Dynamics of relativistic fluids in cosmology

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Disclaimer: These notes are designed as a supplement to a lecture given at Erwin Schrödinger institute in 2023. There is no claim that the results presented are mine (unless stated explicitly). Furthermore note that these notes should give a coarse overview and encourage the reader to dig deeper into the subject, rather than provide a complete or details history of results. Should you find any mistakes, you would be very welcome to send them to me (email on my website).

1 The fluid model in cosmology

Understanding the dynamics of general relativity is quite complicated. Even comparatively simple solutions to the vacuum Einstein equations such as *Minkowski space*, *Schwarzschild* or *Kerr* are still subject of discussion as of today. Naturally, considering the relativistic dynamics of the universe as a whole seems impossible and thus calls for a simplified model.

In analogy to classical theories, investigating a large system of particles calls for a statistical approach. For our model, we will assume that the typical length scale l between different particles is much larger than their respective de-Broglie wave length λ_{DB}, i.e.

$$\frac{\lambda_{DB}}{l} \ll 1,$$

so that a quantum mechanical description is not necessary. However, we will also assume that the length scale L of the system itself is much larger than l. Classically, it is known that if $\frac{l}{L} \ll 1$, a continuous *fluid description* appears to be valid. Heuristically, in a fluid description one can assume the existence of so called *fluid elements*, which are large enough to contain a high number of particles, but small enough such that they can be assumed to be homogeneous with regards to velocity as well as thermodynamic equilibrium.

The description given above appears quite fuzzy but turns out to be very powerful. Indeed it allows for a description in just a few simple quantities, yet yields many accurate results for a wide array of phenomenon.

2 Classical fluids and the Euler equations

For starters we will assume the viscosity of our fluid to negligible. Relaxing this assumption would lead us to the *Navier-Stokes-equations* which will not be treated in our models. Note also that we will here present a relatively straight forward way to derive the classical Euler equations. For a more physically motivated one we refer to [*Relativistic Hydrodynamics*, Rezzolla-Zanotti, 2013] and a more mathematically rigorous and more general one to [*Manifolds, Tensor Analysis and Applications*, Abraham-Marsden-Ratiu, 2003].

2.1 The postulates of inviscid fluids

Let us assume we have a region $D \subset \mathbb{R}^3$ that is filled with a fluid and consider a particle moving through $x \in D$ at time t. Furthermore, we will denote the velocity of the particle at that point in time and space as $u(x,t)$. Hence, u is a vector field on D for every instance of time t. On top of that, for all x and t we assume a well-defined scalar $\rho(x,t)$, the *mass density*. Thus, the *mass* in a region $W \subset D$ at time t is given by

$$m(W,t) = \int_W \rho(x,t).$$

For now, we assume that ρ and u are regular enough so that we have all our analytic tools at our disposal. We will derive the Euler equations from three simple postulate:

1. Mass is conserved;

2. The rate of change of the momentum of a portion of the fluid is equal to the force applied to it (Newton's second law);

3. Energy is conserved.

2.2 Conservation of mass

Let us pick a region $W \subset D$ (with smooth boundary) independent of time. The change of mass in W is given by

$$\frac{d}{dt}m(W,t) = \frac{d}{dt}\int_W \rho(x,t) = \int_W \frac{\partial \rho}{\partial t}(x,t).$$

The mass flow per unit area of ∂W with unit outward normal n is given by $\rho u \cdot n$. Hence we find that

$$\frac{d}{dt}\int_W \rho = -\int_{\partial W} \rho u \cdot n.$$

By the divergence theorem and the fact that W is arbitrary we hence find

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0.$$

This is known as the *continuity equation*.

2.3 Balance of momentum

Let $t \mapsto x(t)$ be the path followed by a particle, so that

$$u(x(t),t) = \frac{dx}{dt}(t).$$

For the acceleration at (x,t) we thus find

$$a(t) = \frac{d}{dt}u(x(t),t) = \partial_t u + \frac{\partial u}{\partial x^i}\dot{x}^i = \partial_t u + (u \cdot \nabla)u =: \frac{D}{Dt}u.$$

