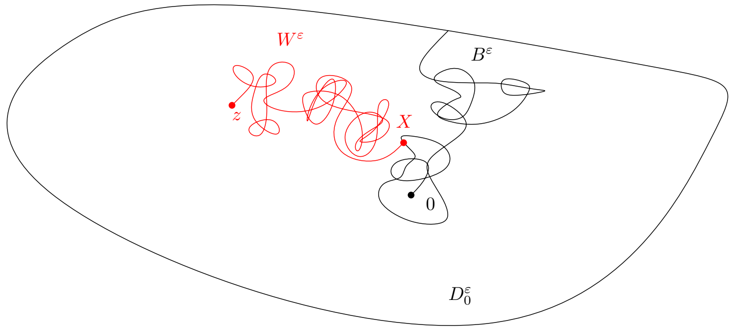

Figure 1: Proof of the variance estimate. X is the first hitting time by W^ε of B^ε. Near X, we locally have some nonintersecting Brownian paths (the union of two black arms not intersecting one red arm).

[Ber13]). Taking the limit (in distribution) as $\varepsilon \to 0$, we obtain a slightly different quantum clock which we will call $(\tilde{\phi}(t), t \leq \tau_x)$.

Then by rotational invariance of the Brownian motion and X,

$$\mathbb{E}\left[1/\tilde{\phi}(\tau_x)\middle|B_{\tau^x} = y\right] = \mathbb{E}\left[1/\tilde{\phi}(\tau_x)\right].$$

By scale invariance and using the finiteness of moments of order -1 proved in [GRV13, Proposition 2.12], we deduce:

$$\mathbb{E}\left[1/\tilde{\phi}(\tau_x)\right] \leq Cr^{-2-\gamma^2},$$

and the lemma follows. □

In the proof of Lemma 5.3 we will need the following quantities and notations. Let $D^\varepsilon = \varepsilon^{-1}D, D_0^\varepsilon = \varepsilon^{-1}D_0, D'^\varepsilon = \varepsilon^{-1}D'$. Let $z \in D'^\varepsilon$ and $|z| \geq \zeta\varepsilon^{-1}$ for some $\zeta > 0$. Let $(B_t)_{t \leq \tau}$ be a Brownian motions starting from 0 run until the hitting time τ of ∂D_0^ε. Let W_t be a Brownian motion starting from z independent of B and h. For all $x \in D_0^\varepsilon$, let h^x be an independent zero boundary GFF on $B_2(x)$ and ϕ^x be its quantum clock for B induced by h^x. Let X be the location where W_t hits $\partial D^\varepsilon \cup B[0, \tau]$ and

$$Y = \int_0^\tau \mathbf{1}_{\{B_s \in B_1(X)\}} d\phi^X(s). \tag{25}$$

Lemma 5.6. *Then $\mathbb{E}[(\log Y)^4] \leq C|\log \varepsilon|^3$, where C is independent of z and ε.*

Remark 5.7. For the variance bound, we will in fact only need that $\mathbb{E}[(\log Y)^2] \leq C|\log \varepsilon|^3$. The stronger bound which is proved here will be needed in Section 5.2, for the covariance bound.

Proof. To control the moments of $\log Y$ we will need to control both a tail at infinty of Y and a tail at zero. Specifically, we will show

$$\mathbb{E}[Y] \leq C|\log \varepsilon|^3. \tag{26}$$

and separately

$$\mathbb{E}[Y^{-1}] \leq C|\log \varepsilon|^6 \tag{27}$$

Since $\log^4 x \leq O(x + x^{-1/6})$ for all $x > 0$, (26) and (27) together Jensen's inequality imply Lemma 5.6.

We start with (26). Conditioning on B and W, let

$$Y' = \lim_{\varepsilon \to 0} \int_{B_1(X)} \exp\{\gamma h_\varepsilon(x) - \frac{\gamma^2}{2} \operatorname{Var}[h_\varepsilon(x)]\} d\sigma(x)$$

where σ is the occupation measure of $B|_{[0,\tau]}$. Then $\mathbb{E}[Y] \leq C\mathbb{E}[Y']$ for some C. By Fatou's Lemma, $\mathbb{E}[Y'] \leq C\mathbb{E}[\sigma(B_1(X))]$. The fine structure of the occupation measure ν is well understood, see e.g. [DPRZ01]. Here we only a very crude bound. From [DPRZ01, Lemma 2.1] and the argument for proving [DPRZ01, Equation (2.12)], there are constants c, C independent of $x \in D_0^\varepsilon, n \in \mathbb{N}$ such that

$$\mathbb{P}[\sigma(B_1(x)) \geq n|\log \varepsilon|^3] \leq C\varepsilon^{cn|\log \varepsilon|}.$$

