Vector valued Fourier transforms and Fourier type operators

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1. Introduction
   - Motivation
   - Abstract harmonic analysis, examples
   - Fourier type operators-Definitions

2. Fourier type $p$ with respect to the Cantor group
   - Old and new results

3. $B$-convexity and Fourier type
   - $B$-convex spaces
   - Bourgain’s Hausdorff-Young inequalities

4. Fourier type 2 operators
   - Kwapień’s result and factorization though a Hilbert space
   - Transference principle for Fourier type 2 operators

5. Final remarks and discussion
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5. Final remarks and discussion
All Banach spaces are equal but some Banach spaces are more equal than others.
## Motivation and Introduction

### Problem

- **Scalar-valued results**
  - EXTENSION ?
- **Vector-valued results**

### Possible answers

- Results remain true for any Banach space,
- Only “trivial” extensions remain true,
- Extension depends on the structure and geometry of Banach space.
Motivation and Introduction

Problem

Scalar-valued results \( \sim \) ? Vector-valued results

Possible answers

- Results remain true for any Banach space,
- Only “trivial” extensions remain true,
- Extension depends on the structure and geometry of Banach space.
Hausdorff-Young inequality

**Fourier transform**

For a function $f \in L_1(\mathbb{R})$ the Fourier transform $\mathcal{F}_\mathbb{R} f$ is given by

$$(\mathcal{F}_\mathbb{R} f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ist} \, dt$$

**Hausdorff-Young inequality**

If $1 \leq p \leq 2$ then we have

$$\|\mathcal{F}_\mathbb{R} f\|_{L'_p(\mathbb{R})} \leq c \|f\|_{L_p(\mathbb{R})} \quad \text{for all} \quad f \in L_p(\mathbb{R}).$$

We study Hausdorff-Young inequalities for vector-valued functions.
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We study Hausdorff-Young inequalities for vector-valued functions.
Abstract harmonic analysis

- We work in the framework of a *locally compact abelian group* $G$ (shortly: lca) which comes equipped with its *Haar measure* $\mu_G$.
- A *character* $\gamma$ on $G$ is a continuous homomorphism from $G$ into the torus $\mathbb{T}$. The collection of characters on $G$ is an abelian group under pointwise multiplication and carries a natural locally compact topology. The resulting lca group is the *dual group* $G'$ of $G$.
- For a function $f \in L_1(G)$, the *Fourier transform* $\mathcal{F}_G f$ is defined by

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(\mathcal{F}_G f)(\gamma) = \int_G f(t)\overline{\gamma(t)} \, d\mu_G(t) \quad \text{for} \quad \gamma \in G'.
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Examples

Integers $\mathbb{Z}$

The characters on $\mathbb{Z}$ are given by $\gamma(k) = z^k$ for some $z \in \mathbb{T}$. It turns out that $\mathbb{Z}' \cong \mathbb{T}$ and the Fourier transform is given by

$$(\mathcal{F}_\mathbb{Z} f)(e^{it}) = \sum_{n \in \mathbb{Z}} f(n)e^{-int} \quad \text{for} \quad e^{it} \in \mathbb{T}.$$
**Integers \( \mathbb{Z} \)**

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**Torus \( \mathbb{T} \)**

The characters on \( \mathbb{T} \) are given by \( \gamma(z) = z^k \) for some \( k \in \mathbb{Z} \). It turns out that \( \mathbb{T}' \cong \mathbb{Z} \) and the Fourier transform is given by

\[
(\mathcal{F}_\mathbb{T} f)(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{it})e^{-int} \, dt \quad \text{for} \quad n \in \mathbb{Z}.
\]
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More examples

Real line \( \mathbb{R} \)

The characters on \( \mathbb{R} \) are given by \( \gamma(x) = e^{ixy} \) with \( y \in \mathbb{R} \). It turns out that \( \mathbb{R}' \cong \mathbb{R} \) and the Fourier transform is given by

\[
(F_{\mathbb{R}}f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iyx} \, dx \quad \text{for} \quad y \in \mathbb{R}.
\]

Cantor group \( \mathbb{D} = \mathbb{Z}_2^\infty = \{0,1\}^\mathbb{N} \)

