MONOTONICITY PRESERVING TRANSFORMATIONS OF MOT AND SEP

MARTIN HUESMANN AND FLORIAN STEBEGG

Abstract. Recently, [5, 6] established that optimizers to the martingale optimal transport problem (MOT) are concentrated on $c$-monotone sets. In this article we characterize monotonicity preserving transformations revealing certain symmetries between optimizers of MOT for different cost functions. Due to the intimate connection of MOT and the Skorokhod embedding problem (SEP) these transformations are also monotonicity preserving and disclose symmetries for certain solutions to the optimal SEP. Furthermore, the SEP picture allows to easily understand the geometry of these transformations once we have established the SEP counterparts to the known solutions of MOT based on the monotonicity principle for SEP which in turn allows to directly read off the structure of the MOT optimizers.

Keywords: Optimal transport, martingale optimal transport, Skorokhod embedding, change of numeraire.

1. Introduction

Given probabilities $\mu$ and $\nu$ on $\mathbb{R}$ and a cost function $c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ the martingale optimal transport problem (MOT) is to minimize

$$\int c(x, y) \, dQ(x, y)$$

among all martingale couplings $Q$ of $\mu$ and $\nu$, i.e. among all couplings satisfying $\int y \, dQ_x(y) = x$ for $\mu$ a.e. $x$, where $(Q_x)_x$ denotes a disintegration of $Q$ wrt $\mu$. [5, 6] showed that similar to the classical optimal transport problem optimizers are characterized by a local optimality condition in that every optimizer is concentrated on a $c$-monotone set $\Xi$. If $c$ is sufficiently nice, also the reverse holds. Every martingale coupling concentrated on $\Xi$ is optimal between its marginals [5, 12]. This indicates that in order to understand the optimizers to MOT it is necessary to understand the geometry of $c$-monotone sets.

The aim of this article is to characterize monotonicity preserving transformations revealing symmetries between solutions to MOT for different cost functions. More precisely, given intervals $I, J, I', J'$ each bijective map $T : I \times J \rightarrow I' \times J'$ together with a positive function $h$ induces a transformation $\tau$ of measures on $I \times J$ to measures on $I' \times J'$ by setting

$$\int g(x', y') \, d\tau(\pi)(x', y') := \int g \circ T(x, y) h(x, y) \, d\pi(x, y)$$

and a transformation of functions $c$ on $I \times J$ to functions $c'$ on $I' \times J'$ by setting

$$c'(x', y') := (f/h) \circ T^{-1}(x', y').$$

The purpose of $T$ is to move mass from a point $(x, y)$ to a point $(x', y')$, whereas $h$ rescales the mass. Because $h$ is taken to be strictly positive, the structure of the 'support' of a measure $\pi$ is only transformed via $T$.

\textit{Date:} September 13, 2016.

We thank Mathias Beiglböck and Alex Cox for numerous valuable discussions. MH gratefully acknowledges support by the CRC1060 and the Hausdorff Center for Mathematics.
Roughly speaking, the pair \((T, h)\) is a monotonicity preserving transformation if it maps a \(c\)-monotone set \(\Xi\) into a \(c'\)-monotone set \(\Xi'\) (for a precise definition we refer to Section 4). It turns out that there are only very few such transformations:

**Theorem 1.1.** Let \(T(x, y) = (s(x, y), t(x, y)) : I \times J \to I' \times J'\) be bijective and \(h : I \times J \to (0, \infty)\). Assume that \((T, h)\) is a monotonicity preserving transformation. Then \(s\) is constant in \(y\) and \(t\) and \(h\) are constant in \(x\). Furthermore

1. either \(h\) is constant and \(t\) is affine in \(y\);
2. or \(h\) is affine in \(y\) \((h(y) = c(y - b))\) and \(t\) is of the form \(t(y) = a/(y - b)\).

We stress that a monotonicity preserving transformation does not need to (and in general will not) preserve the marginals neither the martingale property. However, if we additionally require the transformation to preserve the martingale property, we get a new characterisation of a well known transformation, the change of numeraire transformation which has been studied for MOT in [7].

**Theorem 1.2.** Let \(T : I \times J \to I' \times J'\) be bijective and \(h : I \times J \to (0, \infty)\). Assume that \((T, h)\) is a monotonicity preserving transformation which also preserves the martingale property. Then,

1. either \(T(x, y) = (ax + b, ay + b), h \equiv 1\);
2. or \(h(y) = y\) and \(T(x, y) = (1/x, 1/y)\).

Further examples of monotonicity preserving transformations are the mirror transformations in [15, Remark 5.2]. We emphasize that—using the transformation \(T(x, y) = (-x, y), h \equiv 1\)—our results also reveal certain symmetries between optimizers of \(|x - y|\) and those of \(|x + y|\).

We note that the assumption on the domain of the map \(T\) in Theorem 1.1 is not very restrictive since this is precisely the shape of the domain of an irreducible component of the pair \((\mu, \nu)\) as shown in [5, Theorem 8.4].

Already in one of the first articles on MOT [17] the intimate connection of MOT and the Skorokhod embedding problem (SEP) was used to construct the optimizer for the cost function \(c(x, y) = -|x - y|\). The relation between MOT and SEP was used in [4] to give a short proof of the uniqueness of left monotone martingale couplings. The reason for this fruitful connection is simple: Any discrete time martingale can be interpolated to a continuous time martingale [14] and, hence, by the Dambis-Dubins-Schwarz theorem the optimal SEP can be seen as a continuous time version of MOT.

Recently, [1] established that solutions of the *optimal SEP*, i.e. solutions of SEP optimizing a given cost functional \(\gamma\), are concentrated on \(\gamma\)-monotone sets \(\Gamma\). Understanding these sets for different \(\gamma\) allows to construct various solutions to the optimal SEP including all known solutions as special cases. Viewing the optimal SEP as continuous time version of MOT the transformations characterized in Theorem 1.1 induce transformations of the \(\gamma\)-monotone sets for the optimal SEP for the specific class of functionals \(\gamma\) of the form \(\gamma(g, t) = \gamma(g(0), g(t))\) for \(g \in C[0, t]\) and, thus, Theorem 1.1 discloses symmetries between different solutions of SEP.

Furthermore, the understanding of the geometry of the \(\gamma\)-monotone sets for the SEP counterparts to the MOT problems allows us to give a simple derivation of the known solutions to MOT. From the specific structure of the SEP solutions one can directly read off the defining properties of the optimal martingale couplings. Building on this, it is straightforward to deduce the optimizers for the transformed MOT problem, i.e. for the MOT with respect to the cost function \(c' = (c/h) \circ T^{-1}\).

It is desirable to have a result similar to Theorem 1.1 also for the general optimal SEP. However, it seems that a necessary ingredient is still missing, a 'full monotonicity principle'. We give a conjecture on this principle and observe that the conjecture holds for the special case of cost functions \(\gamma\) considered in this article due to the connection of SEP and MOT.
1.1. Related literature. The MOT was introduced in [3] where also a duality result was established for lsc cost functions. The duality result was extended in [6] to Borel measurable cost functions building on a deep understanding of MOT developed in [5] which develops a theory parallel to classical optimal transport. Additionally, [5] construct the optimizer for cost functions of the form \( c(x, y) = h(y - x) \) with \( h'' > 0 \) (\( h''' < 0 \) resp.), the so called left (right resp.) monotone couplings. It was shown in [15] that the left (right resp.) monotone couplings are also optimizers for the cost functions \( c \) satisfying the generalized Spence Mirrlees condition \( c_{xyy} < 0 \) (\( c_{xyy} > 0 \) resp.). Previously, [17, 16] constructed the optimizers for the cost functions \( c(x, y) = |x - y| \) not satisfying the Spence Mirrlees condition.

In higher dimension, first results have been established by [11, 19]. The duality result was extended in various directions to continuous time, e.g. [10, 8, 9, 18, 2, 13] and references therein.

