Return probability: Exponential versus Gaussian decay

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Abstract
We analyze, both analytically and numerically, the time-dependence of the return probability in closed systems of interacting particles. Main attention is paid to the interplay between two regimes, one of which is characterized by the Gaussian decay of the return probability, and another one is the well-known regime of the exponential decay. Our analytical estimates are confirmed by the numerical data obtained for two models with random interaction. In view of these results, we also briefly discuss the dynamical model which was recently proposed for the implementation of a quantum computation.

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1. Introduction
As is known, in many-body systems the density of energy levels increases extremely fast with an excitation energy. As a result, for highly excited states even a very weak interaction between particles can lead to a strong mixing between the unperturbed basis states, resulting in a complex structure of exact eigenstates. In this case one can speak about chaotic eigenstates since their components can be practically treated as pseudo-random ones. In the dynamics, this fact results in a relaxation of the system to a steady-state distribution. The studies of complex atoms [1], multi-charged ions [2], nuclei [3], Bose–Einstein condensates [4], and spin systems [5,6] have confirmed the dynamical origin of statistical laws in isolated systems (see, also, Refs. [7,8] and the review [9]).

Recently, the theory of many-body chaos has been extended to the models of a quantum computation. In particular, it was argued that due to a very high density of energy levels, any kind of perturbation may lead to decoherence effects thus destroying the operability of a quantum computer. Therefore, it is of importance to search for the conditions when the role of chaos can be significantly reduced [10,11].

So far, the study of the many-body chaos has been mainly restricted by the investigation of statistical properties of the energy spectra and eigenstates. On the other hand, in view of experimental applications, one needs to know what are the dynamical properties of quantum systems with strongly interacting particles. Below, we analyze the time dependence of the widely used quantity, the so-called return probability which determines global properties of the dynamics. Our main interest is in the interplay between the exponential and Gaussian decrease of the return probability, in dependence on the strength of the interaction between particles.

2. Return probability and strength function
In what follows we consider physical systems described by the total Hamiltonian $H$ which can be written in the separable form,

$$H = H_0 + V. \tag{1}$$

Typically, such a representation is used when studying the influence of a perturbation $V$ on a system governed by the unperturbed Hamiltonian $H_0$. In many physical applications this form of $H$ is quite natural, reflecting a different nature of $H_0$ and $V$. However, quite often the separation of $H$ into the two parts is not well defined, as in the case of mean field approaches used to introduce “good” variables in which $H_0$ has a relatively simple
form in comparison with a “residual interaction” $V$. In what follows we discuss some of generic properties of the dynamics of the model (1), by studying specific forms of the Hamiltonian $H$.

It is naturally to represent $H$ in the unperturbed basis $|k\rangle$ of $H_0$. Then the total Hamiltonian is presented by the sum of the diagonal matrix $H_0$ plus the perturbation matrix $V$ with the matrix elements $V_{lk} = \langle l|V|k\rangle$. In order to consider the evolution of wave packets in the basis of $H_0$, one has to express exact eigenstates $|\alpha\rangle$ of the total Hamiltonian $H$ in terms of basis states $|k\rangle$ of $H_0$,

$$|\alpha\rangle = C_k^\alpha |k\rangle.$$  \hfill (2)

The coefficients $C_k^\alpha$ give the expansion of an exact state in terms of the basis states (for $\alpha$ fixed), or the expansion of a basis state in terms of the exact states (for $k$ fixed). In principle, the knowledge of the state matrix $C_n^\alpha$ and the corresponding energy spectrum $E_\alpha$ gives a complete information about the system.

