Systemic risk in banking networks: Advantages of “tiered” banking systems

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Abstract

This paper studies the risk and potential impact of system-wide defaults in a tiered banking network, where a small group of head institutions has many credit linkages with other banks, while the majority of banks have only a few links. A network is random and displays a given distribution of the number of banks’ linkages, known as degree. We model tiering by a negative correlation between degrees of neighboring banks and by a scale-free degree distribution. The main findings of the paper highlight the advantages of tiering. Both the risk of systemic crisis and the potential scope of the crisis are lower in systems with negative correlation of bank degrees than in other types of systems. Similarly, in scale-free networks, the resilience of the system to shocks is increasing with the level of tiering.

1. Introduction

One of the major concerns in recent policy debates over financial stability is how “tiering” in the banking system may affect systemic risk, the large-scale breakdown of financial intermediation due to the domino effect of insolvency.¹ The tiered banking system is commonly defined as an organization of lending–borrowing relations/linkages between banks, where relatively few first-tier or “head” institutions have a large number of interbank linkages, whereas many second-tier or “peripheral” banks have only a few links. First-tier banks are connected to second-tier banks and are also connected with each other, whereas second-tier banks are almost exclusively connected to first-tier banks.² Interbank linkages may act as a device for co-insurance against uncertain liquidity shocks (Bhattacharya and Gale, 1987) and improve market discipline by providing incentives for peer-monitoring (Flannery, 1996; Rochet and Tirole, 1996) but they can also serve as a channel through which problems spread from one bank to another.³

¹ The domino effect of insolvency occurs when the non-repayment of interbank obligations by the failing bank increases the probability that its creditor banks fail to meet their obligations to interbank creditors, and so on.
² For a discussion of the properties of tiered banking systems see Fricke and Lux (2014).
³ For an extensive survey of the literature on financial contagion see Allen and Babus (2009).
Tiered banking systems are found in a range of countries but the empirical evidence of contagion risk in these systems is mixed. In its 2003 Financial System Stability assessment of the United Kingdom, the International Monetary Fund (IMF) highlights the potential contagion risk arising from the highly tiered structure of the UK large-value payment systems. However, several subsequent studies including Wells (2004), Harrison et al. (2005), Lasaosa and Tudela (2008) report relatively limited scope for contagion among UK banks. Boss et al. (2004c) and Degryse and Nguyen (2007) find that tiered banking systems in Austria and Belgium are stable and systemic crises are unlikely. By contrast, Cont et al. (2012) suggests that in the tiered banking system of Brazil, “the risk of default contagion is significant” (p. 5) and the losses resulting from contagion can be large. In addition, somewhat differently from other studies, Elsinger et al. (2006), Mistrulli (2007) and Upper and Worms (2004) find that while contagious failures in tiered Austrian, Italian and German banking systems are relatively rare, large parts of the system are affected in the worst-case scenario.

The controversy in the empirical literature leaves the question of benefits and risks of tiering open for further investigation. In this paper, we aim to shed more light on this issue by proposing what seems to be the first analytical investigation of the effects of tiering and the degree of tiering in the banking system. We develop a stylized theoretical model in which banks are connected by credit linkages and have heterogeneous balance sheets that are determined by the structure of these linkages. Balance sheet interdependence creates the precondition for contagious spread of defaults and can lead to systemic crisis, when large parts of the system are affected. In this setting we study the effects of tiering on the banking system’s susceptibility to systemic crisis and the scope of the crisis. We abstract from the issue of the incentives that drive the formation of tiered banking systems and study the impact of tiering within exogenously given structures.

The banking system is modeled as a network where nodes represent banks and links are interbank exposures. The network is random in the sense that the number of links that a bank has, known as its degree, is determined stochastically, according to a certain degree distribution. Following Newman (2003) and Vega-Redondo (2007), we use the term “random” in a broad sense, assuming that the degree distribution can be of any kind rather than just a binomial or Poisson distribution, as in the Erdős–Renyi random graph. Links in the network are directed: incoming links of a bank reflect its interbank assets/exposures, while outgoing links correspond to its interbank liabilities. An important feature of the links structure is that the number of incoming and outgoing links of each bank is defined by a single random variable and therefore, it is the same for any realization of the random network. As a result, connectivities of banks in the network are determined by a single degree distribution.

To examine the impact of tiering on the stability of the banking system we study the implications of both negative correlation between degrees of the neighboring banks, and of a scale-free distribution of degrees. These two different approaches to modeling tiering are used since there is evidence of both in the empirical literature. For example, the negative degree correlation is found to be a prominent feature of the US banking system (Soramäki et al., 2007; Bech and Atalay, 2010), and the scale-free degree distribution is a common characteristic of banking networks in many other countries (see Boss et al., 2004a, for Austria, Cont et al., 2012, for Brazil, Inaoka et al., 2004, and Souma et al., 2003, for Japan and De Masi et al., 2006, for Italy).

Using the first approach, we model a tiered banking system by a so-called disassortative network, that displays negative correlation between degrees of the neighboring banks, and compare it with other types of structures, showing either a positive degree correlation (an assortative network) or no correlation (a neutral network). In the second approach, we focus on the effects of a variation in the level of tiering of the scale-free network, where the level of tiering is represented by the inverse of the exponent parameter. In both approaches we analyze (i) the resilience of the networks to systemic failure, (ii) the scale of the failure if systemic crisis occurs, and (iii) the extent of necessary bail-outs that would guarantee the global financial stability of the system at minimum costs. The essential findings of both approaches agree. They suggest that as soon as highly connected banks are sufficiently well capitalized, – the condition that is commonly met due to financial regulation and is supported by some of the literature discussed below, – tiering in the banking network improves its financial stability: both the risk of systemic crisis and the scope of the crisis should it occur are lower in a (more) tiered system. Higher tiering also reduces the extent of necessary government intervention as fewer bail-outs are needed to avoid a large-scale breakdown.

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4 A recent summary of empirical studies on the danger of default contagion and the assessment of the simulation methods used in these studies is provided by Upper (2011).
5 An extensive list of related theoretical and empirical literature is discussed in the next section.
6 This is clearly a broad simplification. Some justification of this assumption is provided later in text and a robustness check of the model in the framework where the number of incoming and outgoing links can be different is offered in the Appendix.
7 In Section 3, it is argued that even with negative degree correlation between neighboring nodes, first-tier highly connected banks are still likely to be connected with each other, while the probability of connectedness between second-tier low-degree banks is very small.
8 The exponent parameter of the scale-free degree distribution governs the rate at which the probability that a node has degree k decays with k. Therefore, for smaller values of this parameter, the fraction of highly connected banks in the network is larger and so is the probability that poorly connected banks are linked with highly connected banks rather than with each other.
9 The events triggered by the sub-prime crisis of August 2007 highlighted the importance of these questions. For example, the rescue of some institutions, such as American International Group (AIG), remains a highly disputed issue. As a rule, the main argument of policymakers in favor of these rescues is that many yet unaffected banks (across the national or international financial system) might be exposed to the defaulting institutions. But in fact, no rigorous assessment exists of how far contagion could have spread had AIG been allowed to fail.
The intuition for these results is straightforward. In a disassortative banking network, high-degree banks are broadly distributed over the network and therefore, form links on many paths between other banks. As a consequence, an initial shock hitting a random bank is likely to reach a high-degree bank in a small number of "steps" and gets absorbed by that bank due to its ample capitalization. This implies that disassortative networks are more resilient to systemic crises than assortative and neutral networks. Similarly, scale-free networks with a higher level of tiering have a larger proportion of well-connected banks, so that links emanating from and arriving at these banks span a larger part of the banking system. This implies that initial shocks at any part of the system reach high-degree banks “quickly” and subside at these banks without spreading further. For the same reason, the maximum expected number of required bail-outs is smaller in disassortative networks and in scale-free networks with a higher level of tiering.

These positive effects of tiering on financial stability of the banking network rely on the condition of adequate capitalization of highly connected banks. More specifically, in the model (i) the capital of a bank is increasing in its interbank asset position and (ii) the interbank asset position is proportional to the number of bank’s counterparties (incoming links). The two assumptions imply that capital is increasing in the number of bank’s incoming links.

Admittedly, this is a stylized approach. However, not only does it allow the derivation of results analytically but it also finds support in theoretical and empirical literature. In particular, modeling capital stock of a bank as an increasing function of its total interbank assets is consistent with current bank capital requirements. These requirements specify the minimum amount of bank’s equity that must be held as a percentage of assets. Moreover, while in practice banks are often willing to hold capital above the regulatory minimum, so that the capital requirements are not binding (Flannery and Rangan, 2008; Gropp and Heider, 2010), a range of studies argue that banks’ incentives for capitalization increase in interbank position. For example, an extensive literature on the relation between risk exposure and bank’s capital finds that a larger asset risk exposure induces banks’ capital buildup (Flannery and Rangan, 2008; Shrieves and Dahl, 1992; Gropp and Heider, 2010). And vice versa, high levels of capital serve as a signal of a bank’s financial strength and translate into easier access to interbank market, not only on the liability but also on the asset side (see, for example, Admati et al., 2011; Allen et al., 2011). To solve the model, we employ the probability generating function techniques from the literature on complex networks (Strogatz, 2001; Newman et al., 2001; Vega-Redondo, 2007). We use these techniques to model contagion stemming from an unexpected shock to a single institution in a complex banking network. The banking system and transmission of shocks are modeled similarly to those in Gai and Kapadia (2010). However, the main research question and method used in our paper are very different. First, we focus on the effects of tiering in the banking system that is captured by a negative correlation between degrees of the neighboring banks or by a highly skewed degree distribution. In contrast, Gai and Kapadia (2010) consider a Poisson network, where degrees of all banks are statistically independent and most banks have exactly the average number of connections. We show that in the assessment of the effects of tiering on the network stability, disregarding degree correlations and wide heterogeneity in banks’ degrees may substantially change the predictions. Second, banks in the framework of Gai and Kapadia (2010) hold the same amount of capital and the same amount of interbank assets, independent of the number of incoming links. Instead, in our model, balance sheets of banks are heterogeneous, with higher interbank asset positions and capital buffers at more connected banks. Third, Gai and Kapadia (2010) evaluate the likelihood and extent of contagion by using numerical simulations, whereas all results in our paper are analytical, found either in closed form or numerically.

A natural critique of the approach in this paper and in many other related studies, including Gai and Kapadia (2010), is that we assume that interbank connections are formed randomly and exogenously and are static in nature. This leads to modeling contagion in a relatively mechanical manner, where financial institutions are passive and keep their balance sheets and the structure of interbank linkages fixed as defaults spread through the system. Arguably, though, in crises contagion propagates rapidly through the system and banks have little time to change their behavior before they are affected. Moreover, even if a bank threatened with bankruptcy as a result of a borrower’s default would like to borrow more itself in order to cover the deficit and thus avoid failing, in practice this is rarely possible. Recent turmoil in the financial

10 Strong resilience to shocks of a highly tiered scale-free system is confirmed by a simulation analysis performed for the Austrian banking network (Boss et al., 2004a). In accordance with Boss et al. (2004a, 2004c), we find that given the estimated exponent parameter of the Austrian banking network, large-scale defaults are very unlikely.
11 This increase is either proportional or more or less than proportional.
12 In fact, the traditional view is that, given the high costs of holding capital, capital requirements are binding for banks. This suggests the exact proportional relationship between bank assets and capital (see, for example, Mishkin, 2000). The exact proportionality is, however, not confirmed by recent empirical evidence.
13 The most widely adopted regulations around capital requirements have been the Basel Accords, published by the Basel Committee on Banking Supervision.
14 Furthermore, a positive relation between a bank’s capital and interbank assets is also suggested by other literature. For example, exact proportionality is implied by Gai and Kapadia (2010). They document that capital of a typical large bank constitutes 20% of its total interbank assets, as calibrated from 2005 published accounts of large financial institutions. Similarly, the simulation analysis in Krause and Giansante (2012) assumes that banks’ capital and interbank assets are positively related.
15 Recall that the Poisson distribution is strongly peaked about the mean.
16 The closed form solutions are not feasible in the part of the analysis that addresses the case with internode degree correlations. Yet, the provided numerical results are a solution of the exact system of equations and are found by a numerical iteration from a suitable set of starting values of a variable in question.
market has demonstrated that as soon as defaults show signs of contagion, credit dries up. Furthermore, since banks have no choice over whether to default, strategic behavior on networks of the type assumed in Morris (2000), Jackson and Yariv (2000) is ruled out in the present framework.

The remainder of the paper is organized as follows. Section 2 provides an overview of the related literature on financial contagion together with an outline of the methodologically similar works within the complex network literature. Section 3 introduces the model and describes the generating function approach to measuring the extent of contagion. The main results, offering the comparison of systemic risk in tiered and other banking systems, are presented in Section 4. Finally, Section 5 concludes with a discussion.

