Lecture Notes on “Classical Value Theory or The Linear Model of Production”

Manfred Nermuth
Department of Economics, University of Vienna
Oskar-Morgenstern-Platz 1, A-1090 Vienna (Austria)
manfred.nermuth@univie.ac.at

January 2024

Contents

Lecture 0

Mathematical Notation.

We will use the following notation throughout: A vector x (similarly for matrices) is called *nonnegative*, resp. *semipositive*, resp. *strictly positive*, written $x \geqq 0$, resp. $x \gneqq 0$, resp. $x > 0$, if all its components are nonnegative, resp. all components are nonnegative and at least one component is positive, resp. all components are positive. We also write $x \geqq y$ if $x - y \geqq 0$, etc. The set of nonzero components of a vector $x = (x_i)$ is the *support* of x, denoted by $\mathrm{supp}(x) = \{i \,|\, x_i \neq 0\}$. If J is a subset of K, we write $J \subset K$, and $J \subsetneqq K$ if $J \neq K$ is a proper subset. The unit matrix (of any dimension) is denoted by I. Its rows and columns are unit vectors. We write e_i for the i-th row and e^j for the j-th column of I. The symbol $\mathtt{e} = (1, 1, \ldots, 1)$ denotes a summation vector. The n-dimensional unit simplex is $\Delta = \Delta^n = \{x \in \mathbb{R}^n \,|\, x \geqq 0 \text{ and } \sum_{i=1}^n x_i = 1\}$. Two vectors x, y are *proportional*, written $x \sim y$, if $x = ky$ for some number $k \neq 0$. For a square matrix A we define $A^0 = I$. The transpose of a matrix A is denoted by A^{T}. The transpose of a column vector is a row vector and vice versa. In general, we use column vectors to denote quantities, and row vectors for prices or values. In an expression like Ax or pA, where A is a matrix and x, p are vectors, it is always assumed that x is a column vector and p is a row vector (of suitable dimension). We write $[Ax]_i = \sum_j a_{ij}x_j$ for the i-th component of the column vector Ax, and similarly $[pA]_j = \sum_i p_i a_{ij}$ for the j-th component of the row vector pA.

0 A Little Linear Algebra

A *vector* of length n (or dimension n) is a list of n numbers, $x = (x_1, x_2, \dots x_n)$. The numbers x_i are the components of the vector. We write also $x = (x_i)$ or $x = (x_i)_{i=1,\dots n}$.
The product of a number α and a vector x is again a vector, namely

$$\alpha x = (\alpha x_1, \alpha x_2, \dots \alpha x_n)$$

i.e. all components of x are multiplied by α.
Let $x = (x_1, \dots x_n)$ and $y = (y_1, \dots y_n)$ be two vectors of the same length. The *sum* of x and y is the vector

$$x + y = (x_1 + y_1, x_2 + y_2, \dots x_n + y_n)$$

The *scalar product* $x.y$ is not a vector, but a number (a scalar), namely

$$x.y = x_1 y_1 + x_2 y_2 + \dots x_n y_n$$

A vector x can be written as a row vector

$$x = (x_1, x_2, \dots x_n)$$

or as a column vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

An $(m \times n)$-*matrix* $A = (a_{ij}) = (a_{ij})_{i=1,\dots m;\, j=1,\dots n}$ (or a matrix of dimension $(m \times n)$) is a rectangular array of numbers with m rows and n columns of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The i-th row of the matrix A is the n-dimensional row vector

$$a_i = (a_{i1}, a_{i1}, \dots a_{in})$$

and the j-th column of A is the m-dimensional column vector

$$a^j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix}$$

A row (resp. column) vector can be considered as a matrix with only one row (resp. one column). The *transpose* A^T of an $(m \times n)$-matrix $A = (a_{ij})$ is obtained by interchanging rows and columns, i.e. the elements in the first row of A form the first column of A^T, the elements in the second row of A form the second column of A^T, etc. The transpose of a row vector is a column vector, and the transpose of a column vector is a row vector. An $(m \times n)$-matrix is *square* if $m = n$, i.e. it has the same number of rows and columns. In this case we say that A is a square matrix of dimension n.

Like vectors, matrices can be multiplied with numbers componentwise, i.e. αA is the matrix whose components are αa_{ij}. Two matrices of the same dimension can also be added componentwise. If $B = (b_{ij})$ is a matrix of the same dimension as $A = (a_{ij})$, then $A + B$ is the matrix with components $a_{ij} + b_{ij}$, for $i = 1, \dots m$ and $j = 1, \dots n$.

Matrix multiplication is more complicated. If $A = (a_{ij})$ is an $(m \times n)$-matrix, and $B = (b_{jk})$ is an $(n \times p)$-matrix, then the rows of A have the same length n as the columns of B, and the product $C = A.B$ is defined as follows.

$C = (c_{ik})$ is an $(m \times p)$-matrix, and the element

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk}$$

is the scalar product of the i-th row of A with the k-th column of B, for $i = 1, \dots m$ and $k = 1, \dots p$.

See the examples below for more explanantion.

The n-dimensional *unit matrix* is the $(n \times n)$-matrix

$$I = I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Its rows and columns are *unit vectors*. It is easy to see that $A.I = I.A = A$ for every square matrix A of the same dimension as I. If A is a square matrix, then it can be multiplied with itself. We write $A^2 = A.A$, $A^3 = A.A.A$, etc. We define $A^0 = I$ (when the dimension is clear from the context, we write simply I instaed of I_n). Two square matrices A, B of the same dimension can always be multiplied, but $AB \neq BA$ in general, i.e. matrix multiplication is nor commutative (see the example below).

Inverse Matrix.

A square matrix A is *invertible* if there exists a (square) matrix B such that $A.B = I$. This matrix is called the *inverse* of A and is denoted by $B = A^{-1}$. Not every square matrix has an inverse, but if the inverse of A exists, then we have also $BA = AB = I$. That is $AA^{-1} = A^{-1}A = I$.

A matrix is invertible if and only if its row vectors (equivalently column vectors) are linearly independent. This is the case if (and only if) the determinant $\det(A) = |A|$ of the matrix is not zero. I cannot explain these concepts here, but will show below how to find the inverse of (2×2)- and (3×3)-matrices.

To find the inverse of a (2×2)-matrix A, write now

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

The determinant of A is

$$\det(A) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma \tag{0.1}$$

If this determinant is not zero, the inverse of A is

$$A^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \tag{0.2}$$

(α and δ change places, β and γ change sign). To check formula (0.2), observe that

$$AA^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \alpha\delta - \beta\gamma & -\alpha\beta + \beta\alpha \\ \gamma\delta - \delta\gamma & -\gamma\beta + \delta\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

To find the inverse of a (3×3)-matrix A, write

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Denote by A_{ij} the 2×2-matrix obtained from A be crossing out the i-th row and the j-th column, and write $b_{ij} = \det(A_{ij}) = |A_{ij}|$.

For example if $i = 2$, $j = 1$ we cross out row 2 and column 1:

$$A = \begin{pmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{pmatrix} \quad \text{and obtain:} \quad b_{21} = |A_{21}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{13}a_{32}$$

Compute all these b_{ij} and write down the matrix $B = (b_{ij})$:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

By a well-known theorem (Laplace expansion) the determinant of A is given by, for any $i = 1, 2, 3$:

$$\det(A) = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} b_{ij} \tag{0.3}$$

For example, for $i = 1$ (we get the same result for $i = 2$ or $i = 3$):

$$\det(A) = a_{11}b_{11} - a_{12}b_{12} + a_{13}b_{13} \tag{0.4}$$

(a term $a_{ij}b_{ij}$ gets a negative sign if $i + j$ is odd).
Finally, define the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & -b_{21} & b_{31} \\ -b_{12} & b_{22} & -b_{32} \\ b_{13} & -b_{23} & b_{33} \end{pmatrix} \tag{0.5}$$

C is obtained from B by forming the transpose B^{T} and changing the sign of b_{ij} if $i + j$ is odd. The matrix C is known as the adjugate (or adjoint) matrix of A, denoted by $C = \mathrm{adj}(A)$. It is 'almost' the inverse of A. Multiplying A and C gives

$$A.C = \det(A).I$$

so that the inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)} C \tag{0.6}$$

Linear Equations.

A system of n linear equations in n variables $x_1, \dots x_n$ is of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{12}x_2 + \dots a_{2n}x_n &= b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n &= b_n \end{cases}$$

where the coefficients a_{ij}, b_j are given and the x_i are unknown.

Using the matrix notation explained above, this system of equations can be written as a single linear vector equation of the form

$$Ax = b \tag{0.7}$$

where A is a square matrix, and x and b are column vectors:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$

If the coefficient matrix A is invertible, we can multiply this equation from the left by the inverse A^{-1} and obtain the solution

$$A^{-1}Ax = I.x = x = A^{-1}b$$

That is, solving systems of linear equations is the same as finding inverse matrices.

Equation (0.7) is written in terms of column vectors. This is the usual form. But one can also write a system of linear equations in terms of row vectors. Denote the variables now by $p_1, \dots p_n$, and consider the system of linear equations

$$\begin{cases} p_1a_{11} + p_2a_{21} + \dots p_na_{n1} &= c_1 \\ p_1a_{12} + p_2a_{22} + \dots p_na_{n2} &= c_2 \\ \dots \\ p_1a_{1n} + p_2a_{2n} + \dots p_na_{nn} &= c_n \end{cases}$$

where again a_{ij} and c_i are given coefficients. Using matrix notation this system can also be written as a single vector equation of the form

$$pA = c \tag{0.8}$$

where $A = (a_{ij})$ is a square matrix as before, and $p = (p_1, \ldots p_n)$ and $c = (c_1, \ldots c_n)$ are row vectors. Now the solution can be found by multiplying equation (0.8) by A^{-1} from the right:

$$pAA^{-1} = pI = p = cA^{-1}$$

In the course, we will usually denote prices by row vectors, and quantities by column vectors, and encounter linear equations in both forms (0.7) and (0.8).

Examples.

$n = 2$.

Let A, B and I be (2×2)-matrices, p a row vector (a (1×2)-matrix), and x a column vector (a (2×1)-matrix), as follows.

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 \\ -1 & 6 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p = \begin{pmatrix} 7 & 3 \end{pmatrix}, \quad x = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

Here I is the 2-dimensional unit matrix. Then

$$p.A = (7,3).\begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix} = (7 \times 0 + 3 \times 4,\ 7 \times 1 + 3 \times 2) = (12, 13)$$

$$A.x = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix} . \begin{pmatrix} 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \times 8 + 1 \times 11 \\ 4 \times 8 + 2 \times 11 \end{pmatrix} = \begin{pmatrix} 11 \\ 54 \end{pmatrix}$$

It also is easy to see that $I.x = x$ and $p.I = p$ for all column vectors x and row vectors p. The product of two matrices is

$$A.B = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix} . \begin{pmatrix} 5 & 3 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 18 & 24 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (c_{ij}) = C$$

where c_{ij} is the scalar product of the ith row vector of A with the j-th column vector of B, i.e.

$$c_{11} = 0 \times 5 + 1 \times (-1) = -1, \quad c_{12} = 0 \times 3 + 1 \times 6 = 6$$
$$c_{21} = 4 \times 5 + 2 \times (-1) = 18, \quad c_{22} = 4 \times 3 + 2 \times 6 = 24$$

In the same way one can check that

$$BA = \begin{pmatrix} 5 & 3 \\ -1 & 6 \end{pmatrix} . \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 12 & 11 \\ 24 & 11 \end{pmatrix} \neq AB$$

Note that $AB \neq BA$. Matrix multiplication is not commutative in general. It is also easy to see that $A.I = I.A = A$ always.

To find the inverse of the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$$

compute first the determinant $\alpha\delta - \beta\gamma = 0 - 4 = -4$ and use (0.2) to obtain

$$A^{-1} = \frac{1}{-4} \begin{pmatrix} 2 & -1 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/4 \\ 1 & 0 \end{pmatrix}$$

$n = 3$.
Consider the (3×3)-matrix A:

$$A = (a_{ij}) = \begin{pmatrix} 0 & 3 & -1 \\ 2 & 1 & 4 \\ -2 & 5 & 0 \end{pmatrix}$$

Put $b_{ij} = det(A_{ij})$, where A_{ij} is the (2×2)-matrix obtained from A by crossing out the i-th row and the j-th column. Then $b_{11} = 1 \times 0 - 4 \times 5 = -20$, $b_{12} = 2 \times 0 - 4 \times (-2) = 8$, etc. This gives the matrix B:

$$B = (b_{ij}) = \begin{pmatrix} -20 & 8 & 12 \\ 5 & -2 & 6 \\ 13 & 2 & -6 \end{pmatrix}$$

The determinant of A is, using (0.3) with $i = 1$:

$$\det(A) = a_{11}b_{11} - a_{12}b_{12} + a_{13}b_{13} = 0 \times (-20) - 3 \times 8 + (-1) \times 12 = -36$$

We get the same result for $i = 2$:

$$\det(A) = -a_{21}b_{21} + a_{22}b_{22} - a_{23}b_{23} = -2 \times 5 + 1 \times (-2) - 4 \times 6 = -36$$

To find the inverse of A, compute the adjugate matrix C, using (0.5):

$$\text{adj}(A) = C = \begin{pmatrix} b_{11} & -b_{21} & b_{31} \\ -b_{12} & b_{22} & -b_{32} \\ b_{13} & -b_{23} & b_{33} \end{pmatrix} = \begin{pmatrix} -20 & -5 & 13 \\ -8 & -2 & -2 \\ 12 & -6 & -6 \end{pmatrix}$$

This gives

$$A.C = \begin{pmatrix} 0 & 3 & -1 \\ 2 & 1 & 4 \\ -2 & 5 & 0 \end{pmatrix} \begin{pmatrix} -20 & -5 & 13 \\ -8 & -2 & -2 \\ 12 & -6 & -6 \end{pmatrix} = \begin{pmatrix} -24-12 & -6+6 & -6+6 \\ -40-8+48 & -10-2-24 & 26-2-24 \\ 40-40 & 10-10 & -26-10 \end{pmatrix} =$$

$$= \begin{pmatrix} -36 & 0 & 0 \\ 0 & -36 & 0 \\ 0 & 0 & -36 \end{pmatrix} = (-36). \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det(A).I$$

Therefore the inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)}.C = \frac{1}{-36}\begin{pmatrix} -20 & -5 & 13 \\ -8 & -2 & -2 \\ 12 & -6 & -6 \end{pmatrix} = \frac{1}{36}\begin{pmatrix} 20 & 5 & -13 \\ 8 & 2 & 2 \\ -12 & 6 & 6 \end{pmatrix}$$

Lecture 1

1 The Technology

We consider a closed economy with $n \geqq 1$ produced goods, labeled $i = 1, \dots n$, and one non-produced primary factor $i = 0$, called labor. We write $N = \{1, 2, \dots n\}$. There is exactly one production process for each produced good, with constant returns to scale, and without joint production. Processes are numbered such that process j produces good j, for $j \in N$. Labor is not scarce. There is a common period of production (the "year") for all processes. Inputs (material inputs and labor) are invested in one period, and the output (the "harvest") becomes available in the next period. It can then be used for consumption or investment in this period.

The quantity of each good is measured in certain units. The dimension of these units must be suitable for the good in question (e.g. weight for butter, volume for petrol, time for the amount of work), but the units can otherwise be chosen arbitrarily. The quantity of butter can be measured in kilograms or pounds, petrol in litres or gallons, work in hours or days, and so on. We assume that for each good a suitable unit has been chosen, so that their amounts are well-specified numbers.

A *technology* is given by a pair (a_0, A), where $a_0 \geqq 0$ is an n-vector of labor input coefficients and $A \geqq 0$ is an $(n \times n)$-input coefficient matrix:

$$a_0 = (a_{01}, a_{02}, \dots, a_{0n}), \qquad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The coefficient $a_{ij} \geqq 0$ is the amount of good i needed to produce one unit of good j, for $i = 0, 1, \dots n$ and $j = 1, \dots n$. Process $j \in N$, operated at unit intensity level, transforms the inputs a_{0j}, a_{1j},..., a_{nj} into one unit of good j. Each process has constant returns to scale, i.e. to produce $\alpha \geqq 0$ units of good j, we need the inputs αa_{ij} $(i = 0, 1, \dots, n)$.

We will assume throughout that the technology satisfies the following assumption:

Assumption 1. *The technology* (a_0, A) *is such that*

(a) for every production process $j \in N$ *there is* $i \in \{0, 1, \dots n\}$ *with* $a_{ij} > 0$*, i.e. every production process needs some input*

(b) at least one process needs labor, i.e. $a_{0j} > 0$ *for some* $j \in N$*.*

Condition (a) says that there is *no free production*. Without condition (b), labor would play no role at all in the economy and there would be no reason to include it in our model in the first place. Thus a_0 is not a zero vector, but note that A may be a zero matrix, under Ass. 1.

A *commodity bundle* $d = (d_i)$ contains $d_i \geqq 0$ units of good i, for $i \in N$. It is also referred to as a "composite commodity" or a "basket of commodities". "One unit of good j" corresponds to a commodity bundle $d = e^j$ which contains one unit of good j and nothing else. Commodity bundles are written as column vectors:

$$d = \begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ d_n \end{pmatrix}, \qquad e^j = \begin{pmatrix} 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{pmatrix}$$

The vector e^j (one unit of good j) has 1 in the j-th position and zeros elsewhere.

2 The Price System

The determination of the (relative) prices of all commodities (the "Value Problem") is one of the oldest and most fundamental problems in Economic Theory. We will see that in our Linear Model this problem has a very simple and elegant solution. We begin by determining prices under the assumption of zero profits (price = cost).

We denote the price of (one unit of) good i by p_i. The price of labor p_0 is the wage, and $p = (p_1, \dots, p_n)$ is the price vector. The pair $(p_0, p) = (p_0, p_1, \dots p_n)$ is the price system. For a commodity bundle d, the scalar product $pd = \sum_i p_i d_i$ is the value of d at prices p.

Prices are expressed in some arbitrary unit of account. Multiplying all prices by a positive constant $k > 0$ gives an equivalent price system $(p_0', p') = (kp_0, kp)$. Only relative prices p_i/p_j have economic meaning, and these are independent of the choice of unit of account. A good i is the *numéraire* if its price is one, $p_i = 1$. A commodity bundle d is the numéraire if $pd = 1$.

How should the prices be determined? A classical idea is that the price (or value) of a commodity should be equal to its *cost of production* (i.e. profits are zero). The cost of production of one unit of good j is the cost of the required inputs $a_{0j}, a_{1j}, \dots a_{nj}$, so the price p_j should satisfy the equation

$$p_j = p_0 a_{0j} + p_1 a_{1j} + p_2 a_{2j} + \cdots + p_n a_{nj} \qquad \text{for} \quad j = 1, 2, \dots n \tag{2.1}$$

This is a system of n linear equations in the $n + 1$ unknowns $p_0, p_1, \dots, p_n$. In matrix notation, it takes the form

$$p = p_0 a_0 + pA \tag{2.2}$$

Equation (2.2) is the *Zero-Profit Price Equation.* It is a system of linear equations written in terms of row vectors, i.e. p, a_0, and pA are all row vectors:

$$p = (p_1, p_2, \dots, p_n), \quad a_0 = (a_{01}, a_{02}, \dots, a_{0n}), \quad pA = (pa^1, pa^2, \dots, pa^n)$$

where $pa^j = [pA]_j = \sum_{i=1}^n p_i a_{ij} = p_1 a_{1j} + p_2 a_{2j} + \cdots + p_n a_{nj}$, for $j = 1, 2, \dots n$.
We can rewrite the price equation as

$$p_0 a_0 = p - pA = p(I - A)$$

and obtain the solution

$$p = p_0 a_0 (I - A)^{-1} \tag{2.3}$$

provided the matrix $I - A$ is invertible. Note that we can choose $p_0 > 0$ arbitrarily. This is so because the price equation has one degree of freedom, and determines the prices (p_0, p) only up to multiplication by a constant factor, i.e. up to the choice of numéraire, as it should be.

Example 2.1.

Let $n = 2$ and the technology (a_0, A) be given by

$$a_0 = (a_{01}, a_{02}) = (1,\ 0.5), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.1 \\ 7 & 0 \end{pmatrix}$$

That is, to produce one unit of good 1 we need $a_{11} = 0.2$ units of good 1, $a_{21} = 7$ units of good 2, and $a_{01} = 1$ units of labor. Similarly for good 2, with a_{i2} instead of a_{i1} $(i = 0, 1, 2)$.
The Price Equation $p = p_0 a_0 + pA$ takes the form:

$$\begin{aligned} p_1 &= p_0 a_{01} + p_1 a_{11} + p_2 a_{21} = p_0 + 0.2 p_1 + 7 p_2 \\ p_2 &= p_0 a_{02} + p_1 a_{12} + p_2 a_{22} = 0.5 p_0 + 0.1 p_1 + 0 p_2 \end{aligned}$$

These equations determine (p_0, p_1, p_2) only up to a common factor. It is easy to check that $(p_0, p_1, p_2) = (1, 45, 5)$ is a solution. But any multiple of this is also a solution. For example we could take $(p_0, p_1, p_2) = (3, 135, 15)$. The general solution is $(p_0, p_1, p_2) = (p_0, 45p_0, 5p_0)$, where $p_0 > 0$ is an arbitrary positive number.
If $p_0 = 1$, labor is the numéraire, i.e. $p_1 = 45$ means that one unit of good 1 is worth 45 hours of work, and $p_2 = 5$ means that one unit of good 2 is worth 5 hours of work (all prices are expressed relative to the numéraire).
Alternatively, we can use formula (2.3)

$$p = p_0 a_0 (I - A)^{-1}.$$

We have

$$I - A = \begin{pmatrix} 0.8 & -0.1 \\ -7 & 1 \end{pmatrix}$$

with determinant $\det(I - A) = 0.8 - 0.7 = 0.1 = 1/10$. therefore, by (0.2):

$$(I - A)^{-1} = 10 \begin{pmatrix} 1 & 0.1 \\ 7 & 0.8 \end{pmatrix} = \begin{pmatrix} 10 & 1 \\ 70 & 8 \end{pmatrix}$$

and by (2.3), with $p_0 = 1$:

$$p = (p_1, p_2) = (1, 0.5) \begin{pmatrix} 10 & 1 \\ 70 & 8 \end{pmatrix} = (10 + 35,\ 1 + 4) = (45, 5)$$

the same as above.

