THE LINEAR MODEL OF PRODUCTION

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<u>Remark</u>: The main motivation for this work was the author's wish to clarify (to himself), as completely as possible, the structure of the linear model with joint production, without any unnecessarily restrictive assumptions (like indecomposability), and the relationship of this model to Marx and Sraffa. The paper is therefore essentially a summary, with some occasional generalizations, of known results. It was completed in March 1980, and is reproduced here in the hope that it might still be useful as a reference. I am indebted to G.Clemenz, B.Genser, G.O.Orosel, K.Podczeck and P.Rosner for helpful discussions.

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Abstract:

The paper gives a mathematically self-contained, concise exposition of the general linear model with joint production à la von Neumann, without indecomposability assumptions. Labour values, the rate of exploitation, the capacity growth rate, the warranted rate of profit, the equilibrium price system, and the "Fundamental Marxian Theorem" are studied for both the case where the wage is paid in advance (Marx) and where it is paid post factum (Sraffa). For systems without genuine joint production, both the Quantity and the Price version of the general Nonsubstitution Theorem are proved. The paper summarizes many known results from a large and diverse literature, and contains also some new ones.

Zusammenfassung:

Die Arbeit gibt eine mathematisch exakte, konzise Darstellung des allgemeinen linearen Modells mit Kuppelproduktion à la von Neumann, ohne Unzerlegbarkeitsannahmen. Arbeitswerte, Mehrwert-, Wachstums- und Profitrate, das Gleichgewichtspreissystem und das "Fundamentale Marx'sche Theorem" werden sowohl für den Fall eine vorgeschossenen (Marx) als auch eines im nachhinein bezahlten Lohnes (Sraffa) studiert. Für Systeme ohne echte Kuppelproduktion wird das allgemeine Nonsubstitutionstheorem in beiden Versionen, der Mengenversion und der Preisversion, bewiesen. Die Arbeit faßt viele Resultate aus einer weit verstreuten Literatur zusammen, und enthält auch einige neue Ergebnisse.

C o n t e n t s

§1. Introduction

1.1. Motivation and aim of the present study

The linear model of production is the simplest representation of a fully disaggregated economic system, but at the same time general enough to provide a suitable framework for the analysis of many important and interesting questions. For example, it is the basis for Leontief's input-output analysis and for von Neumann's model of economic growth; and it has also been instrumental for a better understanding of certain problems in capital theory (following the Sraffian critique of 'aggregate' neoclassical theory, in particular the discovery of the "reswitching" phenomenon). Moreover, the linear model of production has been used by many authors to study more rigorously certain 'classical' problems of economic theory already discussed by Ricardo and Marx, especially the influence of changes in the distribution of income upon relative prices (Ricardo's problem of an "invariant standard of value"), and the relationship between equilibrium prices and labour values (Marx' "transformation problem"). Using the mathematical duality between the price and quantity systems, the 'linear' approach has also clarified the relationship between wages and profits (resp. consumption and growth), and the precise conditions under which prices (resp. optimal techniques) can be determined independently of the structure of demand ("nonsubstitution theory").

The relevant results - especially when one includes joint production - are scattered in a large and variegated literature whose fundamental unity is often obscured by the substantial differences in emphasis and terminology among different authors.

The aim of the present paper is to improve this situation by providing a concise, yet mathematically self-contained exposition of the basic formal theory.

The choice of material was motivated mainly by the classical problems mentioned above (cf. also the overview of contents in Sec.1.2). We assume throughout a uniform profit and wage rate, no scarce resources, and only one primary factor, viz. homogenous labour. The number of goods and processes is finite, but we allow arbitrary joint production systems, without further restrictive assumptions like indecomposability etc.

The analysis is based on a suitably modified von Neumann model (i.e. we use <u>in</u>equalities rather than equalities) resp. on methods of Linear Programming. Whether this approach does justice to the classics or not, will not be discussed here[2]; in any case, the classical tradition is followed at least insofar as

(a) the main emphasis is on production,

(b) subjective preferences play no rôle[3], and

(c) the distribution (the "real wage") is exogenous.

Moreover, the price-quantity system is always determined in such a way that

(d) it is compatible with the distribution, and

(e) all quantities are nonnegative.

As already mentioned, the present study concentrates on the <u>formal</u> aspects of the theory. Accordingly - and also for the sake of brevity - economic interpretations have been kept to a minimum. Extended discussions of the economic significance of various results can be found in the large existing literature, to which the reader is referred (see Sec. 1.4).

<u>Remark</u>: Theories which lack the last two properties (d) and (e) must be rejected as unsatisfactory from the economic viewpoint. This is true in particular of Sraffa's price theory and of his 'additive' computation of labour values under joint production. Given the extraordinary influence of this author (Roncaglia's "Bibliography of Writings Relating to 'Production of Commodities by Means of Commodities'" contains more than 4oo items, cf. RONCAGLIA 1978), a more detailed discussion of his theory might seem to be warranted. However, this would be beyond the scope

of this short introduction, and we limit ourselves to the following two observations:

ad (d): Sraffa varies the distribution for a given size and composition of output. This is clearly inadmissible, since a change in distribution, in general, leads to changes in output (e.g. more investment goods, less consumption goods, or vice versa).[4] One might be justified in neglecting these quantity changes, if they left invariant the price equations. But this is not the case, because the coefficients of these equations are precisely the input and output quantities in the various industries. Starting from a certain initial state, Sraffa changes the rate of profit, while keeping the remaining coefficient of his price equations constant. The resulting prices, being computed, so to speak, on the basis of the original quantity system, will, in general, be incompatible with the new quantity system required by the new rate of profit, and, hence, economically meaningless.[5] It is well known that this problem does not arise when there are constant returns to scale in all industries, and if there is no 'genuine' joint production, so that the Nonsubstitution Theorem holds (cf. section 5.1. below). But these are assumptions that Sraffa did not make.

ad (e): The possibility of negative prices and labour values was pointed out by Sraffa himself (cf. also MANARA'1968, SCHEFOLD'1971, STEEDMAN'1977).

1.2. Summary of contents

We give now a brief overview of the main results contained in this paper. Familiarity with the basic concepts of the linear model is here assumed. The technology is given by an input coefficients matrix A, an output coefficients matrix B, and a vector of labour input coefficiets l. Throughout the paper we take the workers' daily consumption bundle c (the "real wage") as the exogenous variable, and determine all other variables (growth rate, profit rate, prices, etc.) as functions of c.

In Sec.2, after some basic definitions, we introduce Marxian labour values v and show that a positive rate of exploitation e is equivalent to productivity of the augmented system (A+cl,B).

In Sec.3 we consider the system when the wage is advanced at the beginning of the production period (as is usual in Marxian theory), and show that for every real wage c (which is feasible and does not consist entirely of free goods) there exists a (in general nonunique) equilibrium price system p and a uniquely determined nonnegative equlibrium rate of profit r. r is equal to the capacity growth rate g, and r is positive iff. the rate of exploitation e is positive ("Fundamental Marxian Theorem"). If the wage increases, the rate of profit goes down, i.e. $pc' > pc$ implies $r' < r$ ("generalized wage-profit frontier"; here r' is the rate of profit associated with c'). If two consumption bundles give rise to the same rate of profit, the corresponding price systems will still in general be different, i.e. the prices depend not only on the distribution of income, measured by r, but also on the structure of demand.

In Sec.4 we consider the same model, but with the wage paid at the end of the production period (à la Sraffa). Surprisingly, with joint production, this tiny modification destroys several of the attractive features of the model considered in Sec.3: the equilibrium rate of profit can become negative, need not be unique, and the Fundamental Marxian Theorem is no longer true. Apart from these differences, Sec.4 parallels Sec.3 very closely.

In Sec.5 we consider the special case of the linear model of production where the so-called "Nonsubstitution Theorem" holds. This case is characterized by the absence of "genuine" joint production (single-product industries are a further special case). We state and prove a general ("dynamic") Nonsubstitution theorem in both the price and the quantity version. It implies that the price system p and the technique used depend both only on the rate of profit r, but not on the structure of demand. Moreover, relative prices are uniquely determined by r, and become proportional to labour values for r=0. For all practical purposes, the theory becomes identical to the one for single-product industries (no joint production). We point out that only under the conditions of the Nonsubstitution theorem is it legitimate to consider the price system and the quantity system separately, and to draw wage-profit curves in the usual way, with the optimal technique depending only on the rate of profit, without worrying about the structure of final demand.

1.3. Some new aspects

Although the present paper is essentially a systematic exposition of known results, it also offers some new, or hitherto neglected, aspects:

(I) The classical authors, in particular Marx, generally assumed (at least in their attempts at formalization) that the wage is paid at the beginning of the production period and forms a part of the capital advanced by the capitalists. In Sraffa's analysis, by contrast, the wage is paid at the end of the production period and is regarded as the share of the workers in the total surplus product. We give a complete and parallel treatment of both cases (§3 and §4). In spite of great similarities, the theoretical results in the two cases are not the same (cf. e.g. point (V) below).

(II) We vary the per-capita consumption of the workers (the 'real wage') in the space of all technically feasible consumption bundles. The wage-profit-curve is thus replaced by a correspondence associating with each rate of profit r the set of all per-capita consumption bundles compatible with r. In the literature, it is usually assumed that workers consume only one consumption good (BURMEISTER & KUGA'1970) or a fixed commodity basket (MORISHIMA'1971

(III) This correspondence between real wage and profit rate has the following strong monotonicity property: Let c be a consumption vector with associated equilibrium rate of profit r and price system p, and let c' be another consumption vector which has a higher value than c, evaluated at the prices p ($pc' > pc$). Then the rate of profit associated with c' is lower: $r' < r$. This result can also be viewed as an optimality property of the price system. Of course it implies immediately the known monotonicity properties of the usual wage-profit curve, even for joint production.

(IV) It is known that the assumption of a positive wage suffices to determine uniquely the rate of profit, even when the technology is decomposable (cf. FUJIMOTO'1975). We show that this result is not true when the wage is paid post factum (cf. Ex.4.3.2).

As a by-product of our main investigation, we obtain also, almost effortlessly, some results on the relationship between the (labour) value system and the price system. Surprisingly, it turns out that

(V) the so-called "Fundamental Marxian Theorem" (MORISHIMA'1974) is not true when the wage is paid 'post factum'. This contradicts an incorrect assertion in WOLFSTETTER'1977 (his "Satz 2", p.51 ff.).

The "Transformation Problem" proper, viz. the transformation of labour values into prices, is not treated at all in this paper, because there is no economically meaningful connection between these two sets of variables, at least not in a joint production system. Cf. Samuelson's famous "erasing algorithm".[6)]

The last section of the paper is devoted to the important special case where there is no "genuine" joint production. Among other things, we prove

(VI) the general[7)] Nonsubstitution Theorem in its two dual versions, the Quantity Version and the Price Version, for any admissible growth resp. profit rate, and without the restrictive assumption that production uses every industry (cf. MIRRLEES'1969, assumption (A'.3) on p.7o; or BLISS'1975, Th.11.3 on p.267).

1.4. Further remarks and references

The mathematical prerequisites for understanding the following pages can be found in any introduction to the theory of linear models, e.g. GALE'1960 or NIKAIDO'1968. For the convenience of the reader, some frequently used results are listed in an Appendix. The various examples in the text are all counterexamples, designed only to demonstrate the invalidity of certain assertions, not to illustrate the positive results in the paper.

Proofs are usually collected in a separate subsection labelled "O" at the end of each section; e.g. the proofs for Sec. 3.3 are to be found in 3.3.O.

The literature on the linear model of production is huge. A small selection: (1) systematic expositions: DORFMAN, SAMUELSON & SOLOW'58, GALE'60, NIKAIDO'68, K.&W.HILDENBRAND'75, BLISS'75, PASINETTI'77. (2) Input-output analysis: LEONTIEF'51, '66. (3) Marxian Econonomics: MORISHIMA'73, '74, NUTZINGER & WOLFSTETTER (eds.)'74, WOLFSTETTER'77, STEEDMAN'77. (4) Neo-Ricardian Theories: SRAFFA'60, SCHEFFOLD'71, RONCAGLIA'78. (5) Questions of Capital theory: HARCOURT & LAING (eds.)'71, BLISS'75. (6) Nonstubstitution theory: BLISS'75. Further references, in paritcular to original articles, can be found in these books.

§2. The linear model of production

2.1. General assumptions

The general linear model of production has the following structure: [8)]

There are $m \geq 1$ produced goods, labelled $i = 1,\dots m$, and one nonproduced ressource, viz. homogeneous labour. Moreover there are $n \geq 1$ technically feasible production processes, labelled $j = 1,\dots n$, all with constant returns to scale. The j-th process, when operated at unit intensity level, transforms an input vector a_j into an output vector b_j, using the amount of labour l_j. Here a_j and b_j are m-dimensional column vectors whose i-th component a_{ij} resp. b_{ij} is the amount of good i used up as an input resp. produced as an output by process j.

The technology (A,B,l) is thus given by an $(m{\times}n)$-dimensional matrix of input coefficients $A = \left[a_{ij}\right]_{\substack{i=1,\dots m\\ j=1,\dots n}} = (a_1,\dots a_n)$ and an $(m{\times}n)$ matrix of output coefficients $B = \left[b_{ij}\right] = (b_1,\dots b_n)$, plus an n-dimensional row vector $l = (l_1,\dots l_n)$ of labour input coefficients. The columns of A and B correspond to processes, and the rows to goods.