2. Related literature

In this section we first provide an overview of the related literature on financial contagion, both theoretical and empirical, and then discuss some contributions in the complex network literature that are most closely related to this paper in terms of methodology.

2.1. Literature on financial contagion

We begin by discussing the theoretical and simulation-based network models of financial contagion, leaving the review of the empirical literature to the end. The large theoretical network literature can be broadly categorized into two groups. The first group of papers studies contagion in the framework, where the financial system is represented by a random network and interactions between banks are determined stochastically. It models contagion as resulting from an initial solvency shock to one or few financial institutions and propagating through the network in a cascade manner. Most of this literature considers the network as exogenous, abstracting from the process of network formation. The present paper falls in this group. The second group of papers evaluates contagion spread in a non-random network setting. It considers the network of interlinked balance sheets as either exogenous or endogenous and examines the impact of initial defaults as predetermined by network externalities. Below we discuss several representative papers within each of the two groups, primarily focusing on their relevance in addressing the specific question of this paper.

The existing literature on financial contagion in a random network setup emphasizes the importance of the credit network topology in evaluating systemic risk. It contributes to our understanding of financial crises by employing models with various degrees of microeconomic detail and a variety of assumptions on properties of banks and contagion mechanism. However, a common feature shared by a great majority of these studies is a focus on one specific factor of credit network topology, network connectivity, i.e., the density of interbank connections. To examine the implications of network connectivity on systemic risk, a large part of the literature resorts to the assumption that any two banks in the network are linked with probability p, independent and identical across pairs of banks (Gai and Kapadia, 2010; Nier et al., 2007; Iori et al., 2006). Naturally, such an assumption rules out the mere possibility of wide disparity in the number of connections of individual banks. May and Arinaminpathy (2010) and Battiston et al. (2012a, 2012b) build on Gai and Kapadia (2010) and Nier et al. (2007) but apply a mean-field approach. In this approach, the approximation for the network is used where banks are essentially identical and have exactly average behavior. In particular, every bank is assumed to be linked to exactly the same, average number of other banks. Similarly, Tedeschi et al. (2012), who study the correlation between bankruptcy cascades and the endogenous business cycle, assume that the banking network is regular, that is, all banks have the same number of links. As a consequence, the resulting network structure in all these papers is “homogenous”, with no possibility of tiering.

Amini et al. (2013) address a wider range of degree distributions by using a configuration model, that generates a network with a given sequence of bank degrees. The drawback of this approach, however, is that the generated network may have loops and multiple links between any pair of nodes. This has unclear interpretation in the financial network context and leads to predictions that may not necessarily hold for a simple graph. More importantly, even though the motivation of Amini et al. (2013) is close to the one in this paper, the impact of tiering/the level of tiering is not discussed. Another recent paper by Elliott et al. (2013) studies contagion by looking at the consequences of integration and diversification in financial network. The closest to addressing tiering in this paper is the numerical example of a core–periphery network, where the core is represented by the set of completely interconnected banks and the peripheral banks each have one connection to a random bank in the core. In this example, the degree structure is fixed but link weights can be reallocated, which changes the relative importance of core and peripheral banks in each bank’s cross-holdings. This is a different approach to studying interdependencies in the financial system than the one pursued in this paper. Furthermore, due to essentially fixed structure of connections, this example does not enable the comparison of networks with different levels of tiering or with different relative arrangement of banks of high and low connectivity (disassortative versus assortative versus neutral networks).

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17 There is a substantial literature that specifically studies the formation of actual network structures observed in reality. A common approach is to implement a preferential attachment rule (due to Barabási and Albert, 1999) or a fitness mechanism (Lenzu and Tedeschi, 2012; De Massi et al., 2006, 2007). However, while this literature allows us to explain the existence of the observed banking network topologies, it usually does not directly address the question of systemic risk. A notable exception is Lenzu and Tedeschi (2012) discussed below.

18 In the network literature such a network is called multigraph.
A distinct and less stylized approach to modeling network structure that offers the possibility of comparing more and less regular topologies is proposed by Lenzu and Tedeschi (2012) and Krause and Giansante (2012). The former paper studies the emergence and evolution of systemic risk in a dynamically changing random network, where at the beginning of each time period banks form (or revise) their financial connections with other banks based on their past performance. The key assumptions that distinguish this work from our model are that the setting is dynamic and network structure is changing over time; the system is perturbed by shocks repeatedly over the course of 1000 periods; in each time period, two random banks are hit by two liquidity shocks of equal magnitude but opposite sign; and the bank hit by the negative shock has an opportunity to raise liquidity from its neighbors in the banking network. While the realism of these assumptions is largely determined by the nature of the banking crisis, bank managers' expectations and many other factors, the implied modeling design appears very different from the one in this paper. Moreover, the analysis of Lenzu and Tedeschi (2012) is fully based on simulations, whereas our (simpler) framework allows us to derive results analytically. The paper by Krause and Giansante (2012), employs both simulations and empirical analysis of the generated data. Its main focus is on the validity of the "too big to fail" paradigm, that is assessed through the examination of several potential determinants of financial contagion: the balance sheet structure, the size of the initially failing bank and different network characteristics, including tiering. Tiering, however, is defined differently than in our paper: it is represented by a combination of factors rather than just by the exponent parameter of the scale-free degree distribution. The authors find that in determining the probability of contagion, the size of the trigger bank is the only economically significant factor. At the same time, for the extent of contagion, the network structure and in particular, tiering matters, with more tiered systems being less susceptible to contagion.

An alternative approach to studying financial contagion, through the examination of network externalities, was introduced by early discussions of systemic risk in Hellwig (1995), Kiyotaki and Moore (2002), Rochet and Tirole (1996) and by the seminal contribution of Allen and Gale (2000) that pioneered the use of explicit network modeling. As in many random-graph models, the key concern of Allen and Gale (2000) and the subsequent stream of literature (Allen et al., 2010; Castiglione and Navarro, 2010) is the relationship between the density of interbank connections and the bankruptcy risk. Some studies employ models with endogenous network formation (Leitner, 2005; Lagunoff and Schreft, 2001; Castiglione and Navarro, 2010; Allen et al., 2010; Cabrales et al., 2013), others examine propagation of bank failures in exogenously given networks (Freixas et al., 2000; Caballero and Simsek, 2010; Acemoglu et al., 2013). These studies are based on networks with rigid, regular or otherwise schematic structures and provide key insights into the mechanism through which the pattern of interconnectedness between banks affects the spread of defaults. However, being restricted to the analysis of simple networks, this literature is limited in addressing the case of real-world contagion in large and complex banking systems. For the same reason the representation of tiered systems in this approach is confined to a very limited set of cases. In contrast, the complex network approach allows for a wide range of network structures with an arbitrarily large number of banks.

To conclude the discussion of the network literature on financial contagion, we emphasize the existence of a substantial body of empirical work. One strand of empirical research studies structural properties of lending networks before and after the recent financial crisis, either at the national level (De Masi et al., 2006; Iori et al., 2007, 2008; Cocco et al., 2009) or more globally (Schiavo et al., 2010; Minoiu and Reyes, 2011; Chinazzi et al., 2012). Another strand of empirical analysis examines financial vulnerability of a banking system in a specific country. This is usually done by calibrating the model to data on real banks' cross-exposures and then simulating the effects of a shock to the system that results from the failure of one or more institutions. The range of examples of such studies, in addition to those that were presented in the introduction, includes Sheldon and Maurer (1998), Toivanen (2013) and Canedo and Jaramillo (2009). Finally, a significant number of studies focus on methodological side of the analysis and develop new analytical tools that should allow us to better identify and monitor systemic risk (Bartram et al., 2007; Sornette and Von der Becke, 2011; Kaushik and Battiston, 2013; Amini et al., 2012).

2.2. Complex network literature

In terms of methodology, our paper is closest to the literature on complex networks (Strogatz, 2001; Newman et al., 2001; Vega-Redondo, 2007). Within this literature, we exploit results from the standard epidemic/information diffusion and

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19 Moreover, not only the structure of links changes but also all defaulting banks are substituted by the new ones at the beginning of every subsequent period.
20 If the illiquid bank manages to attract enough funds, it survives; otherwise, it goes bankrupt and gets replaced by a new bank in the next time period.
21 Consistently with the findings and intuition derived in our theoretical model, more tiered systems display lower fraction of affected banks as the larger size of the most connected banks allows them to absorb any losses – including those that emerge at the periphery – more easily. This limits the contagion spread.
22 An exception from this literature which we are aware of, is the network formation model of Blume et al. (2011). It considers arbitrarily large networks and examines the disparity between socially optimal and stable networks in the presence of contagion risk. The modeling approach and the focus of the paper are, however, very different from the ones in this paper.

For example, one representation of a tiered structure in the deterministic network setting is the “money center bank” structure proposed by Freixas et al. (2000). They consider a model with three banks, where the banks on the periphery are linked to the bank at the center but not to each other. In this framework, they find that for some parameter values the failure of a bank on the periphery will not trigger the breakdown of other institutions while the failure of the money center bank would.
percolation literature. Specifically, the framework builds on the generating function techniques used in the SI(R) models\(^{24}\) of Watts (2002), Callaway et al. (2000), Cohen et al. (2001), and Newman (2002b), and in the structured network model of Newman (2002a). This literature describes the behavior of connected groups of nodes in a random network, with or without internode degree correlations, and characterizes phase transition, i.e., the point at which extensive contagion outbreaks occur, as well as the size of a susceptible cluster beyond that point.

Unlike the generic, undirected network models of Watts (2002) and Newman (2002a), the model in this paper explicitly describes banks’ balance sheets, which specify the direction of links in the financial system. The distinction between incoming links (claims) and outgoing links (obligations) implies that in contrast to epidemiological and percolation models, greater connectivity does not only create more channels through which contagion can spread. It also improves counteracting risk-sharing benefits and, in accordance with our model, leads to higher capital stocks, which strengthens banks’ stability. Moreover, while in most epidemiological models the susceptibility of a node to contagion is determined solely by the total number of its infected neighbors, in the present setup the share of neighbors that default determines the contagion risk.

Last but not least, directed networks are, in general, a more difficult object for the analysis than their undirected counterparts. To start with, as each node in a directed network has not one but two degrees that separately account for the number of incoming and outgoing links, there are also, correspondingly, two degree distributions. In fact, to be completely general, describing a directed network requires defining a joint degree distribution of in- and out-degree. Moreover, since we are interested in the situation where connectivities of neighboring nodes are correlated (to allow for the comparison of assortative, disassortative and neutral networks), a joint degree distribution needs to be defined for the in- and out-degree of two neighboring nodes. This distribution has to satisfy a range of consistency conditions, which apart from the usual conditions on probability distribution functions include constraints due to the specific, network context.\(^{25}\) Thus, to keep the model tractable and representation of assortative, disassortative and neutral networks intuitive, for the main part of our model we resort to a simplifying assumption under which in- and out-degrees of a node, albeit random, are the same. The case where the two degrees are allowed to differ is only considered in the Appendix, in the setting that allows circumventing some of the difficulties outlined above.

3. The model

3.1. Interbank network

Consider a banking system represented by a network where each node is a bank and each link represents a directional lending relationship between two banks. Two crucial properties of this network are that it is (i) directed and (ii) random. The first property implies that links in the network are directed, so that every node has two degrees: an in-degree, the number of links that point into the node, and an out-degree, the number of links that point out. Incoming links of a bank reflect the interbank assets of the bank, that is, funds owed to the bank by a counterparty. In contrast, outgoing links from a bank indicate its interbank liabilities. Randomness of the network means that the connectivity, or degree of each bank is not deterministic but random.\(^{26}\) Specifically, in our case, where the network is directed and two degrees are defined, we consider in- and out-degree of a node as a single random variable. Then the connectivity of a node is described by the probability distribution \(p(k)\), where \(k\) is the node’s in- and out-degree. This means that by construction any realization of the random network will be such that in- and out-degree of each node are the same.

This focus on banking networks with equal number of incoming and outgoing links of a bank does not only make the notions of highly connected and peripheral banks intuitive\(^{27}\) but it also substantially simplifies the discussion and derivation of the results, particularly in the part of the analysis that addresses networks with internode degree correlations. Moreover, while the exact equality of the number of lenders and borrowers of a bank is clearly a broad simplification, an indirect support for this assumption is provided by some empirical findings. For example, recent empirical studies and data sources report a strong correlation between the total size of banks’ interbank assets and liabilities.\(^{28}\) Furthermore, to account for the possibility of a difference in the number of incoming and outgoing links of a bank, in the Appendix we conduct a robustness

\(^{24}\) SI(R) (susceptible-infected (-recovered)) models are canonical epidemiological models, where the life history of each node passes from being susceptible (S), to becoming infected (I) (and, in the SI'R setting, to finally being recovered (R)). The primary theoretical approach used in the SI(R) context is the generating-function analysis.