1.2. Outline. In Section 2 we recall some facts on martingale optimal transport as well as the monotonicity principle of [1] that will allow us to give a simple geometric explanation of the structure of all known solutions to MOT in Section 3. In Section 4 we prove Theorems 1.1 and 1.2 and show that their geometric consequences can be easily unraveled using the SEP picture of martingale optimal transport. We end by stating a conjecture on a full monotonicity principle for Skorokhod embedding in Section 5.

2. Preliminaries

2.1. Martingale optimal transport. Let \( \mu, \nu \in \mathcal{P}(\mathbb{R}) \) (where \( \mathcal{P}(X) \) denotes the set of all probability measures on a space \( X \)). Let \( \mathcal{Cpl}(\mu, \nu) \) be the set of all couplings of \( \mu \) and \( \nu \), i.e. measures on \( \mathbb{R}^2 \) with marginals \( \mu \) and \( \nu \), and let \( \mathcal{MCpl}(\mu, \nu) \subseteq \mathcal{Cpl}(\mu, \nu) \) be the subset of all martingale couplings, i.e. all couplings under which the coordinate process \( (X, Y) \) becomes a martingale, where \( X(x, y) = x \) and \( Y(x, y) = y \). By Strassen’s Theorem \( \mathcal{MCpl}(\mu, \nu) \neq \emptyset \) iff \( \mu \) and \( \nu \) are increasing in convex order, i.e. \( \int \phi \, d\mu \leq \int \phi \, d\nu \) for all convex functions \( \phi \). A measure \( Q \in \mathcal{Cpl}(\mu, \nu) \) is a martingale coupling iff for any disintegration \((Q_x)_x\) of \( Q \) wrt \( \mu \) it holds that \( \mu \)-a.s.

\[
\int y \, Q_x(dy) = x.
\]

Let \( c : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be some cost function. Then, the martingale optimal transport problem is to solve

\[
P^{\text{mg}}_{\mu, \nu}(c) = \inf_{Q \in \mathcal{MCpl}(\mu, \nu)} \int c(x, y) \, Q(dx, dy).
\]  

(MOT)

It is not hard to see that \( \mathcal{MCpl}(\mu, \nu) \) is compact wrt to the weak topology and, hence, (MOT) admits a minimizer if \( c \) is lsc and bounded from below since then the functional

\[
Q \mapsto \int c(x, y) \, Q(dx, dy)
\]

is lsc by the Portmanteau theorem. Beiglböck, Nutz and Touzi showed in [6] the following geometric characterisation of optimisers to (MOT).

**Theorem 2.1** ([6, Corollary 7.8]). Assume \( P^{\text{mg}}_{\mu, \nu}(c) < \infty \). Then there exists a Borel set \( \Xi \subseteq \mathbb{R}^2 \) with the following properties.

(1) A measure \( Q \in \mathcal{MCpl}(\mu, \nu) \) is concentrated on \( \Xi \) iff it is optimal for \( P^{\text{mg}}_{\mu, \nu}(c) \).

(2) Let \( \mu, \nu \) be probabilities increasing in convex order. If \( Q \in \mathcal{MCpl}(\mu, \nu) \) is concentrated on \( \Xi \), then \( Q \) is optimal for \( P^{\text{mg}}_{\mu, \nu}(c) \).

In this article we will use a local version which was derived previously in [5], see also [12] for a small extension. To state it we need to introduce the notion of competitor which is also central in this paper.
Definition 2.2. (1) Let $\alpha$ and $\beta$ be two finite measures on $\mathbb{R}^2$. Then $\beta$ is called competitor of $\alpha$ if it has the same marginals $\alpha_0$ and $\alpha_1$ and for any disintegrations $(\alpha_x)_x$ and $(\beta_x)_x$ of $\alpha$ resp. $\beta$ wrt the first marginal $\alpha_0$ it holds for $\alpha_0$ a.e. $x$ that

\[
\int y \, \alpha_x(dy) = \int y \, \beta_x(dy). \tag{2.1}
\]

(2) A set $\Xi \subseteq \mathbb{R}^2$ is called $c$–monotone (or just monotone) if for any finite measure $\alpha$ concentrated on $\Xi$ with $|\text{supp}(\alpha)| < \infty$ and any competitor $\beta$ of $\alpha$ it holds that

\[
\int c \, d\alpha \leq \int c \, d\beta.
\]

Then we have the following result:

Theorem 2.3 ([5, Lemma 1.11], [12, Theorem 1.3]). Let $Q$ be an optimizer of (MOT) with $P_{\mu,\nu}^Q(c) < \infty$. Then there is a $c$–monotone set $\Xi$ satisfying $Q(\Xi) = 1$. Moreover, if $c$ is upper-semicontinuous and bounded from above by integrable functions (wrt $\mu$ resp. $\nu$) this condition is also sufficient.

2.2. The optimal Skorokhod embedding problem. Fix $\mu, \nu \in \mathcal{P}(\mathbb{R})$ increasing in convex order. For notational convenience we assume that $\nu$ has second moment. For the general case we refer to [1]. Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, \mathbb{P})$ be some stochastic base which is sufficiently rich to support a Brownian motion $B$ starting in $\mu$ and a uniform $\mathcal{G}_0$–measurable random variable independent of $B$. Let $S := \{(f, s) : f \in C([0,s])\}$ and fix $\gamma : S \to \mathbb{R}$. We put $\gamma_t := \gamma((B_u)_{u \leq t}, t)$ and note that this is an optional process. For $(f, s) \in S$ we write $\gamma^{(f,s)}(\cdot) := \gamma(f \circ (B_u)_{u \leq t}, s + t)$.

The optimal Skorokhod embedding problem is to find a minimizer of

\[
\tau \mapsto \mathbb{E}[\gamma_\tau] \tag{OptSEP}
\]

among all $\mathcal{G}$ stopping times $\tau$ such that $B_\tau \sim \nu$ and $(B_{\tau \wedge s})_{s \geq 0}$ is uniformly integrable. We write $\text{Opt}_\gamma$ for the set of all optimizer of (OptSEP). Considering another functional $\tilde{\gamma} : S \to \mathbb{R}$ we say that $\tilde{\tau} \in \text{Opt}_{\tilde{\gamma}}$ is a secondary optimizer if it minimizes

\[
\tau \mapsto \mathbb{E}[\tilde{\gamma}_\tau] \tag{OptSEP_2}
\]

among all $\tau \in \text{Opt}_\gamma$. We will say that (OptSEP) is well posed if $\mathbb{E}[\gamma_\tau]$ exists with values in $(-\infty, \infty)$ for all $\tau$ satisfying $B_\tau \sim \mu$ and $(B_{\tau \wedge s})_{s \geq 0}$ is uniformly integrable and it is finite for one such $\tau$; similarly for (OptSEP_2). It is not hard to see that an optimizer to (OptSEP_2) exists if for example $\gamma, \tilde{\gamma}$ are lsc and bounded from below (see [1, Theorem 4.1]).

Definition 2.4. The pair $((f, s), (g, t)) \in S \times S$ constitutes a stop-go pair, written $((f, s), (g, t)) \in \text{SG}$, iff $f(s) = g(t)$, and for every $(f, s), (g, t) \in \mathcal{G}_{\geq 0}$ stopping time $\sigma$ which satisfies $0 < \mathbb{E}[\sigma] < \infty$,

\[
\mathbb{E}\left[\left(\gamma^{(f,s)}(\cdot)\right)_\sigma\right] + \gamma(g, t) > \gamma(f, s) + \mathbb{E}\left[\left(\gamma^{(g,t)}(\cdot)\right)_\sigma\right], \tag{2.2}
\]

whenever both sides are well-defined, and the left-hand side is finite. We say that $((f, s), (g, t)) \in S \times S$ constitutes a secondary stop-go pair, written $((f, s), (g, t)) \in \text{SG}_{\geq 2}$, iff $f(s) = g(t)$, $\geq$ holds in (2.2) and if

\[
\mathbb{E}\left[\left(\gamma^{(f,s)}(\cdot)\right)_\sigma\right] + \gamma(g, t) = \gamma(f, s) + \mathbb{E}\left[\left(\gamma^{(g,t)}(\cdot)\right)_\sigma\right] \tag{2.3}
\]

then

\[
\mathbb{E}\left[\left(\tilde{\gamma}^{(f,s)}(\cdot)\right)_\sigma\right] + \tilde{\gamma}(g, t) > \tilde{\gamma}(f, s) + \mathbb{E}\left[\left(\tilde{\gamma}^{(g,t)}(\cdot)\right)_\sigma\right], \tag{2.4}
\]

whenever both sides are well-defined and the left-hand side (of (2.4)) is finite.
Figure 1. The Brownian motion travels along lines parallel to the main diagonal and is stopped when it hits the boundary.