If process j is operated at the intensity level $x_j \geq 0$, it transforms the input vector a_jx_j into the output vector b_jx_j, and uses the amount of labour l_jx_j. Collecting the intensities x_j $(j=1,\dots n)$ into an n-dimensional column vector x, we obtain: the economy as a whole transforms the inputs $Ax = \sum a_jx_j$ into the gross output $Bx = \sum b_jx_j$, using the amount of labour $L = lx = \sum l_jx_j$; the net output is $(B-A)x$. x is called intensity or activity vector. The j-th process is called active at x, or x-active, when $x_j > 0$. We denote by $X := \{x \geq 0 \,/\, (B-A)x \gneq 0\}$ the set of all intensity vectors which yield a semipositive net output, and assume that the technology (A,B,l) satisfies the following assumptions:

Assumption I

(i) $A \geq 0$, $B \geq 0$, $l \geq 0$ (nonnegativity)

(ii) $\underline{1}.A \gg 0$, i.e. every column of A contains at least one positive element (every process needs inputs)

(iii) There exists an activity vector $x^o \geq 0$ such that $(B-A)x^o \gg 0$, i.e. it is possible to produce a positive net output of all goods simultaneously (the system is productive)

(iv) $lx > 0$ for all $x \in X$, i.e. labour is indispensable (no full automation)

Ass.I(iii) implies $B.\underline{1} \gg 0$, i.e. every row of B contains at least one positive element.

A set of goods $S \subseteq \{1,..\ m\}$ is called an independent subset for (A,B) if there exists a set of processes $T \subseteq \{1,..\ n\}$ such that $a_{ij} = 0$ for $i \notin S$, $j \in T$; and for every good $i \in S$ there is a process $j \in T$ with $b_{ij} > 0$. I.e. the processes in T need only goods in S as inputs, and every good in S is an output of some process in T. The pair (A,B) is called reducible (or decomposable) if there exists a proper independent subset for (A,B). Otherwise, (A,B) is called irreducible (indecomposable). In what follows, we shall not assume irreducibility, unless explicitly stated otherwise.

Finally, we denote by $p_i \geq 0$ the price of one unit of the i-th good, by $p = (p_1,..\ p_m) \gneq 0$ the price vector, by $w \geq 0$

the wage rate (nominal wage), and by $r \geq 0$ the profit rate (rate of interest). It is assumed that w and r are uniform throughout the economy. A good whose price is zero is called a free good. If the per-capita-consumption of the workers is given by a commodity bundle c (c is an m-dimensional column vector), then we have obviously $w = pc$ (workers do not save).

In this paper we shall determine the price-quantity system (r,p,x) as a function of the consumption bundle (the "real wage") c, i.e. as a function of the distribution of income, for a given technology (A,B,l).

2.2. Labour values

Let an arbitrary commodity bundle $d \geq 0$ be given (d is an m-dimensional column vector). The labour value[9] $V(d)$ of d is defined as the minimum amount of labour necessary to produce d, i.e. the value of the following Linear Programme:

$$\left.\begin{array}{ll} \min\limits_{x} & 1x \\ \text{s.t.} & (B-A)x \geq d, \quad x \geq 0 \end{array}\right\} \qquad (2.2.1)$$

The Dual of (2.2.1) is (with $v:=(v_1, .. \; v_m)$:

$$\left.\begin{array}{ll} \max\limits_{v} & vd \\ \text{s.t.} & v(B-A) \leq 1, \quad v \geq 0 \end{array}\right\} \qquad (2.2.2)$$

Theorem 2.2. (Labour values)
Under Ass.I, every commodity bundle $d \geq 0$ has a uniquely determined, nonnegative labour value $V(d)$. $V(d)$ is a continuous function of d, zero for $d=0$, positive for $d \gneq 0$, and (weakly) monotonically increasing with d, i.e. $d \leq d'$ implies $V(d) \leq V(d')$. Moreover $V(k.d) = k.V(d)$ for every scalar $k \geq 0$; and $V(d) \rightarrow \infty$ for $\sum d_i \rightarrow \infty$.

By Th.2.2., the two programmes (2.2.1) and (2.2.2) have optimal vectors x^* and v^* (cf. Appendix). The components v_i^* of v^* can be interpreted as prices ("shadow prices" or "optimal prices"), with a wage rate $w=1$ and a rate of profit $r=0$. [10] Although the optimal price system v^* for a given commodity bundle d need not be uniquely determined, the labour value $V(d) = v^*.d = \sum v_i^*.d_i$ is uniquely determined. If in particular $d = e_i$, the i-th unit vector [11], then $V(e_i)$ is the labour value of good i, i.e. the amount of labour necessary to produce one unit of good i. The labour value of an arbitrary commodity bundle d is in general smaller than the sum of the labour values of its components: $V(d) \leq \sum d_i . V(e_i)$, with equality in special cases only. The reason for this is of course that, under joint production, the optimal technique (given by the nonzero components of the activity vector x^*) depends on the final demand vector d. We shall see in §5 that in the absence of joint production this is not the case, by the Nonsubstitution Theorem, and labour values can be computed 'additively' (cf. Th.5.2.).

2.2.0

Proof of Th.2.2.: Existence and continuity of $V(d)$ follow immediately from the theory of Linear Programming (cf. Th.A.3 and Th.A.4): Under Ass.I, both programmes (2.2.1),(2.2.2) are feasible. Therefore they have optimal vectors x^*, v^* with $1x^* = v^*(B-A)x^* = v^*d = V(d)$. Clearly $V(0) = 0$; and $V(d) > 0$ for $d \neq 0$ by Ass.I(iv). An increase of d strengthens the constraint in (2.2.1) and increases the maximand in (2.2.2), so that the value of the problem can only increase. Further, linear homogeneity is obvious from (2.2.2). Finally, if $\sum d_i \geq K$ for an arbitrary constant K, then $d_i \geq \frac{K}{m}$ for at least one i. On the other hand, (2.2.2) has at least one feasible vector v with $v_i = V(e_i) > 0$. This implies $\max vd \geq \frac{K}{m}.V(e_i)$.

Q.E.D.

2.3. The set of feasible real wages

Let the per-capita consumption of the workers be given by a commodity bundle $c \geq 0$ (an m-dimensional column vector). A consumption bundle c is called feasible if it can be produced by the technology (A,B,l), i.e. if there exists an activity vector $x \in X$ s.t. net output per worker is at least as large as c:

$$\exists\, x \in X \quad \text{with} \quad (B-A)x \geq c.lx \tag{2.3.1}$$

Condition (2.3.1) can also be written in the form $\left[B - (A+cl)\right] x \geq 0$. The matrix $A+cl$ is called augmented input matrix. It has as typical element $a_{ij} + c_i l_j$ and takes account not only of the physical inputs of production (a_{ij}) but also of the need to feed the workers $(c_i l_j)$.

Lemma 2.3. (Feasible consumption bundles)
A consumption bundle c is feasible if and only if ("iff") its labour value is less than one, $V(c) \leq 1$.

The set of all feasible consumption bundles, denoted by C, is a subset of the commodity space R_+^m. A commodity bundle $c \geq 0$ is certainly feasible if all its components are sufficiently small (by Productivity, Ass.I(iii)), in particular, $c=0$ is feasible. If $c \geq 0$ is feasible, then every smaller vector c' with $0 \leq c' \leq c$ is also feasible. The set C is convex and compact. (This follows immediately from (2.3.1), Th.2.2., and Lemma 2.3.).

2.3.0.

Proof of Lemma 2.3. (by means of Th.2.2):

Let c be feasible, i.e. (2.3.1) is satisfied $\Longrightarrow \exists x$ with $1x = 1$, s.t. $(B-A)x \geqq c$, i.e. $V(c) \leqq 1$.

Conversely, assume $V(c) \leqq 1 \Longrightarrow \exists x$ with $1x \leqq 1$ s.t. $(B-A)x \geqq c \Longrightarrow (B-A)x \geqq c.1x$, i.e. c is feasible.

Q.E.D.

2.4. The rate of exploitation

Let the per-capita consumption of the workers be given by a consumption bundle c. We define the rate of exploitation (for c):

$$\left.\begin{array}{ll} e(c) := \dfrac{1 - V(c)}{V(c)} & \text{for } c \neq 0 \\ e(c) := \infty & \text{for } c = 0 \end{array}\right\} \qquad (2.4.1)$$

$e(c)$ is the ratio of "surplus labour" to "necessary labour" (cf. MORISHIMA'1973 & '1974, WOLFSTETTER'1977). By Lemma 2.3. we have $e(c) \geq 0$ for every feasible c.

Lemma 2.4.

Let c be a feasible consumption bundle. The following conditions are equivalent:

(i) $e(c) > 0$ the rate of exploitation is positive

(ii) $V(c) < 1$ the labour value of c is less than one

(iii) $\exists x \in X$ with $(B-A)x \gg clx$

(iv) $\exists c' \in C$ with $c' \gg c$, i.e. there is a feasible consumption bundle which is strictly greater than c.

(v) The "augmented system" $(A+cl,B)$ is productive in the sense of Ass.I(iii).

2.4.0.

Proof of Lemma 2.4.: we prove only the equivalence of (ii) and (iii), the remaining implications being trivial.

First assume that (iii) is satisfied $\Rightarrow \exists x$ with $1x = 1$ s.t. $(B-A)x \gg c \Rightarrow \exists x$ with $1x < 1$ s.t. $(B-A)x \geq c$, i.e. $V(c) < 1$.

Now assume that (ii) is satisfied, i.e. $V(c) < 1 \Rightarrow \exists x$ with $1x < 1$ s.t. $(B-A)x \geq c \Rightarrow y := (B-A)x - c.1x \geq 0$ and $y_i > 0$ for $c_i > 0$. Choose (by Ass.I(iii)) a sufficiently small (componentwise) activity vector x^o with $(B-A)x^o \gg 0$ and define $z := y + (B-A-c1)x^o$. If $c_i > 0$, then $z_i > 0$ because x^o was chosen sufficiently small; if $c_i=0$, then[12)] $z_i \geq [(B-A-c1)x^o]_i = [(B-A)x^o]_i - \underbrace{c_i.1x}_{=0} > 0$ by def. of x^o.

Therefore $z = (B-A-c1)(x+x^o) \gg 0$, i.e. (iii) is satisfied.

Q.E.D.

§3. The system with the wage paid in advance

In this section we assume that the wage is paid at the beginning of the production period. Wages are part of the capital advanced by the capitalists. In the framework of a von Neumann model, this case was studied, e.g. by MORISHIMA'1974. We assume that the technology (A,B,l) satisfies Ass.I.

3.1. The capacity growth rate

Let $c \geqq 0$ be a feasible consumption bundle. In order to find the largest growth rate of the system, compatible with c, we consider the following nonlinear programme:

Problem I

max g

s.t. $\left[B - (1+g)(A+cl)\right] x \geqq 0, \qquad x \gneqq 0$ (3.1.1)

If this problem has a solution we denote it by $g = g(c)$ and call $g(c)$ the capacity growth rate for c. $g(c)$ is the largest rate at which the system (A,B,l) can grow in a balanced fashion, if the consumption per worker is c and the wage is paid at the beginning of the production period.

Theorem 3.1. (Capacity growth rate)

(i) For every feasible c there is a unique nonnegative capacity growth rate $g(c)$

(ii) $g(c)$ is weakly monotonically decreasing in c, i.e. $c \geqq c'$ implies $g(c) \leqq g(c')$

(iii) $g(c) = 0$ if and only if $e(c) = 0$

(iv) $g(0) =: g_{max}$ is the largest possible growth rate of the system; and for every feasible c we have: $0 \leq g(c) \leq g_{max}$.

(v) The inequality $[B - (1+g(c))(A+cl)]\,x \geq 0$ has at least one solution $x \gneq 0$ with $lx > 0$.

A feasible consumption vector c is called inefficient if all its nonzero components can be increased without lowering the growth rate:

$$\exists\, k > 1 \quad \text{with} \quad g(kc) = g(c) \qquad (3.1.2)$$

c is called efficient if it is not inefficient. Trivially, $c=0$ is inefficient, but not every $c \neq 0$ is efficient, as is shown by the following example:

Example 3.1.1. $A = \begin{pmatrix}1\\1\end{pmatrix}$, $B = \begin{pmatrix}2\\5\end{pmatrix}$, $l = (1)$, $c = \begin{pmatrix}0\\1\end{pmatrix}$

We have $g(c) = g(2c) = 1$.

We shall see in 5.3. that in the absence of genuine joint production every $c \neq 0$ is efficient (cf. Lemma 5.3.1(ii)). Inefficiency means that workers could get more of all goods actually contained in their consumption basket, without lowering the growth rate. This is a degenerate case which can occur only with joint production; intuitively, the workers' consumption consists entirely of "surplus goods". However, these surplus goods can only be produced jointly with some other goods (not consumed by the workers), and it is these goods that the constraint on the growth rate comes from.

3.1.0.

Proof of Th.3.1.: Let c be a feasible consumption bundle.

(i) we denote by $G(c)$ the set of all growth rates g for which the inequality (3.1.1) has a solution $x \gneq 0$. We want to show that $G(c)$ has a greatest element and that this is non-negative. The set $G(c)$ is clearly closed. Moreover, (3.1.1) certainly has a solution for $g \leq 0$, because c is feasible (cf. (2.3.1)), i.e. $G(c)$ is nonempty. Finally, $G(c)$ is bounded above: for sufficiently large g we have $\underline{1}.[B-(1+g)(A+cl)] \ll 0$, because $\underline{1}.A \gg 0$ (Ass.I(ii)), and this implies by Th.A.1 (cf. Appendix) that (3.1.1) has no solution $x \gneq 0$. Therefore the solution of Problem I is given by $\sup G(c) =: g(c) \geq 0$. This proves (i).

(v) If $g(c) = 0$, the assertion follows from feasibility (2.3.1). If $g(c) > 0$, we even have $lx > 0$ for every solution $x \gneq 0$ of (3.1.1). Otherwise $(lx=0)$ (3.1.1) and Ass.I(iv) would imply: $g(c)Ax \leq (B-A)x = 0 \Rightarrow Ax = 0$, contradicting Ass.I(ii). This proves (v).