\(^{25}\) These constraints include a requirement that average in-degree in the network is equal to the average out-degree as well as a set of consistency conditions to account for different ways of calculating, say, a degree distribution of a neighboring node (this can be calculated by summation from a joint degree distribution of the neighboring nodes and should also satisfy the equation similar to (7)). Moreover, in deriving a degree distribution of a neighboring node, one needs to distinguish between two “types” of neighbors: the ones on the incoming links and those on the outgoing links. In the context where in- and out-degree of a node can be different, these distributions are not the same.

\(^{26}\) Restating the remark made in the introduction, this definition of a random network is consistent with the definition of a random network in Newman (2003) and Vega-Redondo (2007). Using the terminology of Newman (2003), it means that we consider a “generalized random graph” or “random graph with arbitrary degree distribution”, rather than the specific Erdős-Rényi random graph, that is often assumed by the term “random network” in other papers.

\(^{27}\) Essentially, both in- and out-degree of highly connected banks are large and both in- and out-degree of peripheral banks are small.

\(^{28}\) The evidence is provided, for example, by Amini et al. (2012) for the banking systems of Austria and Brazil and by the data on UK interbank deposits for 2005–2013 (Source: SNL Financials, http://www.snl.com).
check of the model. We consider a random banking network where in- and out-degree of a bank can be different but their joint probability distribution is such that the sum of the two degrees at any bank follows a scale-free distribution. The advantage of considering this distribution is its empirical validity, on the one hand, and on the other hand, analytical tractability of the model since, given this distribution, the effects of tiering can be studied even in the absence of interbank degree correlations.29

At last, as we have stated in the previous section, this paper belongs to the part of the random network literature that considers the structure of interbank connections as exogenously given, without defining the process that generates this structure. In particular, issues related to endogenous network formation, optimal network structure and network efficiency are left aside. Moreover, in order to enable the use of the probability generating function techniques later in our model, we consider banking networks that are “sufficiently large”, that is, contain a large number of banks n.

3.2. Tiered network structure

A tiered banking network is defined as a network in which relatively few high-degree (first-tier) banks are connected with low-degree (second-tier) banks and are also connected with each other, whereas low-degree banks are almost exclusively connected with high-degree banks. This property of tiering can be modeled by assuming that the banking network displays disassortative mixing on its degrees, that is, degrees of neighboring banks in the network are negatively correlated. It should be noticed that even with negative degree correlations, first-tier, highly connected banks are still likely to be linked with each other. This is explained by the fact that in any random network, the probability of a given source node to be connected to a target node is an increasing function of the degree of the target node.30 Disassortativity just implies that the probability of connections between low-degree banks is smaller and the probability of connections between low-and high-degree banks is larger than in the case where degree correlations between neighboring nodes are either absent or positive. In fact, if the network displays zero or positive internode degree correlations, low-degree banks are likely to be connected with each other, especially if the proportion of these low-degree banks in the network is sufficiently large. Formally, we define disassortative and other networks with internode degree correlations in Section 3.5, while the resilience of these networks to financial contagion is analyzed in Section 4.1.

Alternatively, as suggested by numerous empirical studies (Boss et al., 2004a; Cont et al., 2012; Nier et al., 2007; Soramäki et al., 2007, and others), the same property of tiering in banking networks can be captured by the assumption that banks’ degree distribution is highly right-skewed. One example of such distribution is a power-law, or scale-free,6 since, given this distribution, the effects of tiering can be studied even in the absence of interbank degree correlations.29

Formally, we define disassortative and other networks with internode degree correlations in Section 3.5, while the resilience of these networks to financial contagion is analyzed in Section 4.1.

3.3. Bank’s solvency condition

An individual bank’s assets consist of external assets (investors’ borrowing), denoted by $A^E_i$, and interbank assets (other banks’ borrowing), denoted by $A^B_i$. The total interbank asset position of a bank is assumed to be equally distributed across its counterparties (incoming links), that is, proportional to the number of counterparties:

$$A^B_i = a j_i.$$  

(1)

Here $j_i$ is the number of incoming links of bank $i$ and $a$ is a positive constant.31 In principle, all main results of the paper continue to apply with a more general specification of $A^B_i = a j_i^r$ for any non-negative $r$, at least as long as $r \leq 1.$32 The exact proportionality of interbank assets to connectivity is chosen for the sake of clarity, and the alternative specifications of $A^B_i$ are discussed in the conclusion.

The definition of $A^B_i$ in (1) implies that $a$ is the interbank assets held by $i$ against any of its debtor banks. Also, since this is true for any bank $i$, all links in the network “carry” the same amount of interbank assets. Such representation of interbank exposures, albeit stylized, allows a better understanding of the effects of tiering in the banking network on the potential for contagion spread. In particular, as all links in the network have equal weight, it is precisely the structure of these links as imposed by the degree distribution, that plays a key role in determining network stability. Furthermore, setting any bank’s degree distribution is highly right-skewed. One example of such distribution is a power-law, or scale-free, (Soramäki et al., 2007, and others), the same property of tiering in banking networks can be captured by the assumption that proportionality of interbank assets to connectivity is chosen for the sake of clarity, and the alternative specifications of $A^B_i$ continue to apply with a more general specification of $A^B_i = a j_i^r$ for any non-negative $r$, at least as long as $r \leq 1.$32

29 The discussion of scale-free networks and their properties is provided in the next section and later, in more detail, in Section 4.2 and in the Appendix.
30 See Section 3.5 for more detail.
31 Without loss of generality, each bank is assumed to have at least one incoming link, so that interbank assets of any bank are strictly positive.
32 The particular case of $r = 0$ means that interbank asset positions of banks are independent of the number of incoming links, and hence, identical across all banks in the system. This specification is commonly adopted in earlier random-network literature on financial contagion (see, for example, Gal and Kapadia, 2010; May and Arinaminpathy, 2010). Indirectly, some support for the proportionality between $A^B_i$ and bank’s in-degree is provided by De Masi et al. (2006) and Barrat et al. (2004). For example, De Masi et al. (2006) finds that a bank’s total value of interbank transactions, which in our model can be approximated by $2A^B_i$, is proportional to total degree, which in our model is equal to $2j_i$. 

interbank exposures equal across incoming links allows us to show that even when risk sharing between banks in the system is maximized, widespread contagious defaults may still occur.

A bank’s liabilities are composed of interbank liabilities, denoted by \( L^B_i \), and customer deposits, denoted by \( D_i \). Since every interbank liability of a bank is another bank’s asset, interbank liabilities are determined endogenously. Specifically, given that in- and out-degree of each bank is the same and all links in the network transfer the same amount of interbank assets, total interbank liabilities and interbank assets of each bank are identical, \( L^B_i = A^B_i \). Customer deposits are determined exogenously.

For any bank \( i \) the condition to be solvent can be written as\(^{33}\):

\[
(1 - \delta)A^B_i + A^E_i - L^B_i - D_i \geq 0, \tag{2}
\]

where \( \delta \) is the fraction of banks with obligations to bank \( i \) that have defaulted. Here we assume that when a linked bank defaults, bank \( i \) loses all of its interbank assets held against that bank. This “zero recovery” assumption is rather realistic in the midst of a crisis: when the uncertainty about the rate and timing of borrower’s recovery is high, banks’ managers are likely to assume the worst-case scenario. Alternatively, the solvency condition can be stated as

\[
K_i - \delta A^B_i \geq 0, \tag{3}
\]

where \( K_i = A^B_i + A^E_i - L^B_i - D_i \) is the capital buffer of bank \( i \), or the net worth of a bank, equal to the difference between the book value of a bank’s assets and liabilities.

### 3.4. Transmission of shocks and bank’s vulnerability

Let us examine the consequences of an unexpected idiosyncratic shock hitting one of the banks in the system and leading to its default. This shock can be thought of as resulting from operational risk (fraud) or credit risk.\(^{34}\) Although for credit risk in particular, aggregate or correlated shocks affecting all banks at the same time may be more relevant in practice, idiosyncratic shocks are a clearer starting point for studying the spread of financial contagion. Moreover, a single bank failure may actually result from an aggregate shock which has particularly adverse consequences for one institution.\(^{35}\)

The failure of one bank reduces the interbank asset positions of its creditor banks and if their capital buffer is insufficient to cover the loss on interbank assets, these creditor banks also default. As a result, an initial failure of one bank may give rise to a wave of contagious defaults in the network. In fact, the larger the number of outgoing links of a defaulting institution, the larger the potential for the spread of shocks. On the other hand, the impact of a bank default on the solvency of its creditors is determined according to (3), by creditors’ capital stock and by the losses on their interbank asset position. Given that the losses incurred due to a default of a borrowing neighbor are the same for all banks (and equal to \( a \)), the solvency of a creditor bank essentially hinges on its capital.\(^{36}\) Here we assume that the capital of a bank is increasing in the total amount of its interbank assets. Then the larger the total interbank asset position of a creditor bank, the larger its capital stock and, according to (3), the smaller the risk of its default due to a default of a borrowing neighbor. Formally, let the capital of bank \( i \) be defined as:

\[
K_i = b(A^B_i)\delta, \tag{4}
\]

where \( b \) and \( \delta \) are any positive constants. So, \( K_i \) is increasing in \( A^B_i \) either proportionally or more or less than proportionally. This assumption, albeit stylized, is consistent with current bank capital requirements that minimum capital stocks be higher for banks with larger exposures. Moreover, a positive relation between a bank’s capital and interbank assets is suggested by some literature. For example, Admati et al. (2011) and Allen et al. (2011) show that a larger capital makes banks’ access to interbank market easier, both on the liability and on the asset side,\(^{37}\) and Flannery and Rangan (2008) and Gropp and Heider (2010) suggest that a larger asset risk exposure creates incentives for banks to hold larger capital, especially in the absence of government insurance.\(^{38}\) This positive relation between bank’s capital and its interbank asset position turns out to be key for the results that follow. In this sense, the absence of capital regulations that prescribe banks with larger interbank exposures to hold larger capital could lead to different implications of tiering in the banking network.\(^{39}\)

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33 The price of external assets is fixed at 1 so that liquidity effects associated with the knock-on defaults are ruled out. In principle, Acharya and Yorulmazer (2007) show that the asset price may be depressed (become less than 1) when failed banks’ assets need to be sold but financial markets have limited overall liquidity to absorb the assets.

34 See for example, the case of a failure of Barings in the UK, Drexel Burnham Lambert in the US or Société Générale in France.

35 In this model, the aggregate shocks can be captured through a simultaneous reduction in the capital stocks of all banks, with a major loss for one particular bank.

36 Notice that in calculating the losses of a creditor bank due to a default of a borrower, the opposing effects of in-degree of a creditor on its solvency – risk sharing versus risk exposure – essentially offset each other. This is due to the exact proportionality of bank’s total interbank assets to its in-degree and equality of bank’s interbank exposures across borrowers.

37 For example, Allen et al. (2011) show that high levels of capital allow banks to attract borrowers as when credit markets are competitive, capitalization is a way of commitment for a bank to monitor and offer low interest rate on the loan.

38 Other (indirect) evidence is provided by De Masi et al. (2006), who show that in the Italian interbank market, capital of a bank is strongly correlated with the total amount of daily volume of transactions. Positive relationship is also suggested by Gai and Kapadia (2010) and Krause and Giansante (2012).

39 For example, if the relation between a bank’s capital and interbank assets was negative, such as when \( \delta < 0 \), banks with high connectivity would unambiguously increase contagion risk: not only would they spread shocks to a larger number of other banks but, due to higher interbank asset position...
Recall that $A_{i}^{IB}$ is proportional to the in-degree of bank $i$. Then (4) implies that the capital of a bank is an increasing function of in-degree, $K_{i} = b(a_{ij})^\delta$ where $\delta > 0$. Therefore, the larger the number of incoming links of a bank, the smaller the risk of its default due to contagion. In fact, this inference can be demonstrated by the threshold condition on the number of incoming links. We derive this condition below.

Consider again the solvency condition (3) and rewrite it using the definition of $A_{i}^{IB}$ and $K_{i}$ in terms of the in-degree of bank $i$. Given that a single counterparty default ($\phi = 1/j_{i}$) causes a loss of $(1/j_{i})A_{i}^{IB} = 1/j_{i} \cdot a_{ij} = a$ interbank assets at a creditor bank, the solvency condition becomes:

$$b(a_{ij})^\delta - a \geq 0.$$  

Equivalently, a default can only spread if there is at least one creditor bank for which

$$j_{i} \leq \frac{1}{(a_{ij}^{\delta - 1}b)^{1/\delta}}$$  

Following Gai and Kapadia (2010), bank $i$ is called vulnerable if its in-degree fulfills (6), that is, if $i$ is exposed to the default of a single neighbor. The other banks are called safe. To simplify the notation, we denote by $\bar{K}$ the largest integer that is strictly below $1/(a_{ij}^{\delta - 1}b)^{1/\delta}$. Then condition (6) suggests that given $\bar{K}$, the vulnerability of a bank is fully determined by its in-degree: all banks of in-degree smaller or equal to $\bar{K}$ are vulnerable and banks of in-degree above $\bar{K}$ are safe. In this sense $\bar{K}$ represents the threshold in-degree, separating safe and vulnerable banks.