Of particular interest is the case when the initial state $\Psi(0)$ is a basis state $|k_0\rangle$. Then the evolution of the $\Psi$-function is described by the expression

$$\Psi(t) = \sum_{n,k_0} C_n^\alpha C_{k_0}^{\alpha*} |k_0\rangle \exp(-iE_\alpha t).$$ \hfill (3)

Here and below we assume that $\hbar = 1$. As one can see, the probability $w_k = |A_k|^2 = |\langle k|\Psi(t)\rangle|^2$ to find the system at time $t$ in the state $|k\rangle$ is determined by the amplitude

$$A_k = \langle k|\exp(-iHt)|k_0\rangle = \sum_{\alpha} |C_k^\alpha|^2 \exp(-iE_\alpha t).$$ \hfill (5)

Our main interest is in the return probability $W_0(t)$ which is the probability to find the system at time $t$ in the initial state $|k_0\rangle$. One can see that the return probability is determined by the expression

$$W_0(t) \equiv w_{k_0}(t) = |A_{k_0}(t)|^2,$$ \hfill (6)

where

$$A_{k_0}(t) = \sum_{\alpha} |C_{k_0}^\alpha|^2 \exp(-iE_\alpha t) \approx \int P_{k_0}(E) \exp(-iEt) dE.$$ \hfill (7)

Here we replaced the summation by integration that can be done if the number of large components $C_{k_0}^\alpha$ is large. Indeed, these components strongly fluctuate around their mean values and quite often can be considered as pseudo-random quantities. In fact, this condition of a large number of pseudo-random components in exact eigenstates can be used as the definition of chaos in quantum systems (for details, see Refs. [7,12,13]). In this case, the time dependence of $W_0(t)$ is entirely determined by the Fourier transform of $P_{k_0}(E) = P(E, E_{k_0})$ where $E$ is the energy of exact eigenstates and $E_{k_0}$ is the energy corresponding to the unperturbed state $|k_0\rangle$. This quantity is known in the literature as the strength function (SF) or local spectral density of states,

$$P(E, E_{k_0}) \equiv \overline{|C_{k_0}^\alpha|^2} \rho(E).$$ \hfill (8)

Here $\rho(E)$ is the density of states of the total Hamiltonian $H$, and the average is performed over a number of states with energies close to $E$.

To analyze generic properties of the return probability, let us start with its behavior at small times. According to the perturbation theory, one can easily get the general expression,

$$W_0(t) \approx 1 - \Delta_E^2 t^2,$$ \hfill (9)

where $\Delta_E^2$ is the variance of the strength function in the unperturbed energy space, determined as

$$\Delta_E^2 = \sum_{k \neq k_0} V_{k,k_0}^2.$$ \hfill (10)

Note that the above expression is universal in the sense that it is exact for any kind of the perturbation $V$. Practically, the initial time scale for the perturbative expression (9) to be valid is very small, and the main interest is in the time-dependence of $W_0(t)$ beyond this time scale.

3. Two-body random interaction model

In order to analyze the behavior of $W_0(t)$ on a large time scale, we consider the model which describes a closed system of $N$ Fermi-particles occupying $M$ single-particle levels of energies $\epsilon_s$. The total Hamiltonian can be represented in the form (1) where

$$H_0 = \sum_{s=1}^M \epsilon_s a_s^+ a_s,$$

$$V = \sum_{s_1,s_2,s_3,s_4=1}^M \tilde{V}_{s_1s_2s_3s_4} a_{s_1}^+ a_{s_2}^+ a_{s_3} a_{s_4}.$$ \hfill (11)

Here $H_0$ stands for non-interacting particles, and the interaction $V$ between the particles is expressed in terms of two-body matrix elements $\tilde{V}_{s_1s_2s_3s_4}$. The many-particle basis $|k\rangle$ of $H_0$ is defined by the Slater determinant, $|k\rangle = a_{s_1}^+ \cdots a_{s_N}^+ |0\rangle$, where $a_{s_j}^+$ and $a_{s_j}$ are the creation–annihilation operators, and $|0\rangle$ is the ground state. As one can see, the interaction between particles is assumed to have a two-body nature, therefore, each many-particle matrix element $V_{lk} = \langle l|V|k\rangle$ is a sum of a number of two-body matrix elements $\tilde{V}_{s_1s_2s_3s_4}$ involving at most four single-particle states $|s\rangle$ (for details, see, for instance, Ref. [8]).