Lecture 2

3 The Quantity System.

Let (a_0, A) be a technology. We denote by $a_i = (a_{i1}, a_{i2}, \dots a_{in})$ the i-th row of A, and by $a^j = (a_{1j}, a_{2j}, \dots a_{nj})^\mathsf{T}$ the j-th column of A.

The (gross) output of the economy is represented by a nonnegative column vector $x = (x_j)$, where $x_j \geqq 0$ is the (gross) output of good j, for $j = 1, \dots n$. The required inputs can be found as follows. To produce one unit of good j. the input quantities a_{ij} $(i = 0, 1, \dots n)$ are needed. To produce x_j units of good j. the input quantities $a_{ij}x_j$ $(i = 0, 1, \dots n)$ are needed (constant returns to scale). Thus process j transforms the input vector $a^j x_j = (a_{1j}x_j, a_{2j}x_j, \dots, a_{nj}x_j)^\mathsf{T}$ into x_j units of output of good j, using the amount of labor $a_{0j}x_j$. The total amount of good i needed as input for the output x is

$$y_i = \sum_{j=1}^{n} a_{ij}x_j = a_i x \qquad \text{for } i = 1, 2, \dots n \tag{3.1}$$

The (column) vector of material inputs $y = (y_1, \dots y_n)^\mathsf{T}$ is then given by

$$y = \sum_{j=1}^{n} a^j x_j = Ax \tag{3.2}$$

and the required amount of labor is

$$L = \sum_{j=1}^{n} a_{0j}x_j = a_0 x \tag{3.3}$$

The *net output* of good i is

$$d_i = x_i - y_i \qquad (i = 1, \dots n) \tag{3.4}$$

and the total net output (or net product) is given by the vector $d = (d_1, \dots d_n)^\mathsf{T} = x - y = x - Ax$. Equivalently, we can write

$$x = Ax + d \tag{3.5}$$

Equation (3.5) is the *Quantity Equation.* It is a system of linear equations written in terms of column vectors, i.e. x, d, and $y = Ax$ are all column vectors:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ \dots \\ d_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ a_{21}x_1 + \dots + a_{2n}x_n \\ \dots \\ a_{n1}x_1 + \dots + a_{nn}x_n \end{pmatrix} = Ax \tag{3.6}$$

The part of the economy producing good j is called *sector* j or *industry* j. Given a nonnegative vector $x \geqq 0$, we say that sector j is *active* if $x_j > 0$. The gross output vector x is also called the *activity vector* (or *intensity vector*). Given the technology (a_0, A), the vector x describes the economic activities in our model completely; it is also referred to as a *production plan.* A production plan $x \geqq 0$ is *feasible* (or *sustainable*) if the output is not less than the input, i.e. if $x \geqq Ax$.

In the Quantity equation (3.5) we could also start with an exogenous final demand d (for consumption, say), and ask if we can find gross output levels x which give this d as net output? This question is the *Planning Problem.* Given the technology matrix A and the final demand vector d, the Quantity Equation can be written

$$x - Ax = (I - A)x = d$$

This is a system of n linear equations in the n unknowns $x_1, \ldots .x_n$. The solution is

$$x = (I - A)^{-1} d \tag{3.7}$$

provided the matrix $I - A$ is invertible. The required amount of labor (the "number of workers") is then

$$L = L(d) = a_0 x = a_0 (I - A)^{-1} d \tag{3.8}$$

Stationary State.

The quantity equation (3.5) can be interpreted as follows. Consider a primitive society (a prehistoric village), without social classes and without organized markets. Production and consumption are regulated by custom and tradition, without money and prices. At the beginning of the year, the gross output x (the harvest from the last year) is available. The output x is split in two parts: Ax is used as input for production (seed corn), the rest (the net output) $d = x - Ax$ is consumed. The amount of labor is $L(d) = a_0 x$. All members of society contribute equally to the work effort, and are entitled to an equal share of the net output. The output produced from the inputs Ax and $a_0 x$ is again x, available in the following year. This output is again used for consumption d and investment Ax, and the amount of labor is again $L = a_0 x$. The economy continues in this way indefinitely. The amounts produced and consumed of all commodities are the same in all periods. We are in a *stationary state.*

Remark. The amount $L = a_0 x$ is the amount of labor required per period to enjoy net output d per period, in a stationary state. This is one of several possible definitions of the concept of "labor value" of the commodity bundle d. It is known as the *synchronous* labor value $V^s(d)$ of d. See Sec. 7.

Prices and Quantities.

For a given technology let the quantities x, d and L satisfy the quantity equations $x = Ax + d$ and $L = a_0 x$, and let the prices (p_0, p) satisfy the price equation $p = p_0 a_0 + pA$. If we multiply the quantity equation (from the left) by p, we get

$$px = pAx + pd$$

and if we multiply the price equation (from the right) by x, we get

$$px = p_0 a_0 x + pAx$$

Therefore $pd = p_0 a_0 x$. Since $a_0 x = L$ is the required amount of labor (the "number of workers"), $p_0 a_0 x_0 = p_0 L$ is the sum of all wages. Thus we have proved the following important result:

Theorem 3.1. *If prices satisfy the zero-profit condition $p = p_0 a_0 + pA$, then the value of the net output $d = x - Ax$ is equal to the sum of all wages:*

$$pd = p_0 L(d), \qquad \textit{where } L(d) = a_0 x = a_0 (I - A)^{-1} d$$

The workers' aggregate income is sufficient to buy the entire net output. The Theorem is no longer true when the rate of profit is not zero (see Sec. 12).

The Theorem implies $pd = L(d)$ for $p_0 = 1$, i.e.

Corollary 3.2. *When we choose labor as the numéraire, $p_0 = 1$, then the value pd of a commodity bundle d is equal to the amount of labor $L(d)$ required to produce it. In particular, the price p_i of good i is equal to the amount of labor $L(e^i)$ required to produce one unit of good i.*

For $p_0 = 1$, commodity prices are equal to labor values.

The technology (a_0, A) and the data contained in x and d give a complete description of the working of the economy. The flows of goods and money between the sectors can be described in the form of a Linear Flow Diagram, a Circular Flow Diagram, or an Input-Output Table (see Example 3.3 and Sec. 4 below). Input-Output Analysis was pioneered by Wassily Leontief. The matrix $(I - A)^{-1}$ is known as the "Leontief Inverse" in his honor.

Example 3.3.

Let $n = 2$ and the technology (a_0, A) be given by

$$a_0 = (a_{01}, a_{02}) = (1,\ 0.5), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.1 \\ 7 & 0 \end{pmatrix}$$

This is the same technology as in Example 2.1. There we considered the prices; let us now look at the quantity system.

Consider the following Planning Problem. The Planner wants to produce the net output

$$d = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

to satisfy some external demand (e.g. for consumption). What are the required gross output levels $x = (x_1, x_2)^{\intercal}$? The Quantity Equation $x = d + Ax$ takes the form

$$\begin{aligned} x_1 &= d_1 + a_{11}x_1 + a_{12}x_2 = 10 + 0.2x_1 + 0.1x_2 \\ x_2 &= d_2 + a_{21}x_1 + a_{22}x_2 = 20 + 7x_1 + 0x_2 \end{aligned}$$

By equation (3.7) the solution is

$$x = (I - A)^{-1}d = \begin{pmatrix} 10 & 1 \\ 70 & 8 \end{pmatrix} \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 10 \times 10 + 1 \times 20 \\ 70 \times 10 + 8 \times 20 \end{pmatrix} = \begin{pmatrix} 120 \\ 860 \end{pmatrix}$$

where the inverse matrix $(I - A)^{-1}$ is the same as in Example 2.1. It is easy to check that

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 120 \\ 860 \end{pmatrix}$$

is indeed a solution of the two equations above.

The required inputs are

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Ax = \begin{pmatrix} 0.2 & 0.1 \\ 7 & 0 \end{pmatrix} \begin{pmatrix} 120 \\ 860 \end{pmatrix} = \begin{pmatrix} 24 + 86 \\ 840 + 0 \end{pmatrix} = \begin{pmatrix} 110 \\ 840 \end{pmatrix}$$

and the required amount of labor is

$$L = V^s(d) = a_0 x = (1,\ 0.5) \begin{pmatrix} 120 \\ 860 \end{pmatrix} = 120 + 430 = 550$$

The net output $x - y$ is indeed equal to the desired net output d:

$$d = x - y = x - Ax = \begin{pmatrix} 120 \\ 860 \end{pmatrix} - \begin{pmatrix} 110 \\ 840 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

so that our Planning Problem is solved.

The values for x, y, d just computed describe a stationary state of the economy, in the sense that production and consumption at these levels can be sustained indefinitely. Note that labor is supplied exogenously, in whatever amounts are needed, since by assumption labor is not scarce (the "reserve army").

The flows of goods and labor in the stationary state can be seen in the following *Linear Flow Diagram:*

$$x_1 = 120 \begin{array}{l} \nearrow \; d_1 = 10 \\ \searrow \; y_1 = a_1 x = 110 \begin{array}{l} \nearrow \; a_{11} x_1 = 24 \\ \searrow \; a_{12} x_2 = 86 \end{array} \end{array}$$

$$x_2 = 860 \begin{array}{l} \nearrow \; d_2 = 20 \\ \searrow \; y_2 = a_2 x = 840 \begin{array}{l} \nearrow \; a_{21} x_1 = 840 \\ \searrow \; a_{22} x_2 = 0 \end{array} \end{array}$$

$$L = y_0 = a_0 x = 550 \begin{array}{l} \nearrow \; a_{01} x_1 = 120 \\ \searrow \; a_{02} x_2 = 430 \end{array}$$

In any given period, the gross output $x_1 = 120$ of good 1 (inherited from the previous period) is split in two parts: the amount $d_1 = 10$ is used for consumption, and the rest, $y_1 = 110$ is used as input for production. Of these 110 units of good 1, $a_{11}x_1 = 24$ units serve as input for sector 1, and $a_{12}x_2 = 86$ units serve as input for sector 2. Similarly for the gross output $x_2 = 860$ of good 2: $d_2 = 20$ units are consumed, and the rest is invested: all 840 units in sector 1, since sector 2 needs no input from itself ($a_{22} = 0$). These inputs, together with $L = 550$ units of labor ($a_{01}x_1 = 120$ in sector 1 and $a_{02}x_2 = 430$ in sector 2) are exactly sufficient to produce the same gross output $x = (120, 860)^{\mathsf{T}}$ again, so that the game repeats itself in the next period. In this stationary state, society consumes 10 units of good 1, 20 units of good 2, and expends $L(d) = 550$ units of labor in each period.

Let us also look at the prices. The price equation $p = p_0 a_0 + pA$ takes the form:

$$\begin{aligned} p_1 &= p_0 a_{01} + p_1 a_{11} + p_2 a_{21} = p_0 + 0.2 p_1 + 7 p_2 \\ p_2 &= p_0 a_{02} + p_1 a_{12} + p_2 a_{22} = 0.5 p_0 + 0.1 p_1 + 0 p_2 \end{aligned}$$

This system of equations has the solution $p = (p_1, p_2) = (45p_0, 5p_0)$ (cf. Example 2.1). Thus the value of the stationary state consumption $d = (10, 20)$ is $pd = p_0(45 \times 10 + 5 \times 20) = p_0(450 + 100) = 550p_0$. This is the same as the total wage bill, namely p_0 times the amount $L(d) = 550$ of labor needed to sustain the consumption d, in a stationary state.

4 Input - Output Tables, Circular Flow

The economic activities in our model can be summarized concisely in an *Input-Output Table* or graphically in a *Circular Flow Diagram*, as shown in the following Example.

Example 4.1.

There are two produced goods, good 1 is wheat and good 2 is horses. Labor is good 0. The technology (a_0, A) is given by

$$a_0 = (a_{01}, a_{02}) = (6,\ 6), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

It is easy to see that

$$I - A = \frac{1}{10}\begin{pmatrix} 9 & -4 \\ -3 & 8 \end{pmatrix} \quad \text{and} \quad (I - A)^{-1} = \frac{1}{6}\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix}$$

The price equation $p = p_0 a_0 + pA$ has the solution $p = p_0 a_0 (I - A)^{-1}$, i.e. with $p_0 = 1$:

$$p = (6.6).\frac{1}{6}\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix} = (1,1).\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix} = (11, 13) \tag{4.1}$$

There are $L = 1710$ workers, each supplying one unit of labor. Each worker consumes a certain consumption bundle $c = (c_1, c_2)^{\mathtt{T}}$ and has the budget constraint

$$pc = p_1c_1 + p_2c_2 = 11c_1 + 13c_2 = p_0 = 1$$

Assume that the optimal consumption bundle under this budget constraint, given the workers' preferences, is

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4/57 \\ 1/57 \end{pmatrix} = \frac{1}{57}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

This satisfies the budget constraint because $pc = (11 \times 4 + 13 \times 1)/57 = 1$. The total consumption demand of the workers is thus

$$d = c.L = \frac{1710}{57}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 30.\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 120 \\ 30 \end{pmatrix}$$

The gross output vector x required to produce the net output d is given by

$$x = (I - A)^{-1} d = \frac{1}{6} \begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix} . \begin{pmatrix} 120 \\ 30 \end{pmatrix} = \begin{pmatrix} 180 \\ 105 \end{pmatrix} \tag{4.2}$$

The required labor for this gross output is

$$a_0 x = (6, 6). \begin{pmatrix} 180 \\ 105 \end{pmatrix} = 6 \times 180 + 6 \times 105 = 1710$$

i.e. the firms' labor demand $a_0 x$ is equal to the households' labor supply $L = 1710$.

The flows between the sectors are now given as follows, for $i, j = 1, 2$: The households (sector 0) supply the amount of labor $a_{0j} x_j$ to sector j, and consume the amounts d_i of good i. Sector i delivers the amount $a_{ij} x_j$ of good i to sector j. Sector j pays $p_i a_{ij} x_j$ dollars for this delivery.

These commodity flows and money flows can be summarized in an Input-Output Table, as shown on the next page.

Remark. The tables shown below are similar to, but not quite the same as the actual Input-Output tables compiled by statistical offices for various countries.

First, in *Input-Output analysis*, as pioneered by W. Leontief, one considers a specific period, e.g. a year, and studies the commodity and money flows that take place between the various sectors of the economy *within* the given year. That is, the inputs Ax are supplied out of the current gross output x, the remaining part of x covering the final demand d in the same year. Production lags are not explicitly taken into account. This 'simultaneous' interpretation of Input-Output tables explains also the term "circular flow" (as opposed to the 'linear flow' in Example 3.3).

Moreover, whereas in our theoretical model each sector produces only one good, the 'sectors' in applied input-output analysis are semi-aggregated units, like mining, construction, certain parts of agriculture, etc., each producing many different goods. Thus in practice, only the money flows (aggregated sales and purchases) between sectors are available.

The Input-Output Table in terms of commodity flows.

The entry D_{ij} in cell (i, j) is the amount of good i delivered to sector j $(i, j = 0, 1, 2)$.

		Households	*corn*	*horses*	*sum*
	D_{ij}	$j = 0$	$j = 1$	$j = 2$	
households	$i = 0$	0	$a_{01}x_1 = 1080$	$a_{02}x_2 = 630$	$L = a_0x = 1710$
corn	$i = 1$	$d_1 = 120$	$a_{11}x_1 = 18$	$a_{12}x_2 = 42$	$x_1 = 180$
horses	$i = 2$	$d_2 = 30$	$a_{21}x_1 = 54$	$a_{22}x_2 = 21$	$x_2 = 105$

(4.3)

The row sum in row i is x_i because $d_i + a_{i1}x_1 + a_{i2}x_2 = x_i$ by the quantity equation.

The input-output table in terms of money flows.

The entry M_{ij} in cell (i, j) is the payment of sector j to sector i in dollars. In our example, the prices are $p_0 = 1$, $p = (p_1, p_2) = (11, 13)$. Multiplying the entries in table 4.3 with the appropriate prices gives the money flows:

		Households	*corn*	*horses*	*sum*
	M_{ij}	$j = 0$	$j = 1$	$j = 2$	
households	$i = 0$	0	$p_0a_{01}x_1 = 1080$	$p_0a_{02}x_2 = 630$	$p_0a_0x = 1710$
corn	$i = 1$	$p_1d_1 = 1320$	$p_1a_{11}x_1 = 198$	$p_1a_{12}x_2 = 462$	$p_1x_1 = 1980$
horses	$i = 2$	$p_2d_2 = 390$	$p_2a_{21}x_1 = 702$	$p_2a_{22}x_2 = 273$	$p_2x_2 = 1365$
	sum	$pd = 1710$	$p_1x_1 = 1980$	$p_2x_2 = 1365$	

(4.4)

The column sum in column j is p_jx_j because $p_j = p_0a_{0j}+p_1a_{1j}+p_2a_{2j}$ by the price equation (zero profit condition).) The row sum in row i is p_ix_i because $d_i + a_{i1}x_1 + a_{i2}x_2 = x_i$ by the quantity equation.

The row sums in Table (4.4) show the total income or revenue of the sector, and the column sums show the total expenditure or cost of the sector. For each sector, revenue equals cost. The total income of the households, $p_0L = 1710 = pd$ is the *national income* (net national product).

The information contained in this Table can also be summarized in a *circular flow diagram*:

The Circular Flow Diagram.

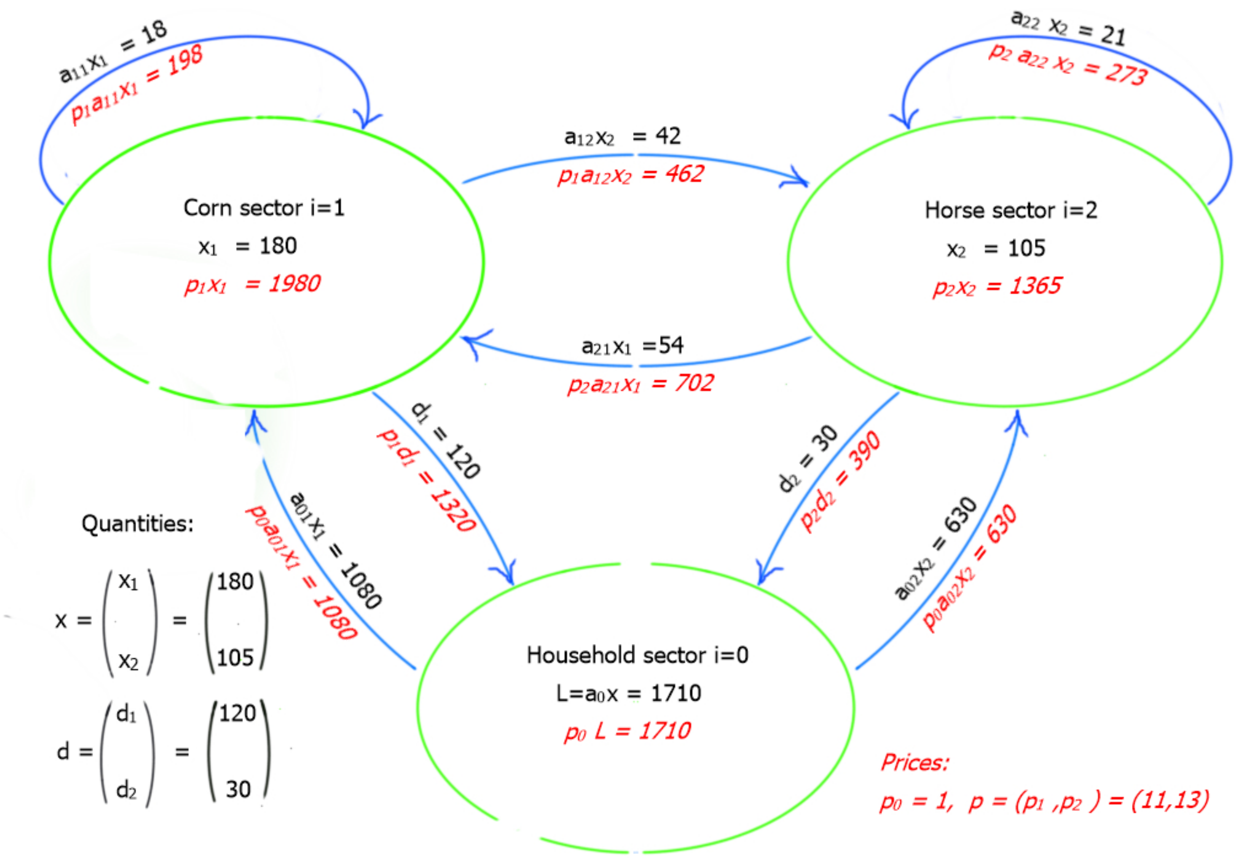

Figure 1: The arrows show the commodity flows between the sectors. The amount of good i delivered from sector $i \in \{0, 1, 2\}$ to sector $j \in \{1, 2\}$ is $a_{ij}x_j$ The amount of good $i \in \{1, 2\}$ delivered from sector i to sector 0 (the households) is d_i These amounts are shown in black. The red entries (in italics) are the values of these deliveries, at prices $p_0 = 1$, $p = (11, 13)$.

