Wavelet Bi-frames with few Generators from Multivariate Refinable Functions

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Abstract

Using results on syzygy modules over a multivariate polynomial ring, we are able to construct compactly supported wavelet bi-frames with few generators from almost any pair of compactly supported multivariate refinable functions. In our examples, we focus on wavelet bi-frames whose primal and dual wavelets are both derived from one box spline function. Our wavelet bi-frames have fewer generators than comparable constructions available in the literature.

Key words: Wavelet bi-frames, multivariate refinable functions, syzygy modules, oblique extension principle

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1 Introduction

Wavelet analysis is nowadays a widely used tool in applied mathematics. The application of wavelets and the fast wavelet algorithms comprises the numerical treatment of operator equations as well as signal and image analysis. Wavelets are commonly derived from *a refinable function* φ:

$$\varphi(x) = |\det(M)| \sum_{k\in\mathbb{Z}^d} a_k \varphi(Mx-k), \qquad a.e.\ x \in \mathbb{R}^d, \tag{1}$$

where $(a_k)_{k\in\mathbb{Z}^d}$ is a finitely supported sequence of real numbers, called a *mask* for φ, and M is a dilation matrix, c.f. Subsection 2.1. The wavelets inherit many properties of φ such as the order of smoothness. In order for wavelet coefficients to capture highly local information of a signal, we need wavelets with small support. Wavelet systems with few generators will reduce the complexity of the associated wavelet transform.

For a pair of biorthogonal wavelet bases, primal and dual wavelets are constructed from two refinable functions φ and $\widetilde{\varphi}$, whose integer shifts are biorthogonal to each other. This strong coupling between φ and $\widetilde{\varphi}$ causes some inconvenient side effects such as the difficulty of incorporating a high order of smoothness with very small supports.

One can circumvent these restrictions by using wavelet frames instead. The frame concept allows for redundancy so that primal and dual refinable functions are decoupled such that we have more flexibility for the construction of wavelets. In the univariate setting, one can derive compactly supported wavelet bi-frames with few generators from any pair of compactly supported refinable functions, see [6]. In arbitrary dimensions, the frame concept has already been applied to construct arbitrarily smooth compactly supported bi-frames with few generators satisfying a variety of optimality conditions, see [8, 9]. There, the underlying refinable function $\varphi = \widetilde{\varphi}$ must be *fundamental*, i.e., $\varphi(k) = \delta_{0,k}$, for all $k \in$

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$\mathbb{Z}^d$, and their Fourier transform must have a certain required factorization. However, these assumptions are quite strong and most box spline refinable functions do not satisfy them. In [3, 13, 14, 15, 20], wavelet bi-frames from box splines have been derived, but the bi-frames have many generators.

In the present work, we construct compactly supported wavelet bi-frames from almost any pair of compactly supported multivariate refinable functions. We follow some ideas in [8], but we remove the assumptions about a fundamental refinable function and the special factor of its mask. At least to a certain extent, this generalizes the univariate results in [6] to the multivariate setting. Moreover, we focus on the construction of wavelet bi-frames with few generators. We apply our findings to box spline refinable functions, and we derive wavelet bi-frames with significantly fewer generators than those in [3, 13, 14, 15, 20].

The paper is organized as follows: In Section 2, we introduce wavelet bi-frames and their construction from refinable functions based on the two-step strategy presented in [8]. Subsections 3.1 and 3.2 are dedicated to verifying that the construction steps can be accomplished for almost any pair of compactly supported refinable functions. Finally, we apply our results to box spline refinable functions in Section 4 and provide some examples of wavelet bi-frames derived from bivariate and trivariate box spline functions.

2 General Setting

2.1 Wavelet Bi-frames

In order to analyze complicated functions or signals, we often decompose them into simple building blocks. In wavelet analysis, building blocks are integer shifts and dilates of a so-called wavelet $\psi : \mathbb{R}^d \to \mathbb{C}$. More precisely, we consider

$$\psi_{j,k}(x) := |\det(M)|^{j/2}\psi(M^j x - k), \quad \text{for } j \in \mathbb{Z}, k \in \mathbb{Z}^d,$$

where M is a *dilation matrix* throughout, i.e., an integer matrix whose eigenvalues are greater than one in modulus. We say that a finite collection $\{\psi^{(1)}, \dots, \psi^{(n)}\}$ of $L_2(\mathbb{R}^d)$-functions *generates a wavelet frame* if there exist two positive constants A and B such that

$$A\|f\|^2_{L_2(\mathbb{R}^d)} \le \sum_{\mu=1}^{n}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d} \left|\langle f, \psi^{(\mu)}_{j,k}\rangle\right|^2 \le B\|f\|^2_{L_2(\mathbb{R}^d)}, \quad \text{for all } f \in L_2(\mathbb{R}^d). \tag{2}$$

Then each function $f \in L_2(\mathbb{R}^d)$ has a series (frame) expansion. We say that two finite collections $\{\psi^{(1)}, \dots, \psi^{(n)}\}$ and $\{\widetilde{\psi}^{(1)}, \dots, \widetilde{\psi}^{(n)}\}$ *generate a wavelet bi-frame* if both generate wavelet frames such that

$$f = \sum_{\mu=1}^{n}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}^{(\mu)}_{j,k}\rangle \psi^{(\mu)}_{j,k}, \quad \text{for all } f \in L_2(\mathbb{R}^d). \tag{3}$$

On the one hand, according to (3), wavelet bi-frames allow for series expansions, where the coefficients can be determined by inner products with wavelets. In this sense, they behave like orthonormal wavelet bases. On the other hand, contrary to bases, frames allow for redundancy. This flexibility can be used to incorporate many additional features such as symmetries, small supports, and a high order of smoothness.