For \( n \in \mathbb{N}_0 \) let \( n = \sum_{k=0}^{\infty} n_k 2^k \) with \( n_k \in \{0,1\} \). The characters on \( \mathbb{D} \) are given by \( \psi_n(x) = (-1)^{\langle n,x \rangle} \) with \( \langle n,x \rangle = n_0x_0 + n_1x_1 + \ldots \) (mod 2) for \( n \in \mathbb{N}_0 \) and \( x \in \mathbb{D} \). It turns out that \( \mathbb{D}' \cong (\mathbb{N}_0, \oplus) \) and the Fourier transform is given by

\[
(F_{\mathbb{D}}f)(n) = \int_{\mathbb{D}} f \psi_n \, d\mu, \quad \text{for} \quad n \in \mathbb{N}_0.
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More examples

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The characters on \( \mathbb{R} \) are given by \( \gamma(x) = e^{ixy} \) with \( y \in \mathbb{R} \). It turns out that \( \mathbb{R}' \cong \mathbb{R} \) and the Fourier transform is given by

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Fourier type of Banach spaces

The Bochner-Lebesgue space

$$L_p^X(G) = \{ f : G \rightarrow X : \int_G \|f(t)\|_X^p \, d\mu_G(t) < \infty \}$$

Definition (J. Peetre 1969 $G = \mathbb{R}$, M. Milman 1984 general case)

A Banach space $X$ has a Fourier type $p$ ($1 \leq p \leq 2$) with respect to $G$ if the operator $\mathcal{F}_G$ originally defined on $L_p(G) \otimes X$ by

$$\mathcal{F}_G (\sum_{i=1}^n \varphi_i x_i) (\gamma) = \sum_{i=1}^n (\mathcal{F}_G \varphi_i) (\gamma) x_i, \quad \varphi_i \in L_p(G), x_i \in X$$

can be extended to a bounded operator $\mathcal{F} : L_p^X(G) \rightarrow L_{p'}^X(G')$. In other words,

$$\|\mathcal{F}f\|_{L_p^X(G')} \leq c \|f\|_{L_p^X(G)}.$$
Fourier type of Banach spaces

The Bochner-Lebegue space

\[ L^X_p(G) = \{ f : G \to X : \int_G \| f(t) \|^p_X \, d\mu_G(t) < \infty \} \]

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can be extended to a bounded operator \( \mathcal{F} : L^X_p(G) \to L^{X'}_{p'}(G') \). In other words,

\[ \| \mathcal{F}_G f \|_{L^{X'}_{p'}(G')} \leq c \| f \|_{L^X_p(G)}. \]
Fourier type operators

**Definition**

An operator $T \in \mathcal{L}(X, Y)$ is said to be of *Fourier type* $p$ ($1 \leq p \leq 2$) with respect to $G$ if the operator

$$\mathcal{F}_G \otimes T : L_p(G) \otimes X \to L'_p(G') \otimes Y$$

extends to a bounded linear operator from $L^X_p(G)$ to $L^Y_p(G')$. In other words

$$\| (\mathcal{F}_G \otimes T)f \|_{L'_p(G')} \leq c \| f \|_{L^X_p(G)}.$$  

The class of all operators of Fourier type $p$ equipped with the operator norm of the extended operator (denoted by $\| \cdot \|_{\mathcal{FT}^G_p}$) is a Banach operator ideal $\mathcal{FT}^G_p$. 
Transference principles

Theorem (M. Milman (1984))

Let $1 < p_1 < p_2 < 2$.

$$\mathcal{FT}^G_{p_2} \subset \mathcal{FT}^G_{p_1} \subset \mathcal{FT}^G_1 \subset \mathcal{FT}^G_1 = \mathcal{L}.$$ 

Problem

Do the ideals $\mathcal{FT}^G_p$ depend at all on the infinite lca group $G$?

More precisely

Let $G_1, G_2$ be infinite lca groups and $p \in (1, 2)$.

- Inclusion: $\mathcal{FT}^{G_1}_p \subset \mathcal{FT}^{G_2}_p$?
- Equality: $\mathcal{FT}^{G_1}_p = \mathcal{FT}^{G_2}_p$?
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Theorem (M. Milman (1984))
Let $1 < p_1 < p_2 < 2$.

$$\mathcal{FT}_2^G \subset \mathcal{FT}_{p_2}^G \subset \mathcal{FT}_{p_1}^G \subset \mathcal{FT}_1^G = \mathcal{L}.$$ 

Problem
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Let $G_1, G_2$ be infinite lca groups and $p \in (1, 2)$.