Definition 2.5. We say that $\Gamma \subseteq S$ is $\gamma$-monotone if

$$SG \cap (\Gamma^\prec \times \Gamma) = \emptyset,$$

where $\Gamma^\prec := \{(f,s) \in S : \exists (g,t) \in \Gamma \text{ extending } (f,s)\}$. We say that $\Gamma \subseteq S$ is $\tilde{\gamma}|\gamma$-monotone if (2.5) holds with $SG_2$ in place of $SG$.

The following theorem is proven in [1].

Theorem 2.6 (Monotonicity Principle). Let $\gamma, \tilde{\gamma} : S \rightarrow \mathbb{R}$ be Borel measurable, suppose that $(\text{OptSEP})$ is well posed and $\hat{\tau}$ is an optimizer. Then there exists a $\gamma$-monotone Borel set $\Gamma \subseteq S$ such that $\mathbb{P}$-a.s.

$$((B_t)_{t \leq \hat{\tau}}, \hat{\tau}) \in \Gamma.$$

If $(\text{OptSEP}_2)$ is well posed and $\hat{\tau}$ is also an optimizer to this problem, then there exists a $\tilde{\gamma}|\gamma$-monotone Borel set $\Gamma \subseteq S$ such that $\mathbb{P}$-a.s.

$$((B_t)_{t \leq \hat{\tau}}, \hat{\tau}) \in \Gamma.$$

3. MOT via SEP

To show the connection between MOT and the SEP, we sketch how the properties of solutions to MOT for well-known cost functions can be derived using the monotonicity results for SEP. We will do this in detail for the case of Spence Mirrlees cost functions. The proof for the form of maximizing and minimizing transports for $|x - y|$ is very similar and will therefore be reduced to a sketch.

Proposition 3.1. Assume that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Then, there exists a unique stopping time $\tau_{BJ}$ which minimizes

$$\mathbb{E}[c(B_0, B_{\tau})]$$

over all solutions to SEP for any cost function $c$ satisfying the Spence Mirrlees condition $c_{xyy} < 0$. It is of the form $\tau_{BJ} = \inf\{t > 0 : B_t - B_0 > \psi(B_t)\}$ a.s., for some measurable function $\psi$. This is the unique solution to SEP of this form.

Proof. Pick $c$ such that $(\text{OptSEP})$ is well posed. Since $c$ is continuous hence lsc there is a minimizer $\hat{\tau}$. Put $\gamma(g,t) = c(g(0), g(t))$. Pick by Theorem 2.6 a $\gamma$-monotone set $\Gamma$ supporting $\hat{\tau}$. We claim that

$$SG \supseteq \{((f,s),(g,t)) \in S \times S : g(t) = f(s), g(0) > f(0)\}.$$
This is represented in Figure 2. To see this, pick some stopping time $\tau$ and observe that for some fixed $y \in \mathbb{R}$ the map
\[
x \mapsto \mathbb{E}[c(x, y + B_{\tau})] - c(x, y)
\]
is strictly decreasing due to the strict concavity of $y \mapsto c_\sigma(x, y)$. Pick some pair $((f, s), (g, t)) \in S \times S$ satisfying $f(s) = g(t)$ and $g(0) > f(0)$. Then (3.2) implies
\[
\mathbb{E}[c(f(0), f(s) + B_{\tau})] + c(g(0), g(t)) > c(f(0), f(s)) + \mathbb{E}[c(g(0), g(t) + B_{\tau})].
\]
Now define
\[
\mathcal{R}_{op} := \{(d, y) : \exists (g, t) \in \Gamma, g(t) = y, d > g(t) - g(0)\}
\]
\[
\mathcal{R}_{cl} := \{(d, y) : \exists (g, t) \in \Gamma, g(t) = y, d \geq g(t) - g(0)\}.
\]
Fix $(g, t) \in \Gamma$. Then, $(g(t) - g(0), g(t)) \in \mathcal{R}_{cl}$. Suppose for contradiction that $\inf\{s \in [0, t] : (g(s) - g(0), g(s)) \in \mathcal{R}_{op}\} < t$. Then there exists $s < t$ such that $(f, s) := (g|_{[0, s]}, s) \in \Gamma^\infty$ and $(f(s) - f(0), f(s)) \in \mathcal{R}_{op}$. By definition of $\mathcal{R}_{op}$, it follows that there exists another path $(k, u) \in \Gamma$ such that $k(u) = f(s)$ and $k(0) > f(0)$. But then $((f, s), (k, u)) \in \mathcal{G} \cap (\Gamma^\infty \times \Gamma)$ which cannot be the case. Hence, $(g, t) \in \Gamma$ implies
\[
\inf\{s \in [0, t] : (g(s) - g(0), g(s)) \in \mathcal{R}_{cl}\} \leq t \leq \inf\{s \in [0, t] : (g(s) - g(0), g(s)) \in \mathcal{R}_{op}\}.
\]
Denote the hitting times of $\mathcal{R}_{op}$ and $\mathcal{R}_{cl}$ by $\tau_{op}$ and $\tau_{cl}$ resp. Pick $\omega$ such that $(g, t) = ((B_s)_{s \leq \tau^\prime}(\omega), \tau^\prime(\omega)) \in \Gamma$. It then follows that
\[
\tau_{cl}(\omega) \leq \tau^\prime(\omega) \leq \tau_{op}(\omega).
\]
To show that $\tau_{cl} = \tau_{op}$ a.s. put $\bar{\psi}(y) := \inf\{d : (d, y) \in \mathcal{R}_{cl}\}$ and note that $\tau_{cl} = \inf\{t \geq 0 : B_t - B_0 \geq \bar{\psi}(B_t)\}$ and $\tau_{op} = \inf\{t \geq 0 : B_t - B_0 > \psi(B_t)\}$. Using the transformation $(d, y) \mapsto (d - y, y)$ and setting
\[
\tilde{\mathcal{R}}_{op} = \{(x, y) : x > \tilde{\psi}(y) := \psi(y) - y\}
\]
\[
\tilde{\mathcal{R}}_{cl} = \{(x, y) : x > \bar{\psi}(y) := \psi(y) - y\}
\]
it is then sufficient to show that the respective hitting times $\tilde{\tau}_{op}$ and $\tilde{\tau}_{cl}$ of the process $(-B_0, B_t)$ are almost surely equal. Since the set
\[
\{\tilde{\psi}(y) : y \text{ is a local minimum}\}
\]
is at most countable we have that $\tilde{\tau}_{op} = 0 \leftrightarrow \tilde{\tau}_{cl} = 0$ almost surely since $\mu$ does not charge atoms. Hence, we can assume that $\tau_{cl}, \tau_{op} > 0$ a.s. Approximating $\tilde{\mathcal{R}}_{cl}$ from outside by sets of the form $\cup_{i=1}^N [a_i, \infty) \times I_i$ for some intervals $I_i$ and from inside by sets of the form $\cup_{i=1}^n [b_i, \infty) \times \{y_n\}$ we can further sandwich $\tilde{\tau}_{cl} \leq \tilde{\tau}_{op}$ between the hitting times $\tilde{\tau} \leq \tilde{\tau}_{cl}$ and $\tau \geq \tilde{\tau}_{op}$ of these sets. Since $P[\tau - \tilde{\tau} - \delta]$ can
be made arbitrarily small by making the approximation of $R_{cl}$ finer and finer the result follows.

Uniqueness of this embedding can be shown by an argument due to Loyines [20] based on previous work by Root [21] for which we refer to [4].

