Lecture Notes on “Classical Value Theory or The Linear Model of Production” *

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Abstract

We give a concise exposition of some aspects of Classical Value Theory and Marxian Economics in the light of modern Mathematical Economics.

1 The Linear Model of Production

There are \( n \geq 1 \) produced goods, labeled \( i = 1, \ldots, n \), and one non-produced primary factor \( i = 0 \), called labor. We write \( N = \{1, 2, \ldots, n\} \). There is exactly one production process for each produced good, with constant returns to scale, and without joint production. Labor is not scarce. There is a common period of production for all processes. The technology is given by a pair \((a_0, A)\), where \( a_0 \) is the \( n \)-vector of labor input coefficients and \( A \) is the \((n \times n)\)-input coefficient matrix:

\[
a_0 = (a_{01}, a_{02}, \ldots, a_{0n}), \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}
\]

The coefficient \( a_{ij} \geq 0 \) is the amount of good \( i \) needed to produce one unit of good \( j \), for \( i = 0, 1, \ldots, n \) and \( j = 1, \ldots, n \). We denote by \( a_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \) the \( i \)-th row of \( A \),

*Preliminary Notes! Please do not quote.
and by \( a^j = (a_{1j}, a_{2j}, \ldots, a_{nj})^T \) the \( j \)-th column of \( A \). If process \( j \) is operated at intensity level \( x_j \geq 0 \), it transforms the input vector \( a^j x_j \) into \( x_j \) units of output of good \( j \), using the amount of labor \( a_{0j} x_j \). Collecting the intensities \( x_j \) into a column vector \( x = (x_1, \ldots, x_n)^T \) we obtain: the economy as a whole transforms the inputs

\[
y = \sum_j a^j x_j = Ax
\]

into the gross output vector \( x \), using the amount of labor

\[
L = \sum_j a_{0j} x_j = a_0 x
\]

The net output is

\[
d = x - Ax = (I - A)x
\]

The part of the economy producing good \( j \) is called sector \( j \) or industry \( j \). Sector \( j \) is active if \( x_j > 0 \). The gross output vector \( x = (x_1, \ldots, x_n)^T \) is also called the activity vector (or intensity vector).

A vector \( x \) (similarly for matrices) is called nonnegative, resp. semipositive, resp. strictly positive, written \( x \geq 0 \), resp. \( x \geq 0 \), resp. \( x > 0 \), if all its components are nonnegative, resp. all components are nonnegative and at least one component is positive, resp. all components are positive. We assume throughout that \( a_0 \geq 0 \) and \( A \geq 0 \), i.e. all input coefficients are nonnegative.

We say that the matrix \( A \) (or the technology represented by it) is productive if it can produce a positive net output of all goods simultaneously, i.e. if there exists an activity vector \( x \geq 0 \) such that \( d = x - Ax > 0 \). We say that labor is indispensable for the technology \((a_0, A)\) if it is impossible to produce a positive net output of any good without labor input, i.e. if \( d = x - Ax \geq 0 \), \( x \geq 0 \) implies \( L = a_0 x > 0 \).

Formally, we assume:

**A.1. (Productivity.)** The matrix \( I - A \) is invertible and \((I - A)^{-1}\) is nonnegative.

**A.2. (Indispensability of labor.)** The vector \( a_0(I - A)^{-1}\) is strictly positive.

**Interpretation:** (i) implies that the equation \( d = (I - A)x \) has the unique nonnegative solution \( x = (I - A)^{-1}d \) for every \( d \geq 0 \), i.e. the technology \( A \) can produce any desired net output \( d \geq 0 \). This implies of course that it is productive in the sense defined above. Th. 5.1 in the Appendix shows that our definition of productivity is in fact equivalent to the apparently stronger condition in **A.1**. The matrix \((I - A)^{-1}\) is also known as the “Leontief inverse” of \( A \). The whole subsequent analysis makes economic sense if and only if the technology is productive. Furthermore, under **A.1** the amount of labor needed to
produce the net output \( d \) is given by \( L = a_0x = a_0(I - A)^{-1}d \). Assumption \( \text{A.2} \) means that this amount is positive for any \( d \gtrsim 0 \).

A nonempty subset \( J \subset N = \{1, 2, \ldots n\} \) of sectors is an autarkic group of sectors if it needs no inputs from the other sectors in \( N \), i.e. if
\[
a_{ij} = 0 \quad \text{for} \quad i \notin J, \ j \in J \quad (i \in N)
\]
An autarkic group of sectors forms a subeconomy that can operate independently of the remaining sectors. Of course the whole set \( N = \{1, \ldots, n\} \) is always autarkic. If this is the only autarkic set, the matrix \( A \) (or the technology described by it) is called indecomposable; otherwise \( A \) is decomposable.

We say that process \( j \) needs input \( i \), directly or indirectly, if there exists a “supply chain” of sectors leading from \( i \) to \( j \) \((i = 0, 1, 2, \ldots; j = 1, \ldots n)\):
\[
i = k_0 \rightarrow k_1 \rightarrow k_2 \rightarrow \cdots k_m = j \quad (1 \leq m \leq n)
\]
such that
\[
a_{k_s, k_{s+1}} > 0 \quad \text{for any two consecutive sectors} \ k_s, k_{s+1}.
\]
That is, sector \( k_1 \) needs input \( k_0 = i \), sector \( k_2 \) needs input \( k_1 \), sector \( k_3 \) needs input \( k_2 \), and so on, and sector \( k_m = j \) needs input \( k_{m-1} \). If \( m = 1 \), sector \( j \) needs input \( i \) directly \((a_{k_0k_1} = a_{ij} > 0)\), if \( m \geq 2 \), sector \( j \) needs input \( i \) indirectly, through the chain (1). Specifically, sector \( j \) needs labor if it needs input \( 0 \).

Assumptions \( \text{A.1} \) and \( \text{A.2} \) have far-reaching consequences. We mention here only the most important ones, which will be used repeatedly in the sequel. For precise statements see the Appendix.

If the technology is productive, we have \( \sum_{t=0}^{\infty} A^t = (I - A)^{-1} \), i.e. the infinite sum \( I + A + A^2 + A^3 + \cdots \) converges and its limit is the Leontief inverse\(^1\). Moreover, with the matrix \( A \) one can associate a nonnegative number \( \lambda(A) \), the so-called Frobenius eigenvalue (or dominant eigenvalue) of \( A \). The technology is productive if and only if \( \lambda(A) < 1 \). Very loosely speaking, a technology is productive if the input coefficients \( a_{ij} \) are not “too large”.