(iii) "$\Rightarrow$": If $e(c) > 0$, then, by Lemma 2.4, the inequality $(B-A-cl)x \gg 0$, $x \gneq 0$ has a solution $\Rightarrow$ for g sufficiently small and positive, (3.1.1) has a solution $\Rightarrow$ $g(c) > 0$.

"$\Leftarrow$": If $g(c) > 0$, then, by (v), there is an x with $lx > 0$ s.t. $y := (B-A-cl)x \geq [B-(1+g(c))(A+cl)]x \geq 0$ and $y_i > 0$ for $c_i > 0$. Choose (Ass.I(iii)) a sufficiently small x^0 with $(B-A)x^0 \gg 0$ and define $z := y + (B-A-cl)x^0$. If $c_i > 0$, then $z_i > 0$ because x^0 was chosen sufficiently small; if $c_i=0$, then $z_i \geq (Bx^0 - Ax^0)_i - \underbrace{c_i .lx}_{=0} > 0$ by def. of x^0 $\Rightarrow$ $z = (B-A-cl)(x+x^0) \gg 0$, $x+x^0 \in X$, i.e. $e(c) > 0$, by Lemma 2.4. This proves (iii). Assertions (ii) and (iv) are obvious from the definitions.

Q.E.D.

3.2. The warranted rate of profit

Let c be a feasible consumption bundle. It is a condition of long-run equilibrium that no process makes profits in excess of the ruling rate of profit. The smallest rate of profit compatible with this condition is called the warranted rate of profit (cf. MORISHIMA'1974) and denoted by $r = r_w(c)$. $r_w(c)$ is the solution (if it exists) of the following nonlinear programme, dual to Problem I of 3.1.:

Problem II

min r

s.t. $p[B - (1+r)(A+cl)] \leqq 0, \qquad p \gneq 0$ (3.2.1)

Theorem 3.2. (Warranted rate of profit)

(i) For every feasible c there is a unique nonnegative warranted rate of profit, $r_w(c)$

(ii) $r_w(c)$ is weakly monotonically decreasing in c, i.e. $c \geqq c'$ implies $r_w(c) \leqq r_w(c')$

(iii) $r_w(c) = 0$ if and only if $e(c) = 0$

(iv) $r_w(0)$ is the largest possible warranted rate of profit and for every feasible c we have $0 \leqq r_w(c) \leqq r_w(0)$.

(v) The inequality $p[B - (1+r_w(c))(A+cl)] \leqq 0$ has a solution $p \gneq 0$ with $pc = 0$ if and only if $r_w(c) = r_w(0)$.

Lemma 3.2. ($g(c)$ and $r_w(c)$)

(i) $r_w(c) \leqq g(c)$ for every feasible c

(ii) $r_w(c) = g(c)$ if (A,B) is irreducible

(iii) $r_w(c) = g(c)$ if $r_w(c) < r_w(0)$

In general, the capacity growth rate and the warranted rate of profit are not equal:

Example 3.2.1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $l = (1,1)$

For $c=0$ we have $g(c) = 2$, $r_w(c) = 1 = r_w(0)$

For $c = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ we have $g(c) = r_w(c) = 1/3 < r_w(0)$.

3.2.0.

Proof of Th.3.2.: Let c be a feasible consumption bundle, and choose (by (2.3.1)) an activity vector $x \in X$ with $(B-A-cl)x \geq 0$.

(i) We denote by $R(c)$ the set of all profit rates r for which the inequality (3.2.1) has a solution $p \gneq 0$. We want to show that $R(c)$ has a smallest element, and that this is nonnegative. The set $R(c)$ is clearly closed. Moreover, for r sufficiently large, (3.2.1) has a solution, e.g. $p = \underline{1}$, because $\underline{1}.A \gg 0$ by Ass.I(ii), i.e. $R(c)$ is nonempty. Finally, for r negative, (3.2.1) has no solution: $r < 0$ implies: $y := [B-(1+r)(A+cl)]x \geq 0$, and $y_i > 0$ for $c_i > 0$. Choose (by Ass.I(iii)) a sufficiently small activity vector $x^o \geq 0$ with $(B-A)x^o \gg 0$, and define $z := y + [B-(1+r)(A+cl)]x^o$. If $c_i > 0$, then $z_i > 0$ because x^o was chosen sufficiently small; if $c_i=0$, then $z_i \geq [(B-A-cl)x^o]_i = [(B-A)x^o]_i \underbrace{-c_i lx^o}_{=0} > 0$ by def. of x^o. $\Longrightarrow$ $z = [B-(1+r)(A+cl)](x+x^o) \gg 0$, $x+x^o \in X$, and this implies by Th.A.1 that (3.2.1) cannot have a solution $p \gneq 0$. Therefore the closed set $R(c)$ is bounded below by zero and has a nonnegative smallest element $\min R(c) =: r_w(c) \geq 0$. This proves (i).

(ii) Obviously the set $R(c)$ can only be enlarged, if anything, by an increase in c; this implies the assertion.

(iv) follows directly from (ii).

(iii) By Lemma 2.4, $e(c) > 0$ if and only if the inequality $(B-A-cl)x \gg 0, \ x \gneq 0$ has a solution. By Th.A.1, this is the case if and only if the inequality $p(B-A-cl) \lneq 0, \ p \gneq 0$ has no solution, i.e. if $0 \notin R(c)$. This proves (iii).

(v) "$\Rightarrow$": $p[B-(1+r_w(c))(A+cl)] \lneq 0, \quad pc = 0, \quad p \gneq 0$

$\Rightarrow \quad p[B-(1+r_w(c))A] \lneq 0 \Rightarrow r_w(c) \in R(0) \Rightarrow r_w(c) \geqq r_w(0)$

$\Rightarrow \quad r_w(c) = r_w(0)$.

"$\Leftarrow$": Assuming for the moment that Lemma 3.2.(i) has already been proved, we have, by Th.3.1, and with $lx > 0$:

$[B-(1+r)(A+cl)]x \geqq 0 \quad \Rightarrow \quad [B-(1+r)A]x \geqq 0$, where $r := r_w(c) = = r_w(0)$. On the other hand we must also have:

$p[B-(1+r)A] \lneq 0 \quad \Rightarrow \quad p[B-(1+r)(A+cl)] \lneq 0$.

$$\Rightarrow \left.\begin{array}{l} p[B-(1+r)A]x = 0 \\ p[B-(1+r)(A+cl)]x = 0 \end{array}\right\} \Rightarrow \quad pclx = 0 \quad \Rightarrow \quad pc = 0.$$

The proof of Th.3.2 will be completed by the proof of L.3.2(i).

Proof of Lemma 3.2

(i) Because $g(c)$ is maximal in Problem I, the inequality $[B-(1+g(c))(A+cl)]x \gg 0$ has no solution $x \geqq 0 \Rightarrow$ (by Th.A.1): the inequality $p[B-(1+g(c))(A+cl)] \lneq 0$ has a solution $p \gneq 0$, i.e. by def.: $g(c) \in R(c) \Rightarrow r_w(c) = \inf R(c) \leqq g(c)$. This proves (i), and completes the proof of Th.3.2.

(ii) By Th.3.1, $[B-(1+g(c))(A+cl)]x \geqq 0, \quad x \gneq 0$, and by Th.3.2, $p[B-(1+r_w(c))(A+cl)] \lneq 0, \quad p \gneq 0$. This implies, by (i): $pBx = (1+g(c))p(A+cl)x = (1+r_w(c))p(A+cl)x$. It suffices to show: $pBx > 0$. The set of all "x-produced goods",

$S := \{i \ / \ (Bx)_i > 0\}$, is an independent subset because $a_{ij} + c_i l_j = 0$ for $i \notin S$, $j \in T$, where $T = \{j \ / \ x_j > 0\}$ is the set of all x-active processes. Irreducibility of (A,B) implies that $S = \{1, \ldots m\}$, i.e. $Bx \gg 0 \Rightarrow pBx > 0$. This proves (ii).

(iii) By Th.3.1(v) $\exists x$ with $[B-(1+g(c))(A+cl)]x \geqq 0$, $lx > 0$. By Th.3.2(v) $\exists p$ with $p[B-(1+r_w(c))(A+cl)] \leqq 0$, $pc > 0$. This implies, by (i): $pBx = (1+g(c))p(A+cl)x = (1+r_w(c))p(A+cl)x$, and this implies the assertion, because $pclx > 0$. This proves (iii).

Q.E.D.

<u>Remark</u>: The proof of Lemma 3.2(ii) shows that it suffices to assume that the pair $(A+cl, B)$ is irreducible.

3.3. The price system

Let c be a feasible consumption bundle, r a profit rate, p a price vector, and $x \in X$ an activity vector. A tripel (r,p,x) is called an equilibrium for c if the following three conditions are satisfied:

$$pB \leqq (1+r)p(A+cl), \qquad p \gneqq 0 \tag{3.3.1}$$

$$Bx \geqq (1+r)(A+cl)x, \qquad x \gneqq 0, \quad lx > 0 \tag{3.3.2}$$

$$pBx > 0 \tag{3.3.3}$$

(r,p,x) is called an equilibrium with positive wage for c if also

$$w = pc > 0 \tag{3.3.4}$$

We shall show that for every feasible c there exists an equilibrium, that $w > 0$ implies $r = g(c)$, and that there exists an equilibrium with positive wage if and only if c is efficient. But first we interpret the four equilibrium conditions (3.3.1)-(3.3.4):

(3.3.1) requires that for every process $j = 1,.. n$:

$$pb_j \leqq (1+r)(pa_j + wl_j) \tag{3.3.1.j}$$

On the left hand side we have the revenue of the j-th process (operated at unit intensity), and on the right hand side we have the costs, viz. the costs of the physical inputs $p.a_j$ plus the wage costs $w.l_j$, both multiplied by the profit factor $(1+r)$. (3.3.1) requires that in every industry j the rate of profit is not higher than the ruling rate of profit. This is certainly a necessary condition for long-run equilibrium and needs no further justification. Process j is called profitable if (3.3.1.j) is satisfied with equality.

(3.3.2) requires that for every good $i = 1,.. m$:

$$(Bx)_i \geq (1+r)\left[(Ax)_i + c_i.lx\right] \qquad (3.3.2.i)$$

On the left hand side there is the aggregate output of good i in a period, and on the right hand side there are those quantities of good i which are used as physical inputs of production $(Ax)_i$, resp. for the consumption of the workers, $c_i.lx$, both multiplied by $(1+r)$. (3.3.2) requires that the economy can grow at least with rate r. Otherwise, it would be impossible for the capitalists - for purely technical reasons - to invest all their profits. They would have to consume at least part of their profits. But then the structure of the capitalits' consumption, i.e. ultimately their preferences, would influence aggregate demand and hence the technique in use and, finally, relative prices. Such a theory, while certainly meaningful in its own right (cf. MORISHIMA'1969, Ch.6), can no longer be considered as "independent of demand factors". Rather, it is an application of General Equilibrium Theory, where prices are determined by the interplay of Supply and Demand. If one wants to find prices that are independent of subjective preferences, and are determined only by the technical conditions of production and the distribution of income (wage and profit rate), then it is best to assume that capitalists use their profits for accumulation. Certainly this assumption is in line with the thinking of the classical authors. Obviously a necessary condition for such an assumption is that (3.3.2) be satisfied. A good i is called a <u>surplus good</u> if (3.3.2.i) is satisfied with strict inequality.

Remark: Insofar as the workers' consumption is determined by their preferences, the theory considered in this paper is also not free from "subjective" influences. However, it may be argued that c is essentially determined by "objective" (physical or social) needs. Such a view - which is certainly consistent with neo-Ricardian price theory, but not with the spirit of General Equilibrium Theory - will be adopted in this paper without further discussion.

(3.3.3) requires simply that the value of total gross output is not zero. This is clearly necessary for any economically meaningful solution and needs no further justification. This condition was first introduced into the von Neumann model by KEMENY, MORGENSTERN & THOMPSON'1956.

(3.3.4) finally requires that the wage is not zero. $w=pc=0$ would mean that the consumption basket of the workers consists entirely of free goods. If this is the case, why should workers work? They might as well consume the free goods on which they subsist (air, water,..) without working. Even if one is willing to assume, for the purposes of an abstract investigation, that workers work at any positive wage, however small, one cannot assume that they work for nothing. Therefore (3.3.4) is also a necessary condition for an economically meaningful equilibrium. We shall see that there are certain consumption bundles whose value at equilibrium is always zero. These are precisely the "inefficient" consumption bundles defined in (3.1.2). In this case condition (3.3.4) cannot be satisfied. On the other hand, it is then always possible to increase the workers' consumption without reducing the growth rate (and the profit rate, cf. Th.3.3). Therefore, the capitalists will not oppose such an increase, until an efficient consumption bundle (and a positive wage) is reached.

The results of this section can be briefly summed up as follows (neglecting inefficient consumption bundles):

a) There exists always an equilibrium (Th.3.3)

b) The rate of profit is uniquely determined by the per-capita consumption of the workers (Th.3.3)

c) The level (not the structure) of the workers' consumption is uniquely determined by the rate of profit (L.3.3.3)

d) The wage is positive and cannot be increased without reducing the rate of profit (Th.3.3, L.3.3.2)

All these results are proved for arbitrary joint production systems without any restrictive assumptions like irreducibility etc.