2.2 A General Construction Recipe

We consider wavelet frames that are derived from a compactly supported refinable function φ. For a function $f \in L_1(\mathbb{R}^d)$, its Fourier transform $\widehat{f}$ used in this paper is defined to be $\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}dx$ for $\xi \in \mathbb{R}^d$. In order to choose φ, one starts with a finitely supported sequence $(a_k)_{k\in\mathbb{Z}^d}$ satisfying $\sum_{k\in\mathbb{Z}^d} a_k = 1$. The *symbol* of the sequence $(a_k)_{k\in\mathbb{Z}^d}$ is the Laurent polynomial

$$a(z) := \sum_{k\in\mathbb{Z}^d} a_k z^k, \qquad z \in (\mathbb{C}\backslash\{0\})^d,$$

and the refinable function φ is given by $\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} a(e^{-2\pi i (M^T)^{-j}\xi})$, $\xi \in \mathbb{R}^d$, where $z^k := z_1^{k_1} \cdots z_d^{k_d}$ for $z = (z_1, \dots, z_d)$ and $k = (k_1, \dots, k_d)$.

Given two compactly supported refinable functions φ and $\widetilde{\varphi}$, one defines wavelets by

$$\psi^{(\mu)}(x) := |\det(M)| \sum_{k\in\mathbb{Z}^d} a_k^{(\mu)} \varphi(Mx-k) \quad \text{and} \quad \widetilde{\psi}^{(\mu)}(x) := |\det(M)| \sum_{k\in\mathbb{Z}^d} b_k^{(\mu)} \widetilde{\varphi}(Mx-k), \tag{4}$$

where $\big(a_k^{(\mu)}\big)_{k\in\mathbb{Z}^d}$ and $\big(b_k^{(\mu)}\big)_{k\in\mathbb{Z}^d}$, $\mu = 1, \dots, n$, are additional finitely supported sequences. In order to guarantee that $\{\psi^{(1)}, \dots, \psi^{(n)}\}$ and $\{\widetilde{\psi}^{(1)}, \dots, \widetilde{\psi}^{(n)}\}$ generate a wavelet bi-frame, one has to impose some conditions on the sequences in (4) as we shall explain next.

Throughout the present paper, we apply the following notations:

$$a(z^M) := \sum_{k\in\mathbb{Z}^d} a_k z^{Mk}, \quad a(\tfrac{1}{z}) := \sum_{k\in\mathbb{Z}^d} a_k z^{-k}.$$

Moreover, let R_M denote a complete set of representatives of $\big(M^{-\top}\mathbb{Z}^d\big)/\mathbb{Z}^d$ with $0 \in R_M$ and let $z_\rho := (z_1 e^{-2\pi i \rho_1}, \dots, z_d e^{-2\pi i \rho_d})$ for $\rho = (\rho_1, \dots, \rho_d) \in R_M$. Given symbols $a^{(0)}$ and $b^{(0)}$, we denote

$$\theta(z) := \sum_{\rho\in R_M} a^{(0)}(z_\rho)\, b^{(0)}(\tfrac{1}{z_\rho}), \qquad \eta(z) := 1 - \theta(z). \tag{5}$$

We also make use of $\mathbf{1} := (1, \dots, 1)$, and we say that a satisfies the *sum rules of order* s if a has a zero of order s at $\mathbf{1}_\rho$ for all $\rho \in R_M \backslash \{0\}$.

The following theorem for the construction of wavelet bi-frames is borrowed from [8]. Its proof is based on the mixed oblique extension principle stated in [7] (also see [4, 6, 12, 19]) and [11].

Theorem 2.1. *Let φ and $\widetilde{\varphi}$ be compactly supported refinable functions in $L_2(\mathbb{R}^d)$ with finitely supported masks $a^{(0)}$ and $b^{(0)}$, respectively such that $\overline{\widehat{\varphi}(0)}\widehat{\widetilde{\varphi}}(0) = 1$. Assume that both $a^{(0)}$ and $b^{(0)}$ satisfy the sum rules of order 1 and $a^{(0)}(\mathbf{1}) = b^{(0)}(\mathbf{1}) = 1$. Moreover, suppose*

(S1) *there are symbols $a^{(\mu)}$, $b^{(\mu)}$ satisfying $a^{(\mu)}(\mathbf{1}) = b^{(\mu)}(\mathbf{1}) = 0$, $\mu = 1, \dots, n_1$, such that*

$$\sum_{\mu=0}^{n_1} a^{(\mu)}(z) b^{(\mu)}(\tfrac{1}{z_\rho}) = \delta_{0,\rho}\theta(z), \quad \text{for all } \rho \in R_M, \tag{6}$$