Remark 3.2. Define $\Xi := \{(x, y) : \exists (g, t) \in \Gamma, g(0) = x, g(t) = y\}$. By the previous theorem, this is precisely a supporting set of the martingale coupling $(B_0, B_{\tau J})$. Moreover, $\tau_{BJ}$ being the hitting time of a right barrier $\mathcal{R}$ (i.e. $(d, y) \in \mathcal{R}, d' > d \implies (d', x) \in \mathcal{R}$) in the $(B_t - B_0, B_t)$-phase space it immediately follows—one can directly read it off from Figure 1—that $\Xi$ is left monotone (as defined in [5]), i.e. if $(x_1, y_1), (x_2, y_2), (x_2, y') \in \Xi$ such that $x_1 < x_2$ and $y_1 < y_2$ then $y' \notin (y_1, y_2)$.

Remark 3.3. Note that to deduce the left-monotone structure of optimal martingale coupling the assumption of $\mu$ not charging atoms is not needed. In fact, it is sufficient to identify the stop-go pairs $SG$ as in (3.1) since observing a path $g$ being stopped at $g(t) = y' > g(0) = x_0$ no path starting at $x_1 < x_2$ can cross the level $y'$ not being stopped without violating (3.1). A similar remark applies to the embedding minimizing $\pm|x - y|$ below.

Remark 3.4. Assuming in Proposition 3.1 $c_{xyy} > 0$ we can run the very same proof to show the existence of a unique solution $\tau'_{BJ}$ of SEP minimizing $E[c(B_0, B_t)]$ and which is of the form $\tau'_{BJ} = \inf\{t > 0 : B_t - B_0 \notin (\psi_1(B_t), \psi_2(B_t))\}$ a.s., for some measurable functions $\psi_1$. This solution corresponds to the hitting time of a left barrier $\mathcal{R}'$ (i.e. $(d, y) \in \mathcal{R}'$, $d' < d \implies (d', x) \in \mathcal{R}'$)

Similarly we can prove the well-known properties for the optimizers in the so-called forward start straddle problem:

Proposition 3.5. Assume that $\mu\{(x)\} = 0$ for all $x \in \mathbb{R}$. There exists a stopping time $\tau_{HN}$ which minimizes

$$E[-|B_t - B_0|]$$

over all solutions to (SEP) and which is of the form $\tau_{HN} = \inf\{t > 0 : B_t - B_0 \notin (\psi_1(B_t), \psi_2(B_t))\}$ a.s., for some measurable functions $\psi_1 \leq \psi_2$.

Proof. The proof runs along similar lines as for Proposition 3.1 once we identified the secondary stop-go pairs. For simplicity we assume that $\nu$ has finite third moment. Consider the functionals $\gamma((f, s)) = -|f(t) - f(0)|$ as well as $\gamma((f, s)) = -(f(s) - f(0))^3$.

We claim that

$$SG_2 \supseteq \{(f, s), (g, t) \in S \times S : g(t) = f(s), 0 < (f(0) - g(0))(f(t) - f(0))\}. \quad (3.3)$$

Fix some $y \in \mathbb{R}$ and a stopping time $\sigma$ with positive and finite expectation and observe that

$$x \mapsto E[|y + B_\sigma - x|] - |y - x|$$

is monotonely increasing on $(-\infty, y]$ and monotonely decreasing on $(y, +\infty)$. Moreover, the map

$$x \mapsto E[-(y + B_\sigma - x)^3] + (y - x)^3$$

is strictly increasing on $(-\infty, y]$ and strictly decreasing on $(y, +\infty)$. Thus, pick some pair $((f, s), (g, t)) \in S \times S$ and $g(0) < f(0) < f(s) = g(t)$. Then,

$$E[|f(s) + B_\sigma - f(0)|] + |g(t) - g(0)| \geq |f(s) - f(0)| + E[|g(t) + B_\sigma - g(0)|],$$

and

$$E[-(f(s) + B_\sigma - f(0))^3] - (g(t) - g(0))^3 > -(f(s) - f(0))^3 - E[(g(t) + B_\sigma - g(0))^3].$$

Moreover, the inequalities remain true for $f(s) = g(t) < f(0) < g(0)$. \qed
Figure 3. The barrier solution optimizing the SEP for the cost functional \(-|B_t - B_0|\) in the phase space \((B_t - B_0, B_t)\).

Figure 4. The barrier solution for the cost functional \(|B_t - B_0|\) in the phase space \((B_t - B_0, B_t)\).

Looking at Figure 3 one directly sees that the martingale coupling \((B_0, B_{\tau_H})\) is concentrated on the graph of two non-decreasing functions.

Furthermore, changing a few signs at the appropriate spaces in the last proof we directly get:

**Proposition 3.6.** Suppose the supports of the marginal measures \(\mu\) and \(\nu\) are disjoint and \(\mu(\{x\}) = 0\) for all \(x \in \mathbb{R}\). There exists a stopping time \(\tau_{HK}\) which minimizes \(E[|B_{\tau} - B_0|]\) over all solutions to (SEP) and which is of the form \(\tau_{HK} = \inf\{t > 0 : B_t - B_0 \in (\psi_1(B_t), \psi_2(B_t))\} \) a.s., for some measurable functions \(\psi_1 \leq 0 \leq \psi_2\). Suppose additionally the existence of an interval \(I\) such that \(\mu(I) = \nu(I^c) = 1\) (cf. [16, Dispersion Assumption 2.1]). Then this is the unique solution to (SEP) of this form.

Looking at Figure 4 we see that the martingale coupling \((B_0, B_{\tau_{HK}})\) is concentrated on the graph of two non-increasing functions.

**Remark 3.7.** The last proposition remains true also if the marginals are not disjoint but the full Dispersion Assumption of [16] holds. In fact, it then follows from the structure of \(\mathcal{SG}_2\) that the common mass \(\mu \wedge \nu\) stays where it is. This is similar to the Rost embedding with general starting law, e.g. [1, Theorem 2.4].
Furthermore the result holds true for the more general class of cost functions $c(x,y) = |x - y|^p$ with $0 < p \leq 1$. Writing up the condition for the stop-go Pairs shows that their stop-go pairs agree with the ones for $|x - y|$.

**Remark 3.8.** The proof of uniqueness of the solution under the strong dispersion assumption $\mu(I) = \nu(I') = 1$ is similar to Loynes’ argument for hitting times of a single barrier. Indeed, if $I = [a, b]$ then there is an ‘upper’ part of the barrier responsible for embedding $\nu'_{(b,\infty)}$ and a ‘lower’ part of the barrier responsible for embedding $\nu'_{(-\infty,a]}$ which do not interfere with each other. On each of these barrier one can run Loynes’ argument.

4. Monotone Transformations

We propose a class of transformations of cost functions that will in particular comprise the transformation in [7] and linear transformations from $h(x - y)$ to $h(x + y)$. The idea is that this allows us to derive monotone sets for the transformed cost function as transformations of monotonicity sets of the original cost function. To prove this monotonicity, we need an accompanying transformation of measures that preserves competitors.

4.1. Competitor-Preserving Transformations. Let us start with the definition of the types of transformations we are interested in (where $\mathcal{E}(X)$ denotes the finite measures on a space $X$):

**Definition 4.1.** We call a transformation $\tau : \mathcal{E}(I \times J) \rightarrow \mathcal{E}(I' \times J')$ competitor preserving, if for given competitors $\alpha'$ and $\beta'$ concentrated on $I' \times J'$ we have that $\tau^{-1}(\alpha')$ and $\tau^{-1}(\beta')$ are also competitors.

We will identify competitor preserving $\tau$ of a specific form:

**Definition 4.2.** Let $T : I \times J \rightarrow I' \times J'$ be a bijective map and $h : I \times J \rightarrow \mathbb{R}^+$. We say the pair $(T, h)$ is a monotonicity preserving transformation if the induced transformation $\tau : \mathcal{E}(\mathbb{R}^2) \rightarrow \mathcal{E}(\mathbb{R}^2)$ defined via

$$\int g(x,y)d\pi(x,y) = \int (g \circ T)(x,y)h(x,y)d\pi(x,y)$$

for all bounded continuous $g$ is competitor preserving.