The operator $\frac{D}{Dt}$ is referred to as *material derivative*. The force S exerted on a region W via *pressure* $p(t,x)$ is given by

$$S = -\int_{\partial W} pn,$$

and for any vector v we have

$$v \cdot S = -\int_{\partial W} pv \cdot n = -\int_W \nabla(pv) = -\int_W (\nabla p)v,$$

and thus

$$S = -\int_W \nabla p.$$

Assuming the force on the whole body is given by

$$F = \int_W \rho b,$$

where $b(x,t)$ is some function describing the force per unit mass, we find that

$$\rho \frac{D}{Dt} u = -\nabla p + \rho b.$$

2.4 Energy conservation

The energy in some region W is given by

$$E_{\text{total}} = E_{\text{kin}} + E_{\text{int}} = \frac{1}{2}\int_W \rho^2 + E_{\text{int}}.$$

Understanding the evolution of the kinetic energy is straightforward, but the internal energy is a priori not known. Hence, considering the problem of energy conservation in full detail is quite a quite extensive task and requires a lot of thermodynamics. Thus we will restrict ourselves to an *isentropic* model ($\frac{Ds}{Dt} = 0$) with equation of state $p = p(\rho)$ to close the system. This is a natural setting, since we do not expect heat flow in an inviscid model and can be shown to be consistent with energy conservation (see [*Fluid Mechanics*, Spurk-Aksel, 1997]).

3 Relativistic fluids

Our ultimate goal is to treat the classical problem described above in a fully dynamic relativistic setting, i.e. we want to analyze solutions to the system

$$R[g]_{\mu\nu} - \frac{1}{2}R[g]g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$

with the appropriate energy momentum tensor $T_{\mu\nu}$. However, due to the complicated nature of the Einstein equations, a simplified approach is to analyze a simpler model first. Since the left hand side of the Einstein equations is divergence free with respect to the Levi-Civita-connection ∇ of g, $T_{\mu\nu}$ must satisfy the condition

$$\nabla_\mu T^{\mu\nu} = 0.$$

As we will see shortly, this will imply a relativistic version of the Euler equations. We shall thus first assume our Lorentzian manifold (M, g) to be *fixed* and try to understand the *problem with backreaction* afterwards.

3.1 The relativistic Euler equations

We are going to use the energy momentum tensor of a so called *perfect fluid* given as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu},$$

where now u describes the fluid 4-velocity that is normalized via $g_{\mu\nu}u^\mu u^\nu = -1$. Note that ρ is *not* the rest mass density but rather the *total energy density*. Also note that this terminology is not standardized and describes different things depending on the author. It is often defined as the energy momentum tensor of an isotropic and isentropic fluid (we will explain why), which is compliant with our models.

In the rest frame of the observer with 4-velocity u,

$$T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p).$$

This makes sense on a heuristic level since

- T_{00} should be the energy density,
- The spatial diagonal should be the pressure which should be isotropic,
- Non-vanishing diagonal entries would lead to unwanted stresses.

We will further motivate this description below. However, the unsatisfied reader may also consult [*Relativistic Hydrodynamics*, Rezzolla-Zanotti, 2013] for further details.

Multiplying $\nabla_\mu T^{\mu\nu} = 0$ with u_ν, we find

$$\begin{aligned} 0 = u_\nu \nabla_\mu T^{\mu\nu} &= u_\nu \big(u^\mu u^\nu \partial_\mu (\rho + p) + (\rho + p)(u^\nu \nabla_\mu u^\mu + u^\mu \nabla_\mu u^\nu) + g^{\mu\nu} \partial_\mu p \big) \\ &= -u^\mu \partial_\mu (\rho + p) - (\rho + p) \nabla_\mu u^\mu + u^\mu \partial_\mu p \\ \Rightarrow 0 &= u^\mu \partial_\mu \rho + (p + \rho) \nabla_\mu u^\mu . \end{aligned}$$

Furthermore we have that

$$\begin{aligned} 0 = \nabla_\mu T^{\mu i} &= u^\mu u^i \partial_\mu (\rho + p) + (\rho + p)(u^i \nabla_\mu u^\mu + u^\mu \nabla_\mu u^i) + g^{\mu i} \partial_\mu p \\ &= \big(u^\mu u^i + g^{\mu i} \big) \partial_\mu p + (\rho + p) u^\mu \nabla_\mu u^i . \end{aligned}$$

Note that this statement would still be true is we put 0 instead of i. However, this does not contribute anything, since u^0 is already determined by the normalization condition. We will refer to the two equations above as the *relativistic Euler equations*. This is a system of 4 equations in 5 equations. To close it we need to impose an *equation of state* $p = p(\rho)$.

Exercise: Using the fundamental thermodynamics relation

$$p = n \frac{\partial \rho}{\partial n} - \rho,$$

show that the relativistic Euler equations imply that particle flow is conserved, i.e. $\nabla_\mu (n u^\mu) = 0$.