Note that the approach we consider here, is also valid for quasi-particles that appear in the mean-field theories. In this case $H_0$ stands for the mean-field part of the Hamiltonian, and $V$ describes a residual interaction. The total Hamiltonian $H$ in the form (11) describes generic properties of such physical systems as complex atoms, nuclei, quantum dots, etc. The energies $\epsilon_s$ in such applications are, in fact, renormalized quasi-particle energies.

In many realistic applications the interaction $V$ between particles (quasi-particles) is so strong and complicated that practically one can describe such an interaction by assuming that all
two-body matrix elements are distributed randomly according to some distribution. Thus, the simplest version of the two-body random interaction model (TBRI) is the Hamiltonian (11) in which all matrix elements $V_{lk}$ are random Gaussian numbers with the zero mean and the variance $\langle \hat{V}^2 \rangle$. It is interesting to note that the many-body matrix elements $V_{lk}$ are, however, weakly correlated, due to the fact that the same two-body matrix elements enter in different many-body matrix elements $V_{lk}$. In general, these correlations can be neglected, however, for specific observables they give rise to unexpected results (for details see Ref. [14]).

Without the loss of generality one can assume that the single-particle spectrum has the constant mean level spacing, $d_0 = \langle \epsilon_{i+1} - \epsilon_i \rangle = 1$; here the brackets $\langle \ldots \rangle$ stand for the average over random values of $\epsilon_i$. The number of many-body states increases very fast with an increase of the number of particles $N$ and number $M$ of single-particles states. This reason even for a relatively small number of particles the exact eigenstates may consist of many unperturbed basis states, thus providing us with a possibility to use statistical methods. In particular, a novel approach has been developed in Refs. [7,12,15], that is based on the chaotic structure of eigenstates in a given basis of unperturbed many-particle states. This approach allows one to relate statistical properties of exact eigenstates in many-body representation directly to the properties of single-particle operators, such as the occupation number distribution of single-particle states.

As was shown above, the time dependence of the return probability is entirely determined by the Fourier transform of the SF. In many applications the SF is known to have the Breit–Wigner (BW) form (or, the same, the Lorentzian) resulting from the application of the random matrix theory, (see, for instance, Ref. [16]). Thus, the half-width $\Gamma_0$ of the BW is given by the Fermi golden rule and in our case reads as

$$\Gamma_0(E) = \Gamma(E, k_0) \simeq 2\pi |V_{0k}|^2 \rho_0(E).$$

Here $|V_{0k}|^2$ is the mean square value of many-body matrix elements (obtained by the average over $k$), and $\rho_0(E)$ is the density of those states which are directly coupled to the basis state $|k_0\rangle$ by the interaction $V$. Note that this density $\rho_0(E)$ is much smaller than total density $\rho_{0N}(E)$ of all many-body states. This fact is manifested by a large number of zero matrix elements for any fixed line in the Hamiltonian matrix $H_{lk}$.

It should be stressed that the above result for the BW form of the SF is based on a non-perturbative approach according to which a large number of many-body states are coupled by a relatively strong interaction. As a result, the decrease of the return probability has the exponential time dependence,

$$W_0(t) \simeq \exp(-\Gamma_0 t),$$

apart from a small time scale $t < t_0$ on which the quadratic decrease (9) occurs.

For a long time it was assumed that the exponential decrease of the return probability is the only regime which is physically relevant to the dynamics of systems with many interacting particles. However, recently it was found that in many situations the form of the SF can be quite close to the Gaussian (see, e.g., [3]). This fact is related to a finite width $\Delta V$ of the interaction in the energy space for isolated systems. For the TBRI model it was shown [17] that if $\Gamma_0$ is much less than $\Delta V$, the form of the SF is, indeed, the BW. However, in the other limit, $\Gamma_0 > \Delta V$, of a very strong interaction, the leading dependence of $W_0(t)$ is the Gaussian,

$$W_0(t) \simeq \exp\left(-\frac{\Delta^2}{\Delta V} t^2\right),$$

and occurs on a time scale $0 < t < t_c$. After, for $t > t_c$, the decay of $W_0(t)$ is described by the exponential function [18].