Using simple algebra, it is easy to show that condition (6) is actually equivalent to $j_{i} < A_{i}^{IB}/K_{i}$. That is, a bank is vulnerable when its in-degree falls below the ratio of interbank assets to capital. According to the figures for developed countries reported by Upper (2011) and to the calibration results in Gai and Kapadia (2010) for published accounts of large financial institutions, the ratio $A_{i}^{IB}/K_{i}$ lies in the range between 5 and 6. In Section 4 we will use this finding to demonstrate some of the model’s results.

Thus, incoming and outgoing links of each bank play opposite roles in determining the extent of contagion. Their joint effect is predetermined by the specific features of the network structure as specified by the degree distribution. In the next section we introduce various types of network structures and later on study the impact of structural characteristics, in particular tiering, on the spread of defaults in more detail.

3.5. Internode degree correlation. Assortative, disassortative and neutral networks

To begin with, we introduce the notion of internode degree correlation and define three types of networks according to the pattern of this correlation.

Let us denote by $\xi(k,k')$ the probability that a randomly selected link goes from a node with in- and out-degree $k$ (node $(k,k)$) to a node with in- and out-degree $k'$ (node $(k',k')$). This quantity obeys the sum rules: $\sum_{k,k'} \xi(k,k') = 1$, $\sum_{k=1}^{\infty} \xi(k,k') = \xi_{d}(k)$ and $\sum_{k=1}^{\infty} \xi(k,k') = \xi_{c}(k')$ for all $k,k' \geq 1$, where $\xi_{d}(k)$ represents the marginal frequency of neighboring debtor banks with in- and out-degree $k$ and $\xi_{c}(k')$ is the marginal frequency of neighboring creditor banks with in- and out-degree $k'$. Observe, however, that every debtor with in- and out-degree $k$ is simultaneously a creditor with the same degree. This implies that the frequency of debtor and creditor banks with degree $k$ is the same, $\xi_{d}(k) = \xi_{c}(k)$. We denote this quantity by $\xi(k)$ and call it the marginal frequency of neighboring nodes with in- and out-degree $k$, or alternatively, the probability that a node reached by following a randomly chosen link on a network has in- and out-degree $k$.

Notice that $\xi(k)$ is different from $p(k)$, the probability that a randomly chosen node has in- and out-degree $k$. Instead, it is biased in favor of nodes of high degree since more edges end (and start) at a high-degree one. Citing Jackson (2010, p. 87), “we are much more likely to find high-degree nodes by following the links in a network than by randomly picking a node”. Therefore, the degree distribution of a neighboring node is not independent of its degree. In fact, it is proportional to $kp(k)$, and the correctly normalized distribution is given by:

$$\xi(k) = \frac{k p(k)}{\sum_{k=1}^{\infty} k p(k)}$$  

If internode degree correlations are absent, the network is called neutral and $\xi(k,k')$ is equal to $\xi(k) \cdot \xi(k')$. Otherwise, $\xi(k,k')$ is different from this value and the network is either assortative or disassortative. The network is called assortative if high-degree nodes show a tendency to be connected to other high-degree nodes. Conversely, the disassortative network is one where high-degree nodes tend to attach to low-degree ones. As we have pointed out earlier, disassortativity in nodes’ degrees can be regarded as tiering. For this reason disassortative networks will be in focus of our analysis below.

(footnote continued)
The amount of assortative/disassortative mixing can be quantified by the degree correlation between neighboring nodes, $\langle kk' \rangle - \langle k \rangle^2 = \frac{1}{\sigma^2_{kk'}} k k' (\xi(k,k') - \xi_0(k) \xi_0(k'))$, where $\langle \cdots \rangle$ indicates an average over links. This correlation is equal to zero for neutral networks and positive or negative for assortative or disassortative networks, respectively. For convenience of comparing different networks, the degree correlation between neighboring nodes can be normalized by dividing it by its maximum value, that it achieves on a perfectly assortative network. In the perfectly assortative network, $\xi(k,k') = \xi_0(k) \delta_{kk'}$, where $\delta_{kk'}$ is the Kronecker delta, i.e., $\delta_{kk'} = 1$ if $k = k'$ and 0 otherwise. The degree correlation between neighboring nodes in such a network is equal to the variance, $\sigma^2_{kk'}$, of the distribution $\xi_0(k)$. Then the normalized correlation is

$$
r = \frac{1}{\sigma^2_{kk'}} \sum_{k,k' = 1} k k' (\xi(k,k') - \xi_0(k) \xi_0(k')),
$$

where $\sigma^2_{kk'} = \sum_{k = 1} k^2 \xi_0(k) - \frac{1}{\sum_{k = 1} k \xi_0(k)}^2$. This is the standard Pearson correlation coefficient of the degrees at either end of a link, and $-1 \leq r \leq 1$.

For the purposes of further analysis, we also use the joint degree distribution of the linked nodes, $\xi_0(k,k')$, to obtain the collection of conditional probability distributions $p(k' | k)$ for any $k \geq 1$. $p(k' | k)$ denotes the probability that a node reached by following a randomly chosen link from a node with in- and out-degree $k$ has in- and out-degree $k'$. This probability is given by

$$
p(k' | k) = \frac{\xi_0(k,k')}{\sum_{k = 1}^\infty \xi_0(k,k)} , \quad k = 1, 2, \ldots
$$

Notice that if internode degree correlations are absent, $p(k' | k) = \xi_0(k')$.

In the next section we study the extent of contagion in the banking network, given a general form of the joint probability distribution for the degrees of the neighboring nodes, $\xi_0(k,k')$. Then in Section 4.1, we consider a special functional form of $\xi_0(k,k')$ that allows for a variation in the pattern of degree correlations between neighboring nodes. For this degree distribution, we examine the effects of different patterns of degree correlations on the spread of contagious defaults.

3.6. The reach of contagious defaults

Generating functions To evaluate the extent of contagion in banking networks, we use the approach based on probability generating functions, a classical technique in statistics to characterize a distribution of a random variable.44 In that, to make exact solutions possible, we restrict attention to networks that contain no cycles, or closed loops, and therefore, are tree-like in structure.45 This means that any component outgoing from a defaulting node, i.e., the set of nodes which can be reached from the defaulting node through a directed path of links, has no closed loops. This property is crucial for the argument that we employ below. Most importantly, the tree-like structure of the components outgoing from the defaulting node guarantees that any bank in the system can be exposed to a default of no more than one borrowing neighbor and no second round of contagion can occur. This implies that safe banks, resistant to a default of a single debtor, never default. As a result, contagion in the banking network propagates only through vulnerable banks: from one vulnerable bank to another.

Given this feature of the model, a key concern is to characterize the distribution of the sizes of components, or clusters, of vulnerable banks that can be reached after an initial default. The size of a component is evaluated in terms of the number of vulnerable banks that these outgoing links lead to is generated by a random link and does not pass the shock any further. In this case the vulnerable component is empty. Or it may lead to a new vulnerable component with size distribution generated by $H_1(y)$.

Notice that the number of outgoing links of our first bank, reached by following a randomly chosen link from a node with in- and out-degree $k$, is distributed according to $p(k' | k)$ from equation (9). Also, by the multiplication property of generating functions, the distribution of the sum of the sizes of $k$ vulnerable clusters that these outgoing links lead to is generated by $(H_1(y))^k$. Thus, when all the different possibilities $k' = 1, 2, \ldots$ are taken into account, the self-consistency condition that

44 By definition, if $[p(k)]_{x = 0}^\infty$ is the distribution of a discrete random variable, then its generating function is given by $G(x) = \sum_{k = 0}^\infty p(k) x^k$, so that the distribution can be fully recovered from $G$ through successive differentiation: $p(k) = (1/k! \frac{d^k G}{dx^k})_{x = 0}$ for any $k \geq 0$. A detailed description of the key properties of the probability generating functions can be found in Newman et al. (2001), Newman (2003) and Vega-Redondo (2007). The two properties that turn out to be particularly useful for the analysis in this paper (and follow immediately from the definition of $G$) are $G(1) = 1$ and $d^k G/dx^k(1)_{x = 0} = G(1) = k$.

45 The tree-like structure emerges when $n = \infty$ as then a chance of a cycle is infinitesimally small, of order $n^{-1}$. The condition of absent closed loops in the financial network is supported by some empirical evidence which implies that clustering coefficients in real-world banking networks are relatively low (see, for example, Cont et al., 2012; Li et al., 2010; Iori et al., 2008; Bech and Atalay, 2010).

46 This approach is described in Newman et al. (2001) and Vega-Redondo, 2007.
functions \( H_1(y|k) \) must satisfy can be written as follows:

\[
H_1(y|k) = \Pr[\text{reach safe bank} | (k, k)] + y \sum_{k'=1}^{\infty} p(k'|k)(H_1(y|k'))^k = \\
= \left(1 - \sum_{k'=1}^{\infty} p(k'|k)\right) + y \sum_{k'=1}^{\infty} p(k'|k)(H_1(y|k'))^k \quad k = 1, 2, \ldots
\]  

(10)

In this equation the leading factor \( y \) accounts for the first vulnerable bank encountered along the initial link, and we have used the fact that the probability of reaching a safe bank is equal to 1 minus the probability of the complementary event that a vulnerable bank is reached.

The size distribution generated by \( H_1 \) is link-based, in the sense that it considers moving along a randomly chosen link. The quantity we actually are interested in is the distribution of the sizes of the vulnerable clusters to which a randomly chosen bank belongs. So, consider a randomly chosen bank, safe or vulnerable, and look at all possible directed paths through which the initial default could spread from this bank in every direction, that is, along each of its outgoing links. The number of links emanating from such a bank is distributed according to the degree distribution \( p(k) \), and every link leads to a vulnerable cluster whose size is drawn from the distribution generated by \( H_1(y|k) \) above. Therefore, the size distribution of the vulnerable clusters to which a randomly selected bank belongs is generated by

\[
H_0(y) = y \sum_{k=1}^{\infty} p(k)(H_1(y|k))^k.
\]

(11)

Notice that such definition of the generating function \( H_0 \) accounts for the possibility that initial random shock can hit a safe bank. In this case, the safe bank defaults and hence, belongs to the vulnerable cluster that it may spread.\(^{47}\) However, by definition, it is then the only safe bank in this vulnerable cluster.

Now one could, in principle, solve the system of equations (10) self-consistently for the collection of \( H_1(y|k), k = 1, 2, \ldots \), and substituting the result into (11), find \( H_0(y) \). This would allow recovering the complete size distribution of (finite) vulnerable components. However, for the purposes of assessing the scope of contagious defaults in the banking network, it is enough to focus on just the first moment of this distribution, that is, the mean size of the finite vulnerable component.

**Average size of the vulnerable component and phase transition** We now derive the expression for the average size of a finite vulnerable component in the banking network and study the conditions for the emergence of the **giant vulnerable component**, when the size of the vulnerable cluster diverges. Formally, the giant component is a unique component whose relative size remains bounded above zero as the number of nodes in the network increases indefinitely. In the framework of this model, the formation of the giant vulnerable component can be interpreted as a system-wide contagion, or "global" banking crisis. It signifies the event when a random initial default of one bank with positive probability causes failure of a substantial fraction of vulnerable institutions in the banking system. Then the relative size of the giant vulnerable component, i.e., the fraction of banks that it contains, can be regarded as a scope of the crisis should systemic failure occur.

Due to a basic property of generating functions, the average size of a finite vulnerable cluster, \( S \), can be computed as:

\[
S = H_0'(1).
\]

(12)

Taking a derivative of \( H_0(y) \) in (11) and evaluating it at \( y=1 \), we obtain that

\[
S = \sum_{k=1}^{\infty} p(k)(H_1(1|k))^k + \sum_{k=1}^{\infty} kp(k)(H_1(1|k))^{k-1}H_1(1|k).
\]

(13)

\( H_1(y|k) \) is a standard generating function, so that \( H_1(1|k) = 1 \) for all \( k \). Also, \( kp(k) = \langle k \rangle \zeta(k) \) as follows from the definition of \( \zeta(k) \) in (7). Therefore, (13) becomes

\[
S = 1 + \langle k \rangle \sum_{k=1}^{\infty} \zeta(k)H_1(1|k).
\]

(14)

Differentiating the expression for \( H_1(y|k) \) in (10) and solving the resulting linear system of equations for the set of values \( H_1'(1|k), k = 1, 2, \ldots \), the average size of the vulnerable cluster can be written in a form of the following matrix expression\(^{48}\):

\[
S = 1 + \langle k \rangle 1 \zeta - \langle k \rangle \tilde{\zeta}' \mathbf{M}^{-1} \tilde{\zeta}.
\]

(15)

Here \( 1 \) is the vector of 1’s and the other three vectors are defined in terms of the joint and marginal degree frequencies of the neighboring nodes: \( \zeta = (\zeta(k))_k=1^\infty \), \( \tilde{\zeta} = (\sum_{k'=1}^{\infty} \xi(k, k'))_k=1^\infty \), and \( \tilde{\zeta}' = (k\zeta(k))_k=1^\infty \). The product \( -\mathbf{M}^{-1} \tilde{\zeta} \) represents the

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\(^{47}\) Consider that \( H_0(0) = 0 \), which by definition of a probability generating function means that with probability one the size of a vulnerable cluster to which a randomly chosen node belongs is larger than zero. This is so because at least one bank, the one that is hit by the initial shock, always defaults, so that the vulnerable component associated with initial default contains at least this one bank.