By a union bound (summing over ε^{-2} balls)

$$\mathbb{P}[\sigma(B_1(X)) \geq n|\log \varepsilon|^3] \leq C\varepsilon^{-2}\mathbb{P}[\sigma(B_1(x)) \geq n|\log \varepsilon|^3] \leq C\varepsilon^{-2}\varepsilon^{cn|\log \varepsilon|}.$$

Therefore,

$$\mathbb{E}(Y) \leq C\mathbb{E}(Y') \leq C\mathbb{E}[\sigma(B_1(X))] \leq C|\log \varepsilon|^3,$$

as desired in (26).

We now turn to the more delicate (27). We first note that $Y \geq \min_{x \in D_0^\varepsilon}\{\int_0^\tau \mathbf{1}_{\{B_s \in B_1(x)\}} d\phi^x(s)\}$ and there is a constant C such that

$$\mathbb{E}\left[\left(\int_0^\tau \mathbf{1}_{\{B_s \in B_1(x)\}} d\phi^x(s)\right)^{-2}\right] \leq C\varepsilon^{-2},\ \forall x \in D_0^\varepsilon.$$

(Since ∂D_0 is smooth, there is no degeneracy at ∂D_0.) Therefore by a union bound

$$\mathbb{E}[Y^{-2}] \leq C\varepsilon^{-4}. \tag{28}$$

Now we will bound $\mathbb{E}[Y^{-1}]$ in five different cases.

Case I: Suppose that $X \in B_{1/4}(z)$, and suppose also the following occurs. Consider the excursions $[\sigma_j, \sigma^j]$ of B between $\partial B_{1/4}(z)$ and $\partial B_{1/2}(z)$, assuming that there are any. (That is, let $\sigma_1 = \inf\{t : |B_t - z| = 1/4\}, \sigma^1 = \inf\{t > \sigma_1 : |B_t - z| = 1/2\}$, and inductively, let

$\sigma_j = \inf\{t > \sigma^{j-1} : |B_t - z| = 1/4\}, \sigma^j = \inf\{t > \sigma_j : |B_t - z| = 1/2\})$. Fix c_1 which will be determined later, then

$$E_1 = \{X \in B_{1/4}(z); \inf\{j : X \in B[\sigma_j, \sigma^j]\} > c_1 |\log \varepsilon|\}.$$

Set $N = c_1|\log\varepsilon|$. We have that on E_1, there are at least N excursions, and the first N excursions all manage to avoid $\{W_t, t \le T\}$ where T is the first time (W_t) exits $B_{1/4}(z)$. We may first condition on $W = \{W_t\}_{t \le T}$. Then by the Beurling estimate [Law05, Theorem 3.76], the conditional probability that the $(n+1)$th excursion avoids W is uniformly bounded above by e^{-c} for some universal $c > 0$, conditionally on W and all past n excursions. Hence taking expectations, $\mathbb{P}(E_1) \le e^{-cN}$. By choosing c_1 large enough we deduce that $\mathbb{P}[E_1] \le C\varepsilon^4$. Applying Cauchy-Schwarz inequality and (28), we have $\mathbb{E}[Y^{-1}; E_1] \le C$.

Case II: Let E_2 be the event that $X \in B_{1/4}(z)$ and $\inf\{j : X \in B[\sigma_j, \sigma^j]\} \le c_1|\log\varepsilon|$ where c_1 is defined as in E_1. Then

$$\begin{aligned}
\mathbb{E}[Y^{-1}; E_2] &\le \mathbb{E}\left[\max_{1 \le j \le c_1|\log\varepsilon|} \left(\int_{\sigma_j}^{\sigma^j} \mathbf{1}_{\{B_s \in B_1(X)\}} d\phi^X s\right)^{-1}\right] \\
&\le \sum_{j=1}^{c_1|\log\varepsilon|} \mathbb{E}\left[\left(\int_{\sigma_j}^{\sigma^j} \mathbf{1}_{\{B_s \in B_1(X)\}} d\phi^X s\right)^{-1}\right] \\
&\le C|\log\varepsilon|
\end{aligned}$$

by reasoning as in Lemma 5.5.