Lecture 3

5 Productivity

Let a technology (a_0, A) be given. We have determined the commodity prices $p = (p_1, \dots, p_n)$ so that they satisfy the zero-profit price equation $p = p_0 a_0 + pA$, where $p_0 > 0$ can be chosen arbitrarily. Given a final demand vector $d \geqq 0$, we have determined the gross output levels $x = (x_1, \dots, x_n)^{\mathsf{T}}$ so that they satisfy the quantity equation $x = Ax + d$.

To be economically meaningful, the prices p_i and the quantities x_i must be nonnegative, for all $i = 1, \dots, n$. When will this be the case?

As a first step, observe that the price equation can be written

$$p(I - A) = p_0 a_0$$

and the quantity equation can be written

$$(I - A)x = d$$

If the Leontief matrix $I - A$ is invertible, these equations have the unique solution

$$p = p_0 a_0 (I - A)^{-1}$$

and

$$x = (I - A)^{-1} d$$

The matrix $I - A$ is known as the Leontief matrix. The matrix $(I - A)^{-1}$ is known as the Leontief inverse of A. We say that a matrix is *nonnegatively invertible* if it is invertible and the inverse is nonnegative.

Lemma 5.1. *The quantity equation $x = Ax + d$ has a unique, nonnegative solution $x \geqq 0$ for every nonnegative final demand $d \geqq 0$ if and only if the matrix $I - A$ is nonnegatively invertible.*

Proof of Lemma 5.1.

The Leontief inverse $(I - A)^{-1}$ is a square $(n \times n)$-matrix. Denote its (ij)-th element by m_{ij} so that $(I - A)^{-1} = M = [m_{ij}]$. The solution of the quantity equation is then $x = Md$ or, explicitly,

$$x_i = m_{i1} d_1 + m_{i2} d_2 + \cdots + m_{in} d_n \qquad (i = 1, \dots n) \tag{$*$}$$

By assumption, $d \geqq 0$, i.e. $d_j \geqq 0$ for all j. Therefore x_i is certainly nonnegative if all coefficients m_{ij} are nonnegative, i.e. if the matrix $(I-A)^{-1} = M$ is nonnegative. Moreover, if we want to be sure that the output levels x_i are nonnegative for *every* possible final demand vector $d \geqq 0$, all coefficients m_{ij} must be nonnegative.

To see this, assume that some $m_{k\ell} < 0$ is negative. Choose $d = e^\ell$ (the ℓ-th unit vector). Clearly this d is nonnegative, and $(*)$ implies

$$x_k = m_{k1}.0 + \cdots + m_{kl}.1 + \cdots + m_{kn}.0 = m_{k\ell} < 0$$

so that x contains a negative element x_k.

■

We conclude that the quantity equation $x = Ax+d$ has a unique, economically meaningful (i.e. nonnegative) solution x for *every* final demand $d \geqq 0$ *if and only if* the matrix $M = (I-A)^{-1}$ is nonnegative. Clearly, in this case, the solution $p = p_0 a_0 (I-A)^{-1}$ of the price equation is also nonnegative, since $a_0 \geqq 0$.

Thus our analysis makes economic sense if and only if the matrix $I - A$ is nonnegatively invertible. When is this the case?

The fundamental theorem (Th. 5.2) below says that this is the case if and only if the technology is productive in the sense of the following definition.

Definition 1. *The matrix A (or the technology represented by it) is* productive *if it can produce a strictly positive net output vector d, i.e. there exists an activity vector $x \geqq 0$ such that $x - Ax = d > 0$.*

We also say that the technology (a_0, A) is productive if its input coefficient matrix A is productive. Productivity means that we can organize production so that the output x_i is bigger than the input $y_i = a_i x = [Ax]_i$, for all goods $i \in N$. Productivity is certainly a necessary condition for the (technical) viability of the system. The following Theorem characterizes productive technologies. It is basic for all that follows.

Theorem 5.2. (Productive Technologies). *Let $A \geqq 0$ be a square matrix. The matrix A is productive if and only if the matrix $I-A$ is invertible and the inverse $(I-A)^{-1}$ is nonnegative. This holds if and only if the infinite series $\sum_{t=0}^{\infty} A^t$ converges. Moreover, in this case, the sum is equal to $(I-A)^{-1}$:*

$$(I-A)^{-1} = \sum_{t=0}^{\infty} A^t = I + A + A^2 + A^3 + \cdots \tag{5.1}$$

Thus the technology A is productive if and only if $I - A$ is nonnegatively invertible. Note that productivity depends only on the matrix A, not on the labor input vector a_0.

Proof of Th. 5.2. The Theorem is an immediate consequence of two lemmas, which are stated and proved in the Appendix. First, the matrix $M = I - A$ satisfies the assumptions of Lemma 19.3. Therefore $I - A$ is nonnegatively invertible if and only if there is $x \geqq 0$ with $(I - A)x > 0$, i.e. A is productive. Second, by Lemma 19.1, $I - A$ is nonnegatively invertible if and only if the sum $I + A + A^2 + A^3 + \cdots$ converges; and in this case formula (5.1) holds.

■

Th. 5.2 implies the following:

Theorem 5.3. *Assume that the technology (a_0, A) is productive. Then*

(a) The quantity equation $x = Ax + d$ has a unique nonnegative solution x for every $d \geqq 0$. The solution is given by $x = (I - A)^{-1}d$

(b) The price equation $p = p_0 a_0 + pA$ has a unique nonnegative solution p for every $p_0 > 0$. The solution is given by $p = p_0 a_0 (I - A)^{-1}$

In fact, these unique nonnegative solutions exist if and *only if* the technology is productive. Moreover. to be fully satisfactory from the economic viewpoint, the required amount of labor should be positive for every $d \gneqq 0$, and the prices p_i of all commodities should also be positive (not just nonnegative). In the next section we show that this is the case if and only if labor is indispensable for production (see Def. 3 and Th. 6.3).

Note also that A is productive if and only if the transpose A^{T} is productive, because formula (5.1) holds for A if and only if it holds for A^{T}. This gives the following Corollary:

Corollary 5.4. *A nonnegative matrix A is productive if and only if there is $p \geqq 0$ such that $p > pA$.*

That is, we can find prices such that the value of the output is larger than the value of the inputs (excluding labor)), in every sector, $p_j > p_1 a_{1j} + p_2 a_{2j} + \ldots p_n a_{nj}$, for all $j \in N$. In this case we also say that the technology is *profitable.* The amount

$$p_j - [p_1 a_{1j} + p_2 a_{2j} + \ldots p_n a_{nj}]$$

is the *value added* in sector j (per unit of output). Profitability means that there exist prices for which the value added is positive in all sectors, so that a surplus is left for paying wages and profits. This is certainly a necessary condition for the (economic) viability of the system.

Proof of Cor. 5.4. As already noted, A is productive if and only if A^{T} is productive. By Def. 1 this means that there is a column vector $x \geqq 0$ such that $x > A^{\mathsf{T}}x$. Transposing this gives the equivalent formula $x^{\mathsf{T}} > x^{\mathsf{T}}A$. If we define the row vector $p = x^{\mathsf{T}}$ this gives $p > pA$, as asserted.

■

We have already seen some productive technologies in the previous sections, for example the following (Example 2.1 and 3.3)

$$a_0 = (a_{01}, a_{02}) = (1,\, 0.5), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.1 \\ 7 & 0 \end{pmatrix}$$

It is not always easy to see if a technology is productive or not. The technology below in Ex. 5.5 is similar to the one above, but not productive.

Example 5.5. *An unproductive technology.*

Let the technology (b_0, B) be given by

$$b_0 = (1,\, 0.5), \quad B = \begin{pmatrix} 0.1 & 0.2 \\ 5 & 0 \end{pmatrix}.$$

We consider again the desired net output $d = (10, 20)^{\mathsf{T}}$ and try to solve the corresponding Planning Problem, as in Example 2.1.
The Quantity Equation $x = d + Bx$ takes the form

$$\begin{aligned} x_1 &= 10 + 0.1x_1 + 0.2x_2 \\ x_2 &= 20 + 5x_1 + 0x_2 \end{aligned}$$

It is easy to check that this system of equations has the unique solution $x = \begin{pmatrix} -140 \\ -680 \end{pmatrix}$.
These negative activity levels are economically meaningless. The net output $d = (10, 20)^{\mathsf{T}}$ cannot be produced with the technology B. Another way to look at this is as follows. The Quantity Equation $d = x - Bx$ implies $-d = -x - B(-x)$, or $d' = x' - Bx'$, where $x' = -x = (140, 680)^{\mathsf{T}}$ is a positive activity vector, but the net output $d' = -d = (-10, -20)^{\mathsf{T}}$ is negative. Suppose we have inherited the output quantities x' from the previous period. To produce the same amounts again in the current period, the inputs $y' = Bx' = (150, 700)^{\mathsf{T}}$ would be required. But this is more (in both components) than the available amounts $x' = (140, 680)^{\mathsf{T}}$, so that the production x' cannot be sustained in a stationary state.
The Price Equation $p = b_0 + pB$ (with $p_0 = 1$) takes the form

$$\begin{aligned} p_1 &= \; = 1 + 0.1p_1 + 5p_2 \\ p_2 &= \; = 0.5 + 0.2p_1 + 0p_2 \end{aligned}$$

Its unique solution is $p = (p_1, p_2) = (-35, -6.5)$. These negative values are also economically meaningless. Thus the matrix B, although superficially not so different from the matrix A in Example 3.3, does not represent an economically viable technology. By Th. 5.3, B is not productive.

Remark. We may also observe that $(I-A)^{-1}$ is nonnegative, but $(I-B)^{-1}$ has negative elements:

$$(I-A)^{-1} = \begin{pmatrix} 10 & 1 \\ 70 & 8 \end{pmatrix} \quad \text{and } (I-B)^{-1} = \begin{pmatrix} -10 & -2 \\ -50 & -9 \end{pmatrix}$$

Therefore the matrix A is productive, but B is not, by Th. 5.2.

6 Dated Inputs

One of the most useful results in the fundamental Th. 5.2 is formula (5.1). In its light, the solutions of the quantity and price equations (3.7) and (3.8) can be understood in terms of "dated inputs" as follows.

Let $d \geqq 0$ be an arbitrary commodity bundle. To obtain output d in the next period, we need the amount of labor $\ell_0(d) = a_0 d$ and the vector Ad of inputs of the other goods (the "means of production") in the current period $t = 0$ (these are the *direct* inputs). The inputs Ad were produced in the previous period, using labor $\ell_1(d) = a_0 Ad$ and physical inputs $A.Ad = A^2 d$. The inputs $A^2 d$ were produced two periods ago, using labor $\ell_2(d) = a_0 A^2 d$ and physical inputs $A.A^2 d = A^3 d$. The inputs $A^3 d$ were produced three periods ago, using labor $\ell_3(d) = a_0 A^3 d$, and so on (these are the *indirect* inputs). The intertemporal structure of production is shown in the following Diagram (periods are counted backwards).

The Intertemporal Diagram.

$$\begin{array}{ccccccccc}
d & \longleftarrow & Ad & \longleftarrow & A^2 d & \longleftarrow & A^3 d & \longleftarrow & \cdots \\
 & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow & \\
 & \ell_0(d) & & \ell_1(d) & & \ell_2(d) & & \ell_3(d) & \\
 & & & & & & & & \\
 & t=0 & & t=1 & & t=2 & & t=3 & \cdots
\end{array} \qquad (6.1)$$

Definition 2. *Let $d \geqq 0$ be a commodity bundle. The vectors $Ad, A^2 d, A^3 d, \ldots$ are the* dated inputs *for d. The* dated labor inputs *are given by*

$$\ell_t(d) = a_0 A^t d \qquad t = 0, 1, 2, \ldots \qquad (6.2)$$

Definition 3. *The production of a commodity bundle d* requires labor *if at least one of the dated labor inputs $\ell_0(d), \ell_1(d), \ell_2(d), \ldots$ is positive. Labor is* indispensable *for production if every nonzero bundle $d \gneqq 0$ requires labor.*

Indispensability of labor means that the production of every good requires labor at some stage, now or in the past, but not necessarily at all stages.

Assume now that A is *productive*. Using formula (5.1), the solutions (3.7) and (3.8) of the quantity equation $x = Ax + d$ can be written as follows.

$$x = (I - A)^{-1} d = (\sum_{t=0}^{\infty} A^t) d = d + Ad + A^2 d + A^3 d + \ldots \qquad (6.3)$$

$$L(d) = a_0(I-A)^{-1}d = a_0(\sum_{t=0}^{\infty} A^t)d = \sum_{t=0}^{\infty} \ell_t(d) = \ell_0(d) + \ell_1(d) + \ell_2(d) + \dots \qquad (6.4)$$

Thus the gross output x needed to obtain the net output d is equal to the sum of d plus all current and past inputs $A^t d$. The total amount of labor $L(d)$ needed to obtain the net output d is equal to the sum $\sum_{t=0}^{\infty} \ell_t(d)$ of all current and past labor inputs.

We say that a commodity bundle $d \gneqq 0$ is *produced by labor alone* if $Ad = 0$. Clearly this implies $A^t d = 0$ for all $t \geqq 1$, i.e. the production of d requires no material inputs at all, neither directly nor indirectly. It also implies $\ell_t(d) = a_0 A^t d = 0$ for $t \geqq 1$, so that the only input needed for d is the direct labor $L(d) = \ell_0(d) = a_0 d$.

Note that Def. 2 and Def. 3 make sense whether or not A is productive. When the technology is not productive, the infinite sum $\ell_0(d)+\ell_1(d)+\ell_2(d)+\dots$ does not converge, but the dated labors $\ell_t(d)$ can still be computed.

Lemma 6.1. *Assume that A is productive. Then labor is indispensable if and only if $a_0(I-A)^{-1} > 0$.*

Proof of Lemma 6.1. By Def. 3 and equation (6.4), labor is indispensable if and only if $a_0(I-A)^{-1}d > 0$ for every $d \gneqq 0$. This is the case if and only if $a_0(I-A)^{-1} > 0$.

■

Lemma 6.2. *Assume that A is productive and let (p_0, p) be a solution of the price equation. Then the value pd of a commodity bundle d is equal to the sum of all wage costs:*

$$pd = \sum_{t=0}^{\infty} p_0 \ell_t(d)$$

Proof of Lemma 6.2. Clear because

$$pd = p_0 a_0 (I-A)^{-1} d = p_0 a_0 \sum_{t=0}^{\infty} A^t d = \sum_{t=0}^{\infty} p_0 a_0 A^t d = \sum_{t=0}^{\infty} p_0 \ell_t(d)$$

■

This allows us to understand the price of a commodity in a very intuitive way.

Th. 5.3 and the above formulas imply immediately:

Theorem 6.3. *Assume that the technology (a_0, A) is productive. Let $L(d)$ be the required amount of labor for a commodity bundle d and let $p = (p_1, \dots p_n)$ be a solution of the price equation $p = p_0 a_0 + pA$ with $p_0 > 0$. Then*

(a) The required amount of labor $L(d)$ is positive for every nonzero final demand $d \gneqq 0$ if and only if labor is indispensable for production.

(b) All prices p_i $(i = 1, \dots n)$ are positive if and only if labor is indispensable for production.

Proof of Th. 6.3. We have $L(d) = a_0(I - A)^{-1}d$ and this is positive for all $d \gneqq 0$ if and only if $a_0(I - A)^{-1}$ is positive.
Moreover, all prices p_i are positive if and only if $pd > 0$ for every $d \gneqq 0$. Since $pd = p_0 a_0 (I - A)^{-1} d$ this is also the case if and only if $a_0(I - A)^{-1}$ is positive.

■

This brings our theory of the Linear Model of Production to a preliminary conclusion: a stationary state quantity system and a zero-profit price system make perfect economic sense in this model if and only if the technology is productive and labor is indispensable for production. Both are very reasonable requirements, and the resulting theory is quite satisfactory. Its mathematical base is Th. 5.2.

Approximation of x and $L(d)$ by sums of dated inputs.
When A is productive, the infinite series $\sum_{t=0}^{\infty} A^t$ is equal to $(I - A)^{-1}$, by (5.1). This means that the partial sums $S_T = \sum_{t=0}^{T} A^t$ converge to $(I - A)^{-1}$:

$$S_T = \sum_{t=0}^{T} A^t \to (I - A)^{-1} \qquad \text{for} \quad T \to \infty$$

Therefore the partial sums of the dated input vectors converge to $x = (I - A)^{-1}d$:

$$\sum_{t=0}^{T} A^t d \to (I - A)^{-1} d = x \qquad \text{for} \quad T \to \infty$$

and the partial sums of the dated labor inputs $\sum_{t=0}^{T} \ell_t(d) = \sum_{t=0}^{T} a_0 A^t d$ converge to $L(d) = a_0(I - A)^{-1}d$:

$$\sum_{t=0}^{T} \ell_t(d) = \sum_{t=0}^{T} a_0 A^t d \to a_0 (I - A)^{-1} d = a_0 x = L(d) \qquad \text{for} \quad T \to \infty$$

Therefore we can approximate x and $L(d)$ by computing these partial sums. If A is a large matrix, direct computation of the Leontief inverse $(I-A)^{-1}$ may be difficult, but computation of the succesive powers of A, i.e. of $I, A`A^2, A^3, A^4, ...,$ and their sums $\sum_{t=0}^{T} A^t$ is relatively easy.

Example 6.4. *(Same technology as Example 4.1)*

There are two produced goods, good 1 is wheat and good 2 is horses. Labor is good 0. The technology (a_0, A) is given by

$$a_0 = (a_{01}, a_{02}) = (6,\, 6), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

We have already seen in Ex. 4.1 that

$$I - A = \frac{1}{10}\begin{pmatrix} 9 & -4 \\ -3 & 8 \end{pmatrix} \quad \text{and} \quad (I-A)^{-1} = \frac{1}{6}\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix}$$

The price equation $p = p_0 a_0 + pA$ has the solution $p = p_0 a_0 (I-A)^{-1}$, i.e. with $p_0 = 1$:

$$p = (6.6).\frac{1}{6}\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix} = (1,1).\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix} = (11, 13) \tag{6.5}$$

Let the final demand be

$$d = \begin{pmatrix} 120 \\ 30 \end{pmatrix}$$

The gross output vector x required to produce the net output d is given by

$$x = (I-A)^{-1}d = \frac{1}{6}\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix}.\begin{pmatrix} 120 \\ 30 \end{pmatrix} = \begin{pmatrix} 180 \\ 105 \end{pmatrix} \tag{6.6}$$

The required labor for this gross output is

$$L(d) = a_0 x = (6,6).\begin{pmatrix} 180 \\ 105 \end{pmatrix} = 6 \times 180 + 6 \times 105 = 1710$$

The following pages show how these values can be approximated by dated inputs.
The tables contain, for $t = 0, 1, 2, \ldots, 10$:
the powers A^t of A: $I, A, A^2, \ldots, A^{10}$

the cumulative sums $\sum_{i=0}^{t} A^i$ (converging to $(I - A)^{-1}$)
the dated inputs $d, Ad, A^2d, \ldots, A^{10}d$
the cumulative sums of these, $\sum_{i=0}^{t} A^i d$ (converging to x)
the dated labor inputs $\ell_t(d) = a_0 A^t d$
the cumulative sums of these, $\sum_{i=0}^{t} \ell_i(d) = \sum_{i=0}^{t} a_0 A^i d$ (converging to $L(d)$)

Datierte Groessen fuer die Perioden t = 0,1,.. 10:

(d.i. Approximation bis auf Sup-Norm(A^T) < 0.001)

Potenzen der Technologiematrix A:

//TableForm=

I =	$\begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix}$	I =	$\begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix}$
A =	$\begin{pmatrix} 0.100 & 0.400 \\ 0.300 & 0.200 \end{pmatrix}$	$I+A$ =	$\begin{pmatrix} 1.100 & 0.400 \\ 0.300 & 1.200 \end{pmatrix}$
A^2 =	$\begin{pmatrix} 0.130 & 0.120 \\ 0.090 & 0.160 \end{pmatrix}$	$I+A+A^2$ =	$\begin{pmatrix} 1.230 & 0.520 \\ 0.390 & 1.360 \end{pmatrix}$
A^3 =	$\begin{pmatrix} 0.049 & 0.076 \\ 0.057 & 0.068 \end{pmatrix}$	$\sum_{i=0}^{3} A^i$ =	$\begin{pmatrix} 1.279 & 0.596 \\ 0.447 & 1.428 \end{pmatrix}$
A^4 =	$\begin{pmatrix} 0.028 & 0.035 \\ 0.026 & 0.036 \end{pmatrix}$	$\sum_{i=0}^{4} A^i$ =	$\begin{pmatrix} 1.307 & 0.631 \\ 0.473 & 1.464 \end{pmatrix}$
A^5 =	$\begin{pmatrix} 0.013 & 0.018 \\ 0.014 & 0.018 \end{pmatrix}$	$\sum_{i=0}^{5} A^i$ =	$\begin{pmatrix} 1.320 & 0.649 \\ 0.487 & 1.482 \end{pmatrix}$
A^6 =	$\begin{pmatrix} 0.007 & 0.009 \\ 0.007 & 0.009 \end{pmatrix}$	$\sum_{i=0}^{6} A^i$ =	$\begin{pmatrix} 1.327 & 0.658 \\ 0.493 & 1.491 \end{pmatrix}$
A^7 =	$\begin{pmatrix} 0.003 & 0.004 \\ 0.003 & 0.004 \end{pmatrix}$	$\sum_{i=0}^{7} A^i$ =	$\begin{pmatrix} 1.330 & 0.662 \\ 0.497 & 1.496 \end{pmatrix}$
A^8 =	$\begin{pmatrix} 0.002 & 0.002 \\ 0.002 & 0.002 \end{pmatrix}$	$\sum_{i=0}^{8} A^i$ =	$\begin{pmatrix} 1.332 & 0.664 \\ 0.498 & 1.498 \end{pmatrix}$
A^9 =	$\begin{pmatrix} 0.001 & 0.001 \\ 0.001 & 0.001 \end{pmatrix}$	$\sum_{i=0}^{9} A^i$ =	$\begin{pmatrix} 1.332 & 0.666 \\ 0.499 & 1.499 \end{pmatrix}$
A^{10} =	$\begin{pmatrix} 0.000 & 0.001 \\ 0.000 & 0.001 \end{pmatrix}$	$\sum_{i=0}^{10} A^i$ =	$\begin{pmatrix} 1.333 & 0.666 \\ 0.500 & 1.499 \end{pmatrix}$
Nullmatrix:	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$(I - A)^{-1}$ =	$\begin{pmatrix} 1.333 & 0.667 \\ 0.500 & 1.500 \end{pmatrix}$

Figure 2: Powers A^t and their sums.