The Frobenius eigenvalue measures, in a sense, the average size of these coefficients.

Labor is indispensable for the technology \((a_0, A)\) if and only if every sector needs labor, directly or indirectly. If the technology \( A \) is indecomposable, then every sector is connected to every other sector in the sense that it needs inputs from it, directly or indirectly. In this case \( \lambda(A) > 0 \). Assumption \( \text{A.1} \) implies that nonnegative prices and labor values can be computed. The formula \( \sum_{t=0}^{\infty} A^t = (I - A)^{-1} \) implies that these prices can be understood in terms of dated labor inputs. Assumption \( \text{A.2} \) implies that all prices and labor values are positive.

\(^{1}\)This formula is analogous to the well-known formula for the geometric series \( 1 + a + a^2 + \cdots = 1/(1-a) \), where \( a \) is a number with \( 0 \leq a < 1 \).
Finally, we note that $A.2$ implies $a_0 \gtrapprox 0$. $A.2$ is trivially satisfied if every sector needs labor directly, $a_0 \gg 0$; and also if $a \gtrapprox 0$ and $A$ is indecomposable (because then $(I - A)^{-1} \gg 0$ by Theorem 5.5).

2 Labor Values

The labor value $V(d)$ of a commodity bundle $d$ is the total amount of labor needed, directly or indirectly, to produce $d$. In our model, this can be formalized as follows: to produce $d$ in the current period $t = 0$, we need the amount of labor $\ell_0(d) = a_0d$ and the vector of inputs of the other goods, $A_d$. The inputs $A_d$ were produced in the previous period, using labor $\ell_1(d) = a_0A_d$ and physical inputs $A.A_d = A^2d$. The inputs $A^2d$ were produced two periods ago, using labor $\ell_2(d) = a_0A^2d$ and physical inputs $A.A^2d = A^3d$. The inputs $A^3d$ were produced three periods ago, using labor $\ell_3(d) = a_0A^3d$, and so on. The current amount $\ell_0(d)$ is the direct labor, and the the past amounts $\ell_1(d), \ell_2(d), \ell_3(d), \ldots$ are the indirect labor. The quantities

$$\ell_t(d) = a_0A^td \quad t = 0, 1, 2, \ldots$$

are known as dated labor inputs. The total amount of labor needed directly or indirectly to produce the commodity bundle $d$ is thus given by the infinite sum $\ell_0(d) + \ell_1(d) + \ell_2(d) + \ell_3(d) + \cdots$:

$$V(d) = \sum_{t=0}^{\infty} \ell_t(d) = \sum_{t=0}^{\infty} a_0A^td = a_0[\sum_{t=0}^{\infty} A^t]d$$

(2)

By Theorem 5.1 the infinite sum $\sum_{t=0}^{\infty} A^t$ converges so that $V(d)$ is a well-defined non-negative quantity for every $d \geq 0$. This definition in terms of dated labor inputs is known as the historical definition of labor values. Moreover, since $\sum_{t=0}^{\infty} A^t = (I - A)^{-1} \geq 0$ by Theorem 5.1 we can write

$$V(d) = a_0(I - A)^{-1}d = vd$$

(3)

where the vector $v$ is defined by

$$v = (v_1, \ldots, v_n) = a_0(I - A)^{-1}$$

(4)

The $i$-th component $v_i$ of $v$ is given by $v_i = a_0(I - A)^{-1}e^i$, where the $i$-th unit vector $e^i$ represents a commodity bundle which contains one unit of good $i$ and nothing else. We call $v_i = V(e^i)$ the labor value of good $i$, and $v$ the vector of labor values. By Assumption 1 all labor values are positive, $v \gg 0$.

Summing up, the vector $v$ of labor values satisfies

$$v = a_0 + vA$$

(5)
and the labor value of a commodity bundle $d$ can be written as

$$vd = \sum_{t=0}^{\infty} \ell_t(d)$$

(6)

where $\ell_t(d) = a_0 A^t d$ is the dated labor input, for $t = 0, 1, 2, \ldots$.

Another way to define labor values is as follows: assume that the economy is in a stationary state: the amounts of goods produced, consumed and invested in any period are constant over time. Amounts produced in one period can either be consumed or invested as inputs for the production in the next period.

We denote by $x_i \geq 0$ the gross output of good $i$ in any period and write $x = (x_1, \ldots, x_n)^T$ for the gross output vector. Similarly, we write $d = (d_1, \ldots, d_n)^T$ for the vector of total consumption (or final demand), where $d_i \geq 0$ is the amount consumed of good $i$; and we write $y = (y_1, \ldots, y_n)^T$ for the vector of investments, where $y_i$ is the amount of good $i$ used as input for the production in the next period. We can now ask the question: “How much labor do we need per period to enjoy the consumption $d$ per period, in a stationary state?”

Since everything is either consumed or invested, we have $x = d + y$. In a stationary state, the amount $y$ invested must be just sufficient to produce the same gross output $x$ in the next period, i.e. we must have $y = Ax$. This gives us the *quantity equation*

$$x = d + Ax$$

(7)

This can be written $(I - A)x = d$ or

$$x = (I - A)^{-1}d$$

(8)

This gives us economically meaningful amounts $x \geq 0$ for every $d \geq 0$ because $I - A$ is nonnegatively invertible. The total amount of labor needed per period to enjoy consumption $d$ in the stationary state is therefore, by (8):

$$L = a_0 x = a_0 (I - A)^{-1}d$$

(9)

This amount is known as the *synchronous* definition of the labor value of the commodity bundle $d$. By assumption A.2 this value is positive for $d \geq 0$. Equation (9) is the same as (3), i.e. the synchronous and the historical definition give the same labor values.

Equation (8) gives a solution to the following *Planning Problem*: “Given a certain desired net output $d = (d_1, \ldots, d_n)^T$, what are the gross output levels $x_i$ needed in the various industries, so that exactly $d$ is left for consumption after subtracting from $x$ the inputs needed to continue production, in a stationary state?”
3 Production prices

Let us now determine the prices in our Linear Model of Production. We denote the price of good $i$ by $p_i$. The price of labor $p_0 = w$ is the wage, and $p = (p_1, \ldots, p_n)$ is the price vector. The pair $(p_0, p)$ is the price system. For a commodity bundle $d$, the scalar product $pd = \sum_i p_i d_i$ is the value of $d$ at prices $p$. Prices are expressed in some arbitrary unit of account. Changing the unit of account (e.g. cents instead of dollars) changes all prices by some common factor $k > 0$ (e.g. $k = 100$) and gives an equivalent price system $(p'_0, p') = (kp_0, kp)$. If we choose the unit of account such that the price of good $i$ is equal to one, $p_i = 1$, we say that good $i$ is the numéraire. More generally, a commodity bundle $d$ is the numéraire if $pd = 1$. Only relative prices $p_i/p_j$ have economic meaning, and these are independent of the choice of unit of account.