Case III: From above we have $\mathbb{E}[Y^{-1}; X \in B_{1/4}(z)] = \mathbb{E}[Y^{-1}; E_1 \cup E_2] \le C|\log\varepsilon|$. Let E_3 be the event that $\operatorname{dist}(X, \partial D_0^\varepsilon) \le 1/2$. We can similarly consider the excursions of B_t between concentric balls of radii $1/4, 1/2$ near ∂D_0. Still by the Beurling estimate, the number of the excursions is dominated by a geometric random variable independent of ε, z. But in this case the excursions are trying to avoid ∂D_0^ε rather than W. When the number of excursions are of bigger than $c|\log\varepsilon|$ for some big enough c, we can apply the Cauchy–Schwarz inequality to control $\mathbb{E}[Y^{-1}]$ as in case I. Otherwise, we can apply the union bound to control Y^{-1} as in case II. To summarize, we have $\mathbb{E}[Y^{-1}; E_3] \le C|\log\varepsilon|$.

Case IV: Let $\sigma = \inf\{t : B_t = X\}, \bar\sigma = \inf\{t > \sigma : |B_t - X| = 1/4\}$. Call W the range of the trajectory of (W_t) until first visiting X. Fixing some c_2 which will be determined later, let

$$E_4 = \left\{\operatorname{dist}(X, \{z\} \cup \partial D_0^\varepsilon) > 1/4,\ \operatorname{dist}(B_t, W) < c_2|\log\varepsilon|^{-1} \quad \forall t \in [\sigma, \bar\sigma]\right\}.$$

We claim that for small enough c_2, $\mathbb{P}[E_4] \le \varepsilon^4$. To see this, observe that we can cover $D_0 \setminus B_{1/4}(z)$ with $N = O(\varepsilon^{-2})$ balls $\{B_{1/10}(x_i) : 1 \le i \le N\}$ such that on E_4, we can find a ball $B_{1/10}(x_i)$ with the following happens: there exists an excursion of B_t from $\partial B_{1/10}(x_i)$ to $\partial B_{2/10}(x_i)$ (recall the definition of excursions in case I) where B_t stays within distance $\delta := c_2|\log\varepsilon|^{-1}$ of W but never intersects W. Let ρ_0 be a starting point of such an excursion. Let $\rho_j = \inf\{t \ge \rho_{j-1} : |B_{\rho_j} - B_{\rho_{j-1}}| = 2\delta\}$ $\forall j \in \mathbb{N}$. Then there exists C such that for

$j < C\delta^{-1}$, $\{B_t\}_{\rho_{j-1}\le t\le \rho_j}$ stays within distance δ of W but never intersects W. By the Beurling estimate, given any realization of W,

$$\mathbb{P}(\text{dist}(\{B_t\}_{\rho_{j-1}\le t\le\rho_j}, W) \le \delta; \{B_t\}_{\rho_{j-1}\le t\le\rho_j} \cap W = \emptyset) \le e^{-c}$$

for some universal $c > 0$. By the Markov property of Brownian motion, iterating this bound $C\delta^{-1}$ many times, we see that we can choose c_2 small enough so that for all x_i such an event occurs with probability less than $C\varepsilon^6$. Summing over x_i yields $\mathbb{P}[E_4] \le C\varepsilon^4$. Combined with (28), we have $\mathbb{E}[Y^{-1}; E_4] \le C$.

Case V: Let $\delta = c_2|\log\varepsilon|^{-1}$, where c_2 is the constant in E_4, and set

$$E_5 = \{\text{dist}(X, \{z\} \cup \partial D_0^\varepsilon) > 1/4,\ \exists t \in [\sigma, \bar\sigma] \text{ s.t. } \text{dist}(B_t, W) > \delta\}\,.$$

On E_5, let

$$\lambda = \inf\{t \ge \sigma : \text{dist}(B_t, W) = \delta\},\ \bar\lambda = \inf\{t \ge \lambda : \text{dist}(B_t, B_\lambda) = \delta/2\}.$$

Conditioned on $X, W, h, B|_{[0,\lambda]}$, and $B|_{[\bar\lambda,\tau]}$, we have that $B|_{[\lambda,\bar\lambda]} - B_\lambda$ is distributed as a Brownian motion starting at 0, stopped at $B_{\delta/2}(0)$, conditioned on visiting $B_{\bar\lambda} - B_\lambda$ upon exiting this ball. Therefore by Lemma 5.5, $\mathbb{E}[Y^{-1}; E_5] \le C|\log\varepsilon|^6$.