\(^{48}\) In detail, the derivation of this result is provided in the Appendix.
vector-form solution for $\langle H_f(1|k) \rangle_{k=1}^{\Gamma}$ and the elements of matrix $M$ are given by

$$m_{k,k'} = \xi(k,k') - \zeta(k') \delta_{kk'}, \quad k, k' = 1, 2, \ldots, \Gamma$$

Expression (15) for the average size of the vulnerable cluster diverges when $\det(M) = 0$. This condition marks the phase transition at which the giant vulnerable component first appears in the network. Above this point insolvency of one bank, by propagating through the network, can affect an extensive number of other vulnerable banks. By studying the behavior of (15) close to the transition, where $S$ must be large and positive in the absence of the giant component, one can show that the giant vulnerable component exists in the network when $\det(M) > 0$.\footnote{See Newman (2002a) and Molloy and Reed (1995) for more details.} In the framework of our model, this means that when $\det(M) < 0$, the vulnerable clusters in the banking system are small and financial contagion stops quickly. However, if $\det(M) \geq 0$, the giant vulnerable cluster arises and a significant fraction of banks can default due to contagion.

The next section presents the analytical base for computing the relative, or fractional size of the giant vulnerable component. We employ this analysis later, in Section 4, to compare the relative size of the giant vulnerable component across tiered and other types of banking networks. This allows us to compare the susceptibility to risk and the scope of global crises in various types of banking structures.

Relative size of the giant vulnerable component When the giant vulnerable component exists in the financial network, it is straightforward to assess its relative size, $\omega$, – the fraction of banks in the network that belong to the giant vulnerable component. To find $\omega$, let us first define $1 - \tilde{\omega}_k$ – the probability that a randomly selected link outgoing from a node $(k,k)$ does not lead to the giant vulnerable component. It is the probability that by following a randomly selected link outgoing from a node $(k,k)$ either none or only a finite number of vulnerable banks can be reached. This means that the endnode of this randomly selected link is either a safe bank or a vulnerable bank that does not span the giant vulnerable component.\footnote{Notice that if the endnode is a safe bank, then with probability one it does not span the giant vulnerable component because the only case in which it may span the giant vulnerable component has the infinitesimal probability $n^{-1}$, the probability to be hit by the initial shock.}

By consistency, in the latter case none of the outgoing links of the endnode bank leads to the giant vulnerable component. This results in the following set of equations:

$$1 - \tilde{\omega}_k = \left(1 - \sum_{k=1}^{\Gamma} p(k'|k)\right) + \sum_{k=1}^{\Gamma} p(k'|k)(1 - \tilde{\omega}_k)^k, \quad k = 1, 2, \ldots$$

Having defined $\{\tilde{\omega}_k\}_{k=1}^{\Gamma}$, let us now determine the probability that a randomly selected bank belongs to the giant vulnerable component, that is, an extensive number of vulnerable banks can be reached from it. This is exactly our magnitude of interest $\omega$, the relative size of the giant vulnerable component. The value $(1 - \omega)$ is then a probability that a randomly selected node does not belong to the giant vulnerable component. It reflects the possibility of two events. First, a randomly selected bank may be safe and hence does not belong to the component with probability one. Alternatively, the bank may be vulnerable but such that none of its outgoing links leads to the giant vulnerable component. The two possibilities lead to:

$$1 - \omega = \left(1 - \sum_{k=1}^{\Gamma} p(k)\right) + \sum_{k=1}^{\Gamma} p(k)(1 - \tilde{\omega}_k)^k.$$  

Although Eqs. (17) and (18) do not usually allow solving for $\omega$ in closed form, the approximate solution can be found by numerical iteration from a suitable set of starting values for $\{1 - \tilde{\omega}_k\}_{k=1}^{\Gamma}$. This approach is used for most of our results in Section 4.1 and for some of the results in Section 4.2 discussed below. It is also employed in Section A.1 of the Appendix.

4. Results

In the following two sections we use the techniques presented above to evaluate the effects of tiering in the banking network on the spread of contagious defaults. First, we compare the phase transition thresholds and the sizes of the giant vulnerable component in assortative, disassortative and neutral networks. After that, we consider scale-free banking networks and examine the effects of a variation in the level of tiering.

4.1. Financial contagion in assortative, disassortative and neutral networks

The numerical analysis is conducted following Newman (2002a). Let us consider the symmetric binomial form of the joint probability distribution for the degrees of the neighboring nodes:

$$\xi(k,k') = N e^{-(k+k')/\nu} \left[ C(\alpha^{k} \beta^{k'}) + C(\alpha^{k'} \beta^{k}) \right],$$

where $\alpha + \beta = 1, \nu > 0$, and $N = 1/2(1 - e^{-1/\nu})$ is a normalizing constant. The choice of this distribution function is explained by its analytical tractability. Moreover, as argued by Newman (2002a), the behavior of this distribution is also quite natural: the
distribution of the sum of the degrees, $k + k'$, decreases as a simple exponential, while that sum is distributed between the two neighboring nodes binomially. The parameter $\alpha$ controls the assortative, disassortative or neutral mixing in the network. Indeed, from Eq. (8), the correlation coefficient of the degrees of the neighboring nodes, $r$, is equal to $r = 8\alpha^2 - 1/2\nu^2 - 1 + 2(\alpha - \beta)^2$, which can be positive or negative and passes through zero when $\alpha = (1/2) + (1/4)\nu^2 \approx 0.1464$ or 0.8536.

Phase-transition threshold $K_{\text{max}}$ and the scope of the crisis Using the specification of the joint degree distribution of the neighboring nodes in (19), and Eqs. (17), (18), we calculate numerically the fractional size of the giant vulnerable component, $\omega$, for $\alpha = 0.05$, where the network is disassortative, $\omega = 0.1464$, where it is neutral, and $\alpha = 0.5$, where it is assortative. In each case, we consider $\omega$ as a function of the degree cutoff parameter $K$. Clearly, when $K = 1$, all banks with in- and out-degree above 1 are safe and the giant vulnerable component is unlikely to emerge ($\omega = 0$). Then, as $K$ increases, more and more banks in the network are characterized as vulnerable and at some point, the giant vulnerable component forms ($\omega$ becomes positive). Let us denote by $K_{\text{max}}$ the largest value of the cutoff parameter $K$ for which no giant vulnerable component exists. This critical value of $K$ is of particular interest to us as it signifies the phase transition, i.e., the threshold above which the risk of global financial breakdown is realized.  

The findings of our numerical analysis for the relative positions of $K_{\text{max}}$ and for the fractional size of the giant vulnerable component in disassortative, neutral and assortative networks are summarized by Result 1.

**Result 1.** The phase transition threshold, $K_{\text{max}}$, is largest in the disassortative network ($\alpha = 0.05$) and smallest in the assortative network ($\alpha = 0.5$). Conversely, the relative size of the giant vulnerable component, $\omega$, is smallest in the disassortative network and largest in the assortative network, for any $K \geq 1$. Thus, both the risk of the systemic crisis and the scope of the crisis should it occur are lowest in the disassortative and highest in the assortative banking network.

These findings are demonstrated by Fig. 1. The three panels of the figure represent cases of different values of the degree scale parameter $\nu$, where $1/\nu$ measures the rate at which the probability of the aggregate degree of neighboring nodes, $k + k'$, declines as $k + k'$ becomes larger.

Result 1 and Fig. 1 highlight two important advantages of disassortativity/tiering in the banking network and disadvantages of the other types of structures, respectively. First, the point of phase transition, at which the giant vulnerable component forms, shifts to the left as the network becomes more assortative. That is, the giant vulnerable component arises more easily and the risk of “global” crisis in the banking system is higher if poorly connected banks preferentially associate with other poorly connected banks. On the contrary, the disassortative banking systems show more resilience to global defaults. For example, when $\nu = 10$ or $\nu = 50$ and $4 \leq K \leq 6$, no risk of system-wide contagion exists in the disassortative banking network but in the assortative network, the risk of global default is positive. Second, for any given value of the degree cutoff $K$, at least in the plausible range of values, the size of the giant vulnerable component as a fraction of the whole banking network, is larger in the assortatively mixed network. This means that as soon as contagious defaults become system-wide (the giant vulnerable component emerges), the fraction of defaulting institutions in the banking network, or the scope of the crisis, is larger if the network is assortative. As a result, in the tiered banking structure, both the risk of systemic crisis and the scope of the crisis are lower than in the other types of structures.

These results are intuitively reasonable. In a disassortative banking network, high-degree banks are broadly distributed over the network and therefore, they presumably form links on many paths between other banks. This implies that with high probability an initial shock hitting a random bank reaches a high-degree bank in a small number of “steps”. Since high-degree banks are relatively resilient to neighbors’ defaults, the shock gets absorbed at that bank and does not spread any further. Therefore, both the risk of systemic defaults and the scale of these defaults in the disassortative network are lower than in the other structures. In fact, this simple intuition suggests that the finding of relative resilience to shocks of disassortative networks, derived here for the specific distribution in (19), can be generalized to a wider range of degree distributions, that is, to a larger class of assortative, disassortative and neutral networks. However, the formal investigation of this question is beyond the scope of this paper.

The extent of necessary bail-outs The comparison of the susceptibility to crises of assortative, disassortative and neutral banking networks allows for the evaluation of the relative effectiveness and costs of the optimal government bail-out strategy in these networks. Of course, in general, random network approach is not well suited for the analysis of regulation and normative measures, primarily due to the lack of behavioral foundations. Yet, we believe that the structural predisposition of assortative and neutral networks to large-scale contagious defaults and relative stability of disassortative networks still provide a basis for some policy lessons.

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51 Strictly speaking, in the setting where all values of $K$ are discrete, the phase transition occurs at $K = K_{\text{max}} + 1$ since by definition of $K_{\text{max}}$, at $K = K_{\text{max}}$ the giant vulnerable component is not yet formed. However, given that in the continuous setting of the next section, the phase transition occurs exactly at $K = K_{\text{max}}$, for consistency we refer to $K_{\text{max}}$ as the phase transition threshold also in this section.  
52 This range of $K$ is consistent with the empirical evidence of Upper (2011) and the calibration results of Gai and Kapadia (2010).  
53 Note that at any level $K$ above phase transition, the giant vulnerable component may include vulnerable banks of any in- and out-degree $k \leq K_{\text{max}}$. That is, it may even include the vulnerable banks with in- and out-degree $k \leq K_{\text{max}}$ that do not span the giant vulnerable component provided that banks with in- and out-degree above theirs are safe; these nodes become “dangerous” and may provoke an infinite number of further contagious defaults when the larger degree banks are not “immune” to defaults themselves.
We focus on targeted bail-outs, when the government only rescues, that is, restores solvency of the defaulting institutions which represent the highest risk in terms of the spread of shocks through the network. In our model, these are the banks with sufficiently large number of interbank connections as their bankruptcy poses a threat to solvency of many other banks and can eventually lead to system-wide breakdown. In the literature, such policy of rescuing banks that occupy key positions in the network is often referred to as “too interconnected to fail” (Markose et al., 2012; Battiston et al., 2012c; Chan-Lau, 2010). Furthermore, we regard the government targeted bail-out strategy as optimal if it minimizes the total number of targeted bail-outs subject to the constraint that the global system stability is preserved. This approach, where the total number rather than the total cost of bail-outs is minimized, though clearly simplistic, is quite natural in our model. Indeed, since any default in the banking network is caused by the failure of a single borrowing neighbor to repay its debt, the loss of any defaulting institution is the same. It is equal to the amount of interbank assets held against a single debtor bank. Then restoring the solvency of any defaulting bank requires the same injection of funds and hence, can be regarded as equally costly. Therefore, minimizing the total cost of bail-outs essentially amounts to minimizing the total number of them.