Datierte Inputvektoren:

d =	$\begin{pmatrix}120.000\\30.000\end{pmatrix}$	d =	$\begin{pmatrix}120.000\\30.000\end{pmatrix}$
A.d =	$\begin{pmatrix}24.000\\42.000\end{pmatrix}$	(I+A).d =	$\begin{pmatrix}144.000\\72.000\end{pmatrix}$
A^2.d =	$\begin{pmatrix}19.200\\15.600\end{pmatrix}$	$\sum_{i=0}^{2} A^i$.d =	$\begin{pmatrix}163.200\\87.600\end{pmatrix}$
A^3.d =	$\begin{pmatrix}8.160\\8.880\end{pmatrix}$	$\sum_{i=0}^{3} A^i$.d =	$\begin{pmatrix}171.360\\96.480\end{pmatrix}$
A^4.d =	$\begin{pmatrix}4.368\\4.224\end{pmatrix}$	$\sum_{i=0}^{4} A^i$.d =	$\begin{pmatrix}175.730\\100.700\end{pmatrix}$
A^5.d =	$\begin{pmatrix}2.126\\2.155\end{pmatrix}$	$\sum_{i=0}^{5} A^i$.d =	$\begin{pmatrix}177.850\\102.860\end{pmatrix}$
A^6.d =	$\begin{pmatrix}1.075\\1.069\end{pmatrix}$	$\sum_{i=0}^{6} A^i$.d =	$\begin{pmatrix}178.930\\103.930\end{pmatrix}$
A^7.d =	$\begin{pmatrix}0.535\\0.536\end{pmatrix}$	$\sum_{i=0}^{7} A^i$.d =	$\begin{pmatrix}179.460\\104.460\end{pmatrix}$
A^8.d =	$\begin{pmatrix}0.268\\0.268\end{pmatrix}$	$\sum_{i=0}^{8} A^i$.d =	$\begin{pmatrix}179.730\\104.730\end{pmatrix}$
A^9.d =	$\begin{pmatrix}0.134\\0.134\end{pmatrix}$	$\sum_{i=0}^{9} A^i$.d =	$\begin{pmatrix}179.870\\104.870\end{pmatrix}$
A^{10}.d =	$\begin{pmatrix}0.067\\0.067\end{pmatrix}$	$\sum_{i=0}^{10} A^i$.d =	$\begin{pmatrix}179.930\\104.930\end{pmatrix}$
	$\begin{pmatrix}0\\0\end{pmatrix}$	Brutto-Output x =	$\begin{pmatrix}180.\\105.\end{pmatrix}$

Figure 3: Dated inputs $A^t d$ and their sums.

Datierte Arbeitsinputs: $a_0 = (6, 6)$

$a_0.d =$	900.000	$a_0.d =$	900.000
$a_0.A.d =$	396.000	$a_0.(I+A).d =$	1296.000
$a_0.A^2.d =$	208.800	$a_0.(\sum_{i=0}^{2} A^i).d =$	1504.800
$a_0.A^3.d =$	102.240	$a_0.(\sum_{i=0}^{3} A^i).d =$	1607.000
$a_0.A^4.d =$	51.552	$a_0.(\sum_{i=0}^{4} A^i).d =$	1658.600
$a_0.A^5.d =$	25.690	$a_0.(\sum_{i=0}^{5} A^i).d =$	1684.300
$a_0.A^6.d =$	12.862	$a_0.(\sum_{i=0}^{6} A^i).d =$	1697.100
$a_0.A^7.d =$	6.428	$a_0.(\sum_{i=0}^{7} A^i).d =$	1703.600
$a_0.A^8.d =$	3.214	$a_0.(\sum_{i=0}^{8} A^i).d =$	1706.800
$a_0.A^9.d =$	1.607	$a_0.(\sum_{i=0}^{9} A^i).d =$	1708.400
$a_0.A^{10}.d =$	0.804	$a_0.(\sum_{i=0}^{10} A^i).d =$	1709.200

Total Labor = 1710.

Figure 4: Dated labor inputs $\ell_t(d) = a_0 A^t d$ and their sums .

7 Labor Values

Definition in Terms of Embodied Labor.

This approach is based on the classical idea that all the labor that has gone into the production of a commodity is somehow contained (or *embodied*) in it. If a good is produced from labor alone, the amount of direct labor needed is embodied in the product and is its labor value. If the production of a good requires also other inputs, the labor embodied in the other inputs is passed on to the output, and also becomes embodied in the final product. The *embodied labor value* V^e of a commodity is then defined as the *total amount of labor embodied in it.* This total amount is equal to the sum of the direct labor needed to produce the commodity, plus the indirect labor embodied in the various inputs.

To formalize these ideas, denote by $v_i \geqq 0$ the embodied labor value of one unit of good i, and write $v = (v_1, \dots v_n)$ for the (row) vector of these values. Then a_{0j} is the direct labor, and $v_i a_{ij}$ is the labor embodied in a_{ij} units of input i ($i = 1, \dots n$). The *embodied labor value* v_j of (one unit of) good j is the sum of the direct labor a_{0j} and the amounts of labor embodied in the other inputs (indirect labor):

$$v_j = a_{0j} + v_1 a_{1j} + v_2 a_{2j} + \cdots v_n a_{nj} \qquad \text{for} \quad j = 1, 2, \dots n \tag{7.1}$$

This is a system of n linear equations in the n unknowns $v_1, \dots, v_n$. In matrix notation, it takes the form

$$v = a_0 + vA \tag{7.2}$$

This is the *Value Equation.* The embodied labor values v_i of the n commodities are given implicitly by the solution of this equation. Equation (7.2) is the same as the zero-profit price equation $p = p_0 a_0 + pA$ with $p_0 = 1$. Therefore the solution is also the same,

$$v = a_0(I - A)^{-1} \tag{7.3}$$

Seen in this light, the embodied labor values are merely a re-interpretation of the zero-profit price equation, with labor as the numéraire.

The *embodied labor value* $V^e(d)$ of a commodity bundle $d \geqq 0$ is then the sum of the labors embodied in its components d_i, i.e.

$$V^e(d) = \sum_{i=1}^{n} v_i d_i = vd = a_0(I - A)^{-1} d \tag{7.4}$$

This is the definition of Labor Values in terms of *embodied labor.*

Synchronous Definition.

The interpretation of the quantity equation as describing a stationary state leads to the following definition of labor values:

The *synchronous Labor Value* $V^s(d)$ of a commodity bundle d is the amount of labor *required per period to enjoy net output d per period, in a stationary state.*

As explained in Sec. 3, it is equal to the required amount of labor $L(d)$ (see 3.8), i.e.

$$V^s(d) = L(d) = a_0(I - A)^{-1}d \tag{7.5}$$

This is known as the "synchronous" definition of labor values, because it refers to an amount of labor that is supplied in the same period (synchronously) as the consumption d.

By (7.4) and (7.5) the embodied value $V^e(d)$ of a commodity bundle d is equal to its synchronous labor value $V^s(d)$:

$$V^e(d) = vd = a_0(I - A)^{-1}d = L(d) = V^s(d) \tag{7.6}$$

In particular, the amount of labor required to produce one unit of good i is given by $V^s(e^i) = L(e^i) = ve^i = v_i$, because "one unit of good i" is represented by the commodity bundle $d = e^i$ (the i-th unit vector).

Historical Definition of Labor Values.

The concept of dated inputs leads to the third definition of labor values:

The *historical labor value* $V^h(d)$ of a commodity bundle $d \geqq 0$ is defined as the *total amount of labor that was expended in the past, directly and indirectly, to produce d.*

It is the sum of all dated labor inputs:

$$V^h(d) = \sum_{t=0}^{\infty} \ell_t(d) \tag{7.7}$$

This definition in terms of dated labor inputs is known as the *historical* definition of labor values because it refers to labor inputs in past periods.

If A is productive, we have by (6.4)

$$V^h(d) = \sum_{t=0}^{\infty} \ell_t(d) = \sum_{t=0}^{\infty} a_0 A^t d = a_0[\sum_{t=0}^{\infty} A^t]d = a_0(I - A)^{-1}d \tag{7.8}$$

That is, the infinite sum of dated labor inputs $\sum_{t=0}^{\infty} \ell_t(d)$ converges and is equal to $a_0(I - A)^{-1}d$, so that $V^h(d)$ is well-defined and nonnegative for every $d \geqq 0$. Clearly, the labor value $V^h(d)$ of a commodity bundle $d \gneqq 0$ is positive if and only if the production of d requires labor. Clearly

$$V^h(d) = a_0(I - A)^{-1}d = V^s(d) = V^e(d) \tag{7.9}$$

i.e. The historical definition gives the same labor values as the other definitions.

Prices and Labor Values.

The special price system $(p_0, p) = (1, v) = (1, v_1, \dots v_n)$ is a solution of the price equation in which the prices of all commodities are numerically equal to their labor values (and labor serves as numéraire). More generally, any solution (p_0, p) of the price equation (2.3) satisfies

$$p = p_0 a_0 (I - A)^{-1} = p_0 v \tag{7.10}$$

i.e., if the interest rate is zero, then the commodity prices $p = p_0 v$ are *proportional* to labor values v (with factor of proportionality $p_0 > 0$). The “Labor Theory of Value” holds. The price equation (2.2) and the value equation (7.2) are really the same equation, and we will use the two terms interchangeably.

Lecture 4

8 Expansion

To study economic growth and positive profit rates, we need some more information on the technology matrix A. Given a technology matrix $A = (a_{ij})$, consider the new technology $B = (b_{ij}) = \alpha A$, for $\alpha > 0$. It is obtained from A by multiplying all coeffients of A by the number α, so that $b_{ij} = \alpha a_{ij}$ for all i, j. When is $B = \alpha A$ productive? Intuitively, one should expect that αA is productive for α sufficiently small, but that αA ceases to be productive if α becomes too large. This intuition is correct, as the following result shows.

Theorem 8.1. . *Let $A \geqq 0$ be a square matrix. Then there is a positive number $\alpha^* = \hat{\alpha}(A)$ (possibly $\alpha^* = \infty$) such that the technology αA is productive if and only if $0 \leqq \alpha < \alpha^*$.*

If the technology αA is productive, then there is an activity vector $x > 0$ such that $x > \alpha Ax$. This condition means that the output x_i is more than α-times the input $y_i = (Ax)_i$ for all goods $i \in N$, i.e. all sectors of the economy expand with a factor greater than α. This is possible if and only if $\alpha < \alpha^*$. We call $\alpha^* = \hat{\alpha}(A)$ the (economic) *expansion factor* for the technology A. It provides an upper bound for the growth rates which the economy can sustain simultaneously in all sectors (see Sec. 13). Clearly, $A \geqq 0$ is productive if and only if $\hat{\alpha}(A) > 1$. The expansion factor is also an upper bound for the profit rates the economy can sustain (in all sectors simultaneously), see Sec. 9.

Proof of Theorem 8.1.

Define the set $M = M(A) = \{\alpha \geqq 0 \,|\, \alpha A \text{ is productive}\}$ and put $\alpha^* = \sup M$. By Definition 1

$$\alpha \in M \quad \Leftrightarrow \quad \exists x \gneqq 0 \text{ with } x > \alpha Ax \tag{$*$}$$

Clearly $0 \in M$. Further, by $(*)$, if $\alpha \in M$ then $\alpha + \epsilon \in M$ for $\epsilon > 0$ sufficiently small. Therefore $\alpha^* > 0$ and $\alpha^* \notin M$. Clearly, if $\alpha \in M$ and $\beta < \alpha$, then $\beta \in M$. Therefore M is a half-open interval of the form $M = [0, \alpha^*)$ and the Theorem is proved.

■

Corollary 8.2. *Let $A \geqq 0$ be a square matrix with expansion factor α^*. Then*

(a) If $x \leqq \beta Ax$, $x \gneqq 0$, then $\beta \geqq \alpha^*$

(b) If $x = \beta Ax$, $x > 0$, then $\beta = \alpha^*$

(c) If $x = \beta Ax$, $x \neq 0$, then $|\beta| \geqq \alpha^*$ and $\alpha^* < \infty$

Proof of Cor. 8.2.

Proof of (a). The assumption means that $(I - \beta A)x \leqq 0$, $x \gneqq 0$, and $\beta > 0$. If β were less than α^*, the matrix $I - \beta A$ would be productive, and $(I - \beta A)^{-1} \geqq 0$. This implies $x = (I - \beta A)^{-1}(I - \beta A)x \leqq 0$, contradicting $x \gneqq 0$. Therefore $\beta \geqq \alpha^*$.

Proof of (b). By (a) we have $\beta \geqq \alpha^*$. If β were greater than α^*, we would have $x = \beta Ax > \alpha^* Ax$ and $\alpha^* A$ would be productive, a contradiction. Therefore $\beta = \alpha^*$.

Proof of (c). Write $x^+ = (|x_1|, \ldots, |x_n|)$. Then $x = \beta Ax$ implies $x^+ \leqq |\beta| Ax^+$, where $x^+ \gneqq 0$ and $|\beta| > 0$. By (a) this implies $|\beta| \geqq \alpha^*$.

■

8.1 The Frobenius-Perron Theorem

Let us recall some important concepts from Linear Algebra.

Definition 4. *Let A be a square $(n \times n)$-matrix (not necessarily nonnegative). If the number λ and the (column) vector x satisfy*

$$Ax = \lambda x, \quad x \neq 0 \tag{8.1}$$

we say that λ is an eigenvalue *of A and x is a (right)* eigenvector *of A, associated with λ (a left eigenvector is a row vector $p \neq 0$ with $pA = \lambda p$).*

Equation (8.1) can be written $(\lambda I - A)x = 0$. This has a nonzero solution $x \neq 0$ if and only if the matrix $(\lambda I - A)$ is not invertible, i.e. if and only if λ satisfies the *characteristic equation*

$$\det(\lambda I - A) = 0 \tag{8.2}$$

This is an algebraic equation of degree n in λ. It has n roots $\lambda_1, \ldots \lambda_n$ (not necessarily distinct, possibly complex). These are the eigenvalues of the matrix A.

The so-called PERRON-FROBENIUS Theorem states that a *nonnegative* matrix A has a special *nonnegative* eigenvalue λ^*, known as the *dominant* eigenvalue (or the FROBENIUS eigenvalue). This λ^* is the inverse of the expansion factor, $\lambda^* = 1/\alpha^*$. It is the greatest eigenvalue of A in absolute value (hence "dominant"), and has associated nonnegative left and right eigenvectors.

Theorem 8.3. (Perron-Frobenius I) *Let $A \geqq 0$ be a square matrix with expansion factor $\alpha^* = \hat{\alpha}(A)$, and define $\lambda^* = \lambda(A) = 1/\alpha^*$ (where $1/\infty = 0$). Then*

(a) λ^* is a nonnegative eigenvalue of A.

(b) There exist nonnegative right and left eigenvectors of A, associated with λ^*.

(c) λ^* is the largest eigenvalue in absolute value: $\lambda^* \geqq |\mu|$ for every eigenvalue μ of A.

Proof of Th.8.3. See Th. 19.4 in the Appendix. ■

Corollary 8.4. *Assume $A \gneqq 0$. If A has a positive eigenvector, then it is associated with λ^*, and $\lambda^* > 0$.*

Proof of Cor. 8.4.
Assume $Ax = \mu x$, $x > 0$. Since $A \gneqq 0$, this implies $\mu > 0$, so that $x = \beta Ax$ where $\beta = 1/\mu$. By Cor. 8.2 this implies $\beta = \alpha^* = 1/\lambda^*$. Hence $\mu = \lambda^*$.

■

The number λ^* is called the *dominant* eigenvalue of A. Since $\lambda^* = 1/\alpha^*$, a technology A is productive if and only if $\lambda(A) < 1$.
Clearly, when $\lambda^* > 0$ (equivalently $\alpha^* < \infty$), then a column vector x [resp. a row vector p] is an eigenvector associated with the dominant eigenvalue if and only if $x = \alpha^* Ax$ [resp. $p = \alpha^* pA$].

Computation of the expansion factor.
Let $A \geqq 0$ be a nonnegative $(n \times n)$-matrix. Write down the characteristic equation $\det(\lambda I - A) = 0$. This is an algebraic equation of degree n in λ. The dominant eigenvalue λ^* of A is the largest nonnegative root of this equation (by Th. 8.3 it has at least one nonnegative root). The expansion factor is given by $\alpha^* = 1/\lambda^*$, where $\alpha^* = \infty$ if $\lambda^* = 0$. Alternatively, one can consider the equation $\det(I - \alpha A) = 0$. The expansion factor α^* is the smallest positive root of this equation. If it has no positive root,[1] then $\alpha^* = \infty$.

[1] One can show that in this case $\det(I - \alpha A) = 1$ is constant, independently of α, so that the equation $\det(I - \alpha A) = 0$ has no solution at all (in particular no positive solution).

Example 8.5.

Consider the technology (a_0, A) introduced in Example 4.1:

$$a_0 = (a_{01}, a_{02}) = (6,\, 6), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

The determinant of the matrix $\lambda I - A$ is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - \frac{1}{10} & -\frac{4}{10} \\ -\frac{3}{10} & \lambda - \frac{2}{10} \end{vmatrix} = (\lambda - \frac{1}{10})\,(\lambda - \frac{2}{10}) - \frac{4}{10}.\frac{3}{10} = \lambda^2 - \frac{3}{10}\lambda + \frac{2}{100} - \frac{12}{100}$$

This gives the characteristic equation

$$\lambda^2 - \frac{3}{10}\lambda - \frac{1}{10} = 0$$

with solution

$$\lambda_{12} = \frac{3}{20} \pm \sqrt{\frac{9}{400} + \frac{40}{400}} = \frac{3}{20} \pm \frac{7}{20}$$

The two roots are $\lambda_1 = \frac{10}{20} = \frac{1}{2}$, and $\lambda_2 = -\frac{4}{20} = -\frac{1}{5}$. These are the eigenvalues of A. The dominant eigenvalue is $\lambda(A) = \lambda_1 = 1/2$, which is nonnegative and the largest eigenvalue in absolute value. Therefore $\alpha^* = \hat{\alpha}(A) = 1/\lambda(A) = 2$.

By Theorem 8.1, the technology αA is productive if and only if $0 \leqq \alpha < 2$. By Th. 5.2 the technology αA is productive if and only if $(I - \alpha A)^{-1}$ exists and is nonnegative. To check this, let us compute $(I - \alpha A)^{-1}$. We have

$$I - \alpha A = \begin{pmatrix} 1 - .1\alpha & -.4\alpha \\ -.3\alpha & 1 - .2\alpha \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 10 - \alpha & -4\alpha \\ -3\alpha & 10 - 2\alpha \end{pmatrix}$$

The inverse is

$$(I - \alpha A)^{-1} = \frac{1}{10 - 3\alpha - \alpha^2} \begin{pmatrix} 10 - 2\alpha & 4\alpha \\ 3\alpha & 10 - \alpha \end{pmatrix}$$

For $0 \leqq \alpha < 2$ the term $10 - 3\alpha - \alpha^2$ is positive and $(I - \alpha A)^{-1}$ exists and is nonnegative. For $\alpha = 2$ the term $10 - 3\alpha - \alpha^2$ is zero and $(I - \alpha A)^{-1}$ does not exist. For $\alpha > 2$ the term $10 - 3\alpha - \alpha^2$ negative, so that the inverse $(I - \alpha A)^{-1}$ exists, but has negative elements. For $\alpha < 0$ the matrix αA is not a technology matrix. Thus αA is a productive technology if and only if $0 \leqq \alpha < \alpha^* = 2$, as it must be.

Example 8.6. *A technology with* $\alpha^* = \infty$.

Let the technology (a_0, A) be given by

$$a_0 = (a_{01}, a_{02}) = (1,\, 1), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Good 1 is produced from labor alone. Process 2 needs labor and good 1 as inputs. The determinant of the matrix $\lambda I - A$ is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = \lambda^2$$

This gives the characteristic equation $\lambda^2 = 0$ with unique solution $\lambda = 0$. The dominant eigenvalue is $\lambda^* = 0$, which is nonnegative and the largest eigenvalue in absolute value. Therefore $\alpha^* = \infty$. The matrix

$$I - \alpha A = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \qquad \text{has the inverse} \quad (I - \alpha A)^{-1} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

which is well-defined and nonnegative for every α with $0 \leqq \alpha < \infty$. Note also that $\det(I - \alpha A) = 1$ for all α, so that the equation $\det(I - \alpha A) = 0$ has no solution.

9 Prices of Production

We have already determined the prices in our Linear Model of Production for the case that the rate of profit is zero. Let us now consider the general case. The key assumption is that *the rate of profit is the same in all sectors.* This assumption can be justified as follows. Since capitalists want to maximize the return on their capital, they will invest their money only in those sectors where the rate of profit is highest. The rate of profit must therefore be the same in all sectors.

We denote this uniform profit rate (or interest rate) by r and write $\rho = 1 + r$ for the *interest factor* or *profit factor.* We want to find a price system such that the profit factor is uniform and equal to a given number ρ in all sectors.