Summing up all five cases, we deduce that $\mathbb{E}[Y^{-1}] \le C|\log\varepsilon|^6$, as claimed. This finishes the proof of the lemma. □

We now return to Lemma 5.3. Let

$$k_\varepsilon(x) = \frac{1}{\gamma}\log\int_0^\tau \mathbf{1}_{\{B_s\in B_\varepsilon(x)\}} d\phi(s) - h_\varepsilon(x). \tag{29}$$

Then $g_\varepsilon(z) = \int k_\varepsilon(x)\omega(z, dx)$. By Cauchy-Schwarz, Lemma 5.3 follows from

Lemma 5.8. *$\exists\, C$ such that $\forall\, z \in D'$*

$$\mathbb{E}\left[\int \widetilde{\text{Var}}[k_\varepsilon(x)]\omega(z, dx)\right] \le C|\log\varepsilon|^3. \tag{30}$$

Proof. Without loss of generality, we assume that $\text{dist}\{D_0, \partial D\} = 2$. To prove that the variance has logarithmic growth, we use the same trick as in Lemma 3.1. Suppose as before, and without loss of generality, that $\varepsilon = 2^{-n}$. For $x \in D_0$, we decompose h into

$$h = \sum_{i=0}^{n-1} h_i^{\text{har}} + h^{n-1}$$

where h_0^{har} is the projection of h to the space of functions in $H_0^1(D)$ that are harmonic on $B_1(x)$, h_i^{har} $(1 \le i \le n-1)$ is the projection of h to the space of functions in $H_0^1(D)$ that are supported on $B_{2^{-i+1}}(x)$ and harmonic on $B_{2^{-i}}(x)$, h_n is the zero boundary Gaussian free field on $B_{2^{1-n}}(x)$. As in the proof of Lemma 3.1,

$$k_\varepsilon(x) = \sum_{i=0}^{n-1}(h_i^{\text{har}}(\xi_i) - h_i^{\text{har}}(x)) + \frac{1}{\gamma}\log\int_0^\tau \mathbf{1}_{\{B_s\in B_\varepsilon(x)\}}d\phi_{n-1}^x(s) - h_\varepsilon^{n-1}(x),$$

where $\xi_i \in B_{2^{-i}}(x)$ comes from the intermediate value theorem, ϕ_{n-1}^x is the quantum clock induced by h^{n-1} and $h_\varepsilon^{n-1}(x)$ is the ε-circle average of h^{n-1}.

Let $X_i(x) = h_i^{\text{har}}(\xi_i) - h_i^{\text{har}}(x)$. Then

$$k_\varepsilon(x) = \sum_{i=1}^{n-1} X_i(x) + \frac{1}{\gamma}\log\int_0^\tau \mathbf{1}_{\{B_s\in B_\varepsilon(x)\}}d\phi_{n-1}^x(s) - h_\varepsilon^{n-1}(x). \tag{31}$$

Let $\overline{\Delta}(x) = \max_{y_1,y_2\in B_\varepsilon(x)}\{h_i^{\text{har}}(y_1) - h_i^{\text{har}}(y_2)\}$. From the proof of Lemma 2.1,

$$\widetilde{\mathbb{E}}[X_i^2(x)] \le \mathbb{E}[\overline{\Delta}(x)^2] \le C4^{i-n} \tag{32}$$

where C is a constant independent of x. Using (32) and the Cauchy–Schwarz inequality, we see that

$$\widetilde{\text{Var}}(\sum_{i=1}^n X_i(x)) \le n\sum_{i=1}^n \widetilde{\mathbb{E}}(X_i^2(x)) \le C|\log\varepsilon|. \tag{33}$$

By Lemma 2.1, $\widetilde{\text{Var}}[h_\varepsilon^{n-1}(x)] = \text{Var}[h_\varepsilon^{n-1}(x)] = O(1)$.