First we observe three trivial, but important properties of any equilibrium (r,p,x) for c:

<u>L e m m a 3.3.1</u>. (Rule of profitability, rule of free goods)
Let (r,p,x) be an equilibrium for c. Then:

(i) $r_w(c) \leq r \leq g(c)$ the profit rate lies between the warranted rate of profit and the capacity growth rate

(ii) $x_j = 0$ if $pb_j < (1+r)p(a_j + cl_j)$ only profitable processes are active ("rule of profitability")

(iii) $p_i = 0$ if $(Bx)_i > (1+r)\left[(Ax)_i + c_i.lx\right]$ surplus goods are free goods ("rule of free goods").

T h e o r e m 3.3. (Existence and Uniqueness)

(i) For every feasible c there is an equilibrium (r,p,x). There is even an equilibrium with $r = g(c)$.

(ii) There is an equilibrium with positive wage for c if and only if c is efficient. $r = g(c)$ for every equilibrium with positive wage for c.

At an equilibrium which does not have the highest possible profit rate the wage is necessarily zero. We shall see in 4.3. that the uniqueness result of Th.3.3(ii) is not true when the wage is paid post factum.

Remark 1: It is possible that the capital advanced consists entirely of wages, i.e. that $pAx = 0$ for every equilibrium for c. This is shown by the following example:

Example 3.3.1. $A = \begin{pmatrix}1\\0\end{pmatrix}$, $B = \begin{pmatrix}2\\1\end{pmatrix}$, $l = (1)$, $c = \begin{pmatrix}0\\1\end{pmatrix}$

It is easy to see that $g(c) = 0$. This implies by (3.3.1):

$p_1 = 0 \Longrightarrow pA = 0 \Longrightarrow pAx = 0$ for every equilibrium for c.

The case $pAx = 0$ cannot be considered as economically meaningless a priori. It means that all physical means of production are free goods. At the beginning of the production period, the capitalists have to advance only the means of subsistence for the workers. By contrast, when the wage is paid at the end of the production period, the condition $pAx > 0$ is necessary for an economically meaningful solution of our model (cf. 4.3., in particular (4.3.3)). In this latter case, $pAx = 0$ would imply that the capitalists do not advance any capital at all (neither physical inputs nor wages). This would be equivalent to an economy without capitalists and without capital, and the concept of a rate of profit would lose all meaning.

Remark 2: Contrary to an erroneous remark in MORISHIMA'74 (p.621, Footnote 10), $w = pc > 0$ does not imply indecomposability. This is shown by

Example 3.3.2. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $1 = (1,1)$, $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The system is decomposable (process 1 uses only good 1 and can be operated independently), but (r,p,x) is an equilibrium with positive wage for c, where $r = r_w(c) = g(c) = 0$, $p = (1,0)$, and $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Lemma 3.3.2 (Monotonicity)

Let c, c' be feasible, and let (r,p,x), (r',p',x') be equilibria for c resp. c'. Then:

(i) $pc' > pc \implies r' < r$

(ii) $pc' = pc \neq 0 \implies r' \leq r$

Lemma 3.3.2 says that the wage cannot be increased without reducing the profit rate. On the other hand we have:

A) $pc' < pc$ implies nothing for the relationship between r and r'

B) $pc' = pc = 0$ implies nothing for the relationship between r and r'

This is shown by the following example:

Example 3.3.3. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 2 \end{pmatrix}$, $1 = (1,1)$, $c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

It is easy to see that $g(c) = 1$. Two possible equilibrium price systems for c are given by:

A) $p = (0,0,1) \implies pc = 1 > 0$

B) $p = (0,1,0) \implies pc = 0$

Moreover it is easy to check that $g(c^o) = 2$ for $c^o = 0$,

$g(c^1) = 0$ for $c^1 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $g(c^2) = 0$ for $c^2 = \begin{pmatrix}0\\0\\2\end{pmatrix}$.

Then we have in Case A): $pc^o < pc$ and $g(c^o) > g(c)$, but $pc^1 < pc$ and $g(c^1) < g(c)$; and in Case B): $pc^o = pc = 0$ and $g(c^o) > g(c)$; $pc^2 = pc = 0$ and $g(c^2) < g(c)$.

Next we turn to the question of efficiency resp. optimality of equilibrium. For this purpose we consider the two dual Linear Programmes (with $L = lx$):

$$\left.\begin{array}{l} \min\limits_{x} \quad wlx \\ \text{s.t.} \quad [B-(1+r)A]x \geq (1+r)Lc, \quad x \geq 0 \end{array}\right\} \tag{3.3.5}$$

$$\left.\begin{array}{l} \max\limits_{p} \quad Lpc \\ \text{s.t.} \quad p[B-(1+r)A] \leq (1+r)wl, \quad p \geq 0 \end{array}\right\} \tag{3.3.6}$$

L e m m a 3.3.3. (Efficiency)

Let c be a feasible efficient consumption bundle, (r,p,x) an equilibrium with positive wage for c, and $L := lx$, $w := pc$. Then:

(i) $r = g(c)$

(ii) x is an optimal vector for (3.3.5)

(iii) p is an optimal vector for (3.3.6)

Lemma 3.3.3 says that the per-capita consumption of the workers can be neither more nor less than c, given the rate of profit r. By (3.3.5), $L = lx$ is the smallest number of workers needed to produce the means of consumption Lc, under the constraint that capitalists accumulate at the rate r. Therefore $\frac{Lc}{lx} = c$ is the largest possible consumption per capita. On the other hand, by (3.3.6), $w = pc$ is the highest value that can be assigned to the commodity bundle c (given the profit rate r and positive nominal wage w) without violating the equilibrium condition that no industry makes profits in excess of the ruling rate of profit. Therefore $\frac{w}{pc} = 1$ is the smallest real wage per capita (expressed with the commodity bundle c as numéraire).

Remark: Of course there are other consumption bundles $c' \neq c$ which are also compatible with the same rate of profit r. However, these consumption bundles contain the various goods in different proportions, representing a different structure of the workers' consumption, and are therefore neither "more" nor "less" than c (but cf. Lemma 3.3.2).

3.3.0.

Proof of Lemma 3.3.1: Obvious from (3.3.1) and (3.3.2)

Proof of Theorem 3.3:

(i) Let $r := g(c)$ and $M := B-(1+g(c))(A+cl)$. By Th.A.2, there are vectors $p \geq 0$, $x \geq 0$, with $pM \leq 0$, $Mx \geq 0$, and $p_i = 0 \Rightarrow (Mx)_i > 0$. By Th.3.1(v) we may assume w.l.o.g. that $lx > 0$. We claim that $pBx > 0$. Otherwise ($pBx = 0$) $p_i > 0$ would imply $(Bx)_i = 0$, and also $[(A+cl)x]_i = 0$. Because $(Mx)_i > 0$ for $p_i = 0$, $g(c)$ would not be maximal, a contradiction. Therefore the tripel (r,p,x) satisfies (3.3.1)-(3.3.3), with $r = g(c)$. This proves (i).

(ii) Denote by $(\hat{r},\hat{p},\hat{x})$ the equilibrium defined in (i). If c is efficient, then there exists an i with $c_i > 0$ and $(M\hat{x})_i = 0 \Rightarrow \hat{p}_i > 0 \Rightarrow \hat{p}c > 0$. If (r,p,x) is another equilibrium for c, with $r < \hat{r} = g(c)$, then we obtain from (3.3.1),(3.3.2): $pB\hat{x} = (1+r)p(A+cl)\hat{x} = (1+\hat{r})p(A+cl)\hat{x} \Rightarrow pcl\hat{x} = 0 \Rightarrow pc = 0$.

If c is inefficient, then there exists a $k > 1$ and an $\bar{x} \in X$ with $M_o.\bar{x} \geq 0$, where $M_o := B-(1+g(c))(A+kcl)$. Now if (r,p,x) is any equilibrium for c, we have: $pM\bar{x} = pM_o\bar{x} = 0 \Rightarrow pcl\bar{x} = k.pcl\bar{x} \Rightarrow pc = 0$.

Q.E.D.

Proof of Lemma 3.3.2:

Define $M := B-(1+r)(A+cl)$, $M' := B-(1+r')(A+c'l)$. We have $pM \leq 0$ by (3.3.1) and $M'x' \geq 0$, $lx' > 0$ by (3.3.2).

(i) $pc' > pc$, but $r' \geqq r$ imply: $y := pM' \lneqq 0$ and $y_j < 0$ for $l_j > 0$. $\Rightarrow$ $yx' = 0$ $\Rightarrow$ $lx' = 0$, contradiction.

(ii) same proof as (i).

Q.E.D.

Proof of Lemma 3.3.3

(i) obvious from Th.3.3

(ii)&(iii): (3.3.1),(3.3.2) imply that the two programmes (3.3.5),(3.3.6) are both feasible; the assertion then follows immediately from $wlx = wL = Lpc$, by the Optimality Criterion of Linear Programming (Th.A.4(i)).

Q.E.D.

3.4. The "Fundamental Marxian Theorem"

Let c be a feasible consumption bundle. The following statement has been called the "Fundamental Marxian Theorem" by, e.g., MORISHIMA'1974:

T h e o r e m 3. 4 (Fundamental Marxian Theorem)
A positive rate of exploitation is necessary and sufficient for positive profits and positive growth, i.e.

$$e(c) > 0 \Leftrightarrow r_w(c) > 0 \Leftrightarrow g(c) > 0.$$

By Lemma 2.4., a positive rate of exploitation means simply that the labour value of the workers' consumption is less than one, in other words, that a worker works more than would be necessary to produce his own means of subsistence. Precisely in this case is the augmented system (A+cl,B,l) productive in the sense of Ass.I(iii), where (A+cl) is obtained from (A,B,l) by adding the necessities of the workers' consumption, $c_i l_j$, to the ordinary input coefficients a_{ij}. Precisely in this case there is a positive surplus product, and, consequently, profits and growth. Viewed in this way, the "Fundamental Theorem" appears almost trivial.

3.4.0.
The proof of Th.3.4. follows immediately from Th.3.1(iii) and Th.3.2(iii).

Q.E.D.

§4. The system with the wage paid post factum

In this section we assume that the wage is paid at the end of the production period ("post factum", cf. SRAFFA'1960,§9). Wages are considered as the workers' share in the surplus product. In the framework of a von Neumann model this case was studied e.g. by WOLFSTETTER'1977. We continue to assume that the technology satisfies Ass.I. §4 is parallel to §3.

4.1. The capacity growth rate

Let $c \geqq 0$ be a feasible consumption bundle. In order to find the largest growth rate of the system, compatible with c, we consider the following nonlinear programme:

Problem III

max g

$$\text{s.t.} \quad [B - (1+g)A - cl]x \geqq 0, \qquad x \gneqq 0 \tag{4.1.1}$$

If this problem has a solution we denote it by $g = g(c)$ and call $g(c)$ the capacity growth rate for c.

Theorem 4.1. (Capacity growth rate)

(i) For every feasible c there is a unique nonnegative capacity growth rate $g(c)$

(ii) $g(c)$ is weakly monotonically decreasing in c, i.e. $c \geqq c'$ implies $g(c) \leqq g(c')$

(iii) $g(c) = 0$ implies $e(c) = 0$

(iv) $g(0) =: g_{max}$ is the largest possible growth rate, and for every feasible c we have: $0 \leqq g(c) \leqq g_{max}$.

(v) The inequality $[B-(1+g(c))A-cl]x \geqq 0$ has at least one solution $x \gneqq 0$ with $lx > 0$.

Theorem 4.1. is literally the same as Theorem 3.1., with the exception of (iii), where the reverse implication is false in Theorem 4.1. When the wage is paid post factum it is possible that $e(c) = 0$, but $g(c) > 0$, as is shown by Example 4.1.1. In 5.4. we shall see that this cannot happen in the absence of genuine joint production (cf. Lemma 5.4.1(iv)).

Example 4.1.1. Let A, B, l, c be as in Ex.3.3.1. Then we have $e(c) = 0$ because $B - A - cl = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (cf. Lemma 2.4(iii)); but the solution of Problem III is $g(c) = 1$, because $B - 2A - cl = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Remark: Ex.4.1.1 shows that the Fundamental Marxian Theorem need not be true when the wage is paid post factum, cf. 4.4. Of course this may also be taken as an indication that perhaps the rate of exploitation should be defined differently in this case. We shall not pursue this matter here.

A feasible consumption bundle c is called inefficient if

$$\exists\, k > 1 \quad \text{with} \quad g(kc) = g(c), \tag{4.1.2}$$

and efficient otherwise.

4.1.0.

The proof of Theorem 4.1. is completely analogous to the proof of Theorem 3.1. and is omitted.

4.2. The warranted rate of profit

Let c be a feasible consumption bundle, and consider the following nonlinear programme, dual to Problem III of 4.1.:

Problem IV

min r

s.t. $p[B - (1+r)A - cl] \leqq 0, \quad p \gneq 0$ (4.2.1)

If Problem IV has a nonnegative solution we denote this solution by $r_w(c)$; otherwise, we define $r_w(c) := 0$. $r_w(c)$ is called the warranted rate of profit for c.

Theorem 4.2. (Warranted rate of profit)

(i) For every feasible c there is a unique nonnegative warranted rate of profit, $r_w(c)$

(ii) $r_w(c)$ is weakly monotonically decreasing in c, i.e. $c \geq c'$ implies $r_w(c) \leqq r_w(c')$

(iii) $r_w(c) = 0$ if and only if $e(c) = 0$

(iv) $r_w(0)$ is the largest possible warranted rate of profit, and for every feasible c we have $0 \leqq r_w(c) \leqq r_w(0)$

(v) The inequality $p[B-(1+r_w(c))A-cl] \leqq 0$ has a solution $p \gneq 0$ with $pc = 0$ if and only if $r_w(c) = r_w(0)$.