The transition from the BW to the Gaussian for the TBRI model has been analyzed in Ref. [19], although the analytical expression in a closed form is unknown. In order to evaluate $W_0(t)$, in Ref. [18] a phenomenological expression was suggested that depends on both parameters, $\Gamma_0$ and $\Delta V$. The analytical expression for the variance $\langle \Delta E^2 \rangle$ of the strength function can be found explicitly [7],

$$\langle \Delta E^2 \rangle = \frac{\Delta^2}{12} = \frac{\rho_0^2}{2} \frac{\Delta V}{12} (N - 1)(M - N)(M - N + 3).$$

Here $[-\nu_0, \nu_0]$ is the range within which the two-body matrix elements are distributed randomly with a constant probability, therefore, $\langle \hat{V}^2 \rangle = \nu_0^2/3$. It is interesting to note that for Fermi-particles the variance $\langle \Delta E^2 \rangle$ turns out to be independent of a specific basis state $|k_0\rangle$.

Thus, in the case of a not very strong perturbation the decrease of the return probability is the exponential one, and with an increase of the interaction $V$, one should expect a quite large time scale on which the Gaussian form (14) occurs. Let us check these predictions by making use of numerical data. In all our calculations we have used $N = 6$ Fermi-particles occupying $M = 12$ single-particle states; this results in the size $924 \times 924$ of the Hamiltonian matrix. For simplicity, the initially excited state was taken at the center of the energy spectrum, $k_0 = 462$, where the density of many-body states is maximal, and the energy spectrum is symmetrical.

For a relatively weak (however, non-perturbative) interaction, $v_0 = 0.12$, the data, indeed, demonstrate a clear exponential dependence, up to some time scale beyond which the finite size effects of the Hamiltonian matrix are important, see Fig. 1. The fit to the exponential dependence (13) gives $\Gamma_0 \approx 0.97$. To compare with the analytical expression (12), one should note that for the TBRI model this expression is difficult to use directly, since the quantity $\Gamma_0(E)$ is not well-defined by Eq. (12). The problem is that the density of directly coupled many-body states strongly changes in dependence on $k$. The rough estimate can be obtained as follows. First, one can relate the term $|V_{0k}|^2$ to $\langle \Delta E^2 \rangle$ in the way, $\Delta^2 = K|V_{0k}|^2$, where $K$ is the number of non-zero elements in any line of the Hamiltonian matrix (which is independent of $k_0$ [7]). Second, the simplest estimate of the mean density $\rho_0$ is due to the average width $\Delta V$ of the interaction in the energy space, $\rho_0 \approx K/\Delta V$. As a result, one gets

$$\Gamma_0 \approx 2\pi \frac{\Delta E^2}{\Delta V}.$$
Finally, it can be shown that the simplest estimate for the width of the interaction reads as \( \Delta V \approx 2d_0(M - N) \). This gives \( \Gamma_0 \approx 1.03 \) which is a good result, taking into account the problems with the evaluation of the expression for \( \Gamma_0 \).

Now let us consider another limit case of a very strong interaction, \( v_0 = 0.5 \), when the SF is quite close to the Gaussian. Numerical data reported in Fig. 2 manifest a long Gaussian decrease of the return probability. Note that the deviation from the Gaussian form of the strength function in dependence on the strength of the interaction reads as \( \Delta V \approx \pi \). This gives \( \Delta E \approx 0.5 \), which roughly corresponds to the data. Note that the critical time \( t_c \) is, in fact, not a well-defined quantity and can be determined up to some numerical factor of the order one. Remarkably, the time of the correspondence of the data to Eq. (14) turns out to be independent of the perturbation strength.