(S2) *there are symbols $\eta^{(\nu)}$, $\widetilde{\eta}^{(\nu)}$ satisfying $\eta^{(\nu)}(\mathbf{1}) = \widetilde{\eta}^{(\nu)}(\mathbf{1}) = 0$, $\nu = 1, \dots, n_2$, such that*

$$\eta(z) = \sum_{\nu=1}^{n_2} \eta^{(\nu)}(z) \widetilde{\eta}^{(\nu)}(\tfrac{1}{z}).$$

For $\nu = 1, \dots, n_2$, let

$$a^{(n_1+\nu)}(z) := \eta^{(\nu)}(z^M) a^{(0)}(z), \qquad b^{(n_1+\nu)}(z) := \widetilde{\eta}^{(\nu)}(z^M) b^{(0)}(z).$$

Then the collections $\{\psi^{(1)}, \dots, \psi^{(n_1+n_2)}\}$ and $\{\widetilde{\psi}^{(1)}, \dots, \widetilde{\psi}^{(n_1+n_2)}\}$ as in (4) generate a wavelet bi-frame.

In Theorem 2.1, the upper frame bound in (2) is guaranteed by [11, Theorem 2.3] and the identity in (3) is deduced from the oblique extension principle. See [12] for recent developments on oblique extension principle and bi-frames. Due to Theorem 2.1, we can obtain a wavelet bi-frame from a pair of refinable functions φ and $\widetilde{\varphi}$ if we can find additional symbols satisfying (S1) and (S2). In order to derive bi-frames with few generators, we have to make sure that n_1 in (S1) and n_2 in (S2) are small. Symbols for (S1) with $n_1 = |\det(M)| - 1$ and (S2) with $2|\det(M)| - 3$ are found in [8], provided that $\varphi = \widetilde{\varphi}$ is *fundamental* and its symbol has a certain factorization. These requirements are very strong and they are not satisfied by most box spline refinable functions. Thus, it is important to remove these assumptions.

3 Two Construction Steps for (S1) and (S2)

In the present section, we verify that, given $a^{(0)}$ and $b^{(0)}$, we can find wavelet symbols in (S1) and (S2).

3.1 Step 1 for (S1)

In [8], (S1) is explicitly solved provided that φ is fundamental. In the present subsection, we remove this assumption.

Throughout the paper, let $m := |\det(M)|$. Given a symbol a, let

$$A_\nu(z) := \sum_{k\in\mathbb{Z}^d} a_{Mk+\rho_\nu^*} z^k, \tag{7}$$

denote its *ν-th subsymbol*, where $R_M^* = \{\rho_0^*, \dots, \rho_{m-1}^*\}$ with $\rho_0^* = 0$ is a complete set of representatives of $\mathbb{Z}^d/M\mathbb{Z}^d$. Hence, a can be decomposed into

$$a(z) = \sum_{\nu=0}^{m-1} A_\nu(z^M) z^{\rho_\nu^*}. \tag{8}$$

For symbols $a^{(\mu)}$ and $b^{(\mu)}$, $\mu = 0, \dots, n_1$, we denote their subsymbols by $A_\nu^{(\mu)}$ and $B_\nu^{(\mu)}$, $\nu = 0, \dots, m-1$, respectively. Moreover, we denote

$$\mathbf{A} := \left(A^{(0)}, A^{(1)}, \dots, A^{(n_1)}\right), \quad \mathbf{B} := \left(B^{(0)}, B^{(1)}, \dots, B^{(n_1)}\right)$$

with

$$A^{(\mu)} := (A_0^{(\mu)}, \dots, A_{m-1}^{(\mu)})^\top, \quad B^{(\mu)} := (B_0^{(\mu)}, \dots, B_{m-1}^{(\mu)})^\top. \tag{9}$$

It is well-known that (6) is equivalent to

$$\mathbf{A}(z)\mathbf{B}^\top(\tfrac{1}{z}) = \Theta_0(z)\mathcal{I}_m, \tag{10}$$

where

$$\Theta_0(z) := \sum_{\nu=0}^{m-1} A_\nu^{(0)}(z) B_\nu^{(0)}(\tfrac{1}{z}) = A^{(0)}(z)^\top B^{(0)}(\tfrac{1}{z}) = B^{(0)}(\tfrac{1}{z})^\top A^{(0)}(z) \tag{11}$$

is the 0-th subsymbol of θ, c.f. [8]. Hence, in order to solve (S1), we have to complete the matrices $\mathbf{A}$ and $\mathbf{B}$ by finding the Laurent polynomial vectors $A^{(1)}, \dots, A^{(n_1)}$ and $B^{(1)}, \dots, B^{(n_1)}$.

Let $\mathbb{C}[z^\pm]$ denote the ring of d-variate Laurent polynomials with coefficients in $\mathbb{C}$. Then $(\mathbb{C}[z^\pm])^m$ is a $\mathbb{C}[z^\pm]$-module and consists of $m \times 1$ column vectors of Laurent polynomials. For $P \in (\mathbb{C}[z^\pm])^m$, denote

$$\mathrm{syz}(P) := \{Q \in (\mathbb{C}[z^\pm])^m \ : \ P^\top Q = 0\}.$$

Then $\mathrm{syz}(P)$ is the (first) syzygy module containing all the *syzygies* of P.