It can also be shown (see e.g. [*Elements of General Relativity*, Chruściel, 2019]) that $u^\mu \partial_\mu s = 0$ and that if $g = \eta$ the Minkowski metric, in the classical limit we recover the Euler equation

$$\rho \frac{D}{Dt} u = -\nabla p.$$

Hence, we appear to have found a suitable relativistic analogue to the classical Euler equations for an isentropic system.

3.2 A side note on well posedness

We would naturally want our system to have (at least a local) solution given initial data in a certain regularity. Luckily, the classical as well as the relativistic Euler together with a suitable equation of state can be cast into the form

$$\partial_t U + A^k(U) \nabla_k U = S(U),$$

where $A^k(U)$ can be shown to be diagonizable with real eigenvalues and has a maximal set of linearly independent eigenvectors. Such a system is referred to as *quasilinear hyperbolic first order system* and can be shown to be at least locally well posed. This means that, if initial data is prescribed on e.g. a certain time, a unique local-in-time solution exists and depends continuously on the prescribed initial data.

For a detailed discussion see [*Relativistic Hydrodynamics*, Rezzolla-Zanotti, 2013] and for the illustrative case of isentropic flows in one spatial dimension see [*A Mathematical Introduction to Fluid Mechanics*, Chorin-Marsden, 1992].

4 Overview: Results in relativistic fluids

From now on we will only consider so called *barotropic* and linear equations of state of the form

$$p = K\rho, \qquad 0 \leq K \leq \frac{1}{3}.$$

The parameter K is taken to be constant and satisfies $c_s^2 = K$, where c_s is the speed of sound of the fluid. This equation of state is often used in cosmology as it provides a simple enough model to describe cosmological phenomena.

We distinguish three cases:

- $K = 0$, referred to as *dust*,
- $K = \frac{1}{3}$, referred to as *radiation*, since it appears by taking $\operatorname{tr} T^{\mu\nu} = 0$,
- $0 < K < \frac{1}{3}$, referred to as *massive fluids.*

Furthermore, for now we consider Lorentzian background metrics of the form

$$g = -dt^2 + a(t)^2\gamma,$$

where $a(t)$ is an increasing function of t and γ is a Riemannian metric. On such backgrounds, we know that we can find a *homogeneous solution*, i.e. a $\rho = \rho(t)$ and $u = u(t)$ that solves the relativistic Euler equations and is global.

Exercise: Under the homogeneity condition $\partial_i \rho = 0$ and assuming that $u^\mu = \delta_0^\mu$, derive that the resulting solution to the relativistic Euler equations with the metric given above and $a(t) = t^\alpha$, is given by

$$\rho = \rho_0 t^{-3(1+K)\alpha}.$$

4.1 The notion of stability

Heuristically speaking, we are interested in the following question:

Given a homogeneous solution, what happens if we perturb the homogeneous initial data a bit? There could be several answers: The new solution could

- *exist globally and stay close to our homogeneous solution (we will refer to this as 'stable').*
- *exist globally but move away from the homogeneous solution.*
- *cease to exist at some point.*

Phenomenologically, we may interpret this in such a fashion: Suppose our universe displays small inhomogeneities in its early state. If our fluid solution can be shown to *homogenize completely*, this solution can not be a good model *for all times*, since it prohibits the formation of structures that we observe today.

Our equations satisfy a

Theorem 1 (Continuation principle)**.** *Suppose that our maximal unique solution s on some interval $[T_0, T_{max})$ obtained be local well posedness is in some Sobolev space H^N for all $t \in [T_0, T_{max})$. Then either $T_{max} = \infty$ or*

$$\limsup_{T \to T_{max}} \|s\|_{H^N} = \infty.$$

This tells us that, should our model universe only exist for finite time, this necessarily leads to infinite growth of the structures inside of it.

4.2 Some recent results

Consider an exponentially expanding model, e.g. the de Sitter spacetime

$$g = -dt^2 + e^{2\sqrt{\Lambda}t}\delta_{ij}dx^i dx^j.$$

Such spacetimes arise as spatially flat FLRW vacuum solutions with non-vanishing positive cosmological constant Λ. Due to works by Rodnianski, Speck, Lübbe, Valiente-Kroon, Hadzic, Oliynyk and Friedrich it is known that all the models we are considering ($0 \leq K \leq \frac{1}{3}$) are *stable under small perturbations* for exponentially expanding spacetimes. This true, even for the fully coupled Euler-Einstein problem. On the contrary, it is known due to work by Christodoulou that solutions to the relativistic Euler equations on a Minkowski background are *unstable.* They develop singularities in finite time. The fact that expanding spacetimes can lead to stability is sometimes referred to as *fluid regularization.*