4. Wigner band random matrices

It is instructive to apply the above analysis to the so-called Wigner band random matrices (WBRM). These matrices are very useful for understanding generic properties of realistic physical systems of interacting particles. The WBRM model is described by the Hamiltonian (1) which consists of two parts, one of which is a diagonal matrix \( H_0 \) with increasing entries \( \epsilon_j \). This part can be treated as the “mean field” part of the total Hamiltonian \( H \). Another part is a banded matrix \( V_{ij} \), which is associated with the interaction between unperturbed basis states. Thus, the model has the following form,

\[
H_{ij} = \epsilon_j \delta_{ij} + V_{ij},
\]

where \( \delta_{ij} \) is the delta-function. It is assumed that random values \( \epsilon_j \) with the mean spacing \( D \) are reordered in an increasing way, \( \epsilon_{j+1} > \epsilon_j \). As for the off-diagonal matrix elements \( V_{ij} \), they are distributed according to the Gaussian distribution (with the zero mean, \( \langle V_{ij} \rangle = 0 \), and the variance \( \langle V_{ij}^2 \rangle = V_0^2 \)) for the matrix elements inside the finite band \(|i - j| \leq b/2\), and zero otherwise.

These matrices have been introduced by Wigner in Ref. [20] in application to nuclear physics. A particular interest was the form of the strength function in dependence on the strength of the interaction \( V \). It was shown that the form of the SF is the critical time \( t_c \), which divides these two regimes,

\[
t_c \approx \frac{\Gamma}{\Delta E} \approx \frac{2\pi}{\Delta V}.
\]

One can see that \( t_c \) is the time to resolve the finiteness of the width of the interaction. If this width is very large, the exponential decrease starts on a small time scale. Contrary, for relatively small values of \( \Delta V \) the Gaussian decrease of \( W_0(t) \) starts from \( t = 0 \) and lasts for a long time. According to this estimate, we have \( t_c \approx 0.5 \) which roughly corresponds to the data.
BW for a moderate (non-perturbative) strength $V$, and the semi-circle for a very strong perturbation. Full analytical treatment of the form of the SF for Eq. (18) is given in Ref. [21] with the use of the modern approach. In particular, it was found that in the transition from the BW to the semi-circle, the form of the SF is very close to the Gaussian.

The condition for the SF to be of the Breit–Wigner form in the WBRM model can be written as follows [21],

$$D \ll \Gamma_0 < \Delta_V, \quad \Delta_V = b D = b \rho_0^{-1},$$  

(19)

where the half-width $\Gamma_0$ is given by the Fermi golden rule,

$$\Gamma_0 = 2\pi \rho_0 V_0^2,$$  

(20)

and $\Delta_V$ is the energy width of the interaction $V$.

The left part of the inequalities in Eq. (19) indicates the non-perturbative situation for which many of unperturbed basis states are strongly coupled by the interaction. On the other hand, the interaction should not be very strong, namely, the width $\Gamma_0$ determined by Eq. (20), has to be less than the width $\Delta_V$ of the interaction in the energy representation. The latter condition is generic for systems with finite range of the interaction $V$. One should stress that, strictly speaking, the BW form of the strength function is not correct in physical applications since its second moment diverges (which assumes an infinite range of the interaction). For the WBRM model with finite values of $b$ it was shown [1,20] that far off the energy range $\Delta_V$ the SF decreases faster than a pure exponent.

In contrast with the TBRI model, in the WBRM model the energy scale $\Delta_V$ is well defined that simplifies our further analysis. One can see that instead of the control parameter $\Delta_V$, one can equivalently use the variance $\Delta_E^2$ of the SF, which can be expressed through the off-diagonal matrix elements of the interaction, $\Delta_E^2 = \sum_j V_{ij}^2$ for $i \neq j$, therefore, $\Delta_E^2 = b V_0^2$. As a result, we have $\Delta_V = 2\pi \Delta_E^2 / \Gamma_0$ and the relation (19) can be written in the form,

$$D \ll \Gamma_0 < \Delta_V \sqrt{2\pi}.$$

(21)

Numerical data for the WBRM model have confirmed that for $\Gamma_0 \approx 2 \Delta_E$ the form of the SF is quite close to the Gaussian. Moreover, it was found that the transition from the BW dependence to the Gaussian-like is very sharp. Note that the extreme limit of a very strong interaction, $\Gamma_0 \gg 2 \Delta_E$, seems to be non-physical, giving rise to the semi-circle form of the SF. The same effect occurs for the TBRI model for which the relation $V \gg H_0$ means that the residual interaction is much stronger than the mean field part.