Since capitalists want to maximize the return on their capital, they will invest their money only in those sectors where the rate of profit is highest. If all sectors are to be active, the rate of profit must therefore be the same in all sectors. We denote this uniform profit rate (or interest rate) by $r \geq 0$.

This means that the price of one unit of good $j$ is equal to $(1 + r)$-times the costs of its inputs, i.e. we obtain the price equations:

$$p_j = (1 + r)[p_0 a_{0j} + p_1 a_{1j} + \cdots + p_n a_{nj}] \quad \text{for } j = 1, \ldots, n$$

or, in matrix notation:

$$p = (1 + r)[p_0 a_0 + pA]$$

This is the same as

$$p = p_0(1 + r)a_0[I - (1 + r)A]^{-1}$$

provided the inverse exists.

Define

$$r^{\max} = r^{\max}(A) = \begin{cases} 1/\lambda(A) - 1 & \text{if } \lambda(A) > 0 \\ \infty & \text{if } \lambda(A) = 0 \end{cases}$$

where $\lambda(A) \geq 0$ is the dominant eigenvalue of $A$ (see Appendix). By Theorem 5.1, $A$ is productive iff $\lambda(A) < 1$.

**Theorem 3.1.** Assume that Assumptions A.1 and A.2 hold. Then $r^{\max} > 0$ is positive and for every $r$ with $0 \leq r < r^{\max}$ the price equations

$$p = (1 + r)[p_0 a_0 + pA]$$

have a strictly positive solution $p \gg 0$ for every positive $p_0 > 0$. If $r = 0$, prices are proportional to labor values: $p = p_0 v$. 


Thus, inherent in the technology $A$ there is a maximal possible rate of profit $r_{\text{max}}(A)$. By definition (13), this $r_{\text{max}}$ is positive if and only if $\lambda(A) < 1$, i.e. $A$ is productive (Theorem 5.1). The price equations have an economically meaningful solution $(p_0, p)$ if the profit rate is not too high. Given $r$, the prices $(p_0, p)$ depend only on the technology $(a_0, A)$. These prices are known as the \textit{prices of production} implied by the profit rate $r$. The whole price system is given by the triple $(r, p_0, p)$. As always, the absolute prices $(p_0, p)$ are determined only up to the choice of numéraire, but relative prices are unique.

If the profit rate is zero, prices are proportional to labor values, but for $r > 0$ this is not the case in general.

\textit{Proof of Theorem.}

The prices $p$ are given by equation (12) (for arbitrary $p_0 > 0$). Choose $r$ with $0 \leq r < r_{\text{max}}$. Define a new, fictitious technology $(b_0, B)$ by

$$b_0 = (1 + r)a_0, \quad B = (1 + r)A$$

Then the price equation (12) becomes

$$p = p_0b_0[I - B]^{-1}$$

We claim that $(b_0, B)$ satisfies $\text{A.1}$ and $\text{A.2}$ if $(a_0, A)$ satisfies these assumptions. First, by Theorems 5.1 and 5.4 the technology $B$ is productive because $\lambda(B) = \lambda((1 + r)A) = (1 + r)\lambda(A)$ and $(1 + r)\lambda(A) < 1$ for $r < r_{\text{max}}$. Therefore $(I - B)^{-1} = \sum_{t=0}^{\infty} B^t$. Second, the fictitious technology $(b_0, B)$ satisfies $b_0 \geq a_0$ and $B \geq A$. Hence $b_0(I - B)^{-1} = b_0[I + B + B^2 + \cdots] \geq a_0[I + A + A^2 + \cdots] = a_0(I - A)^{-1} \gg 0$, and $b_0, B$ satisfies $\text{A.2}$. Therefore $p = p_0b_0[I - B]^{-1} \gg 0$ for every positive $p_0$. If $r = 0$, we have $b_0[I - B]^{-1} = v$ and $p = p_0v$.

\textit{Remark.} The production prices with a positive rate of profit are not proportional to labor values, in general, but the price $pd$ of a commodity bundle $d$ can still be understood in terms of dated labor costs. Indeed, since $(1 + r)A$ is productive, we have

$$[I - (1 + r)A]^{-1} = \sum_{t=0}^{\infty} (1 + r)^t A^t$$

and the price equation (12) gives

$$pd = p_0(1 + r)a_0 \sum (1 + r)^t A^t d = \sum (1 + r)^{t+1} p_0a_0 A^t d$$

or

$$pd = \sum_{t=0}^{\infty} (1 + r)^{t+1} p_0\ell_t(d)$$

(14)
The right-hand side of this equation is the present value of the stream of past labor costs \( (p_0 \ell_0(d), p_0 \ell_1(d), p_0 \ell_2(d), \ldots) \), computed at the interest rate \( r \). This takes account of the fact that production takes time and the labor input \( \ell_t(d) \) is needed \((t+1)\) periods before the output \( d \) becomes available. In the context of an intertemporal price system with interest rate \( r \), the price equation (11) is in fact a zero-profit condition. Of course the formula (14) coincides with (5) for \( r = 0 \) and \( p_0 = 1 \).

The augmented input matrix \( A + ca_0 \).

Consider a profit rate \( r \) with \( 0 < r < r^{\text{max}}(A) \) and associated strictly positive prices \( p_0 > 0, p \gg 0 \), as given in Theorem 3.1:

\[
p = (1 + r)[p_0a_0 + pA]
\]

Let us call a per capita consumption \( c = (c_1, \ldots, c_n)^T \geq 0 \) of the workers feasible for the profit rate \( r \) if the workers can afford it at the prices implied by \( r \), i.e. if \( pc = p_0 \) (workers do not save). Clearly \( c \geq 0 \) because \( p_0 > 0 \). Now let \( c \) be such a feasible consumption. Then we can rewrite the price equation as \( p = (1 + r)[pca_0 + pA] \) or

\[
p = (1 + r)p[A + ca_0]
\]

The matrix \( A + ca_0 \) is called the augmented input coefficient matrix. To understand the matrix \( A + ca_0 \), note that \( ca_0 \) is the product of the column vector \( c \) (an \((n \times 1)\)-matrix) and the row vector \( a_0 \) (a \((1 \times n)\)-matrix). Therefore \( ca_0 \) is an \((n \times n)\)-matrix, with typical element \( c_ia_0j \). This is the amount of good \( i \) consumed by the workers who are needed to produce one unit of good \( j \). The element \( a_{ij} + c_ia_0j \) of the matrix \( A + ca_0 \) is therefore the total amount of good \( i \) needed to produce one unit of good \( j \), including the workers’ consumption.