From this perspective, the optimal targeted bail-out strategy of the government is determined by the threshold $K_{\text{max}}$. That is, given the degree distribution of the banking network, the number of targeted bail-outs that guarantee the global system stability (where with probability 1 the giant vulnerable component does not arise) is minimized whenever the government rescues all defaulting institutions with in- and out-degree above $K_{\text{max}}$ but does not rescue the other institutions. Recall that by definition of $K_{\text{max}}$, banks with in- and out-degree smaller or equal to $K_{\text{max}}$ either do not default or default but do not endanger the global network stability, provided that banks with in-and out-degree above theirs are safe. On the other hand, any default of a bank with in-and out-degree above $K_{\text{max}}$ (but below $K$) provokes a large-scale “chain reaction” even if the banks with in- and out-degree above theirs are safe. Therefore, any threshold degree higher than $K_{\text{max}}$ would not guarantee the stability of the system, whereas any lower threshold would induce unnecessary bail-out costs.

This discussion mainly refers to the case when the degree cutoff $K$, separating safe and vulnerable banks, is high enough, so that $K_{\text{max}} < K$. Clearly, in the other case, when $K_{\text{max}} \geq K$, no bail-outs are required at all since by definition, all banks with in- and out-degree larger than $K$ are safe. For example, according to Fig. 1, when $\nu = 10$ or $\nu = 50$ and $4 \leq K \leq 6$, no bail-outs

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Notice that in the framework of this model, bailing-out defaulting banks simply means “transforming” these initially vulnerable institutions into safe institutions. After having been rescued by the government, they become “immune” to defaults and do not transmit shocks to their neighbors in the banking network.
are needed to preserve the stability of the disassortative banking network; at the same time, the assortative network requires rescues of all failing banks with in- and out-degree exceeding 2.

To formally evaluate and compare the number of optimal targeted bail-outs and hence, the total costs of bail-outs in assortative, disassortative and neutral banking systems, consider the worst-case scenario. Suppose that all banks with in- and out-degree in the “high-risk” range, $K_{\text{max}} < k \leq K$, actually default. In this worst-case scenario, the total share of banks that will need to be bailed-out is given by $\rho = \sum \frac{1}{p(k)}$. This value can be viewed as the maximum expected share of banks that need to be rescued (in case of their default) to preserve the global stability of the system, or the upper bound for the percentage of the required bail-outs. Using our numerical results for the value of the phase transition threshold $K_{\text{max}}$ and the three values of $K = 4, 5, 6$, we evaluate $\rho$ for the assortative, disassortative and neutral network.\footnote{According to Fig. 1, $K_{\text{max}} = 4, 6$ and 12 in the disassortative network at $\nu = 5$, 10 and 15, respectively; $K_{\text{max}} = 3, 5$ and 9 in the neutral network at $\nu = 5, 10$ and 15, respectively; and $K_{\text{max}} = 2$ in the assortative network at all $\nu$.}

**Result 2.** The maximum expected share of banks $\rho$ that need to be rescued upon default according to the optimal targeted bailout strategy is smaller in a disassortative and neutral network ($\alpha = 0.05$ and $\alpha = 0.1464$) than in an assortative network ($\alpha = 0.5$). This result is demonstrated in Table 1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Disassortative ($\alpha = 0.05$)</th>
<th>Neutral ($\alpha = 0.1464$)</th>
<th>Assortative ($\alpha = 0.5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu = 5$</td>
<td>$K = 4$ 0</td>
<td>$K = 5$ 0.0148</td>
<td>$K = 6$ 0.0262</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$K = 5$ 0.0262</td>
<td>$K = 5$ 0.0553</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$K = 6$ 0.0262</td>
<td>$K = 6$ 0.0694</td>
</tr>
<tr>
<td>$\nu = 10$</td>
<td>$K = 4$ 0</td>
<td>$K = 5$ 0.0262</td>
<td>$K = 6$ 0.0553</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$K = 5$ 0.0262</td>
<td>$K = 5$ 0.0553</td>
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<tr>
<td></td>
<td></td>
<td>$K = 6$ 0.0262</td>
<td>$K = 6$ 0.0694</td>
</tr>
<tr>
<td>$\nu = 50$</td>
<td>$K = 4$ 0</td>
<td>$K = 5$ 0.0262</td>
<td>$K = 6$ 0.0694</td>
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<td>$K = 5$ 0.0262</td>
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<td>$K = 6$ 0.0262</td>
<td>$K = 6$ 0.0694</td>
</tr>
</tbody>
</table>

For example, when $\nu = 5$ and $K = 5$, 1.48% of all banks need to be bailed-out in the worst-case scenario if the banking network is disassortative, but this percentage increases to 5.53% if the network is neutral and becomes as large as 20.87% if the network is assortative. Similarly, for larger $\nu$, no bail-outs are required in a disassortative and neutral network, apart from one case, $\nu = 10$ and $K = 6$, when the maximum of 1.87% of banks require rescue in the neutral network. At the same time, in the assortative network, the maximum share of banks to be rescued never falls below 16.71%. Thus, also in terms of the total bail-out costs, disassortativity, or tiering, in the banking network is a desirable property.

4.2. Financial contagion in scale-free networks

Let us now consider an alternative way of representing a tiered banking network that is consistent with the empirical evidence for a range of countries. Namely, let us assume that there are no degree correlations between neighboring nodes but that degree follows a power-law, or scale-free, distribution. The three interrelated features inherent in a scale-free degree distribution, that are widely observed in tiered banking systems, are (i) the presence of a small but non-negligible fraction of nodes with connectivity much larger than the average, (ii) the tendency of highly connected nodes to be linked with poorly connected ones and vice versa, and (iii) the numerical superiority of the low-degree nodes over the high-degree ones. Indirectly the scale-free representation of the banking network is also supported, both theoretically and empirically, by the findings that banks’ capital stocks are distributed according to a power law (Krause and Giansante, 2012; Gabaix, 2009).\footnote{In a broader context, power laws appear widely in economics and other fields. For instance, the distributions of the firm sizes (Luttmer, 2007), city sizes (Gabaix, 1999; Cordoba, 2008), wealth (Benhabib et al., 2010), and productivity of innovations (Chiglino, 2012) all appear to follow power laws.}

This, by means of the assumed relationship between bank’s capital stock and in-degree (cf.(4)), implies a power-law distribution for the degrees.

Formally, the scale-free degree distribution is defined by

$$p(k) = c(\lambda)k^{-\lambda}, \quad k = 1, 2, \ldots \; \text{and} \; 2 < \lambda < 3,$$

where, as before, $p(k)$ is the probability that a randomly chosen node has in- and out-degree each equal to $k$, $c(\lambda)$ is the normalizing constant and $\lambda$ governs the rate at which the probability decays with connectivity. The range of values for the
scaling parameter, $2 < \lambda < 3$, is typical in models that deal with the scale-free distribution.\textsuperscript{57} It means that the degree distribution is so broad that it displays unbounded second moments but still possesses a well-defined average degree. Arguably, it is more reasonable to think of a banking network as being generated by the scale-free degree distribution with such low values of $\lambda$, as this implies that $p(k)$ does not decline “too fast” with $k$ and a substantial share of banks in the network have more than one in- and out-going connection. This consideration is confirmed by some empirical evidence. For example, Boss et al. (2004a) report that the power-law degree distribution of the Austrian interbank market has exponent parameter $\lambda = 2.01$.\textsuperscript{58}

In the framework of this model, $\lambda$ has an important interpretation as the inverse of the level of tiering in the scale-free network. Indeed, as the value of $\lambda$ declines, the fraction of highly connected banks in the network increases and so does the probability that poorly connected banks link with highly connected ones rather than with each other. This implies that by varying the value of $\lambda$, one can study and compare the risk of systemic crisis and the extent of necessary bail-outs in scale-free banking networks with different levels of tiering.

Below, we consider these questions in detail. First, we find the closed-form expression for the phase transition threshold $K_{\text{max}}$ and study the functional dependence of $K_{\text{max}}$ on $\lambda$. After that we numerically evaluate and compare the size of the giant vulnerable component at different levels of $\lambda$. Finally, we discuss the optimal bail-out policy and consider how the level of tiering in the scale-free banking network affects the optimal number of targeted bail-outs.

In what follows it is convenient to approximate the theoretical framework by its continuum counterpart, where the average out-degree of a first vulnerable neighbor.\textsuperscript{59} If this quantity is less than one, all vulnerable clusters in the network are small and contagion dies out quickly. But as the value of $\lambda$ declines, the fraction of highly connected banks in the network increases and so does the probability that poorly connected banks link with highly connected ones rather than with each other. This implies that by varying the value of $\lambda$, one can study and compare the risk of systemic crisis and the extent of necessary bail-outs in scale-free banking networks with different levels of tiering.

Phase-transition threshold $K_{\text{max}}$ and the scope of the crisis To begin with, we observe that the absence of degree correlations between neighboring nodes allows for a simpler representation of the probability generating functions $H_0$ and $H_1$ and the average size of the vulnerable component $S$ derived in Section 3.6. In particular, generating function $H_1$ is now independent of $k$, the in- and out-degree of a node where a randomly chosen link starts. Therefore, $H'(1)$ can be taken out of the sum in Eq. (14) for $S$. Then, given the continuous framework for the degree distribution, the average size of the vulnerable component can be expressed as

$$S = 1 + \langle k \rangle H'(1) \int_1^\infty \zeta(k) \, dk = 1 + \langle k \rangle H'(1).$$

Function $H_1$, in its turn, can be written by analogy with $H_1(y|k)$ in (10) as

$$H_1(y) = \left(1 - \int_1^\infty \zeta(k) \, dk\right) + y \int_1^\infty \zeta(k) H_1(y)^k \, dk,$$

where we have made use of the equality $p(k'|k) = \zeta(k)$, which holds due to the absence of degree correlations. This expression for $H_1$ can be formulated more compactly by means of a new function $G$ that we define as follows:

$$G(y) = \int_1^\infty \zeta(k) y^k \, dk.$$

Then (22) becomes

$$H_1(y) = (1 - G(1)) + y G(H_1(y)),$$

and hence,

$$H'_1(1) = \frac{G(1)}{1 - G(1)}.$$  
(23)

Substituting this value of $H'_1(1)$ into (34), we arrive at the following simple expression for the average size of the vulnerable component:

$$S = 1 + \langle k \rangle \frac{G(1)}{1 - G(1)}.$$  
(24)

To gain insight into this result, consider that $G(1)$, whose value is critical for the definition of $S$, allows the following interpretation. When $K$ is sufficiently large, so that “many” nodes in the network are vulnerable, $G(1)$ approximates the average out-degree of a first vulnerable neighbor.\textsuperscript{50} If this quantity is less than one, all vulnerable clusters in the network are small and contagion dies out quickly. But as $G(1) \nearrow 1$, the average size of the vulnerable cluster, $S$, increases unboundedly. It diverges when $G(1) = 1$. It is at this point that the giant vulnerable component, whose size scales linearly with the size of the whole network, first forms. $G(1) = 1$ is therefore, the phase transition equation and it determines the threshold degree $K_{\text{max}}$ for vulnerable banks, above which systemic risk is realized and massive contagion is likely. Using the definition of $G(y)$

\textsuperscript{57} See, for example, Vega-Redondo (2007), Newman (2005), Boguñá et al. (2003), Clauset et al. (2009).

\textsuperscript{58} Other reported power laws for interbank networks have been within a relatively narrow range around 2.3, both for the in- and out-degree distributions (Soramäki et al., 2007; De Masi et al., 2006; Bech and Atalay, 2010).

\textsuperscript{59} For smaller $K$, $G(1)$ is less than this.
and the degree distribution \( p(k) \) with \( c(\lambda) = \lambda - 1 \), we obtain

\[
G(1) = \frac{1}{\langle k \rangle} \int_1^{K_{\text{max}}} k^2 p(k) \, dk = \frac{\lambda - 1}{\langle k \rangle} \frac{1}{3 - \lambda} K_{\text{max}}^{3 - \lambda} - 1. \tag{25}
\]

Now, given that \( \langle k \rangle = (\lambda - 1)/(\lambda - 2) \), condition \( G(1) = 1 \) leads to a simple closed-form expression for \( K_{\text{max}} \).

**Result 3.** In the scale-free network, the phase transition threshold \( K_{\text{max}} \) is given by

\[
K_{\text{max}} = \left( \frac{1}{\lambda - 2} \right)^{(1/3 - \lambda)}. \tag{26}
\]

\( K_{\text{max}} \) is monotonically decreasing in \( \lambda \) (i.e., increasing in the level of tiering) for all \( 2 < \lambda < 3 \). Therefore, the risk of the systemic crisis is smaller in scale-free banking networks with higher levels of tiering.\(^{60}\)

So, the critical point for the formation of the giant vulnerable component in the scale-free banking network is determined by the level of tiering. Even more important, though, the phase transition threshold, \( K_{\text{max}} \), is increasing in the level of tiering. The functional dependence of \( K_{\text{max}} \) on \( \lambda \) is illustrated in Fig. 2.