The next theorem characterizes all monotonicity preserving transformations $(T, h)$.

**Theorem 4.3.** Let $T(x,y) = (s(x,y), t(x,y))$ be a bijective map and $h : I \times J \rightarrow (0, \infty)$. Then we must have that $s$ is constant in $y$ and $t$ and $h$ are constant in $x$. Furthermore $t$ and $h$ must be of one of the following forms:

1. $h$ is constant and $t$ is affine in $y$;
2. $h$ is affine in $y$ (or $h(y) = c(y - b)$) and $t$ is of the form $t(y) = a/(y - b)$.

The proof of Theorem 4.3 is structured in four preliminary results. The first characterizes functions that cannot distinguish between competitors. The next two describe restrictions on the class of bijective monotonicity preserving transformations arising from their definition. After that we will prove a representation of the inverse transformation of $(T, h)$ in Proposition 4.7 and derive the representation of $T$ from that.

**Lemma 4.4.** Suppose $f : I \times J \rightarrow \mathbb{R}$ for $I$ and $J$ (possibly unbounded) intervals in $\mathbb{R}$ is such that $\alpha(f) = \beta(f)$ whenever $\alpha$ and $\beta$ are competitors. Then we can find a representation of $f$ of the form $f(x,y) = \varphi(x) + \psi(y) + k(x,y)$.

**Proof.** Consider the competitors

$$\alpha := \lambda \delta_{(x_1,y_1)} + (1 - \lambda)\delta_{(x_1,y_2)} + \delta_{(x_2,y')} \quad \beta := \lambda \delta_{(x_2,y_1)} + (1 - \lambda)\delta_{(x_2,y_2)} + \delta_{(x_1,y')}$$
for $y_1 < y' < y_2 \in J$ and $x_1, x_2 \in I$ arbitrary and $\lambda \in (0,1)$ such that $y' = \lambda y_1 + (1-\lambda)y_2$. Then by assumption we must have $\alpha(f) = \beta(f)$ which amounts to

$$\lambda f(x_1, y_1) + (1-\lambda)f(x_1, y_2) - f(x_1, y') = \lambda f(x_2, y_1) + (1-\lambda)f(x_2, y_2) - f(x_2, y').$$

As $y_1 < y' < y_2$ were arbitrary, the non-linear shape of $f$ along $y$ does not depend on $x$ and we can set $g(x, y) := f(x, y) - \psi(y)$ for $\psi(y) := f(x_0, y)$ and $x_0 \in I$ arbitrary but fixed. For any $x \in I$ we then have that $g(x, y)$ is linear in $y$ and the result follows.

Any function $f$ satisfying the assumptions of the last Lemma will be called competitorblind.

Observe that two measures $\alpha$ and $\beta$ are competitors if and only if they cannot distinguish between competitorblind functions. More concretely, they are competitors if for any $g \in \mathcal{C}_0(\mathbb{R})$ we have

\[
\begin{align*}
\int g(x)d\alpha(x, y) &= \int g(x)d\beta(x, y) \\
\int g(y)d\alpha(x, y) &= \int g(y)d\beta(x, y) \\
\int g(x)(y-x)d\alpha(x, y) &= \int g(x)(y-x)d\beta(x, y)
\end{align*}
\]

and sums of these functions (and their limits) are the only functions such that equality holds for arbitrary competitors.

**Lemma 4.5.** Let $\tilde{s} : I' \to I$ and $\tilde{t} : J' \to J$ be non-constant functions and $\tilde{h} : I' \times J' \to (0, \infty)$ be a competitorblind function.

(i) If $g(\tilde{s}(x))\tilde{h}(x, y)$ is competitorblind for any $g \in \mathcal{C}_0(\mathbb{R})$, then $\tilde{h}$ is linear in $y$.

(ii) If $g(\tilde{t}(y))\tilde{h}(x, y)$ is competitorblind for any $g \in \mathcal{C}_0(\mathbb{R})$, then $\tilde{h}$ is constant in $x$.

**Proof.** We can write $\tilde{h}(x, y) = \varphi(x) + \psi(y) + k(x)y$ by assumption. In the case of (i), we always have that $g(\tilde{s}(x))((\varphi(x) + k(x)y)$ is competitorblind. Therefore we must in particular have that $g(\tilde{s}(x))\psi(y)$ is competitorblind. For arbitrary $x_1 < x_2$ and $y_1 < y' < y_2$ with $\lambda \in (0,1)$ such that $y' = \lambda y_1 + (1-\lambda)y_2$ we then have

\[
(g(\tilde{s}(x_1)) - g(\tilde{s}(x_2)))(\lambda \psi(y_1) + (1-\lambda)\psi(y_2) - \psi(y')) = 0
\]

by choosing as above a pair of competitors

\[
\begin{align*}
\alpha := \lambda \delta_{(x_1, y_1)} + (1-\lambda)\delta_{(x_2, y_2)} + \delta_{(x_2, y_1)} \\
\beta := \lambda \delta_{(x_2, y_1)} + (1-\lambda)\delta_{(x_2, y_2)} + \delta_{(x_1, y_1)}.
\end{align*}
\]

As $\tilde{s}$ was assumed to be non-constant, we can choose $x_1$ and $x_2$ such that $\tilde{s}(x_1) \neq \tilde{s}(x_2)$ and therefore $\psi$ has to be linear in $y$ and thus also $\tilde{h}$ has to be linear in $y$.

In the case of (ii), we always have that $g(\tilde{t}(y))\psi(y)$ is competitorblind and hence $g(\tilde{t}(y))((\varphi(x) + k(x)y)$ must be competitorblind. As in part (i) we derive that for any $x_1 < x_2 \in I$ and $y_1 < y' < y_2 \in J$ with $\lambda \in (0,1)$ accordingly, we have that

\[
\begin{align*}
(\varphi(x_1) - \varphi(x_2))[(1-\lambda)g(\tilde{t}(y_1)) + \lambda g(\tilde{t}(y_2)) - g(\tilde{t}(y'))] \\
+ (k(x_1) - k(x_2))[\lambda g(\tilde{t}(y_1))y_1 + (1-\lambda)g(\tilde{t}(y_2))y_2 - g(\tilde{t}(y'))y'] = 0.
\end{align*}
\]

As $\tilde{t}$ is non-constant, we can find $y_1$ and $y_2$ with $\tilde{t}(y_1) \neq \tilde{t}(y_2)$ and can then choose $g$ such that $\lambda g(\tilde{t}(y_1)) + (1-\lambda)g(\tilde{t}(y_2)) - g(\tilde{t}(y')) = 0$ but $\lambda g(\tilde{t}(y_1))y_1 + (1-\lambda)g(\tilde{t}(y_2))y_2 - g(\tilde{t}(y'))y' \neq 0$ which yields that $k$ is a constant function. Now choosing a function $g$ such that $\lambda g(\tilde{t}(y_1)) + (1-\lambda)g(\tilde{t}(y_2)) - g(\tilde{t}(y')) \neq 0$ yields that also $\varphi$ is a constant function.

The proof of the following Lemma is elementary but technical and therefore deferred to the appendix.
Lemma 4.6. Assume we have a function $\tilde{t}(x, y) := \frac{a(x)+b(x)y}{c(x)+d(x)y}$ defined on a rectangle $I' \times J'$ such that $\tilde{h}(x, y) = c(x) + d(x)y > 0$ everywhere and $\tilde{t}(x, y)^k(c(x) + d(x)y)$ is competitorblind for $k = 2, 3$. If $\tilde{t}$ is non-constant in $y$ for all $x \in I'$ then $\tilde{t}$ is constant in $x$. Moreover, we have for some $a, b, c, d \in \mathbb{R}$

- either $\tilde{h}(x, y) = d$ and $\tilde{t}(x, y) = \frac{a}{d}$
- or $\tilde{h}(x, y) = c$ and $\tilde{t}(x, y) = \frac{a+by}{c}$.