That the rate of profit is r means that the price of one unit of good j is equal to $(1+r)$–times the costs of its inputs, i.e. we obtain the *price equations* (with $\rho = 1 + r$):

$$p_j = \rho[p_0 a_{0j} + p_1 a_{1j} + \cdots + p_n a_{nj}] \qquad \text{for } j = 1, \ldots, n \tag{9.1}$$

or, in matrix notation:

$$p = \rho[p_0 a_0 + pA] \tag{9.2}$$

Equation (9.2) is the (general) *Price Equation.* Clearly, if (p_0, p) satisfies equation (9.2), then so does every multiple (kp_0, kp), for $k > 0$. The price equation (9.2) can determine the prices only up to a constant factor (up to the choice of numéraire).

To understand this price equation better, let us define the fictitious technology $(b_0, B) = (\rho a_0, \rho A)$ in which all input coefficients of the original technology are multiplied by the factor ρ. Then (9.2) becomes

$$p = p_0 b_0 + pB \tag{9.3}$$

This equation is nothing other than the price equation (2.2) for the fictitious technology $(b_0, B) = (\rho a_0, \rho A)$. That is, prices for a nonzero interest rate r can be understood as zero-profit prices for a fictitious technology in which all input coefficients have been multiplied by the factor $\rho = 1 + r$. For example, if $r = 0.05 = 5\%$, all inputs are larger by five percent in the fictitious technology than in the original one.

We know from Th. 5.3 and Th. 6.3 that the equation (9.3) has a unique nonnegative solution p for every $p_0 > 0$ if B is productive and that all prices are positive ($p > 0$) if and only if labor is indispensable for the technology (b_0, B). When are these conditions satisfied? First, we have

Lemma 9.1. *Let $\rho > 0$ be a positive number. Then labor is indispensable for the fictitious technology $(b_0, B) = (\rho a_0, \rho A)$ if and only if it is indispensable for the original technology (a_0, A).*

Proof of Lemma 9.1. Denote by $\ell_t^A(d)$ [resp. $\ell_t^B(d)$] the dated labor inputs for the technology (a_0, A) [resp. (b_0, B)]. Then, by (6.2)

$$\ell_t^B(d) = b_0 B^t d = \rho a_0 (\rho A)^t d = \rho^{t+1} a_0 A^t d = \rho^{t+1} \ell_t^A(d)$$

Since $\rho^{t+1} > 0$ for every $t = 0, 1, 2, \ldots$, we have $\ell_t^B(d) > 0$ if and only if $\ell_t^A(d) > 0$. By Def. 3 this proves the asssertion.

■

Second, we know from Theorem 8.1 that $B = \rho A$ is productive if and only if $0 < \rho < \hat{\alpha}(A)$, where $\hat{\alpha}(A)$ is the expansion factor defined in Sec. 8.

We can now apply Ths. 5.3 and 6.3 to the fictitious technology $(b_0, B) = (\rho a_0, \rho A)$ and obtain:

Theorem 9.2. *(*Production Prices*) Let (a_0, A) be a technology with expansion factor $\hat{\alpha}(A) > 0$. Then the price equation*

$$p = \rho(p_0 a_0 + pA) \tag{9.4}$$

has a unique, nonnegative solution $p \geqq 0$ for every $p_0 > 0$ if ρ satisfies $0 < \rho < \hat{\alpha}(A)$. In this case, the solution is given by

$$p = \rho p_0 a_0 [I - \rho A]^{-1} \tag{9.5}$$

where $p_0 > 0$ can be chosen arbitrarily. All prices are positive ($p > 0$) if and only if labor is indispensable for the technology (a_0, A).

Proof of Theorem 9.2. W.l.o.g. put $p_0 = 1$. The price equation (9.4) can be written as $p = p_0 b_0 + pB$, where $(b_0, B) = (\rho a_0, \rho A)$. By Theorem 8.1 $B = \rho A$ is productive if and only if $0 \leqq \rho < \hat{\alpha}(A)$. By Lemma 9.1, labor is indispensable for (b_0, B) if and only if it is so for (a_0, A). The assertion now follows immediately from Ths. 5.3 an 6.3.

■

The prices given by Th. 9.2 are known as the *prices of production* implied by (or associated with) the profit factor ρ (or the profit rate r).

The prices $(p_0, p_1, \ldots p_n)$, are determined only up to a positive factor, but of course relative prices p_i/p_j are uniquely determined by ρ, independently of the choice of the numéraire.

This "classical" price theory is objective in the sense that it does not refer to subjective data like utility or preferences, in contrast to the "neoclassical" theory. The prices (p_0, p) are independent of demand. They depend only on the conditions of production, i.e. the technology (a_0, A), and on the profit factor ρ (resp. the profit rate r).[2]

[2]The actual level of the rate of profit (or interest) is taken as exogenous here. One can think of it as reflecting the relative strength of the the social classes (capitalists and workers). Of course, from a General Equilibrium point of view it would also be endogenously determined and depend on subjective factors like time preferences.

We say that the economy can *sustain the profit rate* r (or the profit factor $\rho = 1 + r$) if we can find positive prices $(p_0, p) = (p_0, p_1, \dots p_n)$ such that, at these prices, the profit rate is uniform and equal to r in all sectors. The theorem shows that, implicit in the technology A, there is an upper bound for the admissible profit factors: ρ must be less than the expansion factor $\hat{\alpha}(A)$.

The lower bound for the profit factor is zero, but note that for $0 < \rho < 1$ the interest rate $r = \rho - 1$ is negative and production is not profitable. If $\hat{\alpha}(A) \leqq 1$, the technology A is not productive and the interest rate is necessarily negative. This case is included in our formal analysis, but of course the economically relevant case is $\hat{\alpha}(A) > 1$ and $\rho \geqq 1$ (i.e. $r \geqq 0$).

Corollary 9.3. *Let (a_0, A) be a technology for which labor is indispensable. Then*

(a) The price equations have solutions for nonnegative interest rates if and only if the technology A is productive.

(b) If the interest rate is zero, prices are proportional to labor values: $p = p_0 v$

Proof of Cor. 9.3.

A nonnegative interest rate r means that $\rho = 1 + r \geqq 1$. This is possible if and only if $\hat{\alpha}(A) > 1$, i.e. A is productive. This proves (a). If $\rho = 1$ the prices are $p = p_0 a_0 [I - A]^{-1} = p_0 v$, by (9.5) and (7.3). This is (b). ■

Remark. The assumption of a uniform rate of profit is a *long-run equilibrium* condition in the following sense. If the rate of profit is higher in some sectors than in others, capital will flow from the less profitable sectors to the more profitable ones, and the output of the more profitable sectors will increase relative to the output of the less profitable ones. This change in supply will lead to a fall in the prices of the more profitable sectors relative to the prices of the less profitable ones, until the rate of profit is the same in all sectors, prices do no longer change, and capitalists have no incentive to switch their investments from one sector to another. The process of price adjustment leading to this kind of equilibrium is not modeled in our theory. It is assumed to have taken place already, so to speak.

10 Prices and dated labor costs

We say that *the Labor Theory of Value holds* (in the strict sense), if prices are proportional to labor values. By Cor. 9.3, this is the case if the profit rate r is zero ($\rho = 1 + r = 1$), but we will see later that it is not true in general for positive rates of profit. As the profit rate rises, the prices of all commodities rise relative to the wage (Th. 10.2), but some prices may rise faster than others, so that the relative prices of goods can no longer be explained by their labor values. See also the example in Section 11.

Even though the production prices with a positive rate of profit are not proportional to labor values, the price pd of a commodity bundle d can still be understood in terms of dated labor costs (cf. Sec. 6):

Theorem 10.1. (Prices and Dated Labor Costs) *Let (p_0, p) be a system of production prices in an economy with interest factor $\rho < \hat{\alpha}(A)$, as in Th. 9.2. Then the value pd of a commodity bundle d is equal to the sum of all past labor costs, compounded with the interest factor ρ:*

$$pd = \sum_{t=0}^{\infty} \rho^{t+1} p_0 \ell_t(d) \tag{10.1}$$

where $\ell_t(d) = a_0 A^t d$ is the dated labor input t periods ago, as defined in Def. 2.

Proof. Since ρA is productive, we have

$$[I - \rho A]^{-1} = \sum_{t=0}^{\infty} \rho^t A^t$$

and the price equation gives

$$pd = p_0 \rho a_0 \sum \rho^t A^t d = \sum \rho^{t+1} p_0 a_0 A^t d = \sum_{t=0}^{\infty} \rho^{t+1} p_0 \ell_t(d)$$

This is equation (10.1) ■

The right-hand side of this equation is the *present value* of the stream of past labor costs $(p_0\ell_0(d), p_0\ell_1(d), p_0\ell_2(d), \dots)$, computed for the interest factor ρ. This takes account of the fact that production takes time and the labor input $\ell_t(d)$ is needed $(t + 1)$ periods before the output d becomes available. In the context of an intertemporal price system with interest rate r, the price equation (9.2) is in fact a zero-profit condition. Of course the formula (10.1) coincides with the labor value $V^h(d) = \sum_{t=0}^{\infty} \ell_t(d)$ for $\rho = 1$ and $p_0 = 1$.

Equation 10.1 shows also that the price pd of a commodity bundle d is positive if and only if d needs labor, i.e. at least one dated labor input $\ell_t(d)$ is positive; and the price pd

is higher than the labor value vd times the wage p_0, for $r > 0$ (of course $pd = p_0vd$ for $r = 0$).

It also shows that all prices increase relative to the wage if the interest factor increases (cf. Th. 10.2 below).

To make the dependence of the prices on ρ explicit, choose an arbitrary numéraire and denote by $p_0(\rho)$, $p(\rho) = (p_1(\rho), \ldots p_n(\rho))$ a system of production prices associated with the profit factor ρ. By (9.5)

$$p(\rho) = p_0(\rho)\rho a_0[I - \rho A]^{-1} \qquad \text{for } 0 < \rho < \hat{\alpha}(A) \tag{10.2}$$

Note that $p_i(\rho)/p_0(\rho)$ is the price of good i relative to the wage.

Theorem 10.2. (Monotonicity) *Let (a_0, A) be a technology with expansion factor $\hat{\alpha}(A)$ and assume that labor is indispensable. Then for each good $i \in N$, the price of i relative to the wage, $p_i(\rho)/p_0(\rho)$, is strictly increasing in ρ, for $0 < \rho < \hat{\alpha}(A)$.*

Proof of Th. 10.2. W.l.o.g. put $p_0(\rho) \equiv 1$, so that $p_i(\rho) = p_i(\rho)/p_0(\rho)$ is the price of good i relative to the wage. By (10.1), the price of an arbitrary commodity bundle $d \gneqq 0$ is (with $p_0 = 1$)

$$p(\rho)d = \sum_{t=0}^{\infty} \rho^{t+1}\ell_t(d)$$

In this sum, each term ρ^{t+1} is strictly increasing in ρ because $t+1 \geqq 1$. Therefore $\rho^{t+1}\ell_t(d)$ is strictly incraesing in ρ, provided $\ell_t(d)$ is positive. Since labor is indispensable, at least one such term is positive, for every nonzero $d \gneqq 0$. Therefore the price $p(\rho)d$ of d increases strictly in ρ, for every $d \gneqq 0$. This implies the assertion, since $p_i(\rho) = p(\rho)e^i$ is the price of the nonzero bundle $d = e^i$, which contains one unit of good i and nothing else.

■

The Transformation Problem.

The connection between the system of labor values and the price system is known as the "transformation problem" in the Marxist literature. Motivated by the belief that labor values were in some sense more fundamental than market prices, or that "values determine prices", some authors tried to explain the latter as deviations from the former, and sought a formula which would "transform" labor values into competitive prices.

Our analysis shows that both labor values and production prices are determined directly by the technological coefficients in (a_0, A) and the profit factor ρ. Both can be understood and interpreted in terms of these data. To deduce prices from labor values via some "transformation algorithm" is unnecessary and misleading. If the interest rate is zero, prices are proportional to labor values (Cor. 9.3), but labor values do not help us to understand the behavior of production prices for other interest rates. Prices are not "transformed labor values".[3]

Having said this, Th. 10.1 can shed some light on the relation between labor values and production prices. Labor values are the simple sum of all past (dated) labor inputs; prices are also a sum of dated labor inputs (or wage costs), but these are compounded at the interest rate r. Since ρ^t increases with t (for $\rho > 1$), this gives higher weight to dated labor inputs in the distant past. Therefore, the prices (unlike labor values) depend not only on the total amount of labor, but also on the intertemporal distribution of the various dated labor inputs (see Sec. 11).

The 'labor theory of value' holds only in a special case, namely when the intertemporal distribution of dated labor inputs is *the same* for all commodities. In Marxist terminology,this means that the 'organic composition' of capital is the same in all sectors (see Sec. 18).

[3]This is expressed succinctly in Samuelson's famous eraser algorithm: ≪... better descriptive words than "the transformation problem" would be provided by "the problem of comparing and contrasting the mutually-exclusive alternatives of 'values' and 'prices'." For when you cut through the maze of algebra and come to understand what is going on, you discover that the "transformation algorithm" is precisely of the following form: "Contemplate two alternative and discordant systems. Write down one. Now transform by taking an eraser and rubbing it out. Then fill in the other one. Voila! You have completed your transformation algorithm."≫ (Samuelson (1971)).

10.1 Example

Example 10.3.

Consider again the technology (a_0, A) introduced in Example 4.1:

$$a_0 = (a_{01}, a_{02}) = (6,\, 6), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

We have shown in Example 8.5 that the expansion factor of A is $\alpha^* = \hat{\alpha}(A) = 2$.

By Th. 9.2, the price equation $p = \rho[p_0 a_0 + pA]$ has a unique positive solution $p > 0$ for every $p_0 > 0$, provided $0 < \rho < \hat{\alpha}(A)$, i.e. for $0 < \rho < 2$. The value $\rho^* = 2$ is an upper bound for the profit rates which the technology can sustain.

The price equations, written out explicitly, are

$$\begin{aligned} p_1 &= \rho[6p_0 + .1p_1 + .3p_2] \\ p_2 &= \rho[6p_0 + .4p_1 + .2p_2] \end{aligned}$$

or equivalently

$$10p_1 = \rho\,[60p_0 + p_1 + 3p_2] \tag{10.3}$$

$$10p_2 = \rho\,[60p_0 + 4p_1 + 2p_2] \tag{10.4}$$

The following table gives the solutions of this system of equations for various values of $\rho = 1 + r$, with $p_0 = 1$:

r	$\rho = 1 + r$	p_1	p_2	p_2/p_1
0%	1	11	13	1.18
1/9 = 11%	10/9	13.63	16.36	1.20
1/4 = 25%	5/4	18	22	1.22
2/3 = 66%	5/3	52.5	67.5	1.28

(10.5)

We know already from Ex. 4.1 that for $\rho = 1$ this system has the solution $p_1 = 11p_0$, $p_2 = 13p_0$. These and the other numbers can easily be checked by substituting them in the equations above. We see that both p_1 and p_2 increase (relative to the wage) when ρ increases, as it must be by Th. 10.2.

Moreover, in this example, the price p_2/p_1 of good 2 relative to good 1 increases also monotonically with ρ. This monotonic behavior of relative prices is not a general feature, see Ex. 11.1 below.

We can also use equation (9.5) to find the general solution for the prices $p(\rho)$. We have

$$p = p_0 \rho a_0 (I - \rho A)^{-1} \qquad (0 < \rho < \alpha^* = 2)$$

This gives

$$\det(I - \rho A) = \begin{vmatrix} 1 - .1\rho & -.4\rho \\ -.3\rho & 1 - .2\rho \end{vmatrix} = (1 - .1\rho)(1 - .2\rho) - .12\rho^2 = 1 - .3\rho - .1\rho^2$$

and

$$(I - \rho A)^{-1} = \frac{1}{1 - .3\rho - .1\rho^2} \begin{pmatrix} 1 - .2\rho & .4\rho \\ .3\rho & 1 - .1\rho \end{pmatrix} = \frac{1}{10 - 3\rho - \rho^2} \begin{pmatrix} 10 - 2\rho & 4\rho \\ 3\rho & 10 - \rho \end{pmatrix}$$

Therefore

$$p = (p_1, p_2) = p_0 \frac{\rho}{10 - 3\rho - \rho^2} (6, 6) \begin{pmatrix} 10 - 2\rho & 4\rho \\ 3\rho & 10 - \rho \end{pmatrix} = p_0 \frac{6\rho}{10 - 3\rho - \rho^2} (10{+}\rho, 10{+}3\rho)$$

so that

$$p_1(\rho) = p_0(\rho) \frac{6\rho}{10 - 3\rho - \rho^2} (10 + \rho), \qquad p_2(\rho) = p_0(\rho) \frac{6\rho}{10 - 3\rho - \rho^2} (10 + 3\rho)$$

The entries in Table 10.5 can easily be verified by putting $p_0 = 1$ and substituting various values of ρ in these equations. For example, for $\rho = 5/4$, we get

$$p_1(5/4) = \frac{30/4}{10 - 15/4 - 25/16} (10 + 5/4) = \frac{120}{75} \times \frac{45}{4} = 18, \quad p_2(5/4) = \frac{120}{75} \times \frac{55}{4} = 22$$

Note that

$$\frac{p_2(\rho)}{p_1(\rho)} = \frac{10 + 3\rho}{10 + \rho} = 1 + \frac{2\rho}{10 + \rho} = 1 + \frac{2}{1 + 10/\rho}$$

is increasing in ρ because $10/\rho$ decreases in ρ.

Fig. 5 shows these prices as functions of ρ, normalized so that the wage satisfies $p_0(\rho) = 1/\rho$, for all ρ. This normalization has no fixed numéraire whose price is always equal to one, but it produces a nice picture in which one can see clearly how the wage decreases relative to all commodity prices when the interest factor increases. Relative prices (the only economically meaningful quantities) are unaffected by the normalization, of course.

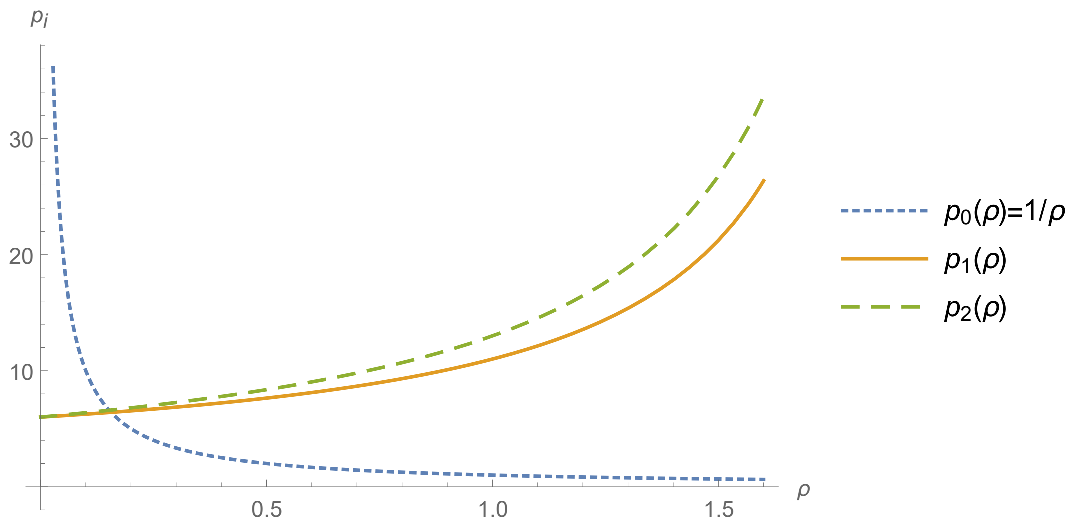

Figure 5: The graph shows the prices $p_0(\rho), p_1(\rho)$, $p_2(\rho)$, for varying ρ. Prices are normalized such that $p_0(\rho) = 1/\rho$.

11 Reswitching

In this section we show by means of an example how production prices change with the profit factor. In particular we see that all commoditiy price rise relative to the wage. But this rise is not uniform in general. It may happen that the price of good 1 increases faster than the price of good 2 in certain ranges of the profit factor ρ, but that the opposite happens in other ranges. It is even possible that good 1 is worth more than good 2 for low values of ρ, worth more for intermediate values, and worth less again for for high values of ρ. This leads to a phenomeneon known as "reswitching", see Sec. 11.1.

Example 11.1. *Prices of Production.*

Consider the following technology, with $n = 3$ goods.

$$a_0 = (1, 0, 5), \quad A = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 1/5 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{Note that} \quad A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } A^t = 0 \quad \text{for } t \geqq 3$$

That is, good 1 needs one unit of labor as input (and nothing else), good 2 needs five units of good 1 as input (and nothing else), and good 3 needs 0.2 units of good 2 and 5 units of labor as inputs. Labor is indispensable in this technology. One can show that $\hat{\alpha}(A) = \infty$.

This example is so simple that it is easy to solve the price equations (9.1) directly. We have

$$p_j = \rho\,[p_0 a_{0j} + p_1 a_{1j} + p_2 a_{2j} + p_3 a_{3j}] \qquad (j = 1, 2, 3)$$

This gives:

$$\left.\begin{array}{rcl} p_1(\rho) & = & \rho p_0 \\ p_2(\rho) & = & \rho 5 p_1 = \rho p_0 5\rho \\ p_3(\rho) & = & \rho[5p_0 + 0.2p_2) = \rho p_0(5+\rho^2) \end{array}\right\} \tag{11.1}$$

If we choose good 1 as numéraire (i.e. if we divide all prices by ρp_0) we obtain

$$p_0(\rho) = 1/\rho, \quad p_1(\rho) = pe^1 = 1, \quad p_2(\rho) = pe^2 = 5\rho, \quad p_3(\rho) = pe^3 = 5 + \rho^2$$

It is easy to see that all commodity prices increase strictly relative to the wage, i.e. $p_i(\rho)/p_0(\rho)$ increases strictly in ρ for $i = 1, 2, 3$, as it must be by Th. 10.2. Also, p_2 and p_3 increase monotonically relative to p_1.