For \( \lambda \) sufficiently close to 2, the threshold degree \( K_{\text{max}} \) is large but it declines sharply with \( \lambda \). For example, when \( \lambda = 2.1 \), \( K_{\text{max}} \) is approximately 12 but it is only around 4 for \( \lambda \) equal to 2.4 and around 3 for \( \lambda \) equal to 2.7. This suggests that the resilience to global crises of the scale-free banking system with high level of tiering (\( \lambda \) is close to 2) is very high but it declines strongly when the level of tiering decreases (\( \lambda \) becomes large). Specifically, since all banks in the network with in- and out-degree greater or equal to \( K \) are safe and \( K \) is estimated to be in the range between 4 and 6 for real-world banking systems, no system-wide contagion occurs in the highly tiered scale-free network with \( \lambda \) close to 2. This result is consistent with the conclusion of the empirical studies by Boss et al. (2004a, 2004c) for the scale-free Austrian banking system where \( \lambda = 2.01 \). They report that “the banking system is very stable and default events that could be classified as a “systemic crisis” are unlikely” (Boss et al., 2004a, c). By contrast, for the scale-free network with lower level of tiering, when \( \lambda \) is around 2.4 or larger, the risk of global contagion exists unless some preventive measures are taken by the government.

These conclusions are confirmed in Fig. 3. More generally, the figure shows that not only are the scale-free networks with higher tiering more resilient to systemic breakdown but they also display a smaller scope of breakdown should a crisis occur. As before, the scope of systemic breakdown is captured by \( \omega \), the fractional size of the giant vulnerable component. On Fig. 3, it is shown as a function of the degree cutoff \( K \), for three different levels of \( \lambda \).\(^{61}\)

The dependence of \( \omega \) on \( \lambda \), demonstrated in Fig. 3, can be summarized as follows:

**Result 4.** For any \( K \geq 1 \) the relative size of the giant vulnerable component, \( \omega \), and hence, the scope of the crisis should it occur is smaller in scale-free networks with higher level of tiering.

Intuitively, a stronger resilience to crises and smaller scope of the crisis in scale-free networks with higher level of tiering can be explained as follows. When the level of tiering is high (\( \lambda \) is small), the fraction of highly connected banks in the network is relatively large and so is the number of paths between other banks in the network which “pass” through these

\(^{60}\) This simple comparative statics result is derived in the Appendix.

\(^{61}\) The size of the giant vulnerable component is computed numerically from the equations analogous to (17) and (18). Under the assumption of absent degree correlations and continuity of the degree distribution, (17) and (18) become

\[
1 - \bar{\omega} = \left( 1 - \frac{1}{\langle k \rangle} \int_1^\infty k p(k) \, dk \right) + \frac{1}{\langle k \rangle} \int_1^\infty k p(k)(1 - \bar{\omega}) \, dk, \tag{27}
\]

\[
1 - \omega = \left( 1 - \int_1^\infty p(k) \, dk \right) + \int_1^\infty p(k)(1 - \omega) \, dk. \tag{28}
\]

As before, \( \bar{\omega} \) represents the fraction of links leading to the giant vulnerable component and \( \omega \) is the fraction of nodes in the giant vulnerable component.
highly connected banks. As a result, in the scale-free network with high level of tiering, initial shocks in any part of the network reach highly connected banks and get absorbed by these banks “faster” than in a less tiered network. Therefore, eventually, a higher level of tiering in the scale-free banking network translates into better financial stability.

These findings shed some light on the controversy in the empirical literature regarding the susceptibility of tiered banking networks to systemic shocks. They suggest that if tiering is represented by the scale-free degree distribution, the extent of resilience to shocks and the scope of contagion are determined by the level of tiering. While highly tiered scale-free networks are extremely resilient to systemic defaults, less tiered structures can be fragile and defaults may propagate through such structures easily.

The extent of necessary bail-outs Now, as earlier in case of assortative, disassortative and neutral networks, we consider the optimal targeted bail-out strategy of the government, that aims at minimizing the number of targeted bail-outs subject to the constraint that the global system stability is preserved. As before, following this policy, the government should recover solvency of any defaulting bank with in- and out-degree in the “dangerous” range of connectivities, between the phase transition threshold $K_{\text{max}}$ and the threshold $\bar{K}$ above which all banks are safe. The exact number of such bail-outs may vary but the worst case scenario, in which all vulnerable banks with in- and out-degree between $K_{\text{max}}$ and $\bar{K}$ default, is easy to assess. To this end, we calculate the maximum expected share of banks, $\rho$, that need to be rescued upon default in order to avoid the systemic breakdown. Given the exact analytical expression for $K_{\text{max}}$ in terms of $\lambda$ (cf. (26)), we find the closed-form expression for $\rho$ and find its dependence on the level of tiering in the scale-free network.

**Result 5.** In the scale-free network, the maximum expected share of banks $\rho$ that need to be rescued upon default according to the optimal targeted bail-out strategy is equal to

$$
\rho = \int_{K_{\text{max}}}^{\bar{K}} p(k) \, dk = \left[ \frac{1}{K_{\text{max}}} \right]^{\lambda - 1} - \left[ \frac{1}{\bar{K}} \right]^{\lambda - 1}.
$$

(29)

For any $\bar{K} \geq 1$, $\rho$ is monotonically increasing in $\lambda$ for all $2 < \lambda < 3$, so that it is smaller in scale-free networks with higher level of tiering.\(^{62}\)

\(^{62}\) The short proof of this result is provided in the Appendix.
So, the upper bound for the percentage of bail-outs in the scale-free networks is increasing in $\lambda$ for any $2 < \lambda < 3$, or equivalently, it is decreasing in the level of tiering. Naturally also, $\rho$ is increasing in $\mathcal{K}$: the higher the threshold degree above which the banks are safe, the larger the gap between $K_{\text{max}}$ and $\mathcal{K}$ and hence, the larger the maximum expected share of targeted bail-outs. Fig. 4 illustrates these insights.

For example, consider the dependence of $\rho$ on $\lambda$ for $\mathcal{K} = 5$ (solid line on Fig. 4). In this case, $\rho$ is equal to 1.3% of all banks if $\lambda$ is 2.4, and it is equal to 6.8% of all banks if $\lambda$ is 2.7. Thus, the total number of bail-outs needed can be as high as 6.8% of the system if $\lambda = 2.7$ but if $\lambda$ is sufficiently close to 2, bail-outs may not be needed at all. Intuitively, this result is an immediate implication of stronger resilience to crises of scale-free networks with higher level of tiering: when the system is more tiered (i.e., smaller), highly connected, safe banks form a larger fraction of the system and therefore, the number of targeted bail-outs that are needed to prevent a massive contagion is smaller.

5. Discussion and conclusions

This paper develops a model of financial contagion in a banking system and studies the effects of tiering in the system on the risk and potential impact of system-wide defaults. High policy relevance of this issue and controversial empirical findings provide strong motivation for this research.

The banking system is represented by a random directed network, where nodes are banks and links indicate claims and obligations of banks to each other. In this framework, the tiered banking system is modeled in two ways: first, by a disassortative network, displaying negative degree correlations between neighboring nodes, and then by a scale-free network. Using the first modeling approach, we compare the resilience to systemic defaults and optimal bail-out strategy of the government in the tiered, disassortative network with those in other types of networks – the assortative network, displaying positive degree correlations, and the neutral network, displaying no correlations. Then, using the second approach, we focus on the effects of the variation in the level of tiering. We argue that in the scale-free network the level of tiering can be approximated by the inverse of the exponent parameter. By changing this parameter, we study the impact of the level of tiering on the susceptibility of the banking system to shocks and the extent of necessary bail-outs.

The key feature of the model that highlights the importance of network structure in determining the spread financial contagion is the counteracting effects of bank connectivity. While greater connectivity increases the spread of contagion in the banking network and generates greater absolute exposure to other banks’ defaults, it also improves risk sharing and guarantees higher capital levels, which amplify banks’ stability. However, in this model, the effect of greater absolute risk exposure associated with a larger number of incoming links exactly offsets the positive effect from greater risk sharing. This is due to the assumption of exact proportionality of a bank’s interbank assets to the number of its incoming links. Then the effects of connectivity that eventually matter for the stability of the system are the remaining counteracting effects: higher potential for the spread of shocks due to a larger out-degree and higher potential for the absorption of shocks due to a larger capital (associated with larger in-degree). These opposing effects of connectivity interact differently in different structures. As a result, the resilience of a banking system to shocks and the optimal number of bank rescues depend on the features of the underlying degree distribution.

For alternative model specifications, where interbank assets increase less or more than proportionally with the number of links, the effects of risk exposure and risk sharing do not offset each other completely. Namely, with less than proportional increase of assets in the number of links, the effects of greater risk exposure are weaker than the effects of better risk-sharing. For example, in an extreme case, when interbank asset positions of banks are independent of the number of links and fixed at the same level for all banks, higher connectivity does not change risk exposure and only improves risk sharing. Then even if the capital is kept constant (does not grow with the number of links), greater connectivity still has the conflicting effects: it creates more channels for contagion but it also augments risk sharing. The existence of these opposing connectivity effects leads essentially to the same conclusions in the model. In fact, as long as the total interbank assets increase less than proportionally with the number of links, all of the paper’s main results continue to apply subject to shifts in the values of degree thresholds $\mathcal{K}$ and $K_{\text{max}}$. On the other hand, if the total interbank asset position increases more than proportionately with the number of links and the capital is kept constant or does not grow “fast enough”, greater connectivity unambiguously increases contagion risk. This latter case however, does not seem to be a plausible description of reality (Gai and Kapadia, 2010).

The findings of the paper suggest that as soon as highly connected banks are sufficiently well capitalized, that is, the capital is increasing in bank’s in-degree, tiering in the banking network improves its financial stability. Namely, we find that the tiered, disassortative banking network is more resilient to shocks and in the event of a crisis, displays a lower number of failures than assortative and neutral networks. Similarly, in the scale-free network, the resilience to contagion is increasing in the level of tiering. Consequently, the threshold degree above which the defaulting banks endanger the stability of the system and require government assistance is higher in a (more) tiered network, so that the overall number of bail-outs tends to be lower.

The advantages of tiering can be explained by the fact that in tiered disassortative networks, high-degree banks, which are able to absorb losses due to their large size, are distributed broadly across the network. Similarly, in scale-free networks, the proportion of these high-degree banks is larger at higher levels of tiering. Clearly, if the capital of a bank was not increasing but decreasing in in-degree, these results would not have held as highly connected banks would have unambiguously eroded system stability. Indeed, not only would they spread the shock to a large number of other banks,
but they would also be particularly vulnerable to default when one of their borrower banks fails to repay its obligations. This scenario, however, is at variance with current bank capital requirements and literature findings which suggest that the capital is positively related to bank’s interbank exposures and those, in turn, are increasing in bank’s in-degree.

The findings of the paper draw attention to some important aspects of the banking system that should be taken into account in policy making. First, relatively high resilience of tiered banking networks to contagion implies that regulators should seek measures that promote the formation of a highly tiered system of interbank relations. While direct meddling in the interbank market may not be feasible, a regulator could design an incentive scheme that induces appropriate decisions by banks in establishing their credit relationships with each other. Second, in line with the current thinking in banking regulation, our model confirms the importance of preventing the failure of large and highly connected banks, whose default threatens the stability of many other financial institutions. This is particularly close to the approach in the “too big to fail” and “too interconnected to fail” policy, that is designed to rescue financially distressed banks which play a key role in the banking network (Freixas et al., 2000; Kaufman, 2002; Haldane and Robert, 2011; Krause and Giansante, 2012; Markose et al., 2012; Battiston et al., 2012c). Preventing failure of these key banks may require tighter regulations for them. This goes along with the recent discussion of the “too big/too interconnected to fail” approach that calls for imposing higher capital requirements for systemically important and interconnected institutions (Acharya and Richardson, 2009; Brunnermeier et al., 2009; Bernanke, 2009; Chan-Lau, 2010; Krause and Giansante, 2012; Krugman, 2010). Third, in case defaults occur and show the signs of contagion, the extent of necessary bail-outs should be determined by explicitly taking into account the features of the banking network. Specifically, if the degree distribution in the banking network is scale-free, the extent of government intervention depends, among other things, on the exponent parameter of the scale-free network.

The model and results presented in this paper suggest some directions for future research. For example, to better understand the implications of the model for real banking networks, one could simulate the model for a large banking system, using real balance sheets and calibrating the joint degree distribution to match the observed data. Another extension is to loosen some of the assumptions of the present theoretical setting, allowing for unequal distribution of interbank exposures across bank’s counterparties and for the possibility of cycles in the banking network. These strands of research would add realism to the model and provide new, potentially valuable insights.