Using this Lemma we can prove the main part of Theorem 4.3. It is easier to prove the shape of the inverse transform first, which will be done in the following Lemma, and deduce the representation of the original transform from it.

Proposition 4.7. Let $\tilde{s} : I' \times J' \to I$ and $\tilde{t} : I' \times J' \to J$ be such that $\tilde{T}(x, y) = (\tilde{s}(x, y), \tilde{t}(x, y))$ maps bijectively from $I' \times J'$ to $I \times J$. Furthermore let $\hat{h} : I' \times J' \to (0, \infty)$ be chosen such that for any $g \in C_0(\mathbb{R})$ we have that $(g \circ \tilde{s}) \cdot \hat{h}, (g \circ \tilde{t}) \cdot \hat{h}$ and $(g \circ \tilde{s}) \cdot (\tilde{t} - \tilde{s}) \cdot \hat{h}$ are competitorblind. Then $\tilde{s}$ is constant in $y$ and $\tilde{t}$ is constant in $x$.

Proof. First assume that $\tilde{s}$ is not constant in $y$. So we find $x_1 \in I'$ and $y_1 < y_2 \in J'$ such that $\tilde{s}(x_1, y_1) \neq \tilde{s}(x_1, y_2)$. Additionally we can find $y_1 \in (y_1, y_2)$ and $x_2 \in I'$ such that $\tilde{s}(x_1, y_1) \neq \tilde{s}(x_2, y_1)$. If this was not the case, $\tilde{s}$ would be constant in $x$ on $I' \times (y_1, y_2)$. Then $\tilde{s}$ can not be constant in $y$ in this area because otherwise $\tilde{t}$ has to be injective in $I' \times (y_1, y_2)$. Considering arbitrary competitors $\alpha$ and $\beta$ concentrated on this area along with an arbitrary bounded continuous function $g$ will entail a contradiction to $g(\tilde{t}(x, y))\tilde{h}(x, y)$ being competitorblind.

As $\tilde{s}$ is not constant in $y$ but constant in $x$, we are in the setting of part (ii) in Lemma 4.5. Therefore $\tilde{h}$ is also constant in $x$ on this interval. With slight abuse of notation we write $\tilde{s}(y) = \hat{\tilde{s}}(x, y)$ and $\hat{\tilde{t}}(y) = \hat{h}(x, y)$ for $(x, y) \in I' \times (y_1, y_2)$. By assumption we also have that $g(\tilde{s}(y))\tilde{t}(x, y)\hat{\tilde{h}}(x, y)$ is competitorblind and then in particular that $g(\tilde{s}(y))\hat{h}(y)\tilde{t}(x, y)$ is competitorblind on $I' \times (y_1, y_2)$. But then, by again using Lemma 4.5 (ii), we can see that $\hat{h}(y)\tilde{t}(x, y)$ is constant in $x$. In particular $\tilde{t}$ must then be constant in $x$ which contradicts the assumption that $\tilde{T}$ is bijective. Hence, there is $y_3 \in (y_1, y_2)$ s.t. $\tilde{s}(x_1, y_3) \neq \tilde{s}(x_2, y_3)$.

Now, taking the usual pair of competitors

$$\alpha := \lambda \hat{h}(x_1, y_1) + (1 - \lambda)\hat{\delta}(x_1, y_2) + \hat{\delta}(x_2, y_1)$$
$$\beta := \lambda \hat{\delta}(x_2, y_1) + (1 - \lambda)\hat{\delta}(x_2, y_2) + \hat{\delta}(x_1, y_3)$$

we have that for arbitrary $g \in C_0(\mathbb{R})$ we must have

$$\lambda g(\hat{s}(x_1, y_1))\tilde{h}(x_1, y_1) + (1 - \lambda)g(\hat{s}(x_1, y_2))\tilde{h}(x_1, y_2) - g(\hat{s}(x_1, y_3))\tilde{h}(x_1, y_3)$$

$$= \lambda g(\hat{s}(x_2, y_1))\tilde{h}(x_2, y_1) + (1 - \lambda)g(\hat{s}(x_2, y_2))\tilde{h}(x_2, y_2) - g(\hat{s}(x_2, y_3))\tilde{h}(x_2, y_3)$$

but there is enough freedom in the choice of $g$ that this leads to a contradiction. Hence, $\hat{s}$ is constant in $y$ and we write with slight abuse of notation $\tilde{s}(x, y) = \hat{s}(x)$.

Then Lemma 4.5 part (i) implies that $\hat{h}$ is linear in $y$. Furthermore, the condition that $g(\hat{s}(x))\tilde{t}(x, y) - \tilde{s}(x))\hat{h}(x, y)$ is competitorblind implies that $g(\hat{s}(x))\tilde{t}(x, y)\hat{h}(x, y)$ is also competitorblind as $g(\hat{s}(x))\tilde{s}(x)\hat{h}(x, y)$ is now trivially competitorblind because $\hat{h}$ is linear in $y$. Therefore we also have that $\tilde{t}(x, y)\hat{h}(x, y)$ is linear in $y$ using part (i) of Lemma 4.5 again. This directly implies that we can write $\tilde{t}(x, y) = c(x) + d(x)y$ for appropriate choice of $a, b, c, d$ and $\hat{h}(x, y) = \hat{c}(x) + d(x)y$ and therefore $c(x) + d(x)y > 0$ everywhere. By assumption we have that $g(\tilde{t}(x, y))\hat{h}(x, y)$ is competitorblind and by approximation then also that $\tilde{t}(x, y)\hat{h}(x, y)$ is competitorblind. We must also have that for all $x$, the function $y \mapsto \tilde{t}(x, y)$ is non-constant, because otherwise $\hat{T}(x, y) = (\hat{s}(x), \tilde{t}(x, y))$ is not injective. Therefore $\tilde{t}$ and $\hat{h}$ satisfy the assumptions of Lemma 4.6 which yields that $\tilde{t}$ is constant in $x$. □
Proof of Theorem 4.3. Let $\alpha', \beta'$ be competitors concentrated on $I' \times J'$. We need that $\alpha = \tau^{-1}(\alpha')$ and $\beta = \tau^{-1}(\beta')$ for $\tau$ as defined above are also competitors. By the definition of $\tau$ we have for a function $f : I \times J$ that $\int f(x,y) d\alpha = \int \frac{f(\tau^{-1}(x', y'))}{h(x', y')} d\alpha'$. Set $h(x', y') := \frac{h(\tau^{-1}(x', y'))}{h(x', y')} \text{ and } T^{-1}(x', y') = (s(x', y'), \tilde{t}(x', y'))$. For $\alpha$ and $\beta$ to be competitors, we need that conditions (C1) – (C3) hold. This gives
\[
\int g(x) d\alpha = \int g(s(x', y')) h(x', y') d\alpha' = \int g(s(x', y')) h(x', y') d\beta' = \int g(x) d\beta
\]
and similarly for (C2) and (C3). In particular we need that $(g \circ \tilde{s}) h_{\tilde{t}}$, $(g \circ \tilde{t}) h_{\tilde{s}}$ and $(g \circ \tilde{s})(\tilde{t} - \tilde{\tilde{s}}) h_{\tilde{t}}$ are competitor-preserving transformations. From Proposition 4.7 we then get that $\tilde{s}$ only depends on $x$, and $\tilde{t}$ and $h$ only depend on $y$ with the representation given by Lemma 4.6. Now, $s$ and $t$ are the inverse functions of $\tilde{s}$ and $\tilde{t}$ and $h(x, y) = 1/h(T(x, y))$.

In the case where we have that $h$ is constant and $\tilde{t}$ is affine, then we also have that $h$ is constant and $t$ is affine.