Applying the arguments given for the TBRI model, the Gaussian decrease of the SF has the form,

$$\ln W_0(t) = -\Delta_E^2 t^2 = -\Gamma_0 t^2 \frac{\pi}{t_c} = -4\pi^2 V_0^2 \frac{\xi}{b D^2} e^2$$

(22)

with $\xi = t / t_c$ and $t_c = 2\pi / b D$.

Numerical data in Fig. 4 obtained for $N = 924$ and different model parameters, are in a good correspondence with our analysis. First, the time dependence of $W_0(t)$ on the time scale $t \lesssim t_c$ is, indeed, of the Gaussian form (22). Second, the critical time

which divides the two characteristic dependencies, nicely corresponds to the expression $t_c \approx 2\pi / b D \approx 1.0$. As one can see, the transition from the Gaussian to the exponential decrease is a quite generic property of the wave packet dynamics in the systems which are described by the TBRI or WBRM models.

5. Dynamical model of a quantum computation

Now we apply our analysis to a physical model of quantum computation which has no random parameters. Although a direct application of the results obtained for random matrix models to the dynamical models is not justified, it is of interest to see whether such a comparison is possible. The model we consider here was recently proposed [22] as a simple realization of a solid-state quantum computation. It describes a one-dimensional chain of interacting 1/2-spins (qubits) that are subject to an external magnetic field. In order to have selective resonant excitation, the time independent part $B^z = B^z(x)$ of a magnetic field is assumed to have a constant gradient along the $x$-direction. This provides different Larmor frequencies for different spins, $\omega_k = \gamma_n B^z = \omega_0 + ak$, where $\gamma_n$ is the spin gyromagnetic ratio and $a$ is proportional to the gradient of the constant part of the magnetic field (see details in [10]).

For a specific pulse of the time-dependent part of the magnetic field, resulting in a single cubit operation, one can derive the time-independent Hamiltonian (1) with

$$H_0 = \sum_{k=0}^{L-1} \left[ -\xi_k I_k^z - 2J I_k^x I_{k+1}^x \right], \quad V = -\sum_{k=0}^{L-1} \omega_k I_k^z,$$

(23)

where $\xi_k = ak$ [10]. Here the frequency $\omega$ is the Rabi frequency of the $p$th pulse, $I_k^{x,y,z} = (1/2)\sigma_k^{x,y,z}$ with $\sigma_k^{x,y,z}$ as the Pauli matrices, and $I_k^z = I_k^z \pm i I_k^y$. It is also assumed that the interaction $J$ between nearest qubits does not depend on the indexes $k$ and $k + 1$.

The unperturbed basis (in which $H_0$ is diagonal) is reordered according to an increase of the index $s$ which is written in the binary representation, $s = i_{L-1}i_{L-2}...i_0$ (with $i_0 = 0$ or 1, depending on whether the single-particle state of the $i$th qubit is the ground state or the excited one). Therefore, the parameter

![Fig. 4. Return probability $W_0(t)$ for the WBRM model (18). Full and open symbols stand for numerical data and theoretical expression (22), respectively.](image-url)
$, \Omega$, corresponds to a non-diagonal coupling, thus, determining the matrix elements $V_{kn} = V_{nk} = -i\Omega/2$ with $n \neq k$. As one can see, in contrast with the TBRI model discussed above, in the chosen representation the interaction between particles is absorbed by $H_0$, and $V$ describes the coupling to the external magnetic field.

The problem studied in Ref. [10] was the analysis of whether the inter-qubit interaction, as well as the interaction of qubits with the external magnetic field, can be a source of a kind of internal decoherence caused by quantum chaos. It is a widespread concern that for many interacting qubits the onset of quantum chaos may occur even for a very weak interaction, see, for instance, [6,23]. This expectation is based on the fact that with an increase of number of qubits the level density of many-body states increases drastically, thus strongly enhancing the delocalization effects due to the interaction between qubits.