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Appendix A

A.1. Relaxing the assumption of equal in- and out-degree

In our presentation of the model, we assumed that in- and out-degree of a node were represented by a single random variable, distributed according to some probability distribution \( p(k) \), so that for any realization of the random network, the number of incoming links of a bank was exactly equal to the number of its outgoing links. In this section we relax this assumption of a perfect match between in- and out-degree. We model each of the two degrees by a separate random variable, distributed according to some probability distribution \( f(k) \) for the in-degree and \( g(k) \) for the out-degree. Furthermore, to enable studying the effects of tiering, we stick to a particular type of the joint probability distribution where the sum of the two degrees at any bank follows a scale-free distribution of the kind analyzed in Section 4.2. This means that most banks in the network have low overall connectivity, with both in- and out-degree being small, but there is also a small but non-negligible fraction of banks whose overall connectivity greatly exceeds the average. We refer to this distribution as a joint scale-free distribution. Formally, it is defined by

\[
p(j, k) = C(\lambda)(j + k - 1)^{-(\lambda + 1)}, \quad k \geq 1, \tag{30}
\]

where \( p(j, k) \) is the probability that a randomly chosen node has in-degree \( j \) and out-degree \( k \). \( C(\lambda) \) is the normalizing constant, and \( \lambda \) is the scaling parameter that controls the rate of decline in probability as the overall bank’s connectivity \( j + k \) increases. The corresponding marginal distributions of in- and out-degrees are also scale-free and given by (20): \( p(j) = c(\lambda)j^{-\lambda} \) for the in-degree and \( p(k) = c(\lambda)k^{-\lambda} \) for the out-degree. As before, we consider \( 2 < \lambda < 3 \), so that the

63 In the literature, it is also called bivariate Pareto distribution of Type 1 (original definition is due to Mardia (1962)).
distribution of the overall bank’s connectivity is so broad that it displays unbounded second moments but the average overall connectivity is still well defined.

Consistently with the analysis in Section 4.2, we study the effects of tiering on the stability of the banking network by comparing the phase-transition thresholds and the sizes of the giant vulnerable components in scale-free networks with different scaling parameter \( \lambda \). Small values of \( \lambda \) correspond to scale-free networks with high level of tiering; on the contrary, when \( \lambda \) is close to 3, the network is practically homogenous as the fraction of highly connected banks is extremely small. In what follows we approximate the actual degree distribution in (30) by its continuum counterpart, where \( j \) and \( k \) are continuous random variables in the range \([1, \infty)\) and \( p(j, k) \) is a continuous density function with \( C(\lambda) = \lambda(\lambda - 1) \).

To find the point of phase transition and the relative size of the giant vulnerable component, we consider the analogues of Eqs. (17) and (18), adjusted to the setting where in- and out-degree are separate random variables and degrees of neighboring nodes are independent:

\[
1 - \hat{\omega} = \left( 1 - \frac{1}{\langle j \rangle} \int_1^{\infty} \int_1^{\infty} dp(j, k) \, dk \right) + \frac{1}{\langle j \rangle} \int_1^{\infty} \int_1^{\infty} dp(j, k)(1 - \hat{\omega})^k \, dk,
\]

\[
1 - \omega = \left( 1 - \int_1^{\infty} \int_1^{\infty} p(j, k) \, dk \right) + \int_1^{\infty} \int_1^{\infty} p(j, k) \, dk (1 - \omega)^k \, dk.
\]

In the first equation, \( \hat{\omega} \) denotes the probability that a randomly selected link in the network leads to the giant vulnerable component. Then the probability of the opposite event accounts for the possibility that this link leads to a safe bank or to a vulnerable bank where none of the outgoing links leads to the giant vulnerable component. By analogy with the previous analysis, the joint in- and out-degree distribution of a neighboring bank, reached by following a randomly selected link is given by \( 1/j \int_1^{\infty} p(j, k) \, dk \), where \( j \) denotes the average in- (and also out-) degree. Using \( \omega \), the second equation determines our main value of interest, \( \omega \), which represents the relative size of the giant vulnerable component, or the probability that a randomly selected bank belongs to the giant vulnerable component. The probability of the opposite event, \( 1 - \omega \), reflects two possibilities: a randomly selected bank is either safe or vulnerable but such that none of its outgoing links leads to the giant vulnerable component.

As in our earlier analysis, \( \omega \) can be considered as a function of the in-degree cutoff parameter \( K \), separating safe and vulnerable banks. It is increasing in \( K \) and changing from zero to positive at \( K = K_{\text{max}} \), which marks the point of phase transition. Fig. 5 demonstrates this functional dependence for three different values of scaling parameter \( \lambda \).

The findings presented on the figure are broadly consistent with those in our prior analysis. They support the advantages of tiering in the banking network. As before, the phase transition threshold \( K_{\text{max}} \) is larger at high levels of tiering, while the relative size of the giant vulnerable component, \( \omega \), is smaller, for any \( K \geq 1 \). For example, when \( K = 5 \), so that all banks with in-degree above 5 are safe and all the other banks are vulnerable, the highly tiered scale-free network corresponding to \( \lambda = 2.1 \) displays strong resilience to shocks: no giant vulnerable component exists and the risk of large-scale contagion is minimal. At the same time, scale-free networks corresponding to \( \lambda = 2.4 \) and \( \lambda = 2.7 \), are not financially stable and initial shock is likely to result in a systemic crisis, affecting a substantial fraction of the banking network. Thus, also in case when in- and out-degree of a bank are not the same, both the risk of the systemic crisis and the scope of the crisis should it occur are lowest in tiered and particularly, in highly tiered banking networks.

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64 In fact, a closer analogue is the set of Eqs. (27), (28) (see footnote 65) for the scale-free networks in Section 4.2.

65 The first term in brackets on the right-hand side of (31) and (32) can be simplified to produce \( 1 - 1/j \int_1^{\infty} p(j) \, dj \) in the first case and \( 1 - \int_1^{\infty} p(j) \, dj \) in the second case, where \( p(j) \) is the marginal distribution of in-degrees implied by the joint distribution of in- and out-degrees \( p(j, k) \).
A.2. Proofs

Derivation of the Eq. (15) Consider the representation of $S$, the average size of the vulnerable component, in Eq. (14):

$$S = 1 + \langle k \rangle \sum_{k=1}^{\infty} \zeta(k) H_1(1|k).$$

The values of $H_1(1|k)$ for all $k = 1, 2, \ldots$ can be obtained by differentiating (10) at $y = 1$. The self-consistency condition produced by this differentiation is

$$H_1(1|k) = \sum_{k=1}^{\infty} p(k'|k) + \sum_{k=1}^{\infty} k' p(k'|k)H_1(1|k') \quad k = 1, 2, \ldots$$

Using this expression for $H_1(1|k)$ and the definition of $p(k'|k) = \xi(k,k')/\zeta(k)$ (cf. (9)), the above equation for $S$ can be rewritten as follows:

$$S = 1 + \langle k \rangle \sum_{k=1}^{\infty} \zeta(k,k') + \langle k \rangle \sum_{k=1}^{\infty} k' \xi(k,k') H_1(1|k').$$

Next, by changing the order of summation and employing the fact that $\sum_{k=1}^{\infty} \xi(k,k') = \zeta(k')$, we readily arrive at

$$S = 1 + \langle k \rangle \sum_{k=1}^{\infty} \zeta(k) + \langle k \rangle \sum_{k=1}^{\infty} k' \xi(k,k') H_1(1|k').$$

Now, in this expression for $S$ both sums on the right-hand side have a finite number of terms, and it remains to find the closed-form solution for the first $K$ elements of the sequence $(H_1(1|k))_k$. These can be easily derived from the first $K$ linear equations in (33). Substituting the result into (34), we then arrive at the following matrix expression for the average size of the vulnerable cluster:

$$S = 1 + \langle k \rangle 1 \zeta - \langle k \rangle \tilde{\zeta} M^{-1} \tilde{\zeta},$$

where $1$ is the vector of 1’s, $\zeta = (\zeta(k))_{k=1}^{\infty}$, $\tilde{\zeta} = (\sum_{k=1}^{\infty} \xi(k,k'))_{k=1}^{\infty}$, and $\tilde{\zeta} = (k \xi(k,k'))_{k=1}^{\infty}$. The product $-M^{-1} \tilde{\zeta}$ is the vector-form solution for $(H_1(1|k))_k$ derived from the equations in (33), where once again, we made use of the definition $p(k'|k) = \xi(k,k')/\zeta(k)$. Therefore, the elements of matrix $M$ are given by

$$m_{k,k'} = k' \xi(k,k') - \zeta(k') \delta_{kk'}, \quad k, k' = 1, 2, \ldots, K$$

Proof of Result 3. $K_{\text{max}}$ is monotonically decreasing in $\lambda$ as soon as its derivative with respect to $\lambda$ is strictly negative for all $2 < \lambda < 3$. To calculate the derivative of $K_{\text{max}}$, notice that from (26):

$$\ln(K_{\text{max}}) = \frac{1}{3-\lambda} \ln \left( \frac{1}{\lambda-2} \right),$$

so that

$$\frac{dK_{\text{max}}}{d\lambda} = \frac{1}{(3-\lambda)^2} \ln \left( \frac{1}{\lambda-2} \right) - \frac{1}{3-\lambda} \cdot \frac{1}{\lambda-2}.$$ 

From this expression it follows that

$$\frac{dK_{\text{max}}}{d\lambda} = K_{\text{max}} \cdot \frac{1}{3-\lambda} \cdot \frac{-(\lambda-2)\ln(\lambda-2)-(3-\lambda)}{(3-\lambda)(\lambda-2)}. $$

Since $2 < \lambda < 3$, this derivative is negative as soon as $f(\lambda) = -(\lambda-2)\ln(\lambda-2)-(3-\lambda) < 0$. Notice that $f(\lambda)$ is monotonically increasing in $\lambda$ for any $2 < \lambda < 3$. Indeed:

$$\frac{df(\lambda)}{d\lambda} = -\ln(\lambda-2) > 0 \quad \forall 2 < \lambda < 3.$$ 

Hence, for all $2 < \lambda < 3$, $f(\lambda) < f(3) = 0$. Therefore, $dK_{\text{max}}/d\lambda$ in (35) is strictly negative for any $2 < \lambda < 3$.

Proof of Result 5. $\rho$ is monotonically increasing in $\lambda$ if its derivative with respect to $\lambda$ is strictly positive for all $2 < \lambda < 3$. I differentiate with respect to $\lambda$ both ratios on the right-hand side of (29). Denote the first ratio by $y$:

$$y = \left[ \frac{1}{K_{\text{max}}} \right]^{\lambda-1} = (\lambda-2)^{1/(3-\lambda)}.$$
Then
\[ \ln(y) = \frac{\lambda - 1}{3 - \lambda} \ln(\lambda - 2) \]
and
\[ \frac{dy}{d\lambda} = (\lambda - 2)^{1/(3 - \lambda)} \left( \frac{2}{(3 - \lambda)^2 \ln(\lambda - 2)} + \frac{\lambda - 1}{(3 - \lambda)(\lambda - 2)} \right). \]
Therefore,
\[ \frac{d\rho}{d\lambda} = (\lambda - 2)^{1/(3 - \lambda)} \cdot \left( \frac{2}{(3 - \lambda)^2} \ln(\lambda - 2) + \frac{\lambda - 1}{(3 - \lambda)(\lambda - 2)} \right) - \left( \frac{1}{\lambda} \right)^{1/(3 - \lambda)} \ln \left( \frac{1}{\lambda} \right) \]
\[ = (\lambda - 2)^{1/(3 - \lambda)} \cdot \frac{1}{3 - \lambda} \cdot \frac{2(\lambda - 2) - (\lambda - 1)(3 - \lambda)}{(3 - \lambda)(\lambda - 2)} - \left( \frac{1}{\lambda} \right)^{1/(3 - \lambda)} \ln \left( \frac{1}{\lambda} \right) \]

Since \( 2 < \lambda < 3 \) and \( K \geq 1 \), this derivative is positive as soon as \( \rho(\lambda) = 2(\lambda - 2)\ln(\lambda - 2) + (\lambda - 1)(3 - \lambda) > 0 \). Notice that \( \rho(\lambda) \) is monotonically decreasing in \( \lambda \) for any \( 2 < \lambda < 3 \). Indeed:
\[ \frac{d\rho(\lambda)}{d\lambda} = 2(\ln(\lambda - 2) - \lambda + 3) < 0 \quad \forall 2 < \lambda < 3, \]
where \( 2(\ln(\lambda - 2) - \lambda + 3) < 0 \) due to the fact that \( 2(\ln(\lambda - 2) - \lambda + 3) \) is monotonically increasing in \( \lambda \forall 2 < \lambda < 3 \) and its maximum value at \( \lambda = 3 \) is equal to 0.

So, \( \rho(\lambda) \) is decreasing in \( \lambda \) for all \( 2 < \lambda < 3 \). Therefore, for \( 2 < \lambda < 3 \), \( \rho(\lambda) > \rho(3) = 0 \). This implies that \( d\rho/d\lambda \) is strictly positive for any \( 2 < \lambda < 3 \). □

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