In the case where we have that $\tilde{h}$ is linear and $\tilde{t}$ is affine in $1/y'$ we now get that $t$ is of the form $t(y) = a/(y-b)$ and $h(y) = c(y-b)$. \qed

4.2. Pushing Monotonicity along Transformations. We are interested in competitor-preserving transformations because they allow us to identify monotonicity sets for new cost functions derived from such transformations in the following way:

Proposition 4.8. Let $(T, h)$ be a monotonicity preserving transformation and let $c$ be a cost function. Define a new cost function $c'(x', y') := \frac{c(T^{-1}(x', y'))}{h(x', y')}$. If $\Xi$ is $c$-monotone, then $\Xi' := T(\Xi)$ is $c'$-monotone.

Proof. Let $\alpha'$ be a finite measure concentrated on $\Xi'$ and $\beta'$ be a competitor of $\alpha'$. Then $\alpha := \tau^{-1}(\alpha')$ and $\beta := \tau^{-1}(\beta')$ are competitors such that $\alpha$ is concentrated on $\Xi$. Therefore we have that $\alpha'(c') = \alpha(c) \geq \beta(c) = \beta'(c')$. As $\alpha'$ and $\beta'$ were arbitrary, we have that $\Xi'$ is a $c'$-monotone set. \qed

Example 4.9. Let $c$ be a Spence Mirrlees cost function (i.e. $c_{xyy} < 0$). We know that a set $\Xi$ is $c$-monotone if and only if it is left-monotone. We consider the simple transformation $T(x, y) = (-x, y)$ and a compensator $h(y) = 1$. Then $(T, h)$ is monotonicity preserving by Theorem 1.1 wrt the transformed cost function $c'(x', y') = c(-x', y')$. Observe that we have $c'_{x'y'y} > 0$. The set $\Xi' := T(\Xi) = \{(x', y') : (-x', y') \in \Xi \}$ is $c'$ monotone by Proposition 4.8. Indeed we can easily check that $\Xi'$ is right-monotone: Suppose we have $(x_1, y_1), (x_1, y_2), (x_2, y') \in \Xi'$ with $x_2 < x_1$ and $y_1 < y' < y_2$, then we have $(-x_1, y_1), (-x_1, y_2), (-x_2, y') \in \Xi$ with $-x_1 < -x_2$ which contradicts the left-monotonicity of $\Xi$.

Remark 4.10. The example above holds true replacing $T$ by $\tilde{T}(x, y) = (s(x), y)$ for some differentiable function $s$ such that $s'(x) < 0$ on the support of $\mu$.

Example 4.11. Let $c(x, y) = -|x - y|$ and again set $T(x, y) = (-x, y)$ and $h(y) = 1$. Then $c'(x, y) = -|x + y|$. If $\mu$ is continuous and $\Xi$ is a monotone set for $c$, then it is concentrated on two graphs of increasing functions as established above. Then $T(\Xi)$ is concentrated on two graphs of decreasing functions which corresponds to the shape of monotone sets for this cost function established in [22].

Example 4.12. The transformation described in [7] is given by $T(x, y) = (1/x, 1/y)$ with $h(y) = y$. For the cost function $c(x, y) = -|x - y|$ this yields the transformed cost $c'(x, y) = -|1/x - 1/y|(1/y) = -|y/x - 1|$ (for $x, y > 0$). Furthermore the graph of an increasing function is transformed into the graph of an increasing
function under this map $T$, which shows that the optimizer of $-|y/x - 1|$ is also concentrated on two graphs of increasing functions. Similarly one can deal with $c(x, y) = |x - y|$ and cost functions of Spence Mirrlees type.

4.3. Transformations of Martingales. The given transformations only transform monotone sets into monotone sets for a modified cost function. They do not necessarily transform optimal martingale transports into optimal martingale transports. It is possible to have a monotone set that is not even capable of supporting a martingale, e.g. $\Xi = \{(0, 1)\}$. Nevertheless if a martingale is concentrated on a $c$-monotone set (for sufficiently nice $c$), it is optimal. For our transformation to preserve martingales we would have to ask for a more stringent condition. Namely we would need $\int g(s(x))(t(y) - s(x))h(y)\,d\tau(x, y) = 0$ to hold for arbitrary martingales and bounded continuous functions $g$. This can only hold if $b(t(y) - s(x))h(y) = r(x)(y - x)$ which by a simple analysis shows that this is only possible for transformations of the form $T(x, y) = (ax + b, ay + b)$, $h(y) = 1$ or $T(x, y) = (1/x, 1/y)$ and $h(y) = y$ which proves Theorem 1.2.

4.4. The Skorokhod picture of monotonicity preserving transformations. In this section we show that there are clear geometric interpretations of monotonicity preserving transformations which can be well understood using the relation of MOT and SEP.

Let us consider a cost function $c$ satisfying the generalized Spence Mirrlees condition $c_{xyy} < 0$. By Proposition 3.1 we know that the corresponding SEP solution is the hitting time of a right barrier (see Remark 3.2) in the phase space $(B_1 - B_0, B_1)$. The map $T(x, y) = (-x, -y)$ the point reflection at $(0, 0)$ transforms the right barrier into a left barrier, similarly for the transformation $T(x, y) = (1/x, 1/y)$. This barrier corresponds to solutions for cost functions $c'$ satisfying $c'_{xyy} > 0$ or equivalently to the maximization problem wrt the cost function $c$ with $c_{xyy} < 0$ (see Remark 3.4).

Considering the transformations $T(x, y) = (-x, y)$, the reflection at the $y$-axis, again a right barrier will be mapped into a left barrier in the $(B_1 - B_0, B_1)$ phase space. The transformed cost function satisfies $c'(x, y) = c(-x, y)$. Specifying $c(x, y) = |y - x|$ we get $c'(x, y) = |y + x|$. Moreover, writing the corresponding SEP solution in the $(B_1 + B_0, B_1)$ phase space we end up with a right barrier. To be more specific, if the original optimal stopping time is $\tau = \inf\{t \geq 0 : B_t - B_0 \geq \psi(B_1)\}$ then the modified stopping rule is given by $\tau' = \inf\{t \geq 0 : B_t + B_0 \geq \psi(B_1)\}$ which can be rewritten to $\tau' = \inf\{t \geq 0 : B_t - B_0 \leq 2B_1 - \psi(B_1)\}$, showing that it is equivalent to a left barrier in the $(B_1 - B_0, B_1)$ phase space revealing a suprising symmetry between solutions for the cost functions $\gamma(f, s) = |f(s) - f(0)|$ and $\gamma'(f, s) = |f(s) + f(0)|$. These transformations are depicted in Figure 5.

Similarly for the cost functional $-|B_0 - B_t|$ we get that the optimal stopping time is given by two barriers in the $(B_1 - B_0, B_1)$ phase space. As before the transformation $T(x, y) = (-x, y)$ transforms these barriers into two barriers in the $(B_1 + B_0, B_1)$ phase space. This is depicted in Figure 6.

We emphasize that this form of transformations does not lead to the optimal stopping time for the same marginals $\mu, \nu$ that the original barrier was constructed for. If we start an arbitrary distribution $\mu$ and stop it at this barrier, it will be the optimal stopping time between its marginals.

5. Towards a 'full' monotonicity principle for SEP

Some transformations discussed in Section 4 have clear analogues for $(\text{OptSEP})$, e.g. $(x, y) \mapsto (1/x, 1/y)$ corresponds to $(f, s) \mapsto (1/f, s)$ or $(x, y) \mapsto (ax + b, ay + b)$ corresponds $(f, s) \mapsto (af + b, s)$, whereas the analogues of other transformations might be less obvious. Nevertheless, the 'good' transformations reveal symmetries...
between solutions to \((\text{OptSEP})\) for different cost functions \(\gamma\). To establish an analogue to Proposition 4.8 for \((\text{OptSEP})\) it seems to be necessary to find and prove a full monotonicity principle. Comparing with MOT we now give a conjecture on this full monotonicity principle. To this end, we need the following definition.
**Definition 5.1.** Let \( \alpha \) and \( \beta \) be two finite measures on \( S \). We say that \( \beta \) is a competitor of \( \alpha \) iff

\[
(h^0)\alpha = (h^0)\beta \tag{5.1}
\]

\[
(h^1)\alpha = (h^1)\beta , \tag{5.2}
\]

where \( h^0(f,s) = f(0) \) and \( h^1(f,s) = f(s) \) and

\[
\int \varphi \, d\alpha = \int \varphi \, d\beta \tag{5.3}
\]

for all martingales \( \varphi \) such that there is a continuous function \( H : S \to \mathbb{R} \) such that \( \varphi = H \circ r \).