From now on we assume

A.3. (Indecomposability.) The augmented input matrix \( A + ca_0 \) is indecomposable.

This assumption is not too restrictive because the input coefficients \( a_{ij} + c_ia_0j \) include also the workers’ consumption. It amounts only to assuming that there are no “luxury goods” which are neither used as inputs for production nor consumed by the workers. If \( A + ca_0 \) is indecomposable, equation (15) has only one positive solution (cf. Th. 5.5) \(^2\), i.e. we must have \( 1/(1 + r) = \lambda(A + ca_0) \) where \( \lambda(A + ca_0) > 0 \) is the dominant eigenvalue of \( A + ca_0 \) and \( p \gg 0 \) is an associated left eigenvector. The augmented matrix is productive because \( \lambda(A + ca_0) = 1/(1 + r) < 1 \).

\(^2\)By A.2 the matrix \( ca_0 \) and hence also \( A + ca_0 \), is not the zero matrix of order 1, so that Theorem 5.5 applies.
Conversely, let us consider the augmented matrix \( A + ca_0 \) for an arbitrary consumption bundle \( c \geq 0 \) and assume that \( A + ca_0 \) is indecomposable. By Theorem 5.5 there exists an essentially unique positive solution of the eigenvalue problem (15). This solution is given by

\[
    r = r(c) = \frac{1}{\lambda(A + ca_0)} - 1
\]

and an associated strictly positive left eigenvector \( p \) (unique up to multiplication by a scalar) of \( A + ca_0 \). If we now define \( p_0 := pc \), the resulting triple \( (r, p_0, p) \) is a production price system of the form given in Theorem 3.1, provided \( 0 \leq r(c) < r_{\text{max}}(A) \). We have \( r(c) < r_{\text{max}}(A) \) because \( \lambda(A + ca_0) > \lambda(A) \geq 0 \) by Theorem 5.5. Moreover, \( r(c) > 0 \) if and only if \( \lambda(A + ca_0) < 1 \), i.e. we have proved

**Lemma 3.2.** Assume that \( A + ca_0 \) is indecomposable. Then the workers’ per capita consumption \( c \) gives rise to a positive rate of profit \( r(c) > 0 \) if and only if \( A + ca_0 \) is productive.

**The quantity system.**

When the rate of profit is positive, the workers consume only a part \( da \) of the total net output \( d \), leaving the surplus product \( ds = d - da \) for the capitalists. This gives a quantity system of the form

\[
    x = Ax + da + ds
\]

where \( x \) is the gross output vector, \( Ax \) represents the inputs, \( da \) is the aggregate consumption of the workers, and \( ds \) is the surplus product. The capitalists may consume or invest the surplus product \( ds \), for the moment let us assume that they consume it all. Total labor is \( L = a_0x \) and the per capita consumption of the workers is

\[
    c = \frac{1}{L} \quad \Leftrightarrow \quad ds = cL = ca_0x
\]

This gives

\[
    x = Ax + cL + ds
\]

or, since \( cL = ca_0x \):

\[
    x = (A + ca_0)x + ds
\]

By definition, the per capita consumption \( c \geq 0 \) is feasible for the technology \((a_0, A)\) if the augmented matrix \( (A + ca_0) \) is productive. This means that \( c \) is feasible iff it is possible to produce a strictly positive surplus vector \( ds \) with \( c \), i.e. \( ds = x - (A + ca_0)x \gg 0 \) for some \( x \geq 0 \).

**Lemma 3.3.** The augmented matrix \( A + ca_0 \) is productive if and only if \( vc < 1 \).
Proof. ”⇒”: if $A + ca_0$ is productive, then there is $x \geq 0$ with $[I - (A + ca_0)]x \gg 0$. Therefore $v(I - A)x - vca_0x > 0$. Noting that $v(I - A) = a_0$ by (5), we obtain $a_0x > vca_0x \Rightarrow vc < 1$ because $a_0x > 0$ by A.2.

”⇐”: Assume $vc < 1$ and choose $x$ such that $(I - A)x = c$. Then $a_0x = vc < 1$. Define $y := (I - A - ca_0)x$. Then $y \geq 0$ and $y_i > 0$ iff $c_i > 0$. Choose $x^0$ with $(I - A)x^0 \gg 0$, and $x^0$ “sufficiently small”. Define $z := y + (I - A - ca_0)x^0$. If $c_i > 0$ then $z_i > 0$ because $x^0$ is sufficiently small. If $c_i = 0$ then $y_i = 0$ and $z = [(I - A - ca_0)x^0]_i = [(I - A)x^0]_i - c_i a_0 x^0 > 0$. Therefore $z = (I - A - ca_0)(x + x^0) \gg 0$, i.e. $A + ca_0$ is productive. ■

National income accounting.

Let $(p_0, p)$ be a price system associated with $r$ resp. $c$, i.e. the wage is $w = p_0 = pc$ and

$$p = (1 + r)[pA + wa_0] = (1 + r)p[A + ca_0],$$

and recall the quantity equation

$$x = Ax + d^a + d^s = Ax + cL + d^s$$

This implies

$$px = (1 + r)[pA + wa_0]x = pAx + wa_0x + r[pAx + wa_0x] = pAx + wL + r[pAx + wL]$$

and

$$px = pAx + pd^a + pd^s = pAx + pcL + pd^s = pAx + wL + pd^s$$

Here $pAx$ is the cost of all inputs, $wL$ is the sum of all wages (total labor costs), and $r(pAx + wL)$ represents total profits. We see that total wages $wL$ are equal to the value $pd^a$ of the workers’ aggregate consumption $d^a$, and total profits are equal to the value $pd^s$ of the capitalists’ consumption.

4 Marxian Economics

In this section we explain some basic concepts of Marxist Economic Theory in the framework of our Linear Model of Production. For a justification of this approach and references to Marx, see Morishima (1973) and Morishima (1974).

We consider again the quantity system (17) of the previous section, but use labor values, not production prices, to evaluate all goods.