Naturally, this poses the following question: *How low can one go in terms of expansion rate, so that the fluid retains its stability properties.* For relativistic Euler on the Euklidian spatial background there are these interesting results [Speck, 2013]:

- Dust: If $a(t)^{-2}$ is integrable on $(1,\infty)$ (e.g. $a(t) = t^{\frac{1}{2}+\epsilon}$) $\Rightarrow$ the solution is stable.
- Radiation: If $a(t)^{-1}$ is integrable on $(1,\infty)$ (e.g. $a(t) = t^{1+\epsilon}$) $\Rightarrow$ the solution is stable.
- Dust: If $a(t)^{-1}$ is not integrable on $(1,\infty)$ (e.g. $a(t) = t^1$) $\Rightarrow$ the solution is unstable (develops singularities in finite time).
- Massive fluid: The region where $a(t)^{-1}$ is integrable is not known in this article.

5 Relativistic fluids in linearly expanding spacetimes

Linearly expanding regimes are especially interesting, as they are the fastest expansion, in which *regions in initial data do not causally decouple.* For details on this we refer to [Ringström, 2008].

We state the first result for massive fluids in a linearly expanding regime:

Theorem 2 (Fajman-Oliynyk-Wyatt, 2021)**.** *The homogeneous solution to the relativistic Euler equations for massive fluids on the background*

$$([T_0, \infty) \times \mathbb{T}^3, -dt^2 + t^2 \delta_{ij} dx^i dx^j)$$

are non-linearly stable.

Note that this result is in an *irrotational* setting. By requiring the vorticity of the fluid to vanish, one may introduce a potential φ, so that ρ and u can be determined in terms of φ.

The first result for linear expansion in the full coupled Euler-Einstein system:

Theorem 3 (Fajman-MO-Wyatt, 2021)**.** *The Milne solution,*

$$(([T_0, \infty) \times M, -dt^2 + \frac{t^2}{9} \gamma_{ij} dx^i dx^j)),$$

where (M, γ) is closed Riemannian Einstein manifold with $R[\gamma]_{ij} = -\frac{2}{9}\gamma_{ij}$, is a non-linearly stable solution to the coupled Einstein-Dust system.

Since this is the fully coupled problem, our solution consists of (g, k, u, ρ) where g is the metric and k is the second fundamental form. For details on the Einstein equations as a Cauchy problem see [3+1 Formalism in General Relativity, Gourgoulhon, 2012].

In our most recent result, we were able to show that even the massive fluid is stable in linear expansion: The first result for linear expansion in the full coupled Euler-Einstein system:

Theorem 4 (Fajman-MO-Oliynyk-Wyatt, 2023)**.** *The Milne solution,*

$$(([T_0, \infty) \times M, -dt^2 + \frac{t^2}{9} \gamma_{ij} dx^i dx^j)),$$

is a non-linearly stable solution to the coupled Einstein-Euler system with linear equation of state and $0 < K < \frac{1}{3}$.

In particular, this result shows that the irrotational linear expanding massive case can be extended to rotational fluids. Together with specks result, this closes the question of stability in the linear expansion case completely.

5.1 Example of a method used in the proof

Obviously, we cannot go through the proofs of these results in detail. However, we are going to explain a bit of machinery used in all of these. To establish *global existence* of the local solutions, we employ the following mechanism:

1. Define energy functionals $E_k[\rho], E_k[u]$ that are coercive to some sufficient Sobolev norm, i.e. for all $k \leq N$

$$\|\rho\|_{H^N} \lesssim E_N[\rho], \qquad \|u\|_{H^N} \lesssim E_N[u].$$

2. The *bootstrap*: Assume a priori, that for some $\epsilon > 0$

$$E_{N-1}[\rho] + E_N[u] < \epsilon.$$

   This always works on *some* maximal nonempty set $[T_0, T_\star)$, by choosing the initial data ρ_0, u_0 small enough and continuous dependence of the local solution.

3. Show that one can improve upon the bootstrap, i.e. for example show that on $[T_0, T_\star)$, we have that

$$E_{N-1}[\rho] + E_N[u] \leq \frac{\epsilon}{2}.$$

   This is usually done controlling the evolution of E, $\frac{d}{dt}E$, *utilizing the bootstrap.*

4. By continuity, we can assume that $T_\star = \infty$.

By the continuation principle, this shows global existence.