Conditions (5.1) and (5.2) are the obvious analogues of the equal marginal constraint in Definition 2.2. The condition (5.3) should be compared to (2.1). Since a full monotonicity principle for (OptSEP) needs to be able to make pathwise comparisons we only require (5.3) for functions which are well defined on a pointwise level. Examples for competitors can be read off from Definition 2.4, i.e. given a stopping time \( \sigma \) with positive and finite expectation we set

\[
\alpha = \delta_{(g,t)} + \int \delta_{(f,s)}\otimes((B_{s+w})_{w \leq \sigma(\omega), \sigma(\omega)}\mathbb{P}(d\omega)
\]

\[
\beta = \delta_{(f,s)} + \int \delta_{(g,t)}\otimes((B_{s+w})_{w \leq \sigma(\omega), \sigma(\omega)}\mathbb{P}(d\omega) .
\]

The following should be compared to Definition 2.4(ii).

**Definition 5.2.** We say that a set \( \Gamma \subseteq S \) is strongly \( \gamma \)-monotone if for any finite measure \( \alpha \) concentrated on \( \Gamma \) such that \((h^0)*\alpha \) is concentrated on finitely many points and any competitor \( \beta \) of \( \alpha \) it holds that

\[
\int \gamma \, d\alpha \leq \int \gamma \, d\beta .
\]

**Conjecture.** Let \( \gamma : S \to \mathbb{R} \) be Borel \( \mu, \nu \) be two probabilities on \( \mathbb{R} \) increasing in convex order. There exists a strongly \( \gamma \)-monotone set \( \Gamma \subseteq S \) such that a solution \( \tau \) of SEP is a solution to (OptSEP) if and only if \( \mathbb{P}((B_{r}, \tau) \in \Gamma) = 1 \).

We only note here that the conjecture holds for functionals \( \gamma(f,s) = c(f(0), f(s)) \) due to the intimate connection of MOT and SEP and the respective result for MOT, Theorem 2.1.

**Appendix A. Proof of Lemma 4.6**

**Proof.** We first write \( t(x,y)^2(c(x) + d(x)y) = \varphi(x) + \psi(y) + k(x)y \) and multiply both sides by \( c(x) + d(x)y \) (which was assumed to be positive everywhere) to obtain

\[
a(x)^2 + 2a(x)b(x)y + b(x)^2y^2 = (c(x) + d(x)y)(\varphi(x) + \psi(y) + k(x)y).
\]

As the left side is a polynomial in \( y \), the right side must also be a polynomial in \( y \) which can only hold if we can write \( \psi(y) = p_{-1}y^{-1} + p_{2}y^2 \) (we can assume that no constant and linear term exists by modifying \( \varphi \) and \( k \) accordingly).

Next observe that we cannot have both \( p_{-1} \neq 0 \) and \( p_{2} \neq 0 \). In this case we must have \( c(x) = d(x) = 0 \) for all \( x \) so that we do not have a term depending on \( y^2 \) nor a term that depends on \( y^{-1} \) on the right side which is necessary for this equality to hold. However, this contradicts the assumption that \( c(x) + d(x)y > 0 \).

Furthermore, it is also impossible to have \( p_{-1} = p_{2} = 0 \). In this case the above equation simplifies to

\[
a(x)^2 + 2a(x)b(x)y + b(x)^2y^2
\]

\[
= c(x)\varphi(x) + [c(x)k(x) + d(x)\varphi(x)]y + d(x)k(x)y^2
\]
which by comparison of coefficients gives the conditions \( a(x)^2 = c(x)\varphi(x) \), \( b(x)^2 = d(x)k(x) \) and \( 2a(x)b(x) = c(x)k(x) + d(x)\varphi(x) \). Suppose for fixed \( x \) that \( c(x) = 0 \), then we also have \( a(x) = 0 \) and therefore \( \tilde{t}(x,y) = b(x)/d(x) \) does not depend on \( y \) which contradicts the assumption that \( t \) is non-constant in \( y \) for this \( x \). Similarly \( d(x) = 0 \) implies \( b(x) = 0 \) and therefore \( \tilde{t}(x,y) = a(x)/c(x) \) which again would be constant in \( y \). Therefore we can assume that \( c(x)d(x) \neq 0 \) and multiply the last condition by this term to obtain \( 2a(x)b(x)c(x)d(x) = c(x)^2k(x)d(x) + d(x)^2\varphi(x)c(x) \). Now substitute the first two conditions to obtain \( 2a(x)b(x)c(x)d(x) = c(x)^2b(x)^2 + d(x)^2a(x)^2 \) which implies that \( b(x)c(x) = a(x)d(x) \) holds. A Möbius-transform with coefficients whose determinant is \( 0 \) is constant in \( y \) which would mean that \( t \) is constant in \( y \).

Now we consider the cases where exactly one of the \( p_i \) is \( 0 \):

If \( p_2 = 0 \) but \( p_{-1} \neq 0 \), we must have \( c(x) = 0 \) for all \( x \) because otherwise the right side of the above equation has a term that depends on \( y^{-1} \) which does not occur on the left side. Then the equation simplifies to

\[
a(x)^2 + 2a(x)b(x)y + b(x)^2y^2 = d(x)p_{-1} + \varphi(x)d(x)y + d(x)k(x)y^2
\]

which by comparing the coefficient of the linear term gives \( a(x)^2 = d(x)p_{-1} \). Substituting \( d(x) \) in the definition of \( \tilde{t}(x,y) \) accordingly gives \( \tilde{t}(x,y) = p_{-1}/a(x)^2 \) for all \( x \) and \( y \).

Now considering \( \tilde{t}(x,y)^3d(x)y = \tilde{t}(x,y)^3a(x)^2/3p_{-1} \) we have that the coefficient of \( y^{-2} \) is given by \( p_{-1}^2/a(x) \) and needs to be constant in \( x \) in order for this function to be competitor-blind which shows that \( a(x) \) is constant in \( x \). Now writing \( a(x) = a \) we also have that the coefficient of \( y^{-1} \) is given by \( 3p_{-1}b(x)/a^2 \) which also needs to be constant in \( x \) and thus \( b(x) \) is constant in \( x \). In conclusion we have that \( \tilde{t}(x,y) \) is constant in \( x \) and affine in \( y^{-1} \).

If \( p_{-1} = 0 \) but \( p_2 \neq 0 \), we must have \( d(x) = 0 \) for all \( x \) because otherwise the right side of the above equation has a term that depends on \( y^3 \) which does not occur on the left side. Then the equation simplifies to

\[
a(x)^2 + 2a(x)b(x)y + b(x)^2y^2 = c(x)\varphi(x) + c(x)k(x)y + c(x)p_2y^2
\]

for which we can compare the coefficient of the square term to obtain the condition \( b(x)^2 = c(x)p_2 \). Substituting for \( c(x) \) in the definition of \( \tilde{t}(x,y) \) then gives \( \tilde{t}(x,y) = p_2b(x)^2/6c(x) + p_2b(x)/6c(x) \). We have that \( \tilde{t}(x,y)^3c(x) = \tilde{t}(x,y)^3b(x)^2/p_2 \) is competitor-blind and thus its coefficient of \( y^3 \) must be constant in \( x \). This coefficient is given by \( p_2^2b(x)/6c(x) \) and therefore \( b(x) = b \) is constant in \( x \). The coefficient of \( y^2 \) of this function is given by \( 3p_2^2b(x)/6c(x) \) and must also be constant in \( x \) which implies \( a(x) = a \) is constant in \( x \). We can conclude as in the first case that \( \tilde{t}(x,y) \) is constant in \( x \) and affine in \( y \).

References


*E-mail address: huesmann@iam.uni-bonn.de*

*E-mail address: florian.stebegg@columbia.edu*