The workers consume only a part $d^a$ of the total net output $d$, leaving the surplus product $d^s = d - d^a$ for the capitalists. This gives a quantity system of the form:

$$x = Ax + d^a + d^s$$

(20)
where \( x \) is the gross output vector, \( Ax \) represents the inputs, \( d^a \) is the aggregate consumption of the workers, and \( d^s \) is the surplus product (consumed by the capitalists). Total labor is \( L = a_0 x \) and the per capita consumption \( c \) of the workers is given by \( d^a = cL = ca_0x \). The wage (in terms of labor values) is \( vc \).

Following Marx, we can think of total labor \( L = a_0 x = vd = vd^a + vd^s \) as being split in two parts: \( L = L^a + L^s \), where \( L^a = vd^a = vcL \) is the necessary labor, and \( L^s = vd^s = vd - vd^a = (1-vc)L \) is the surplus labor. The necessary labor \( L^a \) is the amount of labor needed to produce the workers’ consumption \( d^a \), i.e. the amount of time the workers work for themselves, or the amount of paid labor. The surplus labor \( L^s \) is the amount of time the workers work for the capitalists, or the amount of unpaid labor.

Marx defined the rate of exploitation as the ratio of surplus labor to necessary labor:

\[
\epsilon = \frac{L^s}{L^a}
\]

This implies

\[
\epsilon = \epsilon(c) = \frac{L^s}{L^a} = \frac{1-vc}{vc}
\]

and

\[
1 - vc = \epsilon vc
\] (21)

Workers are exploited if \( \epsilon > 0 \) or, equivalently \( L^s > 0 \) or \( vc < 1 \), i.e. workers work more than would be necessary for their own consumption. By Lemma 3.3 this is the case precisely when the augmented input matrix \( A + ca_0 \) is productive.

**Remark.** We do not discuss here what determines \( c \) resp. the wage \( vc \). One common interpretation is that \( vc \) is a subsistence wage, i.e. the consumption bundle \( c \) is just enough for the workers to survive and reproduce their labor power from one period to the next. This 'subsistence’ consumption need not be biologically determined, but can also represent a socially acceptable minimum standard of living. The latter may depend on the relative strengths of the social classes, etc. In any case, our formal analysis does not depend on such interpretations. We take \( c \) as exogenous.

We proceed to define some further Marxian concepts.

Using (21), the labor value equation (5) can be written

\[
v = vA + a_0 = vA + vca_0 + \epsilon vca_0
\] (22)

or, componentwise

\[
v_j = va_j + vca_{0j} + \epsilon vca_{0j}
\] (23)

The quantity \( K_j = va_j = v_1a_{1j} + v_2a_{2j} + \ldots v_na_{nj} \) is called the constant capital (per unit of output) in sector \( j \). It is the cost (in terms of labor values) of the physical inputs. The
quantity $V_j = V_j(c) = vca_0j$ is called the *variable capital* (per unit of output) in sector $j$. It represents the wage cost. The quantity $S_j = S_j(c) = \varepsilon V_j = \varepsilon vca_0j$ is the surplus (per unit of output) in sector $j$. The value of good $j$ is the sum of these quantities:

$$v_j = K_j + V_j + S_j$$  \hspace{1cm} (24)

We write $K = (K_1, \ldots, K_n) = vA$ for the vector of constant capitals, $V = V(c) = (V_1, \ldots, V_n) = vca_0$ for the vector of variable capitals, and $S = S(c) = (S_1, \ldots, S_n) = \varepsilon V = \varepsilon vca_0$ for the vector of surpluses, so that $v = K + V + S$.

By definition, the rate of exploitation $\varepsilon = S_j/V_j$ is the same in all sectors. Following Marx, we define the *organic composition of capital* in sector $j$ by $q_j = K_j/V_j$, and the *profit rate* in sector $j$ by $\pi_j = S_j/(K_j + V_j)$. These quantities may vary across sectors. Loosely speaking, a sector with a high organic composition of capital uses a lot of physical inputs and little labor. The profit (per unit of output) in sector $j$ is revenue minus cost, i.e. $v_j - (K_j + V_j) = S_j$. The profit rate $\pi_j$ is this profit divided by the cost. Equivalently, we can write

$$v_j = (1 + \pi_j)[(v_1a_{1j} + \ldots v_na_{nj}) + vca_0j] \hspace{1cm} \forall j$$  \hspace{1cm} (25)

**Remark.** The “prices” used in this equation are the labor values $v$ determined by (5) and the wage $vc$ is the labor value of the per capita consumption $c$. When there is no exploitation and workers consume the entire net output ($vc = 1$ or equivalently $d^s = 0$), all profit rates are the same, namely $\pi_j = 0$ for all $j$, and (25) coincides with the value equation (5). But when there is exploitation ($vc < 1$), as we assume in this section, the profit rates $\pi_j$ defined by (25) will be positive and, in general, different for different sectors. We have seen in the previous section that equalization of profit rates across all sectors requires a different price system, namely the production prices of Theorem 3.1, not the labor values used in (25). The relationship between production prices and labor values (the so-called *transformation problem*) has caused a lot of confusion in the Marxist literature.

**Remark.** We have defined constant and variable capital and surplus *per unit of output*. Marx originally defined these concepts with reference to actual output, i.e. he used $K_jx_j$, $V_jx_j$, $S_jx_j$ instead of $K_j$, $V_j$, $S_j$. This gives the same organic compositions and profit rates, of course.

We can also define aggregate versions of these concepts: $Kx = \sum_j K_jx_j$ is aggregate constant capital, $Vx = \sum_j V_jx_j$ is aggregate variable capital, and $Sx = \varepsilon Vx$ is the aggregate surplus (*Mehrwert*). From (24) we get $vx = Kx + Vx + Sx$ for the value of the gross output. Note also that $Kx = vAx$, $Vx = L^n$, and $Sx = L^s$.

Clearly, the rate of exploitation is given by $\varepsilon = Sx/Vx$. The aggregate (or average) *organic composition of capital* is the ratio $\bar{q}(x) = Kx/Vx$, and the aggregate (or average) *rate of profit* is the ratio $\bar{\pi}(x) = Sx/(Kx + Vx)$. Unlike the sectoral quantities $q_j$, $\pi_j$,
which depend only on the technology \((a_0, A)\) and the wage \(vc\), the aggregate quantities \(\bar{q}(x)\), \(\bar{\pi}\) depend also on the activity vector \(x\).