Theorem 3.1. *Let $a^{(0)}$ and $b^{(0)}$ as in Theorem* 2.1. *Then* (S1) *is solvable with $n_1 = m-1$ if and only if all the columns (or rows) of the matrix $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$ are contained in some module with $m-1$ generators. In particular, if the syzygy module* $\mathrm{syz}(A^{(0)})$ *or* $\mathrm{syz}(B^{(0)})$ *is generated by $m-1$ elements, then* (S1) *is solvable with $n_1 = m-1$.*

In the proof of Theorem 3.1, we will make use of the following relations. Let $a^{(0)}$ and $b^{(0)}$ satisfy the sum rules of orders N_1 and N_2, respectively. If $B^{(\mu)}(\frac{1}{z}) \in \mathrm{syz}(A^{(0)})$, then $b^{(\mu)}$ has a zero of order N_1 at $\mathbf{1}$, and if $A^{(\mu)} \in \mathrm{syz}(B^{(0)}(\frac{1}{z}))$, then $a^{(\mu)}$ has a zero of order N_2 at $\mathbf{1}$, c.f. [2].

Proof. Suppose that (S1) is solvable with $n_1 = m-1$, that is, (10) holds with $n_1 = m-1$. Then (10) with $n_1 = m-1$ can be equivalently rewritten as

$$\Big(A^{(1)}(z),\dots,A^{(m-1)}(z)\Big)\Big(B^{(1)}(\tfrac{1}{z}),\dots,B^{(m-1)}(\tfrac{1}{z})\Big)^\top = \Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\tfrac{1}{z})^\top. \tag{12}$$

It follows from (12) that all the columns of $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$ are contained in the module generated by $A^{(1)},\dots,A^{(m-1)}$, and all the rows of $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$ are contained in the module generated by $B^{(1)}(\frac{1}{z})^\top,\dots,B^{(m-1)}(\frac{1}{z})^\top$.

Conversely, suppose that all the columns of $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$ are contained in some module generated by $m-1$ elements $A^{(1)},\dots,A^{(m-1)} \in (\mathbb{C}[z^\pm])^m$. This is equivalent to saying that there are $B^{(1)}(\frac{1}{z}),\dots,B^{(m-1)}(\frac{1}{z}) \in (\mathbb{C}[z^\pm])^m$ such that (12) holds, which is equivalent to (10) with $n_1 = m-1$. Since the matrices on the left-hand side of (10) with $n_1 = m-1$ are square, they commute. Consequently, we have $A^{(1)},\dots,A^{(m-1)} \in \operatorname{syz}(B^{(0)}(\frac{1}{z}))$ and $B^{(1)}(\frac{1}{z}),\dots,B^{(m-1)}(\frac{1}{z}) \in \operatorname{syz}(A^{(0)})$. According to the sum rules of $a^{(0)}$ and $b^{(0)}$, the associated wavelet symbols $a^{(1)},\dots,a^{(m-1)}$ and $b^{(1)},\dots,b^{(m-1)}$ must have a zero at $\mathbf{1}$. Therefore, (S1) is solvable with $n_1 = m-1$. The similar argument applies to the rows of the matrix $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$.

In summary, (S1) is solvable with $n_1 = m-1$ if and only if all the columns (or rows) of the matrix $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$ are contained in some module with $m-1$ generators.

By (11), it is straightforward to see that all the columns of $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$ are contained in the module $\operatorname{syz}(B^{(0)}(\frac{1}{z}))$ and all the rows of $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$ are contained in the module $\{v^\top : v \in \operatorname{syz}(A^{(0)})\}$. Note that $\operatorname{syz}(B^{(0)}(\frac{1}{z}))$ can be generated by $m-1$ elements if and only if $\operatorname{syz}(B^{(0)})$ can be generated by $m-1$ elements. Hence, if the syzygy module $\operatorname{syz}(A^{(0)})$ or $\operatorname{syz}(B^{(0)})$ is generated by $m-1$ elements, then (S1) is solvable with $n_1 = m-1$. $\square$

Let us explain that the desirable module with $m-1$ generators in Theorem 3.1 must be a free submodule (with $m-1$ generators) of either $\operatorname{syz}(A^{(0)})$ or $\operatorname{syz}(B^{(0)}(\frac{1}{z}))$. Suppose that the columns of $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$ are contained in a module generated by $A^{(1)},\dots,A^{(m-1)}$. Then as in the proof of Theorem 3.1, there exist some $B^{(1)},\dots,B^{(m-1)} \in (\mathbb{C}[z^\pm])^m$ such that $\mathbf{B}^\top(\frac{1}{z})\mathbf{A}(z) = \Theta_0(z)\mathcal{I}_m$. In order to show that all $A^{(0)},A^{(1)},\dots,A^{(m-1)}$ must be linearly independent in $\mathbb{C}[z^\pm]$, suppose that there is $C(z) \in (\mathbb{C}[z^\pm])^m$ satisfying $\mathbf{A}(z)C(z) = 0$. By $\mathbf{B}^\top(\frac{1}{z})\mathbf{A}(z) = \Theta_0(z)\mathcal{I}_m$, we must have

$$\Theta_0(z)C(z) = \mathbf{B}^\top(\tfrac{1}{z})\mathbf{A}(z)C(z) = 0.$$

But $\Theta_0 \not\equiv 0$ and $\Theta_0(z)C(z) = 0$ will force $C(z) = 0$. This shows that all $A^{(0)},\dots,A^{(m-1)}$ are linearly independent. In particular, $A^{(1)},\dots,A^{(m-1)}$ are linearly independent and therefore, the submodule generated by $A^{(1)},\dots,A^{(m-1)}$ must be a free module with $m-1$ generators. The similar argument applies to the rows of the matrix $\Theta_0(z)\mathcal{I}_m - A^{(0)}(z)B^{(0)}(\frac{1}{z})^\top$.