Remark: It is possible that Problem IV has no solution or a negative solution if the rate of exploitation is zero, $e(c) = 0$ (cf. Ex.4.2.1 and Ex.4.2.2). In this case we put $r_w(c) = 0$. This definition is justified by the following Lemma:

<u>L e m m a 4.2.1</u>. (Negative profit rates)

Let (i) c be feasible, with $(B-A-cl)x \geqq 0$, $x \in X$, and

(ii) $r < 0$, with $p[B-(1+r)A-cl] \leqq 0$, $p \gneqq 0$.

Then $pBx = pclx > 0$, but $pAx = 0$.

By Lemma 4.2.1 total profits $rpAx$ are equal to zero for every $r \leqq 0$ (provided the system is at least reproducing itself), i.e. we may w.l.o.g. put $r_w = 0$.

<u>Example 4.2.1</u>: Let A,B,l,c be as in Example 3.3.1.

$\Longrightarrow$ $B-(1+r)A-cl = \begin{pmatrix}1-r\\0\end{pmatrix}$. (4.2.1) becomes: $p_1(1-r)+p_2.0 \leqq 0$, $p \gneqq 0$. If $p = (0,1)$ then (4.2.1) is satisfied for every r, and Problem IV has no solution ("$r_{min} = -\infty$").

<u>Example 4.2.2</u>: $A = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $B = \begin{pmatrix}2&1\\1&0\end{pmatrix}$, $l = (1,1)$, $c = \begin{pmatrix}0\\1\end{pmatrix}$

$\Longrightarrow$ $B-(1+r)A-cl = \begin{pmatrix}1-r & 1\\0 & -2-r\end{pmatrix}$. (4.2.1) becomes:

$$\left.\begin{array}{l} p_1(1-r) \leqq 0 \\ p_1-p_2(2+r) \leqq 0 \end{array}\right\}, \quad p \gneqq 0$$

If $p = (0,1)$ then (4.2.1) is satisfied for $r = -2$. It is easy to see that this is the solution of Problem IV.

We shall see in §<u>5.4</u>. that Problem IV does have a nonnegative solution for every feasible c if there are no genuine joint production and no perfectly durable capital goods (cf. Lemma 5.4.2).

<u>L e m m a 4.2.2</u>. ($g(c)$ and $r_w(c)$)

(i) $r_w(c) \leqq g(c)$ for every feasible c

(ii) $r_w(c) = g(c)$ if (A,B) is irreducible

Remark: The analog of Lemma 3.2(iii) is not true. It is possible that $r_w(c) < r_w(0)$, but $r_w(c) \neq g(c)$; even if $r_w(c)$ is in fact the solution of Problem IV. This is shown by

Example 4.2.3: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$, $\underline{1} = (1,1)$, $c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We have $r_w(0) = 1$, $r_w(c) = 0$, but $g(c) = 1$.

4.2.0.

Proof of Theorem 4.2.

(i)&(iii): We denote by $R(c)$ the set of all r for which (4.2.1) has a solution $p \gneq 0$. $R(c)$ is closed (trivial) and nonempty, since for sufficiently large r (4.2.1) has a solution, e.g. $p = \underline{1}$, because $\underline{1}.A \gg 0$ by Ass.I(ii). If $e(c) > 0$, then there is an $x \in X$ with $(B-A-c\underline{1})x \gg 0$, by L.2.4. For $r < 0$ this implies $[B-(1+r)A-c\underline{1}]x \gg 0$, and hence, by Th.A.1, that (4.2.1) has no solution $p' \gneq 0$. Therefore $\inf R(c) =: r_w(c) > 0$. If $e(c) = 0$, then, again by Lemma 2.4, there is no $x \in X$ with $(B-A-c\underline{1})x \gg 0$ $\Longleftrightarrow$ (by Th.A.1): $\exists\, p \gneq 0$ with $p(B-A-c\underline{1}) \leqq 0$, i.e. $0 \in R(c)$. This proves (i),(iii).

(ii),(iv)&(v) are proved as in Th.3.2.

Q.E.D.

Proof of Lemma 4.2.1

(ii) implies $y := p(B-A-c\underline{1}) \leqq 0$, and $yx = 0$ by (i).

If $x_j > 0 \Rightarrow y_j = 0$, i.e. $pa_j + pcl_j = pb_j \underset{\text{by (ii)}}{\leqq} (1+r)pa_j + pcl_j$

$\Rightarrow pa_j = 0$ because $r < 0$. Therefore $pAx = 0$. This implies immediately $pBx = pc\underline{1}x$, and this expression is positive, because $\underline{1}x > 0$; and because $pc = 0$ would imply $p(B-A) \leqq 0$, contradicting (by Th.A.1) Ass.I(iii).

Q.E.D.

Proof of Lemma 4.2.2

(i) Analogous to Lemma 3.2(i)

(ii) By Th.4.1, $[B-(1+g(c))A-cl]x \geq 0$, $x \gneq 0$, and by Th.4.2, $p[B-(1+r_w(c))A-cl] \leq 0$, $p \gneq 0$. This implies by (i): $p(B-cl)x = (1+g(c))pAx = (1+r_w(c))pAx$.

It suffices to show that $pAx > 0$. The set $S := \{i \;/\; (Ax)_i > 0\}$ is an independent subset $\Rightarrow$ $S = \{1, \ldots m\}$, because (A,B) is irreducible $\Rightarrow$ $Ax \gg 0$ $\Rightarrow$ $pAx > 0$. This proves (ii).

Q.E.D.

Remark: It is easy to see that it suffices in Lemma 4.2(ii) if the pair $(A, B-cl)$ is irreducible (of course this pair is not a von Neumann system in the sense of 2.1., because $B-cl$ may contain negative elements).

4.3. The price system

Let c be a feasible consumption bundle. A triple (r,p,x) with $r \geqq 0$ is called an equilibrium for c if it satisfies the following three conditions:

$$pB \leqq (1+r)pA + pcl, \qquad p \gneqq 0 \tag{4.3.1}$$

$$Bx \geqq (1+r)Ax + clx, \qquad x \gneqq 0, \qquad lx > 0 \tag{4.3.2}$$

$$pAx > 0 \tag{4.3.3}$$

(r,p,x) is called an equilibrium with positive wage for c if also

$$w = pc > 0. \tag{4.3.4}$$

The interpretation of the four equilibrium conditions (4.3.1)-(4.3.4) is analogous to the one given for (3.3.1)-(3.3.4) in section 3.3. The only difference is that now the capital advanced by the capitalists consists only of the physical inputs of production. Therefore an economically meaningful solution requires $pAx > 0$ (and not only $pBx > 0$), as explained in Remark 1 after Theorem 3.3. Moreover, when the wage is paid post factum, it is necessary to require explicitly $r \geqq 0$ in the definition of equilibrium, because (4.3.1)-(4.3.4) may be satisfied for negative r as well, as shown in Ex.4.3.1. When the wage is advanced (cf. §3), $r \geqq 0$ is implied automatically by condition (3.3.1), because Problem II (unlike Problem IV) always has a nonnegative solution.

Example 4.3.1: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, $l = (1,0)$, $c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(4.3.1)-(4.3.4) are satisfied for $r = -1$, $p = (0,1)$, $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The results when the wage is paid post factum are similar to the case when the wage is paid in advance, but somewhat less satisfactory. In particular, the rate of profit need not be uniquely determined, i.e. (4.3.1)-(4.3.4) may be satisfied for more than one nonnegative value of r. Moreover, the "Fundamental Marxian Theorem" is not true, and a warranted rate of profit in the strict sense need not exist. (cf. 4.4.; resp. 4.2.). All these difficulties are connected with the presence of genuine joint production (cf. 5.4.).

Neglecting inefficient consumption bundles, the results of this section can be briefly summed up as follows (cf. 3.3.):

a) There exists always an equilibrium (Th.4.3)

b) The rate of profit is not necessarily uniquely determined by the per-capita consumption of the workers (Ex.4.3.2)

c) The level (not the structure) of the workers' consumption is uniquely determined by the rate of profit (Lemma 4.3.3)

d) The wage is positive and cannot be increased without reducing the rate of profit (Th.4.3., Lemma 4.3.2)

Again we have:

L e m m a 4.3.1 (Rule of profitability, rule of free goods)
Let (r,p,x) be an equilibrium for c. Then:

(i) $r_w(c) \leqq r \leqq g(c)$

(ii) $x_j = 0$ for $pb_j < (1+r)pa_j + pcl_j$ ("rule of profitability")

(iii) $p_i = 0$ for $(Bx)_i > (1+r)(Ax)_i + c_i.lx$ ("rule of free goods").

T h e o r e m 4.3. (Existence)

(i) For every feasible c there is an equilibrium (r,p,x). There is even an equilibrium with $r = g(c)$.

(ii) There is an equilibrium with positive wage for c if and only if c is efficient. In this case there is even an equilibrium with positive wage for c with $r = g(c)$.

Unlike the case considered in §3, there may also be equilibria with positive wage for c such that the rate of profit is strictly less than the capacity growth rate, as is shown by Ex. 4.3.2. This indeterminacy of the profit rate is possible only under genuine joint production, cf. Lemma 5.4.3.

Example 4.3.2. (Nonuniqueness of the rate of profit)

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad l = (1,1), \quad c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

It is easy to see that $g(c) = 2$. Two equilibria with positive wage for c are given by: $r = 2$, $p = (1,1,1)$, $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; and $r = 1$, $p = (1,1,0)$, $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

L e m m a 4.3.2. (Monotonicity)

Let c, c' be feasible and let (r,p,x) resp. (r',p',x') be equilibria for c resp. c'. Then:

$$pc' > pc \implies r' < r.$$

This Lemma corresponds to Lemma 3.3.2(i) in §3. Next we consider the two dual Linear Programmes:

$$\left.\begin{array}{ll} \min & wlx \\ \text{s.t.} & [B-(1+r)A]x \geqq Lc, \qquad x \geqq 0 \end{array}\right\} \qquad (4.3.5)$$

$$\left.\begin{array}{ll} \max & Lpc \\ \text{s.t.} & p\left[B-(1+r)A\right] \leqq wl, \qquad p \geqq 0 \end{array}\right\} \qquad (4.3.6)$$

L e m m a 4.3.3. (Efficiency)

Let c be a feasible efficient consumption bundle, (r,p,x) an equilibrium with positive wage for c, and $L:= lx$, $w:= pc$.

Then:

(i) x is an optimal vector for (4.3.5)

(ii) p is an optimal vector for (4.3.6).

The interpretation of Lemma 4.3.3 is similar to the interpretation of Lemma 3.3.3.

4.3.0. Proofs

Lemma 4.3.1 follows immediately from (4.3.1),(4.3.2)

Theorem 4.3.: Analogous to Th.3.3., with $M:= B-(1+g(c))A-cl$.

Lemma 4.3.2: Analogous to Lemma 3.3.2(i)

Lemma 4.3.3: Analogous to Lemma 3.3.3.

4.4. The "Fundamental Marxian Theorem"

When the wage is paid at the end of the production period the "Fundamental Marxian Theorem" does not hold, i.e. it is possible that the rate of profit and the rate of growth are both positive, but the rate of exploitation is zero. This is shown by

Example 4.4.1: Let A,B,l,c be as in Ex.4.1.1. Then we have $g(c) = 1$, but $e(c) = 0$.

This example contradicts an incorrect assertion in WOLFSTETTER'1977, p.62. The implication is correct only in one direction:

T h e o r e m 4.4.
A positive rate of exploitation is sufficient for positive profits and positive growth, i.e.

$$e(c) > 0 \Rightarrow r_w(c) > 0 \text{ and } g(c) > 0.$$

In the absence of genuine joint production, the reverse implication is also true, cf. Lemma 5.4.4.

4.4.0.
Th.4.4. follows immediately from Th.4.1(iii) and Th.4.2(iii).

§5. Nonsubstitution Theory

5.1. Introduction

We consider now an important special case of the general linear production model developed in §§1-4, viz. the case where the so-called "Nonsubstitution Theorem" holds. The Nonsubstitution Theorem says, roughly, that, in equilibrium, both the technique used and the prices of all goods depend only on the rate of profit r, but not on the structure of final demand, i.e. on the composition of the workers consumption basket c, as long as c is compatible with r. Moreover, the relative prices of all actually produced goods are uniquely determined by r. In this case - and only in this case - is it possible to consider the price system and the quantity system separately and to study the "influence of variations in the rate of profit upon relative prices" without explicitly paying attention to the accompanying changes in physical quantities. Again only in this case does it make sense to speak of an 'optimal' or 'profitable' technique, given the rate of profit, and to draw the wage-profit curve as is usually done, namely as a function relating the nominal wage w and r, without worrying about the numéraire, i.e. the physical composition of the workers' consumption. This is so because, under the Nonsubstitution Theorem, a change in the numeraire does not require a change of the optimal technique, given the rate of profit r, and hence leaves invariant the qualitative features of the w-r-curve, in particular its switch-points.

In order to derive the Nonsubstitution Theorem, we need one additional assumption, which amounts to excluding genuine joint production: we shall assume that every process increases the quantity of at most one good (cf. Ass.II below). An important special case is of course the case of single-product industries, where each process produces exactly one good, and completely uses up all other inputs (i.e. there is only circulating capital).

The absence of genuine joint production not only enables us to prove the Nonsubstitution Theorem, but it also makes the theory with the wage paid in advance virtually identical to the theory with the wage paid 'post factum'. In particular, we obtain both uniqueness of equilibrium and the Fundamental Marxian Theorem for the latter case as well (with genuine joint production this need not be true, cf. §4).

§5 forms, so to speak, a small replica of the entire first part of the paper. Section 5.k. corresponds to §k, for $k=1,2,3,4$, and contains the appropriate modifications resp. refinements of the theory when there is no genuine joint production.

5.2. The linear production model without genuine joint production

From now on we shall assume that the technology (A,B,l) satisfies the following assumption, in addition to Ass.I.:

A s s u m p t i o n II (No Genuine Joint Production):
For every process $j = 1,2,.. n$, there exists at most one good i with $b_{ij} > a_{ij}$.