But the relative price of goods 2 and 3 does not change in a monotonic way.

Figure 6 shows that the price $p_3(\rho)$ of good 3 is higher than $p_2(\rho)$ for low interest rates, then falls below $p_2(\rho)$ as ρ increases, but becomes higher than $p_2(\rho)$ again if ρ increases even more.

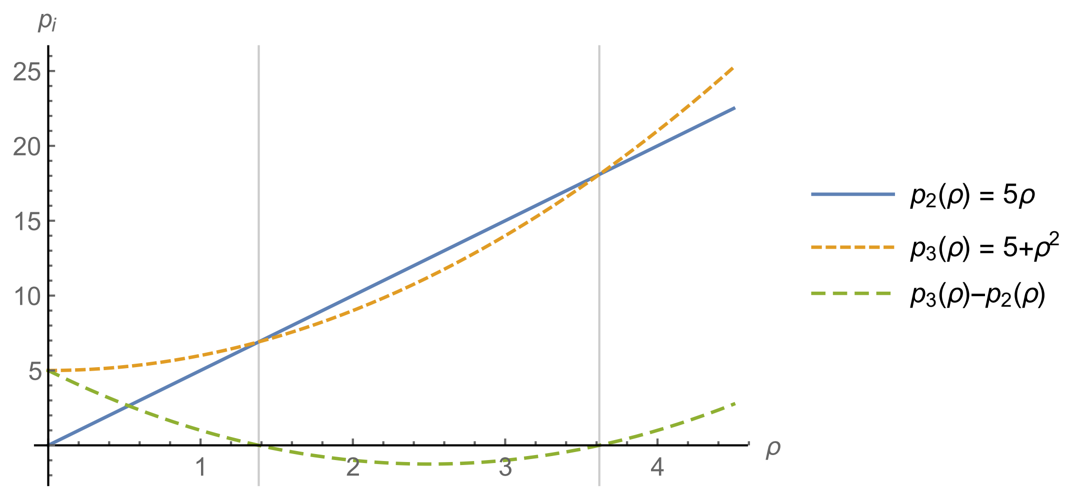

Figure 6: The graph shows the prices $p_2(\rho)$, $p_3(\rho)$, and the difference $p_3(\rho) - p_2(\rho)$, for varying ρ (with good 1 as numéraire). The difference is zero at $\rho_1 = (5 - \sqrt{5})/2 = 1.381966$, and again at $\rho_2 = (5 + \sqrt{5})/2 = 3.618034$. For $\rho < \rho_1$ and for $\rho > \rho_2$ good 3 is worth more than good 2, but for $\rho_1 < \rho < \rho_2$ good 2 is more expensive.

To understand this, recall that by (10.1) the price pd of a commodity bundle d is the present value of the stream of all past wage costs $p_0\ell_t(d)$, for $t = 0, 1, 2, \dots$, compounded with the interest factor ρ, i.e. in our case

$$pd = \rho p_0[\ell_0(d) + \rho\ell_1(d) + \rho^2\ell_2(d) + 0] \tag{11.2}$$

The dated labor inputs $\ell_t(d) = a_0A^td$ for a bundle $d = (d_1, d_2, d_3)^\intercal$ are given by

$$\ell_0(d) = a_0d = d_1{+}5d_3,\ \ell_1(d) = a_0Ad = 5d_2,\ \ell_2(d) = a_0A^2d = d_3,\ \ell_t(d) = a_0A^td = 0 \quad t \geqq 3$$

(this is easy to see because $a_0 = (1, 0, 5)$, $a_0A = (0, 5, 0)$, and $a_0A^2 = (0, 0, 1)$). Therefore the dated labor inputs for the unit bundles e^1, e^2, e^3 are:

$\ell_t(d) = a_0A^td$	$t = 0$	$t = 1$	$t = 2$	$t \geqq 3$
$d = e^1$	1	0	0	0
$d = e^2$	0	5	0	0
$d = e^3$	5	0	1	0

(11.3)

By (11.2) this implies

$$\left.\begin{array}{lllll} p_1(\rho) & = & pe^1 & = & \rho p_0(1 + 0.\rho + 0.\rho^2) \\ p_2(\rho) & = & pe^2 & = & \rho p_0(0 + 5.\rho + 0.\rho^2) \\ p_3(\rho) & = & pe^3 & = & \rho p_0(5 + 0.\rho + 1.\rho^2) \end{array}\right\}$$

These are of course the same prices as in (11.1). Dividing them by the common factor ρp_0 gives the prices shown in Figure 6.

The sequence of dated labor inputs for good 3 is $(5, 0, 1, 0, \dots)$, and the sequence of dated labor inputs for good 2 is $(0, 5, 0, 0, \dots)$. Good 3 needs more total labor (5+1=6 units) than good 2 (5 units), but the distribution of the dated labor inputs is different: good 3 needs 5 units in the current period, and 1 unit two periods ago (the "distant past"). Good 2 needs no labor in the current period, and 5 units one period ago (the "not so distant past"). Therefore the labor value of good 1 is higher, $v_1 = 5 + 1 = 7 > v_2 = 5$. These correspond also to the prices for $\rho = 1$. As the interest rate rises, the five units in the past period for good 2 gain more weight (multiplied with ρ) so that good 2 becomes more expensive than good 3. But eventually, as the interest rate rises even further, the single unit of labor in the distant past for good 3 (multiplied with ρ^2) overtakes everything and good 3 becomes more expensive again.

In other words, the price of a commodity depends not only on the total amount of labor needed for its production, as in the simple Labor Theory of Value, but also on the intertemporal distribution of the various labor inputs. Labor expended in the past is not the same as labor expended in the current period.

11.1 Choice of Technique and Reswitching

Up to now, we have always assumed that there is exactly one production process for each produced good. In order to discuss the phenomenon known as "reswitching", we need a more general model which allows for *choice of technique*. This means that there may be more than one method to produce a certain commodity, and firms producing this commodity must choose between these methods. We will not aim for full generality, but explain the problem by means of a simple example.

Consider again the 3-good technology introduced in Example 11.1, but assume now that there is a fourth good, for which two methods of production are available. To produce one unit of good 4, process 4A needs one unit of good 2 and proces 4B needs one unit of good 3. In addition, both processes require one unit of labor. No process needs good 4 as an input. The two technologies (a_0, A) and (b_0, B) are given by

$$a_0 = (1, 0, 5, 1) \qquad\qquad b_0 = (1, 0, 5, 1)$$

$$A = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 0 & 1/5 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can show that $\hat{\alpha}(A) = \hat{\alpha}(B) = \infty$, so that procduction prices are well-defined for all positive profit factors. The price equations are now

$$p_j = \rho \left[p_0 a_{0j} + p_1 a_{1j} + p_2 a_{2j} + p_3 a_{3j} + p_4 a_{04} \right] \qquad (j = 1, 2, 3, 4)$$

For the first three goods this gives the same prices as in example 11.1 for both technologies, viz. (with $p_0 = 1$):

$$\begin{aligned} p_1(\rho) &= \rho \\ p_2(\rho) &= 5\rho^2 \\ p_3(\rho) &= 5\rho + \rho^3 \end{aligned}$$

and the price of good 4 is given by

$$\begin{aligned} p_4^A(\rho) &= \rho(1 + p_2(\rho)) = \rho + 5\rho^3 \\ p_4^B(\rho) &= \rho(1 + p_3(\rho)) = \rho + 5\rho^2 + \rho^4 \end{aligned} \tag{11.4}$$

in technology A resp. B.

Until now, an equilibrium was simply a price system (ρ, p_0, p) in which all sectors have the same profit rate. Since each sector had only one production process, this meant that all processes have the same rate of profit. If there are several processes to produce the same good, there are more processes than goods, and it is not possible in general to find prices such that all processes generate the same profit factor (more equations than unknowns). Therefore we must modify our definition of equilibrium. Let a profit factor ρ be given. We say that (ρ, p_0, p) is an equilibrium price system if the following two conditions hold, at the prices (p_0, p):

(a) for each commodity $j \in N$, there is one process which has exactly the profit factor ρ

(b) no process has a higher profit factor than ρ

The interpretation is that the n processes in (a) are active and form a technology in the sense of Sec. 1, as before. Prices satisfy the price equations (9.2) for this technology. All processes in (a) have the same profit rate (the "going rate of profit"). Since the profit rates of the processes in (b) are lower than (or at most equal to) the going rate, firms have no incentive to adopt these processes, and the processes in (b) are inactive. Therefore the technology given by the processes in (a) is also known as the *optimal technique* given ρ.

Returning to our example, it is clear from (11.4) that $p_4^A(\rho) < p_4^B(\rho)$ if and only if $p_2(\rho) < p_3(\rho)$. As shown in Fig. 6 this is the case if $\rho < \rho_1 = 1.38$ ("ρ is small") or $\rho > \rho_2 = 3.61$ ("ρ is large") For intermediate values of ρ, $1.38 < \rho < 3.61$, the opposite is the case, $p_4^A(\rho) > p_4^B(\rho)$.

Suppose now $\rho < \rho_1$ is small, technology A is in use, and the price of good 4 is $p_4^A(\rho)$. In this situation, the processes (1, 2, 3, 4A) have the same profit rate ρ, but the profit rate for process 4B is lower, because its unit cost $1+p_3(\rho)$ is higher than the unit cost $1+p_2(\rho)$ in process 4A. Therefore process 4B will not be used. The optimal technology is A. Group (a) consists of the processes (1, 2, 3, 4A) and group (b) contains only process 4B. The same is true if $\rho > \rho_2$ is large.

However, if ρ lies in the intermediate range between ρ_1 and ρ_2, the production cost of process 4B is lower than that of process 4A, so that technology A is no longer an equilibrium, because the firms in sector 4 would have an incentive to switch to process 4B. The equilibrium is now given by technology B, and the price of good 4 is $p_4^B(\rho)$. The active processes in group (a) are now(1, 2, 3, 4B), all with the same profit factor, and group (b) consists of process 4A, which has a lower profit factor.

For $\rho = \rho_1 = 1.38$ and again for $\rho = \rho_2 = 3.61$ the two processes 4A and 4B are equally profitable, so that the firms in sector 4 are indifferent between them. At these points (the switch-points), the optimal technique is not unique (but the equilibrium prices are unique). The firms in sector 4 may use either process 4A or 4B (or even a mixture of the two).

Thus it is possible that a technique A which is optimal for low interest rates ceases to be optimal for higher interest rates, but becomes optimal again for even higher rates.

This phenomenon, that the optimal technique switches from A to B at some interest rate, and then switches back to A again at a even higher interest rate, is known as *Reswitching.*

The possibility of reswitching was pointed out by Sraffa (1960). It came as a surprise to many economists who thought that a rise in the interest rate was equivalent to a rise in the 'price of capital' and would always induce a move to less capital intensive production methods, so that a technique which was once abandoned because the interest rate was too high could never come back at an even higher interest rate.

The example shows that this intuition is not correct. It is not possible in general to rank production processes as more or less "capital intensive" when "capital" consists of many different goods.

12 Profits and Wages

An important feature of the Linear Model of Production is that there is a strict conflict of interest between the social classes (capitalists and workers): higher profit rates imply lower wages (lower consumption for the workers) and vice versa.

To see this, choose an arbitrary numéraire and denote by $p_0(\rho)$, $p(\rho) = p_0 \rho a_0 (I - \rho A)^{-1}$ the production prices associated with the profit factor $\rho = 1 + r$, for $0 < \rho < \hat{\alpha}(A)$. Assume that labor is indispensable, so that all prices are positive.

The number $p_0(\rho)$ is the *nominal* wage. As such, it has no economic significance. To get an idea of what the wage is "really" worth, we must specify some commodity bundle d (a "basket of goods"), and ask how many units of this bundle (how many baskets) a worker can buy with his wage, at the prices $p(\rho)$. This amount is given by

$$w_d = w_d(\rho) = \frac{p_0(\rho)}{p(\rho)d} \tag{12.1}$$

The amount $w_d = w_d(\rho)$ is the *real* wage, in terms of the basket d. The real wage depends on the the profit factor ρ, because the prices depend on ρ, but it is independent of the choice of numeraire. If the profit factor is ρ, a worker can buy $w_d(\rho)$ units of the basket d with his wage. In particular, if d is chosen as numéraire, so that $p(\rho)d = 1$, then the nominal wage is equal to the real wage, $w_d(\rho) = p_0(\rho)$.

The function w_d associates a wage $w_d(\rho)$ with every profit factor ρ. The graph of this function is the *wage - profit curve* for the basket d.

Th. 10.2 implies immediately:

Lemma 12.1. *For every commodity bundle d, the real wage $w_d(\rho)$ is a strictly decreasing function of the profit factor ρ.*

Thus higher profit rates mean lower wages and vice versa. This feature may explain the popularity of the Linear Model (as opposed to other models of General Equilibrium) among left-wing economists.

Remark. The wage-profit curve relates a pure number (the profit rate resp. the profit factor) with a "wage" whose numerical value makes sense only after a numeraire bundle d has been chosen. Thus, for a given technology (a_0, A) there are really infinitely many wage-profit curves, one for each possible numeraire d. All these curves are downward-sloping, but the concrete shape depends on the choice of numeraire. To speak of *the* wage-profit curve, without specifying the numeraire, as is sometimes done in the literature, is thus somewhat dubious from a conceptual point of view.

Example 12.2. *Wage-Profit curves.*

Recall from (11.1) that in the 3-good economy of Example 11.1, the prices were given by

$$\left.\begin{array}{rcl} p_1(\rho) & = & \rho p_0 \\ p_2(\rho) & = & \rho p_0 5\rho \\ p_3(\rho) & = & \rho p_0(5+\rho^2) \end{array}\right\}$$

Therefore the real wage in terms of commodity i (i.e. in terms of the bundle e^i) is given by $w_i(\rho) := w_{e^i}(\rho) = p_0/p_i(\rho)$. This is the number of units of good i a worker can buy with his wage.

$$\left.\begin{array}{rcl} w_1(\rho) & = & 1/\rho \\ w_2(\rho) & = & 1/(5\rho^2) \\ w_3(\rho) & = & 1/(5\rho+\rho^3) \end{array}\right\}$$

Figure 7 shows the wage-profit curves for the bundles $e^1 = (1,0,0)^\intercal$ and $e^2 = (0,1,0)^\intercal$

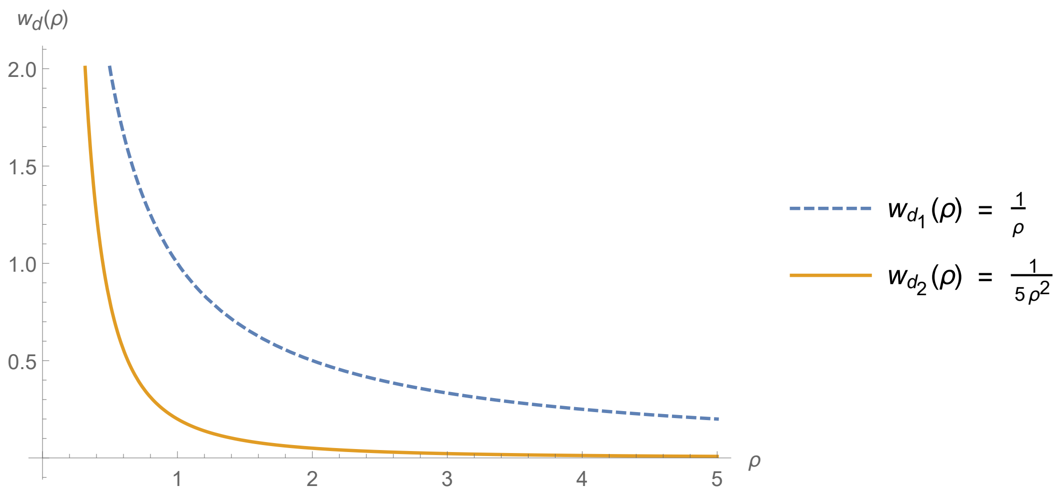

Figure 7: The commodity bundle d_i represents "one unit of good i, for $i = 1, 2$. The dashed line is the wage in terms of good 1, and the solid line is the wage in terms of good 2, for varying values of the profit factor ρ.

Stationary State with Positive Profits.

Consider now a quantity system in a stationary state as in Sec. 3:

$$x = Ax + d, \qquad L(d) = a_0 x \tag{12.2}$$

where d is the net output, x is the gross output, $y = Ax$ is the vector of material inputs, and $L(d) = a_0 x$ is the required amount of labor, all per period.

We saw in Sec. 3 that the value of the net output is equal to the sum of all wages if the rate of profit is zero (Th. 3.1). Assume now that the rate of profit r is positive, so that prices are given by the price equation

$$p = \rho(p_0 a_0 + pA) \tag{12.3}$$

where $\rho = 1 + r$ is the profit factor, and $p_0 > 0$ can be chosen arbitrarily.

If we multiply the quantity equation (12.2) by p (from the left), we get

$$px = pAx + pd \tag{12.4}$$

and if we multiply the price equation (12.3) by x (from the right), we get (using $\rho = 1+r$)

$$px = p_0 a_0 x + pAx + r(p_0 a_0 x + pAx) \tag{12.5}$$

Here px is the value of the gross output, $p_0 a_0 x = p_0 L(d) = W$ is the sum of all wages (the income of the workers), pAx is the total cost of the material inputs, and $p_0 a_0 x + pAx$ is the total investment of the capitalists, i.e. the money advanced at the beginning of the period to pay the wages and buy the means of production. At the end of the production period the capitalists receive the total revenue px. They use it to replace the means of production and advance the wages for the next period. This costs $p_0 a_0 x + pAx$ and leaves $px - (p_0 a_0 x + pAx) = r(p_0 a_0 x + pAx) = \Pi$ as profit ("r percent of total investment"). This is the income of the capitalists. The equations 12.4 and 12.5 imply

$$pd = p_0 a_0 x + r(p_0 a_0 x + pAx) = W + \Pi \tag{12.6}$$

The value pd of the net output (the national income) is equal to the sum of wages and profits. If $r > 0$, then $\Pi > 0$, hence $p_0 a_0 x < pd$, i.e. the workers can no longer buy the entire net output, but only a part of it, and the rest goes to the capitalists. Assuming for the moment that both workers and capitalists spend their income on consumption, the net output is split in two parts:

$$d = d^a + d^s \tag{12.7}$$

where d^a is the consumption of the workers, and d^s is the consumption of the capitalists. Both satisfy their respective budget constraints, i.e.

$$pd^a = W = p_0 a_0 x, \qquad \text{and} \quad pd^s = \Pi = r(p_0 a_0 x + pAx)$$

so that $pd = p_0 a_0 x + r(p_0 a_0 x + pAx)$, as it must be.

Equation (12.6) implies immediately:

Theorem 12.3. *If the profit rate r is positive, then the value of the net output $d = x - Ax$ exceeds the sum of all wages:*

$$pd > p_0 L, \qquad \textit{where } L = a_0 x$$

The aggregate income of the workers is no longer sufficient to buy the entire net output. This should be compared to Th. 3.1.

Remark. If the profit rate r is negative, the revenue px is not enough to replace the means of production of production Ax and pay the wages $p_0 a_0 x$. The consumption of the workers during the period is more than the net output. A stationary state with constant activity levels x as described by equation 12.2 is not possible. It is possible, though, to have a shrinking economy, in which the output and the number of workers shrink from one period to the next.

13 Growth

So far we have only considered stationary states. Let us now look at economic growth. As explained in Sec. 1, there is a common period of production (the "year") for all processes. Inputs (material inputs and labor) are invested in the beginning or during one period. The corresponding output (the "harvest") becomes available only at the end of the period, and can then be used for consumption or investment in the next period.

In a stationary state the quantities consumed and produced of all commodities are the same in all periods, so that there is no need to indicate in the notation to which period a certain amount of a good belongs.

But when the quantities produced and consumed may vary over time, it is necessary to recognize that 'a ton of wheat this year' is not the same as 'a ton of wheat next year', or, more generally, that *goods available in different periods are different economic goods*, even when they are physically identical.

To deal with such situations, we introduce the following notation.

There is an infinite sequence of periods, indexed $t = 0, 1, 2, 3, \dots$. The technology (a_0, A) is the same in all periods, but the production plan may vary. We denote by $x(t) \geqq 0$ the vector of goods available at the start of period t. For $t = 0$, this is an initial endowment, for $t \geqq 1$ it is the gross output produced in the previous period. It can be used in period t for consumption or investment, i.e. we have

$$x(t) = y(t) + d(t) \qquad (t = 0, 1, 2, \dots) \tag{13.1}$$

where $y(t) \geqq 0$ is the total amount of goods invested as inputs for production, and $d(t) \geqq 0$ is the total consumption (of capitalists and workers) in period t. These are column vectors

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}, \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \dots \\ y_n(t) \end{pmatrix}, \quad d(t) = \begin{pmatrix} d_1(t) \\ d_2(t) \\ \dots \\ d_n(t) \end{pmatrix} \tag{13.2}$$

where $x_i(t)$ is the total amount of good i available at the start of period t, for $i \in N$. Similarly $y_i(t)$ and $d_(t)$ represent the total amounts of good i contained in $y(t)$ resp. $d(t)$.

The development of the economy over time is described by a sequence of production plans

$$\tilde{x} = [x(0), x(1), x(2), \dots], \qquad \text{with } x(t) \geqq 0 \text{ for } t = 0, 1, 2, \dots$$

Such a sequence is a *feasible path* (or *feasible development*) if

$$x(t) \geqq Ax(t+1) \qquad \forall t \geqq 0 \tag{13.3}$$

i.e. if the output produced in any period is enough to supply the inputs needed in the following period (otherwise production could not continue).