Remark 3.2. There are several algorithms available in the literature to compute a minimal set of generators for a syzygy module. The cardinalities of any two (minimal) sets of generators for a given submodule generally may differ, but if the submodule is free, then any basis of the submodule (a set of generators that are linearly independent) will have the same cardinality. The freeness of the desirable submodule in Theorem 3.1 facilitates the numerical verification and construction for (S1).

In the following, we discuss several cases, in which the assumptions of Theorem 3.1 are satisfied.

Since the ring $\mathbb{C}[z^\pm]$ is a unique factorization domain, we can find the greatest common divisor (gcd) C of the Laurent polynomial entries of $A^{(0)}$, and let $\widetilde{C}$ denote the gcd of the entries of $B^{(0)}$. For

$$\widetilde{A}^{(0)} := A^{(0)}/C, \quad \text{and} \quad \widetilde{B}^{(0)} := B^{(0)}/\widetilde{C},$$

we still have $\operatorname{syz}(A^{(0)}) = \operatorname{syz}(\widetilde{A}^{(0)})$ and $\operatorname{syz}(B^{(0)}) = \operatorname{syz}(\widetilde{B}^{(0)})$. If all the entries in $\widetilde{A}^{(0)}$ do not have a common zero in $(\mathbb{C}\backslash\{0\})^d$ (note that this is always true for $d = 1$), then it is guaranteed by the

Quillen-Suslin Theorem that $\operatorname{syz}(\widetilde{A}^{(0)})$ is a free module with $m-1$ generators. Therefore, $\operatorname{syz}(A^{(0)})$ has $m-1$ generators, and (S1) is solvable with $n_1 = m-1$ by applying Theorem 3.1. The same argument can be applied to $\widetilde{B}^{(0)}$.

For $d=2$, by observing that the results in [16, Proposition 7] about polynomial syzygies still hold with respect to Laurent polynomial syzygies, the modules $\operatorname{syz}(\widetilde{A}^{(0)})$ and $\operatorname{syz}(\widetilde{B}^{(0)})$ are always generated by $m-1$ elements. So, for $d=2$, (S1) is always solvable with $n_1 = m-1$ by applying Theorem 3.1.

For the case $m=2$ and arbitrary d, no matter whether the entries of $\widetilde{A}^{(0)}$ have a common zero in $(\mathbb{C}\backslash\{0\})^d$ or not, we claim that $\operatorname{syz}(\widetilde{A}^{(0)})$ can be always generated by $(\widetilde{A}_1^{(0)}(z), -\widetilde{A}_0^{(0)}(z))^\top$. In fact, if $(P_0, P_1)^\top \in \operatorname{syz}(\widetilde{A}^{(0)})$, then

$$P_0(z)\widetilde{A}_0^{(0)}(z) = -P_1(z)\widetilde{A}_1^{(0)}(z).$$

Since $\gcd(\widetilde{A}_0^{(0)}, \widetilde{A}_1^{(0)}) \in \mathbb{C}\backslash\{0\}$ and $\mathbb{C}[z^{\pm}]$ is a unique factorization domain, we must have $P_0(z) = \widetilde{A}_1^{(0)}(z)C(z)$ for some $C(z) \in \mathbb{C}[z^{\pm}]$. Therefore, it follows from the above identity that $P_1(z) = -\widetilde{A}_1^{(0)}(z)C(z)$. That is, we indeed have $(P_0(z), P_1(z))^\top = C(z)(\widetilde{A}_1^{(0)}(z), -\widetilde{A}_0^{(0)}(z))^\top$. More explicitly, when $m=2$, (S1) is solvable with $n_1 = 1$ and is given by

$$a^{(1)}(z) = z^{\rho^*} b^{(0)}(\tfrac{1}{z_\rho}), \qquad b^{(1)}(z) = z^{\rho^*} a^{(0)}(\tfrac{1}{z_\rho}), \tag{13}$$

where $\rho \in M^{-\top}\mathbb{Z}^d \setminus \mathbb{Z}^d$ and $\rho^* \in \mathbb{Z}^d \setminus M\mathbb{Z}^d$, also see [8] for this choice.

We finally consider the special case $b^{(0)} = a^{(0)}$ with respect to dyadic dilations, see [17] for the following result:

Lemma 3.3. *Denote $M = 2\mathcal{I}_d$. For a given symbol $a^{(0)}$ which is real-valued on $|z|=1$, we set $b^{(0)} = a^{(0)}$. Suppose that there exists a function $\vartheta : \{0,1\}^d \to \{0,1\}^d$ with $\vartheta(0) = 0$ such that, for all $\rho^*, \tilde{\rho}^* \in \{0,1\}^d$, $\rho^* \neq \tilde{\rho}^*$, the usual inner product on $\mathbb{R}^d$*

$$\big(\vartheta(\rho^*) + \vartheta(\tilde{\rho}^*)\big) \cdot \big(\rho^* + \tilde{\rho}^*\big) \tag{14}$$

is odd. Let $\{0,1\}^d = \big\{0, \rho_1^*, \ldots, \rho_{2^d-1}^*\big\}$ and $\rho_\mu := \frac{1}{2}\rho_\mu^*$, then the wavelet symbols

$$a^{(\mu)}(z) := b^{(\mu)}(z) := z^{\vartheta(\rho_\mu^*)} a^{(0)}(z_{\rho_\mu}), \qquad \textit{for } \mu = 1, \ldots, 2^d - 1$$

solve (S1) *with $n_1 = 2^d - 1$.*

The function ϑ can be explicitly determined for $d = 1,2,3$, and it does not exist for $d \geq 4$, see [17]. Summarizing the above results, we have