Therefore it remains to show that for some constant $C > 0$,

$$\mathbb{E}\left[\int \omega(z,dx)\widetilde{\text{Var}}\left[R_{n,\varepsilon}(x)\right]\right] \le C|\log\varepsilon|^3 \tag{34}$$

where

$$R_{n,\varepsilon}(x) = \frac{1}{\gamma}\log\int_0^\tau \mathbf{1}_{\{B_s\in B_\varepsilon(x)\}}d\phi_{n-1}^x(s). \tag{35}$$

Note that

$$\mathbb{E}\left[\int \omega(z,dx)\widetilde{\text{Var}}\left[R_{n,\varepsilon}(x)\right]\right] \le \mathbb{E}\left[\int \omega(z,dx)\widetilde{\mathbb{E}}\left[R_{n,\varepsilon}^2(x)\right]\right] = \mathbb{E}\left[\int \omega(z,dx)R_{n,\varepsilon}^2(x)\right]. \tag{36}$$

Thus set

$$Y = \int_0^{\tau^\varepsilon} \mathbf{1}_{\{B_s^\varepsilon\in B_1(X)\}}d\phi^X(s). \tag{37}$$

where, using the notations from Lemma 5.6, B^ε is a Brownian motion run from 0 until hitting ∂D_0^ε, X is the point where the Brownian motion W^ε (started from $z^\varepsilon = \varepsilon^{-1}z$) hits the range of B^ε for the first time.

By the conformal invariance of Liouville Brownian motion [Ber13, Theorem 1.3],

$$\gamma^2\int \omega(z,dx)R_{n,\varepsilon}^2(x) \overset{d}{=} (\log Y + Q\log\varepsilon)^2. \tag{38}$$

Therefore (34) follows immediately from the stronger Lemma 5.6. □

5.2 The covariance estimate

Let $q \in (0,1)$ be a constant that will be determined later. Recalling (29), write

$$\mathbb{E}\left[\widetilde{\mathrm{Cov}}[g_\varepsilon(z_1), g_\varepsilon(z_2)]\right] = M(z_1,z_2) + R(z_1,z_2),$$

where

$$\begin{aligned} M(z_1,z_2) &= \mathbb{E}\left[\int\int_{|x_1-x_2|\geq \varepsilon^q} \widetilde{\mathrm{Cov}}\left[k_\varepsilon(x_1), k_\varepsilon(x_2)\right]\omega(z_1,dx_1)\omega(z_2,dx_2)\right], \\ R(z_1,z_2) &= \mathbb{E}\left[\int\int_{|x_1-x_2|< \varepsilon^q} \widetilde{\mathrm{Cov}}\left[k_\varepsilon(x_1), k_\varepsilon(x_2)\right]\omega(z_1,dx_1)\omega(z_2,dx_2)\right]. \end{aligned} \tag{39}$$

We claim that:

Lemma 5.9. *Fix $\zeta > 0, k > 0, q \in (0,1)$.*

$$M(z_1,z_2) = o_\varepsilon(1) \tag{40}$$

uniformly in $z_1, z_2 \in D'$ for $|z_1|, |z_2| > \zeta, |z_1 - z_2| \geq |\log\varepsilon|^{-k}$,.

Lemma 5.10. *Fix $\zeta > 0, k > 0, q \in (0,1)$. $\exists\, q \in (0,1)$ such that*

$$R(z_1,z_2) = o_\varepsilon(1) \tag{41}$$

uniformly in $z_1, z_2 \in D'$ for $|z_1|, |z_2| > \zeta, |z_1 - z_2| \geq |\log\varepsilon|^{-k}$,.

Lemma 5.9 and Lemma 5.10 together will imply Lemma 5.4.

Proof of Lemma 5.9. For $x_1, x_2 \in D, |x_1 - x_2| > \varepsilon^q$, let L be the segment dividing D into two components and bisecting $\overline{x_1 x_2}$ as in the proof of Lemma 3.2. Let U_i be the connected component of $D \setminus L$ containing x_i, $i = 1, 2$. Let h^i be the projection of h onto U_i and $\bar{h}^{\mathrm{har}} = h - h^1 - h^2$ which is harmonic on U_1 and U_2. Let ϕ_i $(i = 1, 2)$ be the quantum clock of B in U_i induced by h_i .

Let

$$\begin{aligned} \psi_\varepsilon(x_i) &= \frac{1}{\gamma}\log\int_0^\tau \mathbf{1}_{B_s\in B_\varepsilon(x_i)}d\phi_i(s) - h^i_\varepsilon(x_i) \\ \Delta_i &= \frac{1}{\gamma}\log\int_0^\tau \mathbf{1}_{B_s\in B_\varepsilon(x_i)}e^{\gamma\bar{h}^{\mathrm{har}}(B_s)}d\phi_i(s) - \frac{1}{\gamma}\log\int_0^\tau \mathbf{1}_{B_s\in B_\varepsilon(x_i)}d\phi_i(s) - \bar{h}^{\mathrm{har}}(x_i). \end{aligned}$$

We remark that $\widetilde{\mathrm{Var}}[\phi_\varepsilon(x_1)], \widetilde{\mathrm{Var}}[\phi_\varepsilon(x_2)], \widetilde{\mathrm{Var}}[\Delta_1], \widetilde{\mathrm{Var}}[\Delta_2]$ are all random functions of (x_1, x_2) that are measurable w.r.t. $B[0,\tau]$.