**Relations between the system of labor values and the price system.**

Recall from the definitions that the labor values \(v\) satisfy \(v = vA + a_0\) and the rate of exploitation satisfies \((1 + \varepsilon)vc = 1\). Recall also \(K = vA\), \(V = vca_0\), \(S = \varepsilon V\), and \(q_j = K_j/V_j\). The price system \((r, p_0, p)\) satisfies \(p = (1 + r)p(A + ca_0)\), where \(p_0 = pc\). Both \(\varepsilon\) and \(r\) depend on \(c\); when we want to make this dependence explicit, we write \(\varepsilon(c)\) and \(r(c)\) (of course the prices \((p_0, p)\) also depend on \(c\), but we do not make this explicit in the notation).

**Theorem 4.1.** Assume that \(A + ca_0\) is indecomposable. Then

(i) the rate of profit \(r(c)\) implied by \(c\) is positive if and only if the rate of exploitation \(\varepsilon(c)\) is positive.

(ii) \(r(c) < \varepsilon(c)\). More precisely, \(\varepsilon(c) = r(1 + \bar{q}(y))\), where the activity vector \(y \gg 0\) satisfies \(y = (1 + r)(A + ca_0)y\), i.e. \(y\) is a right eigenvector of \(A + ca_0\) associated with the dominant eigenvalue \(\lambda(A + ca_0) = 1/(1 + r)\).

That is, positive exploitation \((\varepsilon > 0)\) is necessary and sufficient for the capitalist system to be profitable \((r > 0)\). This observation has been called the “Fundamental Marxian Theorem” by Morishima (1974).

**Proof.** (i) follows immediately from Lemmas 3.2 and 3.3, because both \(r(c)\) and \(e(c)\) are positive if and only if the matrix \(A + ca_0\) is productive.

(ii) By Theorem 5.5 there exists an activity vector \(y \gg 0\) with \(y = (1 + r)(A + ca_0)y\). This implies

\[(K + V + S)y = vy = (1 + r)v(A + ca_0)y = (1 + r)(K + V)y\]

and hence \(\varepsilon Vy = S y = r(K + V)y\). This proves (ii).

**Theorem 4.2.** Assume that \(A + ca_0\) is indecomposable. Then

(i) \(p_j/p_0 > v_j\) for all \(j = 1, \ldots n\). All production prices (relative to the wage) are higher than the corresponding labor values.

(ii) \(p = kv\) for some constant \(k > 0\) if and only if \(q_j = K_j/V_j = q\) for all \(j = 1, \ldots n\). Production prices \(p\) are proportional to labor values \(v\) if and only if the organic composition of capital \(q_j\) is the same in all sectors. Moreover, in this case, \(p/p_0 = v(1 + \varepsilon)\) and the common organic composition satisfies \(\varepsilon = r(1 + q)\).
Proof. (i) The price equation implies:

\[
\frac{1}{p_0} p = (1 + r)\left[\frac{1}{p_0} pA + a_0\right] > \frac{1}{p_0} pA + a_0
\]

and therefore

\[
\frac{1}{p_0} p[I - A] > a_0 \Rightarrow \frac{1}{p_0} p > a_0 (I - A)^{-1} = v
\]

This proves (i).

(ii)

"⇒": Assume that \( v \) is proportional to \( p \). This implies that \( v \) is also a left eigenvector of \( A + ca_0 \), associated with the eigenvalue \( 1/(1+r) \), i.e. \( v = (1+r)v(A + ca_0) = (1+r)(K + V) \). On the other hand, \( v = K + V + S \), hence \( r(K + V) = S = \varepsilon V \). Therefore \( rK = (\varepsilon - r)V \), so that \( K = qV \) where the common organic composition is \( q = (\varepsilon - r)/r \) or \( r(q + 1) = \varepsilon \).

"⇐": Assume that \( K = qV \), where \( q \) is the common organic composition of capital in all sectors. This implies that \( v = K + V + S = (q + 1 + \varepsilon)V \) and \( v(A + ca_0) = K + V = (q + 1)V \). Therefore \( v \) is proportional to \( v(A + ca_0) \), so \( v \) is be a strictly positive left eigenvector of \( A + ca_0 \), hence \( v \) is proportional to \( p \), by Th. 5.5.

More precisely,

\[
v = (1 + \frac{\varepsilon}{1 + q})v(A + ca_0)
\]

i.e. the dominant eigenvalue of \( A + ca_0 \) is \( 1/(1+r) \), where \( r = \frac{\varepsilon}{1 + q} \).

Finally, if prices are proportional to labor values, \( p = kv \), then \( p_0 = pc = kvc = k/(1 + \varepsilon) \), so that \( p/p_0 = v(1 + \varepsilon) \). This proves (ii).
5 Appendix: Nonnegative Matrices

5.1 Productive Matrices

Let $A \geq 0$ be a square matrix of input coefficients. The following Theorem characterizes productive matrices.

Theorem 5.1. Let $A$ be a nonnegative square matrix. Then the following statements are equivalent:

(i) $A$ is productive, i.e. $x \gg Ax$ for some $x \geq 0$

(i') $A$ is “profitable”, i.e. $p \gg pA$ for some $p \geq 0$

(ii) the technology $A$ can produce any net output vector $d$, i.e. for every $d \geq 0$ there exists $x \geq 0$ such that $x - Ax = d$ (the planning problem always has a solution)

(iii) the value equations always have an economically meaningful solution, i.e. for any $a_0 \geq 0$ there exists $v \geq 0$ such that $v - vA = a_0$

(iv) the matrix $(I - A)$ is nonnegatively invertible

(v) $\sum_{t=0}^{\infty} A^t$ is convergent

(vi) $(I - A)^{-1} = \sum_{t=0}^{\infty} A^t$ (and the sum converges)

(vii) the largest eigenvalue of $A$ (the Frobenius eigenvalue) is less than one: $\hat{\lambda}(A) < 1$

(viii) all principal minors of $I - A$ are positive (HAWKINS-SIMON conditions)

Proof. Condition (viii) is mentioned only for completeness’ sake. We do not consider it. We prove

(ii) $\Rightarrow$ (i) obvious.

(i) $\Leftrightarrow$ (i'): this follows from the fact that $A$ and its transpose $A^T$ have the same eigenvalues.

(i) $\Leftrightarrow$ (vii): see Theorem 5.3(iii').

(v) $\Rightarrow$ (iv),(v): obvious

(v) $\Rightarrow$ (iv): Define the partial sums $M_s = \sum_{t=0}^{s} A^t$ for $s = 0, 1, 2, \ldots$. This implies

$$(I - A)M_s = M_s(I - A) = I - A^{s+1} \leq I \quad \forall s \quad (*)$$

If $M_s$ converges to the limit $M = \sum_{t=0}^{\infty} A^t$, then $A^{s+1} \to 0$ and by passing to the limit in (*) this implies $M = (I - A)^{-1}$. Clearly $M$ is nonnegative because $A$ is nonnegative.
(iv) ⇒ (vi): If $I - A$ is nonnegatively invertible, then multiplying (*) by $(I - A)^{-1}$ gives $M_s \leq (I - A)^{-1}$ for all $s$, i.e. the sequence $(M_s)$ is bounded above. It is also increasing (because $M_s \leq M_{s+1}$), hence convergent. By the argument above, it must converge to $(I - A)^{-1}$.