Corollary 3.4. *If one of the following conditions is satisfied,*

(i) $d = 1, 2$,

(ii) $m = 2$,

(iii) *the subsymbols of $a^{(0)}$ (or $b^{(0)}$) divided by their* gcd *have no common zero in* $(\mathbb{C}\backslash\{0\})^d$,

(iv) $d = 3$ *with $M = 2\mathcal{I}_3$ and $a^{(0)} = b^{(0)}$ which is real-valued on $|z| = 1$,*

then (S1) *is solvable with $n_1 = m-1$.*

3.2 Step 2 for (S2)

Since $a^{(0)}$ and $b^{(0)}$ satisfy the sum rules of order 1 and $a^{(0)}(\mathbf{1}) = b^{(0)}(\mathbf{1}) = 1$, the Laurent polynomial η has a zero at $\mathbf{1}$. In order to solve (S2), we require that this zero should be of order at least 2. Note that this extra assumption on the double zero of η at $\mathbf{1}$ is also a necessary condition for us to construct a wavelet bi-frame using θ, defined in (5), in the oblique extension principle. In the following, we explore the algebraic structure of η in more detail.

Given a ring $\mathcal{R}$ and elements $q_1, \dots, q_n \in \mathcal{R}$, we denote $\langle q_1, \dots, q_n \rangle_{\mathcal{R}}$ the ideal generated by $q_1, \dots, q_n$ in $\mathcal{R}$. Let $\mathbb{C}[z]$ denote the ring of d-variate polynomials with coefficients in $\mathbb{C}$. As a standard result in algebra, $f \in \mathbb{C}[z]$ has a zero of order N at 0 if and only if

$$f \in \left\langle z^\alpha : \alpha \in \mathbb{N}_0^d, |\alpha| = N \right\rangle_{\mathbb{C}[z]},$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. By shifting and multiplying with a monomial of a sufficiently high degree, this implies that η has a zero of order N at $\mathbf{1}$ iff

$$\eta \in \left\langle (z - \mathbf{1})^\alpha : \alpha \in \mathbb{N}_0^d, |\alpha| = N \right\rangle_{\mathbb{C}[z^\pm]}. \tag{15}$$

Now we establish the main result of this subsection on (S2):

Theorem 3.5. *Under the assumptions of Theorem* 2.1 *, if* $1 - a^{(0)}(z) b^{(0)}(\frac{1}{z})$ *has a zero of order at least* 2 *at* $z = \mathbf{1}$*, then* (S2) *is solvable with* $n_2 = d$.

Proof. Note that both $a^{(0)}$ and $b^{(0)}$ satisfy the sum rules of order at least one. By $\eta(z) = 1 - a^{(0)}(z) b^{(0)}(\frac{1}{z}) - \sum_{\rho \in R_M \setminus \{0\}} a^{(0)}(z_\rho) b^{(0)}(\frac{1}{z_\rho})$, one can verify that η has a zero of order 2 at $z = \mathbf{1}$. According to (15), there are Laurent polynomials $p^{(\nu,i)}$ such that

$$\eta(z) = \sum_{\nu=1}^{d} \sum_{i=\nu}^{d} (z_\nu - 1)(z_i - 1) p^{(\nu,i)}(z). \tag{16}$$

Hence, the choice

$$\eta^{(\nu)}(\tfrac{1}{z}) := (z_\nu - 1), \quad \widetilde{\eta}^{(\nu)}(z) := \sum_{i=\nu}^{d} (z_i - 1) p^{(\nu,i)}(z)$$

provides a solution to (S2) with $n_2 = d$. □

Since $a^{(0)}(\mathbf{1}) = 1$ and $a^{(0)}(z) a^{(0)}(\frac{1}{z})$ is real-valued on $|z| = 1$, the term $1 - a^{(0)}(z) a^{(0)}(\frac{1}{z})$ has a double zero at $\mathbf{1}$. Thus, the assumptions of Theorem 3.5 are satisfied for the canonical choice $b^{(0)} = a^{(0)}$.

3.3 Vanishing Moments

In order to provide a high approximation order, wavelets must have a high number of vanishing moments, i.e., their symbols have a zero of high order at $\mathbf{1}$. In case of orthogonal and biorthogonal wavelet bases, the sum rule orders of the symbols of the underlying refinable functions completely determine the number of vanishing moments of the primal and dual wavelets. But for frames, this implication no longer holds in its generality, c.f. [2, 3, 4, 6, 7, 10, 11, 12, 18, 19, 20].