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Old and new results

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Known result


\[ \mathcal{F}T_{p}^\mathbb{R} = \mathcal{F}T_{p}^\mathbb{Z} = \mathcal{F}T_{p}^\mathbb{T} = \mathcal{F}T_{p}^\mathbb{R}^n = \mathcal{F}T_{p}^\mathbb{Z}^n = \mathcal{F}T_{p}^\mathbb{T}^n \]

Cantor group

\[ \mathbb{D} = \{ x = (x_n)_{n \in \mathbb{N}} : x_n \in \{0, 1\} \} \]

Its continual analogue

\[ \mathbb{F} = \{ x = (x_n)_{n \in \mathbb{Z}} : x_n \in \{0, 1\} \text{ and } x_n \to 0 \text{ for } n \to -\infty \} \]
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\[ \mathcal{FT}_p^\mathbb{R} = \mathcal{FT}_p^\mathbb{Z} = \mathcal{FT}_p^\mathbb{T} = \mathcal{FT}_p^{\mathbb{R}^n} = \mathcal{FT}_p^{\mathbb{Z}^n} = \mathcal{FT}_p^{\mathbb{T}^n} \]

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Its continual analogue

\[ F = \{ x = (x_n)_{n \in \mathbb{Z}} : x_n \in \{0, 1\} \text{ and } x_n \to 0 \text{ for } n \to -\infty \} \]
Let $1 < p < 2$. For an operator $T \in \mathcal{L}(X, Y)$ the following statements are equivalent:

- $T$ has Fourier type $p$ with respect to group $\mathcal{D}$.
- $T$ has Fourier type $p$ with respect to group $\mathcal{D}^m$ for all $m \in \mathbb{N}$.
- $T$ has Fourier type $p$ with respect to group $\mathcal{F}$.
- $T$ has Fourier type $p$ with respect to group $\mathcal{F}^m$ for all $m \in \mathbb{N}$.

Moreover, in this case all norms coincide.

\[ \mathcal{F}T_p^{\mathcal{D}} = \mathcal{F}T_p^{\mathcal{F}} = \mathcal{F}T_p^{\mathcal{D}^m} = \mathcal{F}T_p^{\mathcal{F}^m} \]
New result

**Theorem**

Let $1 < p < 2$. For an operator $T \in \mathcal{L}(X, Y)$ the following statements are equivalent

- $T$ has Fourier type $p$ with respect to group $\mathbb{D}$.
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Rademacher type and $B$-convex spaces

**Definition (Rademacher type)**

A Banach space $X$ has the Rademacher type $p$ ($1 \leq p \leq 2$), if there is a constant $c > 0$ such that for any $x_1, \ldots, x_n \in X$

$$\left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_{L^2_X} \leq c \left( \sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}.$$

**Theorem (G. Pisier, B. Maurey):**

A Banach space $X$ is $B$-convex if and only if it has some nontrivial Rademacher type if and only if it does not contain the spaces $\ell_1^n$ uniformly.
Rademacher type and $B$-convex spaces

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Bourgain’s Hausdorff-Young inequality for cyclic groups

Theorem (J. Bourgain (1988)): A Banach space $X$ is B-convex if, and only if, it has some nontrivial Fourier type with respect to the classical groups or the Cantor group.

Theorem: Let $m$ be a power of a prime. A Banach space $X$ is B-convex if, and only if, it has some nontrivial Fourier type with respect to $\mathbb{Z}_m^\infty$. 
Bourgain’s Hausdorff-Young inequality for cyclic groups

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Kwapień’s result and factorization though a Hilbert space

Theorem (S. Kwapień (1972):)

A Banach space $X$ has Fourier type 2 with respect to some infinite lca group if and only if it is isomorphic to a Hilbert space.

Open question

Let $\mathcal{H}$ denote the class of all operators $T$ factoring through a Hilbert space. If $G$ is infinite group, is it true that

$$\mathcal{F}T^G_2 = \mathcal{H}?$$

In other words, does every operator of Fourier type 2 with respect to $G$ factor through Hilbert space?
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Open question

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$$\mathcal{FT}_2^{G} = \mathcal{H}?$$

In other words, does every operator of Fourier type 2 with respect to $G$ factor through Hilbert space?
Theorem (A. Hinrichs, M.P.): Any operator of Fourier type 2 with respect to the classical groups has Fourier type 2 with respect to all lca groups. More precisely,

\[ \mathcal{FT}^T_2 \subseteq \mathcal{FT}^G_2 \quad \text{and} \quad \| T | \mathcal{FT}^G_2 \| \leq \| T | \mathcal{FT}^T_2 \| \]

holds for all lca groups G and all \( T \in \mathcal{FT}^T_2 \).
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Research questions

2. Vector–valued Fourier multiplier theorems, pseudodifferential operators with operator valued symbols
THANK YOU FOR YOUR ATTENTION