Ass.II means that the net output vector $b_j - a_j$ has at most one positive component. In other words, every process increases the quantity of at most one good. When Ass.II is satisfied we say that there is "no genuine joint production".
If $b_{ij} - a_{ij} > 0$, we say that the j-th process produces good i $(1 \leq i \leq m)$. The set of all such processes forms the i-th industry, denoted by $T_i := \{ j \,/\, b_{ij} > a_{ij},\ j=1,.. n \}$. By productivity (Ass.I.iii), T_i is nonempty for every good $i = 1,.. m$.

Recall from 2.2. that the labour value $V(d)$ of a commodity bundle d is the minimum amount of labour needed to produce d.

Theorem 5.2. (Labour values)
Let (A,B,l) satisfy Ass.I and Ass.II. Then for every commodity bundle $d \geq 0$, its labour value is given by $V(d) = \sum_{i=1}^{m} V(e_i) . d_i$, where $V(e_i) > 0$ is the labour value of one unit of the i-th good $(1 \leq i \leq m)$.

This result should be compared with the remarks after Th.2.2.

Next we observe that, when the technology is reducible and the growth rate sufficiently high, then, in general, not all goods can be produced in positive quantities. This causes a certain difficulty in establishing the Nonsubstitution Theorem (cf. BLISS'1975, pp.266-268). In order to overcome this difficulty, we introduce now some auxiliary concepts.

For a nonnegative number $g \geqq 0$ we call good i <u>g-producible</u> if there exists an activity vector $x \geqq 0$ such that $y := [B-(1+g)A]x \geqq 0$ and $y_i > 0$. The set of all g-producible goods is denoted by $S := S(g)$. A g-producible good is a good of which a net surplus ("for consumption") can be produced, over and above the amount required for investment at growth rate g.

For an arbitrary pair of nonnegative $(m*n)$-matrices (A,B) satisfying Ass.II, but not necessarily Ass.I, we give the following definitions: (A,B) is called <u>semiproductive</u> if there exists an $x \geqq 0$ s.t. $(B-A)x \gneqq 0$, i.e. if $X = \{x \,/\, (B-A)x \gneqq 0\} \neq \emptyset$. Good i is called <u>producible</u> in (A,B) if there is an $x \in X$ s.t. $(Bx-Ax)_i > 0$. Process j is called <u>productive</u> in (A,B) if it produces a producible good and if there is an $x \in X$ with $x_j > 0$.

<u>Lemma 5.2.</u> (Semiproductive systems)

Let (A,B) be semiproductive and satisfy Ass.II. Then the productive processes in (A,B) need no net inputs of nonproducible goods, i.e. $b_{ij} = a_{ij}$ for $i \notin S$, $j \in T$, where S is the set of producible goods and T is the set of productive processes in (A,B).

5.2.0.

Proof of Theorem 5.2.

First choose a strictly positive $\bar{d} \gg 0$, and corresponding optimal vectors $\bar{x}$, $\bar{v}$ for the Linear Programs (2.2.1), (2.2.2). Such optimal vectors exist, by the proof of Th.2.2. By the Basis Theorem of Linear Programming (cf. Th.A.3), $\bar{x}$ can be chosen so that it has at most m positive components. Define $T := \{ j \,/\, \bar{x}_j > 0 \}$, and denote[13)] by $M := B_T - A_T$ the square submatrix obtained by striking out all processes not contained in T. By the Inversion Lemma (Th.A.5), M is nonnegatively invertible, because $M\bar{x}_T \geq \bar{d} \gg 0$, and all off-diagonal elements of M are nonpositive, by Ass.II (w.l.o.g. we may arrange the processes in T in the same order as the goods they produce).

Now let d be any commodity bundle, and define $x := (x_T, 0)$, where $x_t := M^{-1}.d$. Then $(B-A)x = d$, i.e. x is feasible for (2.2.1), and $\bar{v}$ remains feasible for (2.2.2). Moreover, the pair $(x, \bar{v})$ satisfies the complementary slackness condition (Th.A.4.ii), and hence is optimal. $\Rightarrow$

$$\Rightarrow \quad V(d) = \bar{v}d = lx = l_T.M^{-1}d = \sum_{i=1}^{m} V(e_i)d_i, \quad \text{where}$$

$\bar{v}_i = V(e_i) = l_T.M^{-1}.e_i > 0$ is the labour value of one unit of good i.

Q.E.D.

Proof of Lemma 5.2.

Take an arbitrary productive process $j \in T$, producing, say, good s (i.e. $j \in T_s$). There exists an $x \geq 0$ with $x_j > 0$ s.t.

$$\sum_{k=1}^{n} b_{sk}x_k > \sum_k a_{sk}x_k \tag{1}$$

$$\sum_k b_{hk}x_k \geq \sum_k a_{hk}x_k \qquad \text{for all goods } h=1,\ldots m \tag{2}$$

If $i \notin S$ is a nonproducible good, we must have:

$$\sum_k b_{ik}x_k = \sum_k a_{ik}x_k \qquad (3)$$

Now for ε sufficiently small and positive we may replace x_j by $x_j - \varepsilon$ without disturbing inequality (1), while inequalities (2) are, if anything, strengthened, because $b_{hj} \leq a_{hj}$ for $h \neq s$, by Ass.II. If $a_{ij} > b_{ij}$ then (3) implies

$$\sum_k b_{ik}x_k - \varepsilon b_{ij}x_j > \sum_k a_{ik}x_k - \varepsilon a_{ij}x_j .$$

But this would mean that good i is producible, contrary to hypothesis. Therefore $a_{ij} = b_{ij}$.

Q.E.D.

This proof is adapted from Gale's argument for the single-product case (GALE'1960, p.298).

5.3. The case of the wage paid in advance

In this section we study the model of §3 for the special case where the technology satisfies Ass.I and Ass.II, i.e. there is no genuine joint production. Definitions and notation are taken from §3. In particular, the wage is paid at the beginning of the production period, and the capacity growth rate $g(c)$ and equilibrium (r,p,x) are defined accordingly.

Let $c \geq 0$ be a feasible consumption bundle, and $g(c)$ the capacity growth rate for c. By Th.3.1.(v), the inequality

$$Bx \geq (1+g(c))(A+cl)x, \quad x \gneq 0, \quad lx > 0 \tag{5.3.1}$$

has at least one solution x.

<u>Lemma 5.3.1</u> (Capacity growth rate)

Let (A,B,l) satisfy Ass.I and Ass.II.

(i) $Bx = (1+g(c))(A+cl)x$ for every solution x of (5.3.1)

(ii) $g(c) < g(c')$ for $c \gneq c'$. In particular, every $c \neq 0$ is efficient.

(iii) If $c \neq 0$, then every solution x of (5.3.1) uses only $g(c)$-producible goods, i.e. $a_{ij} = b_{ij} = 0$ for $i \notin S(g(c))$, $x_j > 0$.

<u>Theorem 5.3.</u> (Nonsubstitution Theorem)

Let (A,B,l) satisfy Ass.I and Ass.II and let $0 \leq r = g < g_{max}$. Then there exists a set of processes $\hat{T} = \hat{T}(g)$ containing exactly one process $j \in T_i$ for every g-producible good $i \in S(g)$, and a price vector $\hat{p} = \hat{p}(r)$ with positive prices $\hat{p}_i > 0$ for all $i \in S(g)$, s.t. for every feasible c with $g(c) = g$ the following is true:

(i) There exists an activity vector $\hat{x}$, using only processes in $\hat{T}$ (i.e. $\hat{x}_j = 0$ for $j \notin \hat{T}$), such that $(r, \hat{p}, \hat{x})$ is an equilibrium for c, with $\hat{w} = \hat{p}c = 1$.

(ii) If (r, p, x) is any equilibrium for c, with $w = pc > 0$, then $p_i = w.\hat{p}_i$ for all i with $(Bx)_i > 0$, i.e. the prices of all actually produced goods are proportional to those given by $\hat{p}$.

The set $\hat{T} = \hat{T}(g)$ is called an optimal technique for g. Usually $\hat{T}$ is uniquely determined by g; a value of g for which two or more techniques are optimal is called a switch-point. The existence of an optimal technique $\hat{T}$ means that a change in final demand c, does not require a change of technique $\hat{T}$, i.e. does not require substitution among technical processes, as long as the rate of growth g remains constant. This statement is the Nonsubstitution Theorem in its Quantity version.

The dual statement, the Price version of the Nonsubstitution Theorem, says that relative prices are independent of demand, i.e. a change in c does not lead to a change in the price vector $\hat{p} = \hat{p}(r)$, as long as r remains constant. Relative prices are determined exclusively, and in fact even uniquely, by the rate of profit.

Example 5.3.1 shows that both the Quantity version and the Price version of the Nonsubstitution Theorem are false if the economy violates Ass.II, i.e. if there is genuine joint production.

<u>Example 5.3.1</u> $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 4 & 3 \\ 4 & 0 & 3 \end{pmatrix}$, $l = (1,1,1)$

Consider the three feasible consumption vectors:
$c^1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $c^2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $c^3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. We have $g(c^k) = 0$ for $k=1,2,3$; but if (r,p,x) is an equilibrium for $c = c^k$, then

$c = c^1$ implies: x uses only the first process, and $p_1 < p_2$

$c = c^2$ implies: x uses only the second process, and $p_1 > p_2$

$c = c^3$ implies: x uses only the third process.

<u>Lemma 5.3.2</u> (Labour values and prices)
Let (A,B,l) satisfy Ass.I and Ass.II. Then $\hat{p}_i(0) = V(e_i)$ for all goods $i=1,..$ m, i.e. when the rate of profit is zero, then prices are equal to labour values, provided the wage is taken as numéraire $(w=1)$.

5.3.0

Proof of Lemma 5.3.1

(i) Let $g := g(c)$ and assume, indirectly, that the system $((1+g)(A+cl),B)$ is semiproductive. Denote by S the set of all producible goods and by T the set of all productive processes for this system. There exists an $x \geqq 0$, with $x_j > 0$ iff. $j \in T$, s.t.

$$\sum_{j=1}^{n} b_{ij}x_j > (1+g) \cdot \sum_j (a_{ij}+c_i l_j)x_j \qquad \text{for } i \in S \qquad (1)$$

$$\sum_j b_{ij}x_j = (1+g) \cdot \sum_j (a_{ij}+c_i l_j)x_j \qquad \text{for } i \notin S \qquad (2)$$

By Lemma 5.2.,

$$b_{ij} = (1+g)(a_{ij}+c_i l_j) \qquad \text{for } i \notin S, \ j \in T \qquad (3)$$

If $g > 0$, then (3) and Ass.II imply $b_{ij} = a_{ij}+c_i l_j = 0$ for $i \notin S$, $j \in T$; and from (1),(2) one gets a contradiction to the maximality of $g=g(c)$.

If $g=0$, then (3) and Ass.II imply, in particular, $c_i=0$ for $i \notin S$. Choose (by Productivity, Ass.I.iii) an $x^o \geqq 0$ sufficiently small and s.t. $(B-A)x^o \gg 0$, and define $u := (B-A-cl)(x+x^o)$. For $i \in S$, $u_i > 0$ by (1) and because x^o is sufficiently small; for $i \notin S$, $u_i \geqq (Bx^o-Ax^o)_i - \underbrace{c_i l x^o}_{=0} > 0$ by def. of x^o

$\Longrightarrow$ $u \gg 0$, contradicting $g=g(c)=0$.

This proves (i).

(ii) Let $c \gneqq c'$, but assume indirectly $g := g(c) \geqq g(c') =: g'$. For any solution x of (5.3.1) we have, by (i):

$0 = Bx - (1+g)(A+cl)x \leqq Bx - (1+g')(A+c'l)y =: y$ and $y_i > 0$ for $c_i > c_i'$, contradicting (i).

This proves (ii).

(iii) If $g(c)=0$ the assertion is trivial because every good is 0-producible, by Productivity (Ass.I.iii).

If $g:=g(c)>0$, and x is a solution of (5.3.1), then $1x>0$, and hence, by (i), $c_i=0$ for $i \notin S:=S(g)$. Note that $c\neq 0$ implies that S is nonempty and $g<g_{max}$, by (ii).

We denote by T the set of all productive processes for the system $((1+g)A,B)$, and obtain from Lemma 5.2.:

$b_{ij}=(1+g)a_{ij}$ for $i\notin S$, $j\in T$. By Ass.II, this implies

$$b_{ij}=a_{ij}=0 \quad \text{for } i\notin S,\ j\in T \qquad (*)$$

Now write x in the form $x=(x_T,x_U)$, where $U:=\{1,\ldots n\}\setminus T$ is the set of unproductive processes for $((1+g)A,B)$, and define $x':=(x_T,0)$, $x'':=(0,x_U)$. Obviously $x \gneq 0$ because $c\neq 0$; we want to show that $x_U=0$ (the case where U is empty is trivial, like the case where $g(c)=0$). If a process $j\in U$ is active, it certainly cannot produce a good $i\in S$. Therefore (5.3.1) and $(*)$ imply: $Bx' \geq (1+g)(A+c1)x'$. By (i), there must be equality, and hence also: $Bx''=(1+g)(A+c1)x''$.

Because $c_i>0$ for at least one $i\in S$, and by definition of x'', this implies $1x''=0$. By Ass.I(iv), $Ax''=Bx''=(1+g)Ax''$ $\Rightarrow$ $Ax''=0$, and therefore, by Ass.I(ii), $x''=0$.

This proves (iii) and Lemma 5.3.1.

Q.E.D.

Proof of Theorem 5.3.

Let $0\leq r=g<g_{max}$.