The simple estimate [6] shows that generically the threshold for the onset of chaos decreases as $J_{cr} \sim 1/L$, where $J_{cr}$ is a critical inter-qubit interaction above which the eigenstates are extended over unperturbed many-body states.

However, as was found in Ref. [10], in the case of an external magnetic field with a constant gradient along the chain of qubits, the onset of chaos is strongly suppressed. Although in this case the total energy width may be large (proportional to $L$), the estimate [24] shows a feasibility of an experimental realization for a quite large number of qubits. Another principally different scheme that allows to avoid strong delocalization/chaos effects, is suggested and analyzed in Ref. [11] for a quantum computer based on electrons on helium.

Our interest below is to see whether the results for the return probability, obtained above in terms of random matrix models, can be applied to the dynamical model (23). Note that a strong decrease of the return probability $W_0(t)$ can serve as an effective measure of a stability of a quantum computation. It can be shown that the delocalization effects are directly governed by the time-dependence of $W_0(t)$. Strictly speaking, the effectiveness of a quantum computation should be analyzed for the time-dependent model with a large number of pulses. However, if for a single pulse the effects of quantum chaos are strong, they will be generically enhanced in the presence of many pulses. This is why below we restrict ourselves by a consideration of the simplified time-independent Hamiltonian (23). We hope that our analysis may have also an interest in view of general problems of the dynamics of systems with a large number of interacting spins.

First, we start with the so-called non-selective regime which is defined by the conditions, $\Omega \gg a \gg J$. This inequality provides the simplest way to prepare a homogeneous superposition of $2^L$ many-body states, which is needed to start with the implementation of the Shor’s or Grover’s algorithm. The analytical and numerical treatment of the model (23) in this regime has revealed [10] that the constant gradient magnetic field (with $a \neq 0$) strongly reduces unwanted effects of quantum chaos. Specifically, it was shown that in this case the chaos border turns out to be independent of the number $L$ of qubits, in contrast to the models thoroughly studied before [23]. In particular, the quantum chaos may occur only for a large coupling $\Omega$ and strong interaction $J$ between qubits. Another new effect which was found in Ref. [10], is that the border of quantum chaos does not coincide with the border of delocalization. This peculiarity is important in view of applications to integrable or nearly integrable models for which the quantum chaos is absent, however, the delocalization effects can be very strong.

As was shown in Ref. [10] in the non-selective regime with $\Omega = 100$, the delocalization effects start to play essential role for $J > J_c \approx 10$. On the other hand, the quantum chaos effects, such as the Wigner–Dyson distribution for the spectrum statistics, occurs for $J > J_I \approx 100$. Since in this case both the inter-qubit interaction and the perturbation due to the magnetic field are strong, one can analyze the dynamics of the return probability $W_0(t)$ in connection with the previously discussed results.

Numerical data presented in Fig. 5 show a strong decrease of $W_0(t)$ for three values of $J$. As one can see, in all cases the return probability becomes smaller as $a$ increases. The correspondence with theoretical predictions should be treated as a good one, taking into account the dynamical character of the considered model. It is important that the perturbation $\Omega$ due to the magnetic field is strong, therefore, the results are almost insensitive to the inter-qubit interaction in a very wide region of $J$.

Now we analyze the regime of selective excitation which is characterized by the following range of parameters, $\Omega \ll J \ll a$. In this regime each pulse acts selectively on a chosen qubit thus resulting in a resonant transition. In Ref. [25] this regime was analyzed in detail with the main interest to the fidelity of some quantum protocol (for the time-dependent Hamiltonian with many pulses). It was shown that in this regime for a relatively large gradient of the magnetic field there is no any danger of quantum chaos, and the dynamics can be analyzed on the base of the perturbation theory. It is instructive now to see how the return probability $W_0(t)$ corresponds to these results.