(iv) ⇒ (ii)+(iii): if $(I - A)$ is nonnegatively invertible, then for any $d \geq 0$ the activity vector $x = (I - A)^{-1}d$ is nonnegative and satisfies $(I - A)x = d$. Similarly, for any $a_0 \geq 0$ the value vector $v = a_0(I - A)^{-1}$ is nonnegative and satisfies $v(I - A) = a_0$. Therefore (iv) implies (ii) and (iii).

(ii) ⇒ (iv) and (iii) ⇒ (iv): assume that for every $d \in \mathbb{R}_+$ there exists $x \geq 0$ such that $(I - A)x = d$. This implies that $I - A$ has full rank, and the inverse $(I - A)^{-1}$ exists. Thus for every $d \in \mathbb{R}_+$ the corresponding $x$ is given by $x = (I - A)^{-1}d$. If $(I - A)^{-1}d$ contained a negative element, say in position $(ij)$, we could find a $d \in \mathbb{R}_+$, for example the $j$-th unit vector $d = e_j$ such $x_i = [(I - A)^{-1}d]_i < 0$. Therefore (ii)⇒(iv). The proof that (iii)⇒(iv) is analogous.

It remains to show that (i) implies (iv). This is a consequence of the following Claim.

**Claim.** If $A$ is productive and $x \geq Ax$, then $x \geq 0$.

**Proof of Claim.** Assume that $A$ is productive. By definition, there exists $\bar{x} \geq 0$ such that $\bar{x} \gg Ax$. Since $Ax \geq 0$, this implies $\bar{x} \gg 0$.

Suppose now that $x$ satisfies $x \geq Ax$, but not $x \geq 0$. Then some coordinate of $x$ is negative. Let $\lambda > 0$ be the largest number such that $x' = \lambda x + \bar{x} \geq 0$. Then some component of $x'$ is equal to zero (such a $\lambda$ exists because $x' \gg 0$ for $\lambda = 0$ and $x'$ has negative components for $\lambda$ sufficiently large). On the other hand, $x' = \lambda x + \bar{x} \gg \lambda x + Ax \geq 0$ so that $x' \gg 0$, a contradiction. This proves the Claim.

(i) ⇒ (iv): Assume that $A$ is productive. If $(I - A)x = 0$ then $-(I - A)x = 0$, and by the Claim this implies that $x \geq 0$ and $-x \geq 0$, therefore $x = 0$. Thus $I - A$ is invertible. We have to show that $(I - A)^{-1}$ is nonnegative. Because $(I - A)(I - A)^{-1} = I$, the $i$-th column of $(I - A)^{-1}$ is the vector $x$ such that $(I - A)x = e_i$ (the $i$-th unit vector). Since $e_i \geq 0$, this implies $x \geq 0$ by the Claim. This proves that (i) ⇒ (iv).

The following Theorem characterizes technologies for which labor is indispensable.

**Theorem 5.2.** Consider a technology $(a_0, A)$ where $A$ is productive. The following statements are equivalent:

(i) labor is indispensable for $(a_0, A)$, i.e. $a_0x > 0$ for $x - Ax \geq 0$ and $x \geq 0$

(ii) every sector $j \in N$ needs labor (directly or indirectly)

(iii) every autarkic subgroup needs labor directly, i.e. if $J \subset N$ is autarkic, then $a_{0j} > 0$ for at least one $j \in J$
(iv) the vector \( v = a_0(I - A)^{-1} \) is strictly positive.

**Proof of Theorem 5.2**

Since \( A \) is productive, the matrix \( I - A \) is nonnegatively invertible and

\[
(I - A)^{-1} = \sum_{t=0}^{\infty} A^t = I + A + A^2 + A^3 + \cdots
\]

"(i)⇔(iv)" We have \( x - Ax = d \iff x = (I - A)^{-1}d \); hence \( L = a_0x = a_0(I - A)^{-1}d = vd \) is positive for every \( d \geq 0 \) iff \( v = a_0(I - A)^{-1} \gg 0 \).

"(ii)⇔(iv)" we have

\[
v = a_0(I - A)^{-1} = a_0 \sum_{t=0}^{\infty} A^t = a_0 + a_0A + a_0A^2 + a_0A^3 + \cdots
\]

Denote by \( a_{ij}^{(t)} \) the \((i,j)\)-th element of the matrix \( A^t \). Then

\[
v_j = a_{0j} + \sum_i a_{0i}a_{ij}^{(1)} + \sum_i a_{0i}a_{ij}^{(2)} + \sum_i a_{0i}a_{ij}^{(3)} + \cdots
\]

Clearly, \( v_j > 0 \) iff either \( a_{0j} > 0 \) or at least one of the terms \( a_{0i}a_{ij}^{(t)} \) in this sum is positive \((i = 1, 2, \ldots n, t = 1, 2 \ldots)\). From the definition of matrix multiplication,

\[
a_{ij}^{(1)} = a_{ij} \\
a_{ij}^{(2)} = \sum_k a_{ik}a_{kj} \\
a_{ij}^{(3)} = \sum_k a_{ik}a_{kj}^{(2)} = \sum_k \sum_l a_{ik}a_{kl}a_{lj}
\]

Therefore \( a_{ij}^{(t)} > 0 \) if and only if there is a chain of sectors \( i = k_0, k_1, k_2, \ldots, i = j \), leading from \( i \) to \( j \) such that \( a_{k_s,k_{s+1}} > 0 \) for any two consecutive sectors \( k_s, k_{s+1} \). Therefore \( v_j > 0 \) iff either \( a_{0j} > 0 \) or there exists \( i \in \{1, \ldots n\} \) such that \( a_{0i} > 0 \) and \( a_{ij}^{(t)} > 0 \). But this means that sector \( j \) needs labor, either directly or indirectly.