(i) If c is feasible with $g(c)=g$, then c must be efficient and $c\neq 0$, by Lemma 5.3.1(ii). Moreover $c_i=0$ for $i\notin S:=S(g)$.

Consider the two dual Linear Programmes:

$$\left.\begin{array}{ll} & \min\ 1x \\ \text{s.t.} & [B-(1+r)A]x \geqq (1+r)c, \qquad x \geqq 0 \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{ll} & \max\ pc \\ \text{s.t.} & p[B-(1+r)A] \leqq (1+r)1, \qquad p \geqq 0 \end{array}\right\} \quad (2)$$

By Lemma 3.3.3, the programmes (1),(2) have optimal vectors x, p with $1x = pc = 1$. Moreover, if x is an optimal vector for (1), we must have, by Lemma 5.3.1(iii):

$$a_{ij} = b_{ij} = 0 \qquad \text{for } i \notin S, \quad x_j > 0 \qquad (*)$$

I.e. the non-g-producible goods do not occur at all in the processes used by x, and we may therefore strike out all processes which either produce or use a good that is not g-producible. The restricted programme thus obtained has the same optimal vectors (up to certain omitted zero components) as the original programme (1). Moreover, it has only as many genuine constraints as there are elements in S, the rows corresponding to other goods consisting entirely of zeros, by (*). Therefore, by the Basis Theorem of LP (cf. Th.A.3), there exists an optimal vector $\hat{x}(c)$ with at most card(S) [14] nonzero components.

Now choose a $\hat{c}$ with $\hat{c}_i > 0$ for all $i \in S$. By Ass.II, the set of active processes for $\hat{x} = \hat{x}(\hat{c})$ must in fact contain exactly one process from each industry T_i, $i \in S$. Define $T := \hat{T} := \{j \,/\, \hat{x}_j(\hat{c}) > 0\}$. The matrix $M := B_{S,T} - (1+r)A_{S,T}$ is square and nonnegatively invertible by Th.A.5, because $M\hat{x}_T \geqq (1+r)\hat{c}_S \gg 0$, and all off-diagonal elements are nonpositive by Ass.II (w.l.o.g. we may assume the processes in T to be arranged in the same order as the goods in S they produce).

Now choose an optimal vector $\hat{p} = \hat{p}(r)$ for (2). By complementary slackness (Th.A.4.ii), $\hat{p}[b_j - (1+r)a_j] = (1+r)l_j$ for $j \in T$. $\Rightarrow$ (by (*)) $\hat{p}_S.M = (1+r)l_T \Rightarrow \hat{p}_S = (1+r)l_T.M^{-1}$.
$\hat{p}_S$ must be strictly positive, for if $p_i=0$ for some $i \in S$, then the corresponding process in $T \cap T_i$ would make a loss, by Ass.II and Ass.I(ii),(iv).

Next choose any feasible c with $g(c) = g$. We claim that T is an optimal technique, and $\hat{p}$ an optimal price vector for c. Define $x := (x_T, 0)$, where $x_T := M^{-1}.(1+r)c_S$. $\Rightarrow$ $[B - (1+r)A]x = (1+r)c$. The pair $(x,\hat{p})$ is optimal for (1),(2) by Th.A.4(ii); in particular $lx = \hat{p}c = 1$. But this means that $(r,\hat{p},x)$ is an equilibrium for c with the desired properties. This proves (i).

(ii) If (r,p,x) is any equilibrium for c, with $w = pc > 0$, observe first that we can replace x by a vector y using only the processes in T, s.t. (r,p,y) is also an equilibrium for c, with $ly = 1$, and with the same set of actually produced goods, $R := \{i \,/\, (Bx)_i > 0\} = \{i \,/\, (By)_i > 0\}$. Denote by $U := \{j \,/\, y_j > 0\}$ the set of y-active processes, and by $N := B_{R,U} - (1+r)A_{R,U}$ the corresponding submatrix of M.
It is clear that U needs no goods outside R, i.e.

$$a_{ij} = b_{ij} = 0 \qquad \text{for } i \notin R,\ j \in U \qquad (**)$$

Therefore the matrix M is of the form:

$$M = \begin{pmatrix} N & * \\ 0 & * \end{pmatrix}, \quad \text{with} \quad M^{-1} = \begin{pmatrix} N^{-1} & * \\ 0 & * \end{pmatrix}.$$

The vectors y, $\frac{p}{w}$ are optimal for (1),(2) by Lemma 3.3.3, and by complementary slackness we have:
$\frac{p}{w}.[b_j - (1+r)a_j] = (1+r)l_j$ for $j \in U$. By (**), this implies

$p_R.N = w.(1+r)1_u \quad\Rightarrow$

$p_R = w(1+r)1_U.N^{-1} = w.\left[(1+r)1_T M^{-1}\right]_R = w.\hat{p}_R.$

This proves (ii) and the Theorem.

Q.E.D.

Proof of Lemma 5.3.2

Let c be a feasible consumption bundle, with $r = g(c) = 0$, and let x,v be optimal vectors for the LP's (2.2.1),(2.2.2), where $d = c$. By Lemma 2.4., Th.3.1, $V(c) = vc = 1$; and by Th.5.2., $v_i = V(e_i)$. By def., the tripel $(0,v,x)$ is an equilibrium with positive wage for c (cf. Footnote 1o). By Th.5.3(ii), this implies $v_i = \hat{p}_i(0)$.

Q.E.D.

5.4. The case of the wage paid post factum

In this section we study the model of §4 for the special case where the technology satisfies Ass.I and Ass.II, i.e. there is no genuine joint production. Definitions and notation are taken from §4. In particular, the wage is paid at the end of the production period, and the capacity growth rate $g(c)$, the warranted rate of profit $r_w(c)$, and equilibrium (r,p,x) are defined accordingly.

Recall from §4 that with unrestricted joint production, the model where the wage is paid post factum has a number of unattractive features, compared to the case of the wage advanced: zero exploitation does not imply zero growth (Ex.4.1.1), Problem IV need not have a nonnegative solution (Ex.4.2.1, Ex.4.2.2), and the equilibrium rate of profit need not be uniquely determined (Ex.4.3.2). We shall see in the present section that all these difficulties disappear when we rule out genuine joint production (and, in one case, perfectly durable capital goods as well, cf. Ass.II' below). Section 5.4. is parallel to, but considerably longer than, section 5.3.

Let $c \geqq 0$ be a feasible consumption bundle, and $g(c)$ the capacity growth rate for c. By Theorem 4.1., the inequality

$$Bx \geqq [(1+g(c))A + cl]x, \qquad x \gneq 0, \quad lx > 0 \tag{5.4.1}$$

has at least one solution x.

<u>Lemma 5.4.1.</u> (Capacity growth rate)

Let (A,B,l) satisfy Ass.I and Ass.II.

(i) $Bx = [(1+g(c))A + cl]x$ for every solution x of (5.4.1)

(ii) $g(c) < g(c')$ for $c \gneq c'$. In particular, every $c \neq 0$ is efficient.

(iii) If $c \neq 0$, then every solution x of (5.4.1) uses only $g(c)$-producible goods, i.e. $a_{ij} = b_{ij} = 0$ for $i \notin S(g(c))$, $x_j > 0$.

(iv) $g(c) = 0 \Leftrightarrow e(c) = 0$.

Assertions (i) - (iii) are analogous to the corresponding assertions in Lemma 5.3.1. Assertion (iv) should be compared with Theorem 4.1.(iii) for the joint production case, where the implication "$e(c) = 0 \Rightarrow g(c) = 0$" is not true.

We know from <u>4.2.</u> that in the case of joint production the inequality

$$p[B - (1+r)A - cl] \leqq 0, \qquad p \gneq 0 \tag{5.4.2}$$

may have solutions $p \gneq 0$ even for negative values of r. This caused some difficulties for the definition of the warranted rate of profit, $r_w(c)$. We shall see now that these difficulties disappear when the technology satisfies the following assumption, slightly stronger than Ass.II:

<u>Assumption II'</u> (No Genuine Joint Production and No Perfectly Durable Capital Goods): For every process $j = 1,.. n$, there exists at most one good i with $b_{ij} \geqq a_{ij} \neq 0$.

Recall that Ass.II required that for every j there is at most one good i with $b_{ij} > a_{ij}$. This leaves the possibility that $b_{ij} = a_{ij}$ for more than one good i, i.e. the process j uses these goods as perfectly durable capital goods. Under Ass.II', this is not possible: a process j either does not use a good at all ($b_{ij} = a_{ij} = 0$) or decreases its quantity ($b_{ij} < a_{ij}$), except of course for the single good the process produces (if it produces anything at all, which is not required by either Ass.II or Ass.II').

Lemma 5.4.2. (Warranted rate of profit)
Let (A,B,l) satisfy Ass.I and Ass.II'. Then for every feasible c, Problem IV has a solution, and this solution is equal to the warranted rate of profit, as defined in 4.2. (in particular, it is nonnegative).

If Ass.II' is replaced by the (weaker) Ass.II, then Lemma 5.4.2 is not true, as the following example shows:

Example 5.4.1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, $l = (1,0)$, $c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The technology (A,B,l) satisfies Ass.I and Ass.II, but Problem IV does not have a nonnegative solution: The inequality $p[B-(1+r)A-cl] \leqq 0$ has a solution $p = (0,1) \geqq\!\!\!\!\!\neq 0$ for $r = -1$. Obviously process $j=1$ violates Ass.II'.

In 4.3. we have seen that in the case of joint production the four equilibrium conditions (4.3.1) - (4.3.4) could be satisfied for more than one profit rate $r \geqq 0$. This indeterminacy did not occur when the wage was paid in advance. We show now that Ass.II suffices to remove this indeterminacy for the case when the wage is paid post factum as well, and that the only possible equilibrium profit rate is the capacity growth rate.

Lemma 5.4.3. (Uniqueness of equilibrium)
Let (A,B,l) satisfy Ass.I and Ass.II, let $c \neq 0$ be a feasible consumption bundle, and let (r,p,x) satisfy (4.3.1)-(4.3.4), with $r \geq 0$. Then $r = g(c)$.

Next we state the Nonsubstitution Theorem. Its interpretation is is analogous to the one given for Theorem 5.3.

Theorem 5.4. (Nonsubstitution Theorem)
Let (A,B,l) satisfy Ass.I and Ass.II and let $0 \leq r = g < g_{max}$. Then there exists a set of processes $\hat{T} = \hat{T}(g)$ containing exactly one process $j \in T_i$ for every g-producible good $i \in S(g)$, and a price vector $\hat{p} = \hat{p}(r)$ with positive prices $\hat{p}_i > 0$ for all $i \in S(g)$, s.t. for every feasible c with $g(c) = g$ the following is true:

(i) There exists an activity vector $\hat{x}$, using only processes in $\hat{T}$ (i.e. $\hat{x}_j = 0$ for $j \notin \hat{T}$), such that $(r,\hat{p},\hat{x})$ is an equilibrium for c, with $\hat{w} = \hat{p}c = 1$.

(ii) If (r,p,x) is any equilibrium for c, with $w = pc > 0$, then $p_i = w.\hat{p}_i$ for all i with $(Bx)_i > 0$, i.e. the prices of all actually produced goods are proportional to those given by $\hat{p}$.

We conclude this section with a last look at the connection between the value and the price systems.

Lemma 5.4.4. (Labour values and prices)

Let (A,B,l) satisfy Ass.I and Ass.II. Then

(i) $\hat{p}_i(0) = V(e_i)$ for all goods $i = 1,..\ m$, i.e. when the rate of profit is zero, then prices are equal to labour values, provided the wage is taken as numéraire (w=1).

(ii) $e(c) > 0 \Leftrightarrow r_w(c) > 0 \Leftrightarrow g(c) > 0$, i.e. a positive rate of exploitation is necessary and sufficient for positive profits and positive growth ("Fundamental Marxian Theorem").

As we already know, both statements of the Lemma are false if we allow genuine joint production.

5.4.0.

Proof of Lemma 5.4.1.

(i)-(iii) are proved by reducing the present case to the case considered in Lemma 5.3.1. Let us temporarily denote by $\bar{g}(c)$ the capacity growth rate for c when the wage is paid in advance, as in 5.3. When c is feasible, then $\bar{c} := \frac{1}{1+g(c)}.c \leq c$ is also feasible (cf.2.3.). We write $g := g(c)$, $\bar{g} := \bar{g}(\bar{c})$, and claim that $g = \bar{g}$. Obviously (5.4.1) can equivalently be written in the form:

$$Bx \geq (1+g)(A+\bar{c}l)x, \quad x \gneq 0, \quad lx > 0 \tag{1}$$

Since (5.4.1) has a solution, (1) has also a solution, and therefore $g \leq \bar{g}$, by def. of $\bar{g}$. Assume indirectly $g < \bar{g}$. There exists an $\bar{x} \gneq 0$, with $l\bar{x} > 0$, such that:

$$B\bar{x} \geq (1+\bar{g})(A+\bar{c}l)\bar{x} \iff B\bar{x} \geq \Big[(1+\bar{g})A + \underbrace{\frac{1+\bar{g}}{1+g}}_{>1}.cl\Big]\bar{x} \geq \Big[(1+\bar{g})A+cl\Big]\bar{x}$$

$\Rightarrow$ $\bar{g} \leq g$, by def. of g, a contradiction. Therefore $g = \bar{g}$, as asserted, and (5.4.1) can also be written in the form:

$$Bx \geq (1+\bar{g}(\bar{c}))(A+\bar{c}l)x, \quad x \gneq 0, \quad lx > 0 \tag{2}$$

But the last inequality is exactly of the type (5.3.1) considered in Lemma 5.3.1. Assertions (i) and (iii) of the present Lemma follow immediately from Lemma 5.3.1(i),(iii), because x is a solution of (5.4.1) iff. it is a solution of (2); and (ii) follows from Lemma 5.3.1(ii) because $\bar{c}$ increases iff. c increases. This proves (i) - (iii).