For a feasible development $\tilde{x} = (x_i(t))_{t=0,1,2,...}$ we define

$$y(t) = Ax(t+1), \quad d(t) = x(t) - Ax(t+1) \qquad (t \geqq 0) \tag{13.4}$$

and obtain from (13.1)

$$x(t) = Ax(t+1) + d(t) \qquad (t = 0, 1, 2, \dots) \tag{13.5}$$

This is the *General Quantity Equation.* The required number of workers in period t is the total amount of labor needed to produce $x(t+1)$:

$$L(t) = a_0 x(t+1) \tag{13.6}$$

We will not study feasible paths in general, but focus on the following special case.

Definition 5. *A feasible path $\tilde{x} = (x(t))_{t=0,1,...}$ is a* balanced growth path *with growth factor $\gamma > 0$ (growth rate $g = \gamma - 1$) if all quantities increase by the factor γ from one period to the next, i.e. if*

$$x(t+1) = \gamma x(t) \qquad (t = 0, 1, 2, \dots) \tag{13.7}$$

For example, if the growth rate g is two percent ($g = 0.02$), then the growth factor is $\gamma = 1 + g = 1.02$, and all sectors grow at the rate of 2 percent *per annum.* In a balanced growth path the only thing that changes from one period to the next is the scale of production, but the technology and the structure of the economy (the relative sizes of the various sectors) remain the same. Every period is like every other, except possibly for a change of scale. We say that the economy is in a *semi-stationary state*, and refer to balanced growth also as *semi-stationary* growth. The special case of *no growth* ($g = 0$ or $\gamma = 1$) corresponds to the 'stationary state' introduced in Sec. 3.

First we note that if $x(t)$ grows with factor γ, then so do all the other quantities, $y(t), d(t), L(t)$. Indeed, suppose that $\tilde{x} = (x(t)$ is a balanced growth path with growth factor $\gamma > 0$. Equation (13.7) implies, for all $t = 0, 1, 2, \dots$:

$$x(t) = \gamma x(t-1) = \gamma^2 x(t-2) = \gamma^3 x(t-3) = \cdots = \gamma^t x(0)$$

Therefore

$$y(t) = Ax(t+1) = \gamma^t Ax(1) = \gamma^t y(0)$$

and

$$d(t) = x(t) - y(t) = \gamma^t [x(0) - y(0)] = \gamma^t d(0)$$

and also

$$L(t) = a_0 x(t+1) = \gamma^t a_0 x(1) = \gamma^t L(0)$$

Define $x = x(0)$, $y = y(0)$, $d = d(0)$, $L = L(0)$. Then

$$x(t) = \gamma^t x, \quad y(t) = \gamma^t y, \quad d(t) = \gamma^t d, \quad L(t) = \gamma^t L \qquad (t = 0, 1, 2, \dots) \tag{13.8}$$

The quantity equation (13.5) now takes the form

$$\gamma^t x = \gamma^{t+1} A x + \gamma^t d$$

or, dividing by γ^t

$$x = \gamma A x + d \tag{13.9}$$

This is the *Growth Equation.* It is the same as $x(0) = Ax(1) + d(0)$. This is the quantity equation (13.5) with $t = 0$, for the path $x(t) = \gamma^t x$. Also, since $L(t) = a_0 x(t+1)$

$$L = \gamma a_0 x \tag{13.10}$$

The growth equation implies

Lemma 13.1. *A sequence $\tilde{x} = (x(t))_{t=0,1,2,\dots}$ is a balanced growth path with growth factor $\gamma > 0$ if and only if it is of the form $x(t) = \gamma^t x$, where $x \geqq 0$ satisfies the growth equation $x = \gamma A x + d$, for some nonnegative vector $d \geqq 0$.*

Proof of Lemma 13.1.

The "only if" part follows from the analysis above.

Conversely, if $x \geqq 0$ and $x = \gamma A x + d$ for some $d \geqq 0$, then $x(t) = (\gamma^t x)$ is balanced by definition, and feasible because $x(t) - Ax(t+1) = \gamma^t(x - \gamma A x) = \gamma^t d \geqq 0$ for all t.

■

Therefore γ is a possible growth factor of the technology (a_0, A), given d, if and only if equation (13.9) has a nonnegative solution $x \geqq 0$. The vector d specifies the structure of exogenous demand (consumption) in all periods, because $d(t) = \gamma^t d$ is proportional to d for all t. Let us first consider growth factors which are feasible for arbitrary nonnegative d.

Theorem 13.2. *Let (a_0, A) be a technology with expansion factor $\hat{\alpha}(A)$. Then a number $\gamma > 0$ is a possible growth factor of this technology, for all $d \geqq 0$, if and only if $0 < \gamma < \hat{\alpha}(A)$.*

Proof of Th. 13.2. The growth equation can be written $(I - \gamma A)x = d$, with solution

$$x = (I - \gamma A)^{-1} d \tag{13.11}$$

This solution exists and is nonnegative for arbitrary d if and only if the matrix $I - \gamma A$ is nonnegatively invertible, i.e. (by Th.5.2) if and only if the matrix γA is productive.

By Th. 8.1 this is the case if and only if $\gamma < \hat{\alpha}(A)$. The theorem now follows from Lemma 13.1.

■

The expansion factor $\hat{\alpha}(A)$ of the technology is an upper bound for the possible growth factors of the economy. Note that this is the same as the upper bound for the possible profit factors in Th. 9.2.

Profits and Growth

Let (a_0, A) be a technology with expansion factor $\alpha^* = \hat{\alpha}(A)$. Let γ be a growth factor,and ρ a profit factor, with $0 < \gamma, \rho < \alpha^*$. Which values of γ and ρ are compatible with each other in a semi-stationary state?

Let x and d be a semipositive solution of the growth equation

$$x = \gamma Ax + d \tag{13.12}$$

and consider the balanced growth path $\tilde{x} = (x(t)) = (\gamma^t x)$. Then $d(t) = \gamma^t d$ is total consumption in period t, and $x(t) = Ax(t+1) + d(t)$ for all $t = 0, 1, 2, \ldots$. As in Sec. 12, let us split d in two parts, $d = d^a + d^s$, where d^a is the workers' and d^s is the capitalists' consumption, and assume that these terms also grow with factor γ, so that $d^a(t) = \gamma^t d^a$ [resp. $d^s(t) = \gamma^t d^s$] is the workers' [resp. the capitalists'] consumption in period t. The growth equation takes the form

$$x = \gamma Ax + d^a + d^s \tag{13.13}$$

The prices (p_0, p) are the same in each period and are given by the price equation

$$p = \rho(p_0 a_0 + pA) \tag{13.14}$$

We assume that labor is indispensable for production, so that all prices are positive. Equations (13.13) and (13.14) imply

$$px = \gamma pAx + pd^a + pd^s \tag{13.15}$$

and

$$px = \rho(p_0 a_0 x + pAx) \tag{13.16}$$

so that

$$\gamma pAx + pd^a + pd^s = \rho(p_0 a_0 x + pAx) \tag{13.17}$$

If we want growth, then only a part of current income can be used for consumption, and the other part must be saved to finance growth (to buy the additional inputs needed).

Assume first that workers do not save, and capitalists do not consume. This means that the workers' consumption is equal to total consumption, $d^a = d$, and $d^s = 0$, i.e. the capitalists save all their profits to finance growth.

The workers' budget constraint is $p_0 L(t) = p_0 a_0 x(t+1) = pd^a(t) = pd(t)$, for all t, or, since $d^s = 0$,

$$\gamma p_0 a_0 x = pd = pd^a + pd^s$$

By (13.17) this implies $\gamma pAx + \gamma p_0 a_0 x = \rho(p_0 a_0 x + pAx)$, hence

$$\gamma = \rho \quad \Leftrightarrow \quad g = r$$

If the capitalists spend all their profits for accumulation, then the growth rate g is equal to the profit rate r.

Given r, this maximizes growth, but it results in a semi-stationary state in which the capitalists never get to consume anything. If we think of the capitalists as individuals who derive also utility from consumption, like the workers, this scenario is not plausible.

Let us assume therefore that, along our semistationary path, the total consumption contains also some consumption of the capitalists, i.e. $d^s \gneqq 0$. Workers still spend their entire income on consumption, subject to the budget constraint $p_0 L(t) = pd^a(t)$ or $\gamma p_0 a_0 x = pd^a$. Equation (13.17) Then takes the form

$$\gamma pAx + \gamma p_0 a_0 x + pd^s = \rho(p_0 a_0 x + pAx)$$

This implies

$$pd^s = (\rho - \gamma)(p_0 a_0 x + pAx))$$

If $d^s \gneqq 0$, this implies

$$\gamma < \rho$$

More precisely, in period t, the firms have invested

$$z(t) = p_0 L(t) + py(t) = p_0 a_0 x(t+1)) + pAx(t+1)$$

dollars. This investment results in the total output $x(t+1)$ at the end of period t, giving the firms the revenue $px(t+1)$. Using (13.16) this revenue is equal to

$$px(t+1) = (1+r)z(t) = z(t) + r.z(t) = z(t) + g.z(t) + (r-g)z(t)$$

dollars. This money is used in period $t+1$ as follows: The first term $z(t)$ is used to replace the inputs $y(t)$ and re-hire the workers $L(t)$ who were employed in period t. The second term $r.z(t)$ is the profit. It is split in two parts: the amount $g.z(t)$ is saved and used to finance growth: it is invested to buy additional inputs $gy(t)$ and hire additional workers $gL(t)$. This gives $y(t+1) = (1+g)y(t)$ and labor $L(t+1) = (1+g)L(t)$. The second part of the profit, $(r-g).z(t)$, is used for the capitalists consumption $d^s(t+t)$. It is easy to check that the quantity equation and all budget constraints are satisfied along such a balanced path. Summing up, we have proved:

Theorem 13.3. *Consider a balanced growth path with growth factor γ and profit factor ρ. The rate of profit is an upper bound for the rate of growth. The two are equal if and only if the capitalists spend their entire profits on accumulation, otherwise the growth rate is smaller.*

Remark. In this scenario, workers do not save at all, and the capitalists save a constant fraction s of their income (of profits), namely

$$s = \frac{g.z(t)}{r.z(t)} = \frac{g}{r}$$

This is in line with the so-called *classical savings assumption*: " Under this assumption all income accruing to labour is spent immediately on consumption goods - the only source of saving is profits. And ... the amount of profit saved is a constant fraction of the total" (Bliss (1975), p. 123).

In our exposition, we have started with a given growth rate and profit rate, and adjusted the savings rate s accordingly. One can also take the capitalists' savings rate and the rate of profit as given, and derive the growth rate $g = s.r$ from this. Obviously $g < r$ for $s < 1$, and $g = r$ for $s = 1$.

One can also show that such a balanced growth path is a General Equilibrium in the sense of Arrow-Debreu, for suitably specified endowments and preferences. In this equilibrium, the capitalists own the initial endowment $x(0)$ and their rate of time preference is such that a semi-stationary consumption path is optimal. We do not pursue this matter here.

Autarkic Subsystems.

Remark. Higher growth factors may be possible for certain subgroups of sectors. A matrix A is *decomposable* if it there exists a nonempty subgroup $J \subsetneqq N$ of sectors which needs no inputs from the other sectors:

$$a_{ij} = 0 \qquad \text{for } i \notin J,\ j \in J \tag{13.18}$$

Such a set J is called autarkic. An autarkic group of sectors forms a subeconomy which can operate independently of the other sectors. In this case, only the sectors in J are active, and all other sectors are inactive ($x_i = 0$ and hence also $d_i = 0$ for $i \notin J$). Such a subeconomy cannot produce a positive output of all goods, but it can have higher growth factors than the whole economy, see Example 13.4.

Considering autarkic subeconomies makes sense mainly if one wants to maximize growth *per se*, given the technology, in a purely technocratic manner, without regard for demand conditions, as in the von Neumann model. But it is problematic if one also considers demand. After all, the restriction to a subset of sectors makes sense only if nobody wants to consume the goods produced by the other sectors. But if this is so, why include these other sectors in the description of the economy in the first place?

One can study such cases by applying our theory to the various autarkic subgroups, considered as independent economies, but we will not pursue this here. We only indicate the possibility by the following simple example.

Example 13.4. *An autarkic subset with growth factor higher than $\hat{\alpha}(A)$.*

In this example with two goods, sector 1 needs no input from the other sector, so that the $J = \{1\}$ is an autarkic subset. We will see that sector 1 alone can have higher growth rates than $\hat{\alpha}(A)$, but only if sector 2 is inactive.

Let (a_0, A) be given by

$$a_0 = (1,\, 1), \qquad A = \begin{pmatrix} 2/10 & 1/10 \\ 0 & 3/10 \end{pmatrix}$$

The characteristic equation is $\det(\lambda I - A) = (\lambda - 2/10)(\lambda - 3/10) = 0$ with solutions $\lambda_1 = 2/10$ and $\lambda_2 = 3/10$. The dominant eigenvalue is $\lambda^* = 3/10$ and the expansion factor is $\alpha^* = 1/\lambda^* = 10/3$. Therefore growth with factor γ is possible for all $\gamma < 10/3$.

For example, if $\gamma = 2$, the matrix $I - \gamma A$ is nonnegatively invertible:

$$(I - \gamma A)^{-1} = \begin{pmatrix} 5/3 & 5/6 \\ 0 & 5/2 \end{pmatrix}$$

and the quantity equation $x = \gamma(Ax + d)$ has the nonnegative solution $x = (I - \gamma A)^{-1}\gamma d$ for every $d \geqq 0$. If we choose $d = (4, 1)^{\mathsf{T}}$, this gives $x = (15, 5)^{\mathsf{T}}$. We can check that the growth equation $x = 2.(Ax + d)$ is satisfied by observing that

$$Ax + d = \begin{pmatrix} 2/10 & 1/10 \\ 0 & 3/10 \end{pmatrix} . \begin{pmatrix} 15 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 15/2 \\ 5/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 15 \\ 5 \end{pmatrix} = \frac{1}{\gamma} x$$

If we put $x(t) = 2^t x$ for $t = 0, 1, 2, \ldots$, we obtain a balanced growth path with $d(t) = 2^t d$, so that both goods are consumed in positive amounts in all periods.

On the other hand, when $\gamma > \alpha^* = 10/3$, the matrix $I - \gamma A$ is no longer nonnegatively invertible, and the growth equation has no nonnegative solution in general. But it may have a solution if some sectors are inactive. Along such a growth path, not all goods can be consumed in positive amounts.

To see this, consider the growth equation (13.9) for our example:

$$\begin{aligned} x_1 &= \gamma(.2x_1 + .1x_2 + d_1) \\ x_2 &= \gamma(0.3x_2 + d_2) \end{aligned}$$

For $\gamma > \alpha^* = 10/3$, the only nonnegative solution of the second equation is $x_2 = 0$, $d_2 = 0$. Therefore sector 2 must be inactive. For sector 1 this implies

$$x_1 = \gamma(.2x_1 + d_1) \quad \Leftrightarrow \quad x_1 = \frac{1}{1 - .2\gamma} d_1$$

This has a nonnegative solution for positive d_1 if and only if $\gamma < 5$. For example, we can choose $\gamma = 4$, $d = (1,0)^{\mathtt{T}}$, $x = (20,0)^{\mathtt{T}}$. This satisfies the growth equation $x = 4.(Ax + d)$ because

$$Ax + d = \begin{pmatrix} 2/10 & 1/10 \\ 0 & 3/10 \end{pmatrix} \cdot \begin{pmatrix} 20 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 20 \\ 0 \end{pmatrix} = \frac{1}{\gamma} x$$

The autarkic subsystem $J = \{1\}$ consisting only of sector 1 grows with factor $\gamma = 4 > \alpha^*$, but the rest of the economy is inactive.

Lecture 5 MARX

In this Lecture we explain some basic concepts of Marxist Economic Theory in the framework of our Linear Model of Production. For a justification of this approach and references to Marx, see Morishima (1973) and Morishima (1974).

We assume here that the technology (a_0, A) is productive and all sectors need labor directly, i.e. $a_0 > 0$. This ensures that all labor values are positive and concepts like the organic composition of capital (see below) are always well-defined. From the Marxian point of view, this assumption is certainly no restriction.

14 The Rate of Exploitation

Consider a quantity system describing a stationary state ("simple reproduction" in Marx) as in Sec. 3:

$$x = Ax + d, \qquad L = L(d) = a_0 x$$

Here $x = (I - A)^{-1}d$ is the gross output vector, $y = Ax$ is the vector of inputs, d is the net output vector, and $L = L(d) = a_0 x = a_0(I - A)^{-1}d$ is the required amount of labor. We assume that one worker supplies one unit of labor, so that $L(d)$ is the "number of workers" needed to produce d. Recall that $L(d) = vd$ is the labor value of the commodity bundle d, where $v = a_0(I - A)^{-1}$ is the solution of the value equation $v = a_0 + vA$.

Assume that the workers consume only a part d^a of the total net output d, leaving the *surplus product* $d^s = d - d^a$ for the capitalists. This gives a quantity system of the form:

$$x = Ax + d^a + d^s \tag{14.1}$$

where $d = d^a + d^s$ is the net output vector, d^a is the aggregate consumption of the workers, and d^s is the surplus product (which goes to the capitalists). We have seen in Sec. 12 that such a situation can arise in a stationary state when the rate of profit is positive, but here we simply consider the quantity system 14.1, without production prices. We assume that both d^a and d^s are non-zero. This ensures that the various concepts introduced below (rate of exploitation, organic composition of capital, etc.) are always well-defined. Consideration of the limiting cases $d^a = 0$ or $d^s = 0$ is not difficult, but left to the interested reader.

Following Marx, we can think of total labor $L = a_0 x = vd = vd^a + vd^s$ as being split in two parts: $L = L^a + L^s$, where $L^a = vd^a$ is the *necessary labor*, and $L^s = vd^s$ is the *surplus labor*. The necessary labor L^a is the amount of labor needed to produce the workers' consumption d^a, i.e. the amount of time the workers work for themselves, or the amount of *paid* labor. The surplus labor L^s is the amount of time the workers work for the capitalists, or the amount of *unpaid* labor.

Marx defined the *rate of exploitation* ε ("Mehrwertrate") as the ratio of surplus labor to necessary labor:

$$\varepsilon = \frac{L^s}{L^a} = \frac{vd^s}{vd^a} \tag{14.2}$$

It is also possible to define the rate of exploitation in *per capita* terms. Total labor is $L = vd$ and the per capita consumption c of the workers is given by $c = d^a \frac{1}{L}$, so that

$$d^a = cL = ca_0x, \qquad d^s = d - d^a \tag{14.3}$$

This implies $L^a = vd^a = vcL$ and $L^s = vd^s = vd - vd^a = L - vcL = (1 - vc)L$. We write $v_0 = vc$ for the labor value of a worker's consumption. This is the wage in terms of labor values. Note that $v_0 > 0$ because $v > 0$ and $c = d^a \frac{1}{L} \gneqq 0$. Thus (14.2) can also be written

$$\varepsilon = \frac{L^s}{L^a} = \frac{1 - vc}{vc} = \frac{1 - v_0}{v_0} \tag{14.4}$$

To understand this 'per capita' characterization of the rate of exploitation note that, by definition, a single worker supplies one unit of labor, and consumes the commodity bundle c. The labor value $v_0 = vc$ is the amount of time needed to produce c. In other words, $v_0 = vc$ is the amount of labor needed to (re-)produce one unit of labor power (the "labor value of labor"). If vc is less than one unit of labor, the worker works more than would be necessary to produce his own means of subsistence. Thus vc is the necessary labor and the rest $1 - vc$ is the surplus labor *per worker*. Workers are *exploited* if $\varepsilon > 0$ (or, equivalently $L^s > 0$ or $vc < 1$).

In the classical literature, the per capita consumption c is is usually taken as given and interpreted as some kind of *subsistence* consumption. The bundle c contains the goods a worker needs to survive and be able to work. The bare biological needs of a worker certainly provide a lower bound for c, but it may also be the case that c reflects some socially acceptable minimum standard of living. Whatever the case may be, we do not discuss here what determines c, but will simply explore the implications of different possible per capita consumptions.

Note on terminology. The classical economists use the word *social product* for the excess of outputs over inputs, i.e. for the net product $d = x - Ax$. The workers' consumption $d^a = ca_0x$ is also called *necessary consumption* (reflecting the idea of a 'subsistence' consumption). What is left of the social product after subtracting the necessary consumption is the *surplus product*

$$d^s = x - Ax - ca_0x = x - (Ax + ca_0x) \tag{14.5}$$

The surplus product is also known as the "classical net product" or "produit net". The labor value vd^s of the surplus product is the *surplus value* ("Mehrwert"). The social product is divided between the classes: the necessary consumption serves to feed the workers, and the surplus is at the disposal of the capitalists. Of course c must be such that a surplus remains after feeding the workers, i.e. c must not be too large. In Sec. 15 we show that a positive surplus exists if and only if $vc < 1$. In the following discussion, we will always assume that this is the case.

Remark. The Marxian concept of "exploitation" as defined here means simply that the workers do not consume the entire net output, i.e. the economy produces a surplus over and above the workers' immediate consumption needs. This surplus can be used for the capitalists' personal consumption of luxury goods, or it can be used productively as investment to achieve economic growth (to feed a growing population, for example). Thus whether exploitation in the sense of (14.2) is a good or a bad thing, may depend on what the surplus is used for.

Exploitation and Profits.

What is the relation between the rate of exploitation ε and the rate of profits r? Denote by $c \geqq 0$ the per capita consumption of the workers.

By definition, $\varepsilon = (1 - vc)/vc$. Note that the labor value vc of c is positive because $v > 0$ by indispensability of labor.

Let $\rho = 1 + r$ be a profit factor with $0 < \rho < \hat{\alpha}(A)$, and let (p_0, p) be associated positive prices. These prices are unique up to multiplication by a scalar and satisfy the price equation

$$p = \rho[p_0 a_0 + pA] \tag{14.6}$$

Given these prices (or given the profit rate r), a bundle $c \geqq 0$ is a *feasible per capita consumption* for the workers if it satisfies the budget constraint $pc = p_0$.