In Fig. 6 we present numerical data demonstrating the dependence of $W_0$ for some time $t_0 \approx 3.0$. First, one can see that for small values of $a$ the perturbation turns out to be very strong.
leading to a strong decrease of $W_0$. This result demonstrates that even a small interaction between qubits gives rise to a strong leakage of the probability from an initially excited state. The origin of this phenomenon is the degeneracy (for $a = 0$) or quasi-degeneracy (for small $a$). With an increase of the gradient of the magnetic field, the dynamics turns out to be very stable, as is manifested by the values of $W_0(t_c)$ close to one. The analytical estimate of the critical value of the inter-qubit coupling above which the fidelity is very high, was found [25] to be $a_{ct} \approx 4J$. Our data, however, clearly show a slightly different value $a_{ct} \approx J$. A more careful inspection of the numerical data of Ref. [25] confirms that, indeed, the fidelity starts to decrease at smaller than $a \approx 4$ values. Our data in Fig. 6 also reveal a quite interesting resonance effect that occurs for specific values $a = 1$ and $a = 2$. This effect has the same origin as that found in Ref. [25].

In order to see more clearly the role of the gradient magnetic field, we performed an additional check of the time dependence of the return probability for small, $a = 0.05$, and large, $a = 3.0$, values of $a$. The results in Fig. 7 manifest a very different behavior of the return probability for these two representative values. If for the small value of $a$ the decrease of $W_0(t)$ is very fast and seems to be non-recurrent, for large $a = 3.0$ the return probability remains very close to one and shows a clear recurrence. It is interesting to note that for $a = 0.05$ one can see two regimes, the Gaussian one (for $t < 2.0$), and the exponential one (for $t > 3.0$). Note that due to a specific character of the selective excitation (quasi-degeneracy for $a = 0.05$ and very weak perturbation for $a = 3.0$), the comparison with the discussed above analytical estimates in this case is not valid.

6. Conclusion

In conclusion, we have studied the return probability $W_0(t)$ in three models of strongly interacting particles. The first model is the model with random two-body interaction, typically used to describe the many-body systems of Fermi-particles such as heavy nuclei, many-electron atoms, quantum dots, etc. As usual, the two-body random matrix elements are assumed to be the random entries, which is known to be reasonable when the interaction is strong and has a complicated form. The main result of our analytical and numerical study is that with an increase of the interaction, the Gaussian decrease of the return probability can last for a long time. The typical picture is the following: on some time scale $t < t_c$ the decrease of $W_0(t)$ is the Gaussian, and for $t > t_c$ it is the exponential one. The Gaussian decrease on the scale $0 < t < t_c$ can be either weak or strong, depending whether the interaction is strong or very strong, respectively. Thus, the standard exponential decrease, associated with the Fermi golden rule, is not correct for a very strong perturbation. Numerical data confirm the analytical predictions.

In order to elucidate the meaning of our analytical estimates, we have also analyzed the model of Wigner band random matrices, which captures essential features of the chaotic systems of interacting particles. This model turns out to be very effective since it demonstrates in a very transparent way the dependence of the dynamics on few global parameters of physical significance. Our numerical data reflect the generic properties of the interplay between the Gaussian and exponential decrease of the return probability.

Finally, we have analyzed the model which was recently proposed as an implementation of a quantum computer. In contrast with the random matrix models, this model is purely dynamical one (without any random parameters). As is now well understood, the effects of quantum chaos and/or delocalization may also arise in the models of quantum computation, and lead to a kind of internal decoherence. Our numerical data show that in the region of parameters that corresponds to a strong delocalization and quantum chaos, the return probability has the Gaussian form of the decay for a quite a long time, similar to what occurs in the random matrix models. Additional study of the role of a non-zero gradient magnetic field has shown that the fast decrease of the return probability corresponds to earlier results for a low fidelity of some quantum protocol with many pulses. Therefore, the return probability can be considered as a quite sensitive quantity for establishing the region of parameters where one can expect stable operability of quantum computers.
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References