(iv) By Th.4.1.(iii) we have only to show: $g(c) > 0 \Rightarrow e(c) > 0$. Assume that $g(c) > 0$. By Th.4.1(v) there is an $x \gneq 0$ with $lx = 1$ s.t. $y := (B-A-cl)x \geq Bx - (1+g(c))Ax - clx \geq 0$, and even $y \gneq 0$, because $Ax \gneq 0$, by Ass.I(ii). $\Rightarrow$ $c' := (B-A)x \gneq c\underbrace{lx}_{=1} = c$ $\underset{\text{Th.5.2.}}{\Longrightarrow}$ $V(c) < V(c') \leq 1$ $\underset{\text{L.2.4.}}{\Longrightarrow}$ $e(c) > 0$. This proves (iv).

Q.E.D.

Proof of Lemma 5.4.2.

In view of Th.4.2. and its proof it suffices to show that the set $R(c)$ is bounded below by zero even if $e(c) = 0$. In other words, if c is a feasible consumption bundle with $e(c) = 0$, then the inequality (5.4.2) has no solution $p \gneq 0$ for $r < 0$. By Th.A.1. it suffices to show that the inequality

$$[B - (1+r)A - cl]\, x \gg 0, \qquad x \geqq 0 \qquad (*)$$

has a solution for $r < 0$.

Now choose an $r < 0$ and a feasible c with $e(c) = 0$. Then there exists an $x \geq 0$ with $lx = 1$ and s.t. $(B-A)x = c$. W.l.o.g. we may assume that all x-active processes are productive. $\Longrightarrow$ $[B-(1+r)A]x = c - rAx \gneq c$ because $r < 0$ and $Ax \gneq 0$ by Ass.I(ii). Denote by $S := \{i \,/\, (Ax)_i > 0\}$ the (nonempty) set of "capital goods" used by x, and by $T := \{j \,/\, x_j > 0$, and $j \in T_i$ for some $i \in S\}$ the set of x-active processes producing these capital goods. By Ass.II', T is nonempty, and by Ass.I(iv), $l_T x_T > 0$. Now "scale down" all processes in T by a small factor k, $0 < k < 1$, i.e. replace x_j by $y_j := (1-k)x_j$ for $j \in T$, and leave all other intensities unchanged, $y_j = x_j$ for $j \notin T$. Then $z := [B-(1+r)A]y = [B-(1+r)A]x - [B-(1+r)A].(kx_T, 0) =$ $= c - rAx - k.u$, where $u := \sum_{j \in T} (b_j - (1+r)a_j)x_j$.

By construction $(Ax)_i = u_i = 0$ for $i \notin S$; and $-r(Ax)_i - ku_i \geq 0$ for $i \in S$ and k sufficiently small, so that in any case $z_i \geq c_i$. Moreover $ly = lx - kl_T x_T < 1$. Choose x^0 with $(B-A)x^0 \gg 0$, $lx^0 + ly = 1$ (cf. Ass.I(iii)). $\Longrightarrow$ $[B-(1+r)A](y+x^0) \gg c \geq c.l(y+x^0) \Longleftrightarrow [B-(1+r)A-cl](y+x^0) \gg 0$, i.e. $(*)$ has a solution.

Q.E.D.

Proof of Lemma 5.4.3.

Assume indirectly that $r < g := g(c)$, and choose (by Th.4.1) an $x' \geq 0$ with $[B-(1+g)A-cl]x' \geq 0$. Denote by $T := \{j \,/\, x'_j > 0\}$ the set of x'-active processes, by $S := \{i \,/\, (Ax')_i > 0\}$ the set of all "capital goods" used by x', and by $U := \{j \in T \,/\, j \in T_i \text{ for some } i \in S\}$ the set of x'-active processes producing these capital goods. By Ass.II, and because $g > 0$, U is nonempty. Moreover, $(B-A)(x'_U, 0) \gneq 0$, and hence, by Ass.I(iv), $l_U \neq 0$.

On the other hand, (4.3.1),(4.3.2) imply, as in the proof of Th.3.3(ii): $pAx' = 0$, i.e.

$$pa_j = 0 \quad \text{for } j \in T; \quad \text{and} \quad p_i = 0 \quad \text{for } i \in S. \qquad (*)$$

Now take any process $j \in U$, producing a good $i \in S$. By "complementarity" we have $p[b_j-(1+g)a_j-cl_j] = 0$. By Ass.II, $(*)$ implies $pb_j = p_i b_{ij} = 0$, and hence, by (4.3.4), $l_j = 0$. But this contradicts $l_U \neq 0$.

Q.E.D.

Proof of Theorem 5.4.

We prove the theorem by reducing it to the case considered in Th.5.3. For this purpose, denote temporarily by $\bar{T}, \bar{p}$ the optimal technique resp. price vector corresponding to $r = g$, as defined in Th.5.3., and define $\hat{T} := \bar{T}$, $\hat{p} := \frac{1}{1+g} \cdot \bar{p}$.

For an arbitrary feasible c with $g(c) = g$, denote temporarily the capacity growth rate when the wage is paid in advance by $\bar{g} := \bar{g}(c)$, and define $\bar{c} := \frac{1}{1+g} \cdot c$. From the proof of Lemma 5.4.1. we know that $g = \bar{g}$.

(i) By Th.5.3. there exists an $\hat{x}$, using only processes in $\hat{T}$, s.t. $(r, \bar{p}, \hat{x})$ solves $[B-(1+r)(A+\bar{c}l)]\,x \geq 0$, $x \gneq 0$, $lx > 0$; and $p[B-(1+r)(A+\bar{c}l)] \leq 0$, $p \gneq 0$, $p\bar{c} > 0$; with $\bar{p}\bar{c} = 1$,

and $\bar{p}_i > 0$ for all actually produced goods. But this implies immediately that $(r,\hat{p},\hat{x})$ satisfies (4.3.1)-(4.3.4), i.e. is an equilibrium for c, with $\hat{w} := \hat{p}c = \frac{1}{1+g}.\bar{p}.(1+g)\bar{c} = 1$.
This proves (i).

(ii) If (r,p,x) is any equilibrium for c, with $pc = w > 0$, then (r,p,x) is an equilibrium for $\bar{c}$ in the sense of Th.5.3., with $\bar{w} := p\bar{c} = \frac{1}{1+g}.w$. Therefore, by Th.5.3(ii),
$p_i = \bar{w}\bar{p}_i = \frac{1}{1+g}.w.(1+g)\hat{p}_i = w\hat{p}_i$ for all i with $(Bx)_i > 0$.
This proves (ii).

Q.E.D.

Proof of Lemma 5.4.4.

(i) follows from Lemma 5.3.2 because, when r=0, then an equilibrium with the wage paid post factum is also an equilibrium with the wage paid in advance.

(ii) follows from Lemma 5.4.1(iv).

Q.E.D.

Appendix

Let A be an arbitrary $(m \times n)$-matrix. x denotes an m-dimensional row vector and y denotes an n-dimensional column vector.

Theorem A.1. (Semipositive solutions of homogeneous inequalities)
Exactly one of the following alternatives holds:

either (i) $\exists\, x \gneq 0$ with $xA \leqq 0$

or (ii) $\exists\, y \geqq 0$ with $Ay \gg 0$.

Theorem A.2. (Complementary solutions of homogeneous inequalities)
There is an $x \geqq 0$ with $xA \geqq 0$, and $y \geqq 0$ with $Ay \leqq 0$, s.t.

(i) $x_i = 0 \Rightarrow (Ay)_i < 0$

(ii) $y_j = 0 \Rightarrow (xA)_j > 0$.

Now let b be an m-dimensional column vector and c an n-dimensional row vector. We consider the two dual Linear Programmes:

$$\left.\begin{array}{l} \min\ xb \\ \text{s.t.}\quad xA \geqq c, \quad x \geqq 0 \end{array}\right\} \qquad (A.1)$$

$$\left.\begin{array}{l} \max\ cy \\ \text{s.t.}\quad Ay \leqq b, \quad y \geqq 0 \end{array}\right\} \qquad (A.2)$$

$x \geqq 0$ is called a <u>feasible vector</u> for (A.1) if it satisfies the constraint $xA \geqq c$. The programme (A.1) is <u>feasible</u> if it has a feasible vector. x is a <u>basic vector</u> if it has no more than n nonzero components (n is the number of constraints in $xA \geqq c$). A feasible vector x^* is called <u>optimal</u> for (A.1) if $x^*b \leq xb$ for all feasible x, i.e. x^* minimizes the <u>objective function</u> xb. Then x^*b is called the <u>value</u> of the programme (A.1). Analogous definitions apply to (A.2).

<u>Theorem A.3</u>. (Duality theorem and Basis theorem of LP)
If (A.1) and (A.2) are both feasible, then there exist optimal vectors, even optimal basic vectors, x^*, y^*, and $x^*b = x^*Ay^* = cy^*$. If one of the two programmes is not feasible, then neither has an optimal vector.

<u>Theorem A.4</u>. (Optimality criterion and Equilibrium theorem of LP)
Two feasible vectors x, y are optimal for (A.1) resp. (A.2) if and only if one of the following conditions is satisfied:

(i) $xb = cy$ (Optimality criterion)

(ii) $x_i = 0$ for $\sum_{j=1}^{n} a_{ij}y_j < b_i$, and

$y_j = 0$ for $\sum_{i=1}^{m} x_i a_{ij} > c_j$. I.e. if a constraint is not binding, then the corresponding variable is equal to zero ("complementary slackness").

<u>Theorem A.5</u>. (Inversion Lemma)
Let $M = \left[m_{ij}\right]_{\substack{i=1,\dots n\\ j=1,\dots n}}$ be a square matrix, with $m_{ij} \leq 0$ for $i \neq j$, and assume that $\exists\, x \geq 0$ with $Mx \gg 0$. Then M is invertible and $M^{-1} \geq 0$.

<u>Proofs</u>: Th.A.1: GALE'1960 (Th.2.1o); Th.A.2: NIKAIDO'1968 (Cor.3 of Th.3.7); Th.A.3 and Th.A.4: GALE'1960, Ch.3, or NIKAIDO'1968, §9.1. (the Basis Theorem is only in GALE); Th.A.5: MIRRLEES'1969, pp.68-69.

F o o t n o t e s

1) There is no Footnote 1).

2) Cf., e.g. the Sraffa - discussion in the Journal of Economic Literature, in particular the contribution by EATWELL'1977; also MORISHIMA'1974,pp.611-616, WOLFSTETTER'1977,pp.66-67, the books by STEEDMAN'1977, RONCAGLIA'1978, etc.

3) In this sense the resulting price theory is 'cost oriented', not 'demand oriented'. An exception is MORISHIMA'1969,Ch.VI, where preferences are introduced into the von Neumann model.

4) A theoretically conceivable exception would be the case where both workers and capitalists always consume only more or less of a certain fixed commodity basket, and where this commodity basket could also serve as an investment good. But this would be analytically equivalent to a One-good-model ('corn'), precisely what Sraffa did not want to investigate.

5) This argument remains valid even if we accept the interpretation of J.ROBINSON'1961 (".. we need not take the word 'change' literally. We are only to compare the effects of having differing rates of profit, with the same technical conditions and the same composition of output."). Of two 'islands' with the same composition of output, but differing profit rates, at least one is, in general, an a priori impossible construct, simply because any given quantity system is, in general, consistent with at most one rate of profit.

6) SAMUELSON'1971,p.4oo: " ..the 'transformation algorithm' is precisely of the following form: 'contemplate two alternative and discordant systems. Write down one. Now transform by taking an eraser and rubbing it out. Then fill in the other one. Voilà! You have completed your transformation algorithm.'"

7) also known as the "dynamic" Nonsubstitution Theorem (MIRRLEES'1969). However, this is an unfortunate terminology (cf.BLISS'1975, p.26o)

8) The following mathematical notation is used: a vector x (similarly for matrices) is called nonnegative, resp. semi-positive, resp. strictly positive, written $x \geqq 0$, resp. $x \gneq 0$, resp. $x \gg 0$, if all its components are nonnegative, resp. all components are nonnegative and at least one component is positive, resp. all components are positive. The symbol $\underline{1}$ denotes a summation vector, i.e. a row or column vector of suitable dimension, all of whose components are unity.

9) The problem of the definition of labour values under joint production is discussed, e.g. in MORISHIMA'1973,Ch.14, MORISHIMA'1974, and STEEDMAN'1977. Our approach is Morishima's.

1o) If $d \neq 0$, then the tripel $(r,p,x) = (0,v^*,x^*)$ is an equilibrium with positive wage $w=pc=1$ for the per-capita consumption vector $c = (V(d))^{-1}.d$, in the sense of 3.3. (not necessarily in the sense of 4.3., cf. Ex.4.4.1)

11) The i-th component of e_i is equal to one, all other components are zero.

12) $(..)_i$ or $[..]_i$ denotes the i-th component of the vector in brackets

13) For an $(m*n)$-dimensional matrix A and subsets $S \subseteq \{1,.. m\}$, $T \subseteq \{1,.. n\}$ we denote by $A_{S,T} = [a_{ij}]_{\substack{i \in S \\ j \in T}}$ the submatrix obtained from A by striking out all rows resp. columns whose indices do not belong to S resp. T. Similarly, for an n-vector x, we denote by $x_T = (x_i)_{i \in T}$ the subvector obtained from x by striking out all components whose index does not belong to T. $y := (x_T, 0)$ denotes the n-vector obtained from x by setting all components whose index does not belong to T equal to zero, i.e. $y_i = x_i$ for $i \in T$, and $y_i = 0$ for $i \notin T$.

14) card(S) is the number of elements contained in S.

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