In the following, we explain that our approach still allows for transforming sum rule orders into vanishing moments. Let symbols $a^{(0)}$ and $b^{(0)}$ as in Theorem 2.1 satisfy the sum rules of orders N_1 and N_2, respectively. With $n_1 = m - 1$ in (S1), according to the results in [2], all the primal wavelet symbols $a^{(1)}, \dots, a^{(m-1)}$ have a zero of order N_2 at $\mathbf{1}$. Similarly, all the dual wavelet symbols $b^{(1)}, \dots, b^{(m-1)}$ have a zero of order N_1 at $\mathbf{1}$.

The orders of zeros at $\mathbf{1}$ of the other wavelet symbols $a^{(m-1+\nu)}$ and $b^{(m-1+\nu)}$ in (S2), $\nu = 1, \dots, n_2$, are closely related to the order of zero at $\mathbf{1}$ of η (c.f. [4, 6, 7, 12]). Assume that η, defined in (5), has a zero of order N_3 at $\mathbf{1}$. Let N_4 and N_5 be any positive integers such that $N_4 + N_5 = N_3$. By (15) and a splitting similar to the ideas in the proof of Theorem 3.5, we can easily obtain the wavelet symbols $a^{(m-1+\nu)}$ and $b^{(m-1+\nu)}$, $\nu = 1, \dots, n_2$ with $n_2 \leq \mathrm{card}(\{|\alpha| = N_3\}) = \binom{d-1+N_3}{N_3}$, such that all the primal wavelet symbols $a^{(m-1+\nu)}$, $\nu = 1, \dots, n_2$, have a zero of order N_4 at $\mathbf{1}$ and all the dual wavelet symbols $b^{(m-1+\nu)}$, $\nu = 1, \dots, n_2$, have a zero of order N_5 at $\mathbf{1}$. Therefore, all the primal wavelets have $\min(N_2, N_4)$ vanishing moments and all the dual wavelets have $\min(N_1, N_5)$ vanishing moments. Note that the number n_2 could be reduced by appropriately regrouping the factors and symbols. The following special case is derived by factoring out the terms $(z_1 - 1)^N, \dots, (z_d - 1)^N$:

Lemma 3.6. *If η has a zero of order* $\max(dN, 2N)$ *at* $\mathbf{1}$*, then* (S2) *is solvable with* $n_2 = d$ *such that all the primal and dual wavelet symbols in* (S2) *have a zero of order* N *at* $\mathbf{1}$*.*

4 Examples

Given a collection of nonzero *direction vectors* Ξ in $\mathbb{R}^d$, which span $\mathbb{R}^d$ and allow for multiples of the same vector, the *box spline* φ_Ξ is defined by its Fourier transform

$$\widehat{\varphi_\Xi}(\xi) := \prod_{v\in\Xi} \frac{1-e^{-2\pi i\xi\cdot v}}{2\pi i\xi\cdot v}, \qquad \xi \in \mathbb{R}^d.$$

It turns out that φ_Ξ is refinable with respect to the dyadic dilation $M = 2\mathcal{I}_d$ and the symbol

$$a_\Xi(z) := \prod_{v\in\Xi} \frac{1+z^v}{2},$$

c.f. [1]. In the following, we focus on the bivariate setting with direction vectors

$$\begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\-1\end{pmatrix} \tag{17}$$

of multiplicities ℓ, p, q, and r, respectively, with $\ell, p > 0$. We apply the short-hand notation $\varphi_{(\ell,p,q,r)}$ for this 4-directional box spline, which is refinable with respect to

$$a_{(\ell,p,q,r)}(z) := \Big(\frac{1+z_1}{2}\Big)^\ell \Big(\frac{1+z_2}{2}\Big)^p \Big(\frac{1+z_1z_2}{2}\Big)^q \Big(\frac{1+z_1/z_2}{2}\Big)^r.$$

For $r = 0$, we have a 3-directional box spline, and we write $\varphi_{(\ell,p,q)}$ as well as $a_{(\ell,p,q)}$.

The construction of compactly supported wavelet bi-frames such that both the primal and dual wavelets are derived from one box spline function has been addressed in [3, 13, 15]. For 3-directional box splines, they need 26 generators in [3], 13 generators in [15], and 11 generators in [13]. The number of generators is even higher for 4-directional box splines. For specific choices of ℓ, p, q, r, they can reduce this number, but each of the above mentioned approaches requires at least 6 or even more generators. By applying our approach, we obtain wavelet bi-frames with at most 5 generators for all 3- and 4-directional box splines, see Table 1. Note that we can even reduce this number for specific 4-directional box splines as we shall explain next.

The *box spline dilation matrix* and the *quincunx dilation matrix* are given by

$$M_b = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad M_q = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},$$

respectively. It turns out that $\varphi_{(\ell,p,\ell,p)}$ is not only refinable with respect to the dyadic dilation, but also with respect to M_b. In [20], wavelet bi-frames with $\ell + p$ generators have been derived from $\varphi_{(\ell,p,\ell,p)}$ with respect to the box spline matrix M_b. The number of generators in [14] is independent of the choice of ℓ and p, but the approach still requires 6 generators.

Both matrices M_b and M_q satisfy $m = 2$. Hence, provided that $\ell = q$ and $p = r$, we can solve (S1) with $n_1 = 1$ as in (13). Finally, we derive wavelet bi-frames with only 3 generators. Except for some pathological cases, this number is significantly smaller than the number of generators in [3, 13, 14, 15, 20], see Table 1 for more details.