As in the proof of Lemma 3.2,

$$k_\varepsilon(x_i) = \psi_\varepsilon(x_i) + \Delta_i. \tag{42}$$

By the independence of h_1, h_2 under $\widetilde{\mathbb{P}}$, for $|x_1 - x_2| > \varepsilon^q$,

$$\widetilde{\mathrm{Cov}}\left[\psi_\varepsilon(x_1), \psi_\varepsilon(x_2)\right] = 0.$$

Therefore by (42)

$$M(z_1, z_2) = \mathrm{I}(z_1, z_2) + \mathrm{II}(z_1, z_2) + \mathrm{III}(z_1, z_2)$$

where

$$\mathrm{I}(z_1, z_2) = \mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Cov}}\left[\psi_\varepsilon(x_1), \Delta_2\right]\omega(z_1, dx_1)\omega(z_2, dx_2)\right],$$

$$\mathrm{II}(z_1, z_2) = \mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Cov}}\left[\Delta_1, \psi_\varepsilon(x_2)\right]\omega(z_1, dx_1)\omega(z_2, dx_2)\right],$$

$$\mathrm{III}(z_1, z_2) = \mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Cov}}\left[\Delta_1, \Delta_2\right]\omega(z_1, dx_1)\omega(z_2, dx_2)\right].$$

By the intermediate value theorem and Lemma 2.1,

$$\widetilde{\mathrm{Var}}[\Delta_i] \leq C\varepsilon^2/|x_2 - x_1|^2 \tag{43}$$

where C is a constant that only depends on D, D', D_0. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
\mathrm{I}(z_1, z_2) &\leq \mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Var}}^{1/2}[\psi_\varepsilon(x_1)]\widetilde{\mathrm{Var}}^{1/2}[\Delta_2]\omega(z_1, dx_1)\omega(z_2, dx_2)\right] \\
&\leq C\varepsilon^{1-q}\mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Var}}^{1/2}[\psi_\varepsilon(x_1)]\omega(z_1, dx_1)\omega(z_2, dx_2)\right] \\
&\leq C\varepsilon^{1-q}\mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Var}}[\psi_\varepsilon(x_1)]\omega(z_1, dx_1)\omega(z_2, dx_2)\right]^{1/2}.
\end{aligned}$$

A similar estimate holds for $\mathrm{II}(z_1, z_2)$. Moreover, $\mathrm{III}(z_1, z_2) \leq C\varepsilon^{2-2q}$.

To finish the proof it suffices to show that for $|z_1 - z_2| \geq |\log\varepsilon|^{-k}$,

$$\mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Var}}[\psi_\varepsilon(x_1)]\omega(z_1, dx_1)\omega(z_2, dx_2)\right] \leq C|\log\varepsilon|^3 \tag{44}$$

where C is a constant only depending on D, D', D_0.

In fact by (42),

$$\begin{aligned}
&\mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Var}}[\psi_\varepsilon(x_1)]\omega(z_1, dx_1)\omega(z_2, dx_2)\right] \\
\leq& 2\mathbb{E}\left[\int \widetilde{\mathrm{Var}}[k_\varepsilon(x_1)]\omega(z_1, dx_1)\right] + 2\mathbb{E}\left[\int\int_{|x_1-x_2|\geq\varepsilon^q} \widetilde{\mathrm{Var}}[\Delta_i]\omega(z_1, dx_1)\omega(z_2, dx_2)\right] \\
\leq& 2\mathbb{E}\left[\int \widetilde{\mathrm{Var}}[k_\varepsilon(x_1)]\omega(z_1, dx_1)\right] + 2C\varepsilon^{1-q},
\end{aligned}$$

where C is the constant in (43). Now (44) hence Lemma 5.9 follows from Lemma 5.8. ☐