"(iii)⇒(ii)" Fix an arbitrary \( j_0 \in \{1, 2, \ldots, n\} \). If \( a_{ij_0} = 0 \) for all \( i = 1, \ldots, n \) the singleton set \( \{j_0\} \) is autarkic and \( a_{0j_0} > 0 \) by (iii). If \( a_{ij_0} > 0 \) for at least one \( i \), we define

\[
K_1 = \{ i \in N \mid a_{ij_0} > 0 \} \\
K_2 = \{ i \in N \mid a_{ik} > 0 \text{ for some } k \in K_1 \} \\
K_3 = \{ i \in N \mid a_{ik} > 0 \text{ for some } k \in K_2 \} \\
\ldots
\]
Note that \( K_1 \) is nonempty. The sequence of sets \( K_1, K_1 \cup K_2, K_1 \cup K_2 \cup K_3, \ldots \) is (weakly) increasing and becomes constant after at most \( n \) steps. Denote the limit set by \( J = \bigcup_{t=1}^{\infty} K_t \) . By construction, \( j_0 \) needs every input \( i \in J \), and \( J \) is autarkic. By (iii) there is \( i_0 \) with \( a_{0i_0} > 0 \). Therefore sector \( j_0 \) needs labor.

"(ii) \( \Rightarrow \) (iii)":

Assume that (iii) is false, i.e. there is an autarkic group \( J \) with \( a_{0j} = 0 \) for all \( j \in J \). But then \( a_{0i} \) can be positive only for \( i \notin J \), and because \( a_{ij} = 0 \) for all \( i \notin J, j \in J \), no supply chain leading from 0 to any \( j \in J \) can exist.

\[ \blacksquare \]

5.2 The Frobenius-Perron Theorem

Let \( A = (a_{ij}) \) be a square matrix.

A number \( \lambda \) is an eigenvalue of \( A \) if the matrix \( A - \lambda I \) is not invertible. A nonzero (column) vector \( x \) is a (right) eigenvector of \( A \), associated with the eigenvalue \( \lambda \), if \( Ax = \lambda x \). A nonzero (row) vector \( p \) is a (left) eigenvector of \( A \), associated with the eigenvalue \( \lambda \), if \( pA = \lambda p \). The eigenvalues of a matrix are the roots of the characteristic equation \( \det(A - \lambda I) = 0 \). A matrix has \( n \) eigenvalues in general (not necessarily distinct, and possibly complex). For each eigenvalue \( \lambda \) there exist right (and left) eigenvectors, because the equation \( (A - \lambda I)x = Ax - \lambda x = 0 \) has a nonzero solution \( x \) precisely when \( (A - \lambda I) \) is not invertible (and similarly for left eigenvectors). If \( x \) is an eigenvector for an eigenvalue \( \lambda \), then so is \( \alpha x \) for every number \( \alpha \), i.e. eigenvectors are determined only up to multiplication by a constant.

Nonnegative matrices have special properties regarding their eigenvalues and eigenvectors.

**Theorem 5.3.** Let \( A \geq 0 \) be a nonnegative square matrix. Then

(i) The matrix \( A \) has a nonnegative eigenvalue \( \hat{\lambda} = \lambda(A) \geq 0 \), and associated nonnegative right and left eigenvectors \( \hat{x} \geq 0, \hat{p} \geq 0 \) with \( A\hat{x} = \hat{\lambda}x \), \( \hat{p}A = \hat{\lambda}p \).

(ii) \( \hat{\lambda} \geq |\lambda| \) for all eigenvalues \( \lambda \) of \( A \)

(iii) \( \rho I - A \) is nonnegatively invertible iff \( \rho > \hat{\lambda} \)

(iii') \( A \) is productive iff \( \lambda(A) < 1 \)

(iv) If \( Ay \geq \mu y \) for a real number \( \mu \) and a semipositive vector \( y \geq 0 \), then \( \hat{\lambda} \geq \mu \)

This maximal nonnegative eigenvalue \( \lambda(A) \) is called the **FROBENIUS-PERRON** eigenvalue or dominant eigenvalue of \( A \). Assertion (iii') follows from (iii) by taking \( \rho = 1 \).

The dominant eigenvalue \( \lambda(A) \) has the following properties:
Theorem 5.4. Let $\lambda(A)$ be the dominant eigenvalue of a nonnegative square matrix $A \geq 0$. Then

(i) $\lambda(A) = \lambda(A^T)$, where $A^T$ is the transpose of $A$

(ii) $\lambda(\alpha A) = \alpha \lambda(A)$ for $\alpha \geq 0$

(iii) $\lambda(A^k) = (\lambda(A))^k$ for any positive integer $k$

(iv) $\lambda(A) \geq \lambda(B)$ if $A \geq B \geq 0$

(v) $\lambda(A) = 0$ iff $A^k = 0$ for some positive integer $k$

If the matrix $A$ is indecomposable, Theorem 5.3 can be strengthened to

Theorem 5.5. Let $A \geq 0$ be an indecomposable matrix. Then

(i) $\lambda(A) > 0$, and $\lambda(A)$ is a simple root of the characteristic equation.

(ii) Any nonnegative eigenvector associated with $\lambda(A)$ is strictly positive: $\hat{x} \gg 0$, $\hat{p} \gg 0$. Moreover, these eigenvectors are unique up to multiplication by a scalar.

(iii) Any other eigenvector (associated with another eigenvalue) contains negative components.

(iv) If $(\rho I - A)x \geq 0$ for an $x \geq 0$, then $\rho I - A$ is nonnegatively invertible.

(v) If $\rho I - A$ is nonnegatively invertible, then $(\rho I - A)^{-1} \gg 0$

(vi) If $A \geq B \geq 0$, and one of $A$ or $B$ is indecomposable, then $\lambda(A) > \lambda(B)$

Proof of Theorem 5.5. All assertions except (iii) are in Nikaido (1968) (Thms. 7.3 and 7.4). To prove (iii), assume that there is an eigenvalue $\lambda \neq \hat{\lambda}$ with associated right eigenvector $x \geq 0$, so that $Ax = \lambda x$ and $x$ has no negative component. Now let $\hat{\lambda} > 0$ be the dominant eigenvalue and $\hat{p} \gg 0$ an associated left eigenvector of $A$, so that $\hat{p}A = \hat{\lambda}\hat{p}$. Then $\hat{\lambda}\hat{p}x = \hat{p}Ax = \hat{p}\lambda x$ implies $\hat{p}x = 0$, contradicting $\hat{p} \gg 0$, $x \geq 0$.

Remark. Properties (i) and (ii) in Theorem 5.5 characterize indecomposable matrices. A nonnegative matrix $A$ with dominant eigenvalue $\lambda(A)$ is indecomposable if and only if $\lambda(A)$ is a simple root of the characteristic equation and has strictly positive right and left eigenvectors. See Gantmacher (1966), p. 70.
References