Example 4.1. Let $a^{(0)} = b^{(0)} = a_{(2,2,2)}$ be the symbol of the 3-directional box spline with equal multiplicity 2. Applying Lemma 3.3 to the symbol $a^{(0)}(z) \cdot \big(\frac{1}{z_1z_2}\big)^2$, we obtain a solution of (S1) with $n_1 = 3$ wavelet symbols. Note that both (ii) and (iv) of Corollary 3.4 are satisfied. In order to solve

number of generators						box splines
in our approach	in [3]	in [13]	in [14]	in [15]	in [20]	(ℓ, p, q, r)
5	7	9	6	6	–	$(1,1,1)$
	≤ 17	9	–	6	–	$(2,2,1)$
	≤ 26	11	–	7	–	$(2,2,2)$
	≤ 26	11	–	≤ 13	–	(ℓ, p, q)
	≤ 35	12	–	8	–	$(2,2,1,1)$
	≤ 80	19	–	≤ 26	–	(ℓ, p, q, r)
3	15	12	6	6	2	$(1,1,1,1)$
	≤ 80	19	6	≤ 26	$\ell + p$	(ℓ, p, ℓ, p)

Table 1: The number of generators of bi-frames derived from 3- and 4-directional box splines.

(S2), we apply the computer algebra software Singular. A computation yields that there are Laurent polynomials f_1 and f_2 such that η in (S2) can be expanded by

$$\eta(z) = (z_1 - 1)^2 f_1(z) + (z_1 - 1)(z_2 - 1) f_1(z) + (z_2 - 1)^2 f_2(z).$$

Hence, we can choose

$$\eta^{(1)}(\tfrac{1}{z}) := (z_1 - 1) + (z_2 - 1), \qquad \widetilde{\eta}^{(1)}(z) := (z_1 - 1) f_1(z),$$
$$\eta^{(2)}(\tfrac{1}{z}) := (z_2 - 1), \qquad \widetilde{\eta}^{(2)}(z) := (z_2 - 1) f_2(z),$$

where the Laurent polynomials f_1 and f_2 are given by

$$\begin{aligned}
f_1(z) = \tfrac{z_1^{-4} z_2^{-4}}{128} \Big(& - 6z_1^6 z_2 + 5z_1^6 - 11z_1^5 z_2 + 9z_1^5 - 155z_1^4 z_2 + 124z_1^4 - 270z_1^3 z_2 + 210z_1^3 - 124z_1^2 z_2 \\
& + 93z_1^2 - 7z_1 z_2 + 5z_1 - 3z_2 + 2 \Big), \\
f_2(z) = \tfrac{z_1^{-4} z_2^{-4}}{4096} \Big(& - 3z_1^8 z_2^6 - 10z_1^8 z_2^5 + 4z_1^7 z_2^6 - 35z_1^8 z_2^4 - 12z_1^7 z_2^5 - 18z_1^6 z_2^6 - 64z_1^8 z_2^3 + 12z_1^7 z_2^4 - 76z_1^6 z_2^5 \\
& + 4z_1^5 z_2^6 - 96z_1^8 z_2^2 - 4z_1^7 z_2^3 - 452z_1^6 z_2^4 - 32z_1^5 z_2^5 - 3z_1^4 z_2^6 - 128z_1^8 z_2 - 972z_1^6 z_2^3 \\
& + 76z_1^5 z_2^4 - 26z_1^4 z_2^5 - 160z_1^8 - 1810z_1^6 z_2^2 - 76z_1^5 z_2^3 - 367z_1^4 z_2^4 - 4z_1^3 z_2^5 + 192z_1^7 - 2688z_1^6 z_2 \\
& + 32z_1^5 z_2^2 - 968z_1^4 z_2^3 + 32z_1^3 z_2^4 - 3424z_1^6 - 4z_1^5 z_2 + 1489z_1^4 z_2^2 - 76z_1^3 z_2^3 - 18z_1^2 z_2^4 + 4608z_1^5 \\
& + 3686z_1^4 z_2 + 76z_1^3 z_2^2 - 76z_1^2 z_2^3 + 9245z_1^4 - 32z_1^3 z_2 - 452z_1^2 z_2^2 - 4z_1 z_2^3 - 4668z_1^3 - 972z_1^2 z_2 \\
& + 12z_1 z_2^2 - 5554z_1^2 - 12z_1 z_2 - 3z_2^2 - 124z_1 - 10z_2 - 131 \Big).
\end{aligned}$$

Therefore, we obtain a wavelet bi-frame with 5 generators.

The following example deals with trivariate box splines:

Example 4.2. Let a_Ξ be the trivariate symbol associated to the direction vectors

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\0\\1\end{pmatrix} \tag{18}$$

with certain multiplicities such that each of the first three vectors appears in Ξ at least once. In [13], a bi-frame with 56 generators has been derived from this box spline. We can significantly reduce the number of generators as we shall explain next.

Since the matrix formed by the vectors in (18) is unimodular, the subsymbols of a_Ξ have no common zero in $(\mathbb{C} \setminus \{0\})^3$, c.f. [1, 5]. Thus, by Corollary 3.4, (S1) is solvable with $n_1 = 7$. Finally, (S2) can be easily solved by Singular for $b^{(0)} = a^{(0)} = a_\Xi$ and we obtain a trivariate bi-frame with 10 generators.

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