Proof of Lemma 5.10. Let

$$\begin{aligned}
\mathrm{I}'(z_1,z_2) &= \mathbb{E}\left[\int\int \mathbf{1}_{\{|x_1-x_2|<\varepsilon^q\}}\omega(z_1,dx_1)\omega(z_2,dx_2)\right],\\
\mathrm{II}'(z_1,z_2) &= \mathbb{E}\left[\int\int_{|x_1-x_2|<\varepsilon^q} \widetilde{\mathrm{Var}}\left[k_\varepsilon(x_1)\right]\omega(z_1,dx_1)\omega(z_2,dx_2)\right],\\
\mathrm{III}'(z_1,z_2) &= \mathbb{E}\left[\int\int_{|x_1-x_2|<\varepsilon^q} \widetilde{\mathrm{Var}}\left[k_\varepsilon(x_2)\right]\omega(z_1,dx_1)\omega(z_2,dx_2)\right].
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
R^2(z_1,z_2) \leq& \mathbb{E}^2\left[\int\int_{|x_1-x_2|<\varepsilon^q} \widetilde{\mathrm{Var}}^{1/2}\left[k_\varepsilon(x_1)\right]\widetilde{\mathrm{Var}}^{1/2}\left[k_\varepsilon(x_2)\right]\omega(z_1,dx_1)\omega(z_2,dx_2)\right]\\
\leq& \mathrm{II}'(z_1,z_2)\mathrm{III}'(z_1,z_2). \qquad (45)
\end{aligned}$$

For each $x \in D_0$, define $R_{n,\varepsilon}(x)$ as in (39). Recall (31) and (33), we have

$$\widetilde{\mathrm{Var}}[k_\varepsilon(x) - R_{n,\varepsilon}(x)] \leq C|\log\varepsilon|$$

Then

$$\mathrm{II}'(z_1,z_2) \leq C|\log\varepsilon|\mathrm{I}'(z_1,z_2) + 2\mathbb{E}\left[\int\int_{|x_1-x_2|<\varepsilon^q} \widetilde{\mathrm{Var}}\left[R_{n,\varepsilon}(x_1)\right]\omega(z_1,dx_1)\omega(z_2,dx_2)\right]. \qquad (46)$$

For the second term in the right hand side of (46), by Cauchy-Schwarz inequality,

$$\mathbb{E}^2\left[\int\int_{|x_1-x_2|<\varepsilon^q} \widetilde{\mathrm{Var}}\left[R_{n,\varepsilon}(x_1)\right]\omega(z_1,dx_1)\omega(z_2,dx_2)\right] \leq \mathbb{E}\left[\int \widetilde{\mathrm{Var}}^2\left[R_{n,\varepsilon}(x_1)\right]\omega(z_1,dx_1)\right]\mathrm{I}'(z_1,z_2).$$

Similar as in (36),

$$\mathbb{E}\left[\int \widetilde{\mathrm{Var}}^2\left[R_{n,\varepsilon}(x_1)\right]\omega(z_1,dx_1)\right] \leq \mathbb{E}\left[\int R^4_{n,\varepsilon}(x_1)\omega(z_1,dx_1)\right].$$

Recall (38) and Lemma 5.6,

$$\mathbb{E}\left[\int \widetilde{\mathrm{Var}}^2\left[R_{n,\varepsilon}(x_1)\right]\omega(z_1,dx_1)\right] \leq C(\mathbb{E}[\log^4 Y] + |\log\varepsilon|^4) \leq C|\log\varepsilon|^4 \qquad (47)$$

where Y is defined as in (38). (46) implies that $\mathrm{II}'(z_1,z) \leq C|\log\varepsilon|^4\mathrm{I}'(z_1,z_2)$. Similarly, $\mathrm{III}'(z_1,z_2) \leq C|\log\varepsilon|^4\mathrm{I}'(z_1,z_2)$. Therefore by (45), $R(z_1,z_2) \leq C|\log\varepsilon|^4\mathrm{I}'(z_1,z_2)$. Now Lemma 5.10 follows from Lemma 5.11 below. □

Lemma 5.11. *Fix $q \in (0,1], \zeta, k \in \mathbb{N}$. Let $\mathrm{I}'(z_1, z_2)$ be defined as above. For some $C, r > 0$*

$$\mathrm{I}'(z_1, z_2) \leq C\varepsilon^r$$

uniformly $z_1, z_2 \in D'$ such that $|z_1|, |z_2| > \zeta, |z_1 - z_2| \geq |\log \varepsilon|^{-k}$.

Proof. Without loss of generality we can assume $q = 1$. Suppose we have three independent Brownian motions B_t, W_t^1, W_t^2 with $B_0 = 0, W_0^1 = z_1, W_0^2 = z_2$. Let τ be the hitting time of ∂D_0 for B_t. If $W_i (i = 1, 2)$ hits $B[0, \tau]$ before hitting exiting D, let X_1 and X_2 be the hitting position respectively. Then $\mathbb{P}[|X_1 - X_2| \leq \varepsilon] = I'(z_1, z_2)$.

We consider three cases that will cover the event $|X_1 - X_2| \leq \varepsilon$ and show that each of their probability has a power law decay. We first fix a $p \in (0, 1)$ that will be determined later.

Case I: $|X_1 - X_2| \leq \varepsilon$ and $X_1, X_2 \in B_{\varepsilon^p}(z_1) \cup B_{\varepsilon^p}(z_2) \cup B_{\varepsilon^p}(0)$.
Note that $\{|X_1 - X_2| \leq \varepsilon, X_1, X_2 \in B_{\varepsilon^p}(z_1)\} \subset \{B_t, W_t^2$ do not intersect before hitting $B_{\varepsilon^p}(z_1)\}$. Recall the whole plane Brownian motion intersection exponent $\zeta(m, n)$ defined in [LSW01b, Equation (1.2)]. We have that

$$\mathbb{P}[B_t, W_t^2 \text{ do not intersect before hitting } B_{\varepsilon^p}(z_1)] = \varepsilon^{2p\zeta(1,1) + o_\varepsilon(1)}.$$

The same estimate holds for $B_{\varepsilon^p}(z_2)$ in place of $B_{\varepsilon^p}(z_1)$. Similarly,

$$\{|X_1 - X_2| \leq \varepsilon, X_1, X_2 \in B_{\varepsilon^p}(0)\} \subset \{W_t^1, W_t^2 \text{ do not intersect before hitting } B_{\varepsilon^p}(0)\}$$
$$\mathbb{P}[W_t^1, W_t^2 \text{ do not intersect before hitting } B_{\varepsilon^p}(0)] = \varepsilon^{2p\zeta(1,1) + o_\varepsilon(1)}.$$

All of the three estimates are uniform in z_1, z_2 satisfying the condition in Lemma 5.11. By [LSW01b, Theorem 5.1], $\zeta(1, 1) = 5/4$. Therefore we have established the power law decay in Case I.

Case II: $\{|X_1 - X_2| \leq \varepsilon, \mathrm{dist}(X_1, \partial D_0) \wedge \mathrm{dist}(X_2, \partial D_0) \leq \varepsilon^p$.
From the same argument as in Case I, the power decay probability also comes from non-intersecting exponent. In fact, since ∂D_0 is smooth, we can replace the whole plane non-intersection exponent by the half-plane non-intersection exponent ([LSW01a]) and argue in the same way as in Case I. We omit the details in this case due to the similarity.

Case III: $|X_1 - X_2| \leq \varepsilon$ and $\mathrm{dist}(\{X_1, X_2\}, \{z_1, z_2, 0\} \cup \partial D_0) \geq \varepsilon^p$.
We can cover D_0 using $N = O(\varepsilon^{-2})$ number of balls of radius 10ε such that X_1, X_2 belongs to one of these balls. Denote these balls by $\{B_{10\varepsilon}(x_i) : 1 \leq i \leq N\}$. If Case III happens, then there exists $1 \leq i \leq N$ such that $X_1, X_2 \in B_{10\varepsilon}(x_i), \mathrm{dist}(x_i, \{z_1, z_2, 0\} \cup \partial D_0) \geq \varepsilon^p$. Still by non-intersection exponent consideration,

$$\mathbb{P}[X_1, X_2 \in B_{10\varepsilon}(x_i), \mathrm{dist}(x_i, \{z_1, z_2, 0\} \cup \partial D_0) \geq \varepsilon^p] \leq \varepsilon^{2(1-p)\zeta(2,2) + o_\varepsilon(1)}$$

uniformly in z_1, z_2. Therefore the probability that Case III happens is less than

$$N\varepsilon^{2(1-p)\zeta(2,2) + o_\varepsilon(1)} = \varepsilon^{2(\zeta(2,2) - 1) - 2p\zeta(2,2) + o_\varepsilon(1)}.$$

By [LSW01b, Theorem 1.2], $\zeta(2, 2) = 35/24 > 1$. By choosing p small enough we establish the power law decay in Case III. ∎

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