In principle, both ε and r can be positive or negative. The following analysis covers both cases, but the economically relevant case is probably the one where these numbers are positive.

Theorem 14.1. [Fundamental Marxian Theorem 1] *Let (p_0, p) be production prices associated with the profit factor $\rho = 1 + r$, where $0 < \rho < \hat{\alpha}(A)$, and let $c \gneqq 0$ be a feasible per capita consumption for these prices, $pc = p_0$. Let $\varepsilon = (1 - vc)/vc$ be the rate of exploitation. Then*

(a) if $Ac = 0$ (c is produced by labor alone), then $\varepsilon = r$

(b) if $Ac \gneqq 0$ (c requires some material inputs), then

$$\begin{aligned} \varepsilon > r \quad &for \quad r > 0 \\ \varepsilon = r \quad &for \quad r = 0 \\ \varepsilon < r \quad &for \quad r < 0 \end{aligned}$$

In particular, the rate of profit is positive if and only if the rate of exploitation is positive. This observation was called the "Fundamental Marxian Theorem" by Morishima: "This result ... may be claimed as the Fundamental Marxian Theorem, because it asserts that the exploitation of labourers by capitalists is necessary and sufficient for the existence of a price-wage set yielding positive profits or, in other words, for the possibility of conserving the capitalist economy (Morishima (1973), p. 53).

Proof of Th. 14.1. W.l.o.g. let $p_0 = 1$. Then $pc = p_0 = 1$ and the price equation (14.6) implies

$$1 = pc = \rho a_0 c + \rho p A c$$

Similarly, the value equation $v = a_0 + vA$ implies

$$\rho v c = \rho a_0 c + \rho v A c$$

Therefore

$$1 - \rho v c = \rho(p - v)Ac$$

By definition, $1 = (1 + \varepsilon)vc$. Therefore

$$(1 + \varepsilon)vc - (1 + r)vc = (\varepsilon - r)vc = \rho(p - v)Ac \qquad (*)$$

If $Ac = 0$ this implies $r = \varepsilon$, independently of r. This proves assertion (a).

If $Ac \gneqq 0$, recall that $p = v$ for $r = 0$ by (7.10) (since $p_0 = 1$). Therefore, by Th. 10.2, $p - v > 0$ for $r > 0$, and $p - v < 0$ for $r < 0$. Assertion (b) then follows immediately from $(*)$.

■

15 The Augmented Input Matrix

Consider the quantity equation

$$x = Ax + d, \qquad L = a_0 x \tag{15.1}$$

and denote by c the per capita consumption of the workers. Then the total consumption of the workers is $d^a = c.L = ca_0x$, and $d^s = d - d^a$ is the surplus. Assume that $d^s > 0$. This implies

$$x = Ax + d^a + d^s = Ax + ca_0x + d^s = (A + ca_0)x + d^s \tag{15.2}$$

The matrix $A + ca_0$ is the *augmented input coefficient matrix.* Its (ij)-th element is $a_{ij} + c_i a_{0j}$. This is the amount of good i needed to produce one unit of good j, including the workers' consumption. Indeed, to produce one unit of good j, we need the direct input a_{ij} of good i, and a_{0j} units of labor. One unit of labor (one worker) consumes c_i units of good i, so that a_{0j} workers consume $c_i a_{0j}$ units of good i. Therefore $a_{ij} + c_i a_{0j}$ is the total amount of good i needed to produce one unit of good j, including the workers' consumption of good i. It is as if the capitalists, instead of paying the workers a money wage, gave them their consumption goods directly (like food for the cattle).

The augmented matrix $A+ca_0$ can be considered as another technology matrix, including the food for the workers. The economy can produce a positive surplus $d^s > 0$ over and above the workers' subsistence needs, if and only if the equation

$$x = (A + ca_0)x + d^s$$

has a nonnegative solution $x \gneqq 0$. This is the case if and only if the augmented matrix is productive, i.e. the inverse $[I - (A + ca_0)]^{-1}$ exists and is nonnegative. In this case, the solution is

$$x = [I - (A + ca_0)]^{-1} d^s$$

This is analogous to equation (3.7) in Sec. 3.

The following Lemma says that the augmented matrix is productive if and only if the labor value vc of the workers' per capita consumption c is less than one, i.e. if and only if the rate of exploitation $\varepsilon = (1 - vc)/vc$ is positive.

Lemma 15.1. *Assume that A is productive. Then $A + ca_0$ is productive if and only if $vc < 1$.*

Proof of Lemma 15.1.

Let $vc < 1$. Then there is $c' > c$ with $vc' < 1$. Let $x \geqq 0$ such that $x = Ax+c'$. This implies $V(c') = a_0x = vc' < 1$. Therefore $x = Ax + c' > Ax + c'a_0x > Ax + ca_0x = (A + ca_0)x$, i.e. $A + ca_0$ is productive.

Conversely, assume that $A + ca_0$ is productive. Then there is $x \geqq 0$ such that $x = (A + ca_0)x + d = Ax + ca_0x + d$ with $d > 0$, i.e. $a_0x = v(ca_0x + d) = vca_0x + vd$ with $vd > 0$. This implies $vc < 1$. ■

By Def. 1 and Cor. 5.4 the augmented matrix $A + ca_0$ is productive if and only if the following two equivalent conditions are satisfied:

(a) There exists $x \gneqq 0$ such that $x > (A + ca_0)x$

(b) There exists $p \gneqq 0$ such that $p > p(A + ca_0)$

Condition (a) means that there is a growth factor $\gamma = 1+g > 1$ such that $x > \gamma(A+ca_0)x$, i.e. the output is more than γ-times the input (including the workers' consumption). The economy can grow with a positive growth rate g, even if it provides consumption c for the workers.

Condition (b) means that there is a profit factor $\rho = 1+r > 1$ such that $p > \rho p(A+ca_0)$, i.e. the value of the output is more than ρ-times the costs (including wages). The economy can sustain a positive profit rate r, even if it pays a wage that allows the workers to consume c.

By Lemma 15.1 both conditions are satisfied if and only if $vc < 1$. Therefore exploitation is necessary and sufficient not only for positive profits, but also for positive growth. This gives the following "generalized" Marxian Theorem:

Theorem 15.2. [GENERALIZED FUNDAMENTAL MARXIAN THEOREM]

Both positive growth and positive profits are possible if and only if the rate of exploitation is positive.

16 Constant and Variable Capital, etc.

We proceed to define some further Marxian concepts.

Following Marx, we use *labor values*, not production prices, to evaluate all goods. Commodity prices are given by their labor values, $v = (v_1, \ldots, v_n)$. The wage in terms of labor values is $v_0 = vc$, i.e. it is the labor value of a worker's consumption.

Nota bene. The system $(v_0, v) = (v_0, v_1, \ldots, v_n)$ with $v_0 < 1$ is NOT a production price system in the sense of Th. 9.2! There we have seen (Cor. 9.3) that commodity prices coincide with labor values, $p = v$ if the profit rate is zero and labor is chosen as the numéraire ($p_0 = 1$). This gives the production price system $(p_0, p) = (1, v) = (1, v_1, \ldots, v_n)$, where the wage is equal to one, the rate of profit is the same (zero) in all sectors, and workers are not exploited. In the present context, commodity prices equal labor values, but the wage (in terms of labor value) is less than one. If we use the Marxian "price system" $(v_0, v_1, \ldots v_n)$, with $v_0 < 1$, then workers are exploited, all sectors make positive profits, and the rate of profit is not uniform, but different in different sectors (see below).

Formula (14.4) implies

$$v_0 + \varepsilon v_0 = v_0(1 + \varepsilon) = 1 \tag{16.1}$$

Using this, the labor value equation (7.2) can be written

$$v = vA + a_0 = vA + v_0 a_0 + \varepsilon v_0 a_0 \tag{16.2}$$

or, componentwise

$$v_j = va^j + v_0 a_{0j} + \varepsilon v_0 a_{0j} \tag{16.3}$$

The quantity $K_j = va^j = v_1 a_{1j} + v_2 a_{2j} + \ldots v_n a_{nj}$ is called the *constant capital* (per unit of output) in sector j. It is the cost (in terms of labor values) of the physical inputs.

The quantity $V_j = V_j(c) = v_0 a_{0j}$ is called the *variable capital* (per unit of output) in sector j. It represents the wage cost (again in terms of labor values). Note that $V_j > 0$ for all j because $v > 0$ and $a_0 > 0$ by assumption.

The quantity $S_j = S_j(c) = \varepsilon V_j = \varepsilon v_0 a_{0j}$ is the *surplus value* (per unit of output) in sector j. By (16.3) the value of good j is the sum of these quantities:

$$v_j = K_j + V_j + S_j \tag{16.4}$$

We write $K = (K_1, \ldots K_n) = vA$ for the vector of constant capitals, $V = V(c) = (V_1, \ldots V_n) = v_0 a_0$ for the vector of variable capitals, and $S = S(c) = (S_1, \ldots S_n) = \varepsilon V = \varepsilon v_0 a_0$ for the vector of surplus values, so that $V + S = a_0$ by (16.1), and

$$v = K + V + S \tag{16.5}$$

By definition, the rate of exploitation $\varepsilon = S_j/V_j$ is the same in all sectors.

Following Marx, we define the *organic composition of capital* in sector j as the ratio of constant to variable capital, i.e. by $q_j = K_j/V_j$.

We also define the *profit rate* in sector j by $\pi_j = S_j/(K_j + V_j)$.

The *profit* (per unit of output) in sector j is revenue minus cost, i.e. $v_j - (K_j + V_j) = S_j$. The *profit rate* π_j is this profit divided by the cost. Equivalently, we can write

$$v_j = (1 + \pi_j)[(v_1 a_{1j} + \ldots v_n a_{nj}) + v_0 a_{0j}] \quad \forall j \tag{16.6}$$

The "Marxian" profit rate π_j is defined in terms of labor values, and should not be confused with the "normal" profit rate r, which is defined in terms of production prices. When there is no exploitation and workers consume the entire net output ($v_0 = 1$ or equivalently $d^s = 0$), all profit rates are the same, namely $\pi_j = 0$ for all j, and (16.6) coincides with the value equation (7.2). But when there is exploitation ($v_0 < 1$), as we assume in this section, the profit rates π_j defined by (16.6) will be positive and, in general, different for different sectors (see Sec. 17). We have seen in the previous section that equalization of profit rates across all sectors requires a different price system, namely the production prices of Theorem 9.2, not the labor values used in (16.6). The relationship between production prices and labor values (the so-called *transformation problem*) has caused some confusion in the Marxist literature. See also the remarks at the end of Sec. 10.

Remark. We have defined constant and variable capital and surplus value *per unit of output*. Marx originally defined these concepts with reference to actual output, i.e. he used $K_j x_j$, $V_j x_j$, $S_j x_j$ instead of K_j, V_j, S_j. This gives the same organic compositions and profit rates, of course.

We can also define aggregate versions of these concepts: $Kx = \sum_j K_j x_j$ is aggregate constant capital, $Vx = \sum_j V_j x_j$ is aggregate variable capital, and $Sx = \varepsilon V x$ is the aggregate surplus value (MEHRWERT). From (16.5) we get $vx = Kx + Vx + Sx$ for the value of the gross output. Note also that $Kx = vAx$, $Vx = L^a = vd^a$, and $Sx = L^s = vd^s$.

Clearly, the rate of exploitation is given by $\varepsilon = Sx/Vx$. The aggregate (or average) *organic composition of capital* is the ratio $\bar{q}(x) = Kx/Vx$, and the aggregate (or average) *rate of profit* is the ratio $\bar{\pi}(x) = Sx/(Kx + Vx)$. Unlike the sectoral quantities q_j, π_j, which depend only on the technology (a_0, A) and the wage v_0, the aggregate quantities $\bar{q}(x)$, $\bar{\pi}$ depend also on the activity vector x.

Remark. Given the per-capita consumption c of the workers, the following quantities are determined:

- the wage v_0 by $v_0 = vc$
- the rate of exploitation ε by $\varepsilon = (1 - v_0)/v_0$
- the variable capital V_j by $V_j = v_0 a_{0j}$ for $\forall j$
- the organic composition q_j by $q_j = va^j/(v_0 a_{0j})$ for $\forall j$
- the (Marxian) profit rate π_j by $v_j = (1 + \pi_j)[va^j + v_0 a_{oj}]$ for $\forall j$

Clearly, any other consumption c' with $vc' = vc$ gives the same values for $v_0, \varepsilon, V_j, q_j, \pi_j$. Conversely, any one of the quantities $v_0, \varepsilon, V_j, q_j, \pi_j$ (any j) determines all the others uniquely, by the formulae given above. All these numbers contain the same information.

Remark on Terminology. The distinction between "constant" and "variable" capital refers to the different roles played by material inputs and labor in the creation of "value" in Marxist theory. When the inputs $a_{0j}, a_{1j}, \ldots a_{nj}$ are transformed into one unit of good j in the process of production, the (embodied) labor value of the output $v_j = a_{0j} + v_1 a_{1j} + \ldots v_n a_{nj} = a_{0j} + K_j$ (recall equation (16.3)) is larger than the value of the inputs $V_j + K_j$ (the sum of constant and variable capital), because $V_j = v_0 a_{0j} < a_{0j}$ if workers are exploited ($v_0 < 1$). The interpretation is that the value contained in the material inputs, namely the constant capital $K_j = v_1 a_{1j} + \cdots + v_n a_{nj}$, is simply passed on to the output without change (remaining "constant"), but the value of the labor input, namely the variable capital $V_j = v_0 a_0$, increases in the process of production (is "variable") and contributes more than V_j, namely a_{0j}, to the value of the output. Labor (and only labor) creates value. Note also that this Marxian terminology has nothing to do with the common distinction between "fixed" and "circulating" capital. Fixed capital refers to means of production that last for several periods, like buildings and machinery. Circulating capital refers to means of production that are used up in one period. In our model, there is only circulating capital. To take proper account of fixed capital, one would have to introduce joint production.

17 An Extended Example

The following example illustrates the concepts introduced in the preceding sections.

Example 17.1.

Let (a_0, A) be the already familiar technology from examples 4.1 and 10.3:

$$a_0 = (a_{01}, a_{02}) = (6,\ 6), \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

Then

$$I - A = \frac{1}{10}\begin{pmatrix} 9 & -4 \\ -3 & 8 \end{pmatrix} \quad \text{and} \quad (I - A)^{-1} = \frac{1}{6}\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix}$$

We know also from Ex. 8.5 that $\lambda(A) = 1/2$ and $\alpha^* = \hat{\alpha}(A) = 2$.
Labor values $v = (v_1, v_2)$ are given by

$$v = a_0(I - A)^{-1} = (6, 6).\frac{1}{6}\begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix} = (11, 13)$$

Now consider the price equation

$$p = \rho[p_0 a_0 + pA] \qquad (0 < \rho < \alpha^* = 2)$$

and assume $\rho = 1 + r = 5/4$, so that the profit rate is $r = 1/4$. Put $p_0 = 1$. This gives the prices (see Ex. 10.3)

$$p = (p_1, p_2) = \rho a_0[I - \rho A]^{-1} = (18, 22)$$

Let $c \gneqq 0$ be a per capita consumption bundle which satisfies the workers' budget constraint $pc = p_0 = 1$, e.g.

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/18 \\ 0 \end{pmatrix}, \qquad pc = (18, 22)\begin{pmatrix} 1/18 \\ 0 \end{pmatrix} = 18 \times \frac{1}{18} + 0 = 1 = p_0$$

Then

$$ca_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(a_{01}, a_{02}) = \begin{pmatrix} c_1 a_{01} & c_1 a_{02} \\ c_2 a_{01} & c_2 a_{02} \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 0 & 0 \end{pmatrix}$$

and the augmented matrix is

$$A + ca_0 = \frac{1}{10}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 1/3 & 1/3 \\ 0 & 0 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 13 & 22 \\ 9 & 6 \end{pmatrix}$$

This gives

$$I - (A + ca_0) = \frac{1}{30}\begin{pmatrix} 17 & -22 \\ -9 & 24 \end{pmatrix}, \quad \det[I - (A + ca_0)] = \frac{7}{30}$$

and the Leontief inverse of the augmented matrix

$$[I - (A + ca_0)]^{-1} = \frac{1}{7}\begin{pmatrix} 24 & 22 \\ 9 & 17 \end{pmatrix}$$

This is nonnegative, so the augmented matrix is productive.
Since $pc = p_0$, the price equation implies

$$p = \rho[p_0 a_0 + pA] = \rho[pca_0 + pA] = \rho p[A + ca_0]$$

i.e. the price vector is a left eigenvector of the augmented matrix (associated with the eigenvalue $1/\rho = 4/5$). We can check this:

$$\rho p[A + ca_0] = \frac{5}{4}(18, 22)\frac{1}{30}\begin{pmatrix} 13 & 22 \\ 9 & 6 \end{pmatrix} = \frac{1}{24}(18\times 13 + 22\times 9, 18\times 22 + 22\times 6) = (18, 22) = p$$

as it must be.
The labor value of the workers' consumption and the rate of exploitation are

$$v_0 = vc = (11, 13)\begin{pmatrix} 1/18 \\ 0 \end{pmatrix} = \frac{11}{18}, \quad \varepsilon = \frac{1 - vc}{vc} = \frac{7}{11} = 0.6363$$

By Lemma 15.1 this shows again that the augmented matrix is productive, i.e. it can produce a positive surplus d^s (in fact, any positive surplus, given enough labor). For example, choose

$$d^s = \begin{pmatrix} d_1^s \\ d_2^s \end{pmatrix} = \begin{pmatrix} 25 \\ 30 \end{pmatrix}$$

and consider the "augmented" quantity equation

$$x = (A + ca_0)x + d^s$$

The solution is

$$x = [I - (A + ca_0)]^{-1} d^s = \frac{1}{7}\begin{pmatrix} 24 & 22 \\ 9 & 17 \end{pmatrix}\begin{pmatrix} 25 \\ 30 \end{pmatrix} = \begin{pmatrix} 180 \\ 105 \end{pmatrix}$$

and the required amount of labor (the number of workers) is (cf. Ex. 6.4)

$$L = a_0 x = (6,6)\begin{pmatrix} 180 \\ 105 \end{pmatrix} = 1710$$

The workers' total consumption is

$$d^a = cL = \begin{pmatrix} 1/18 \\ 0 \end{pmatrix} \times 1710 = \begin{pmatrix} 95 \\ 0 \end{pmatrix}$$

The material inputs are

$$Ax = \frac{1}{10}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 180 \\ 105 \end{pmatrix} = \begin{pmatrix} 60 \\ 75 \end{pmatrix}$$

One can check that $x = Ax + d^a + d^s$:

$$\begin{pmatrix} 180 \\ 105 \end{pmatrix} = \begin{pmatrix} 60 \\ 75 \end{pmatrix} + \begin{pmatrix} 95 \\ 0 \end{pmatrix} + \begin{pmatrix} 25 \\ 30 \end{pmatrix}$$

The total income of the workers is $p_0 L = 1710$. The value of their total consumption is the same:

$$pd^a = (18,22)\begin{pmatrix} 95 \\ 0 \end{pmatrix} = 1710$$

The total investment of the capitalists consists of the cost pAx of the aggregate inputs and the advanced wages $p_0 L$, i.e. it is

$$p(A + ca_0)x = pAx + pca_0 x = pAx + p_0 L = (18,22)\begin{pmatrix} 60 \\ 75 \end{pmatrix} + 1710 = 2730 + 1710 = 4440$$

The total profit is $r \times$ this amount, i.e.

$$rp(A + ca_0)x = \frac{1}{4} \times 4440 = 1110$$

This is equal to the value of the surplus d^s:

$$pd^s = (18,22)\begin{pmatrix} 25 \\ 30 \end{pmatrix} = 1110$$

Finally, necessary labor L^a and surplus labor L^s in this economy are

$$L^a = vd^a = (11, 13)\begin{pmatrix} 95 \\ 0 \end{pmatrix} = 1045, \quad L^s = vd^s = (11, 13)\begin{pmatrix} 25 \\ 30 \end{pmatrix} = 665$$

with $L^a + L^s = 1045 + 665 = 1710 = L$. The rate of exploitation, as defined as the ratio of these quantities, is

$$\varepsilon = \frac{L^s}{L^a} = \frac{665}{1045} = \frac{7}{11} = 0.6363$$

the same as $(1 - vc)/vc$ before.

Using the labor values, one can also compute various other "Marxian" quantities for this example: the vector of constant capitals $K = (K_1, K_2) = vA$, the vector of variable capitals $V = (V_1, V_2) = v_0 a_0$, the vector of surpuses $S = (S_1, S_2) = \varepsilon V$ (all per capita), the organic compositions $q_j = K_j/V_j$ and the "Marxian" profit rates $\pi_j = S_j/(K_j + V_j)$ for the two sectors $j = 1, 2$.

This gives, with $v_0 = \frac{11}{18}$, $\varepsilon = \frac{7}{11}$, and $v = (v_1, v_2) = (11, 13)$:

$$K = (K_1, K_2) = vA = (11, 13)\frac{1}{10}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \frac{1}{10}(50, 70) = (5, 7)$$

$$V = (V_1, V_2) = v_0 a_0 = \frac{11}{18}(6, 6) = (\frac{11}{3}, \frac{11}{3})$$

$$S = (S_1, S_2) = \varepsilon V = \frac{7}{11}(\frac{11}{3}, \frac{11}{3}) = (\frac{7}{3}, \frac{7}{3})$$

One can check that $K + V + S = v$:

$$(5, 7) + (\frac{11}{3}, \frac{11}{3}) + (\frac{7}{3}, \frac{7}{3}) = (5, 7) + (6, 6) = (11, 13) = (v_1, v_2)$$

The organic compositions $q_j = K_j/V_j$ in the two sectors are

$$q_1 = \frac{K_1}{V_1} = \frac{5}{11/3} = \frac{15}{11} = 1.3636, \quad q_2 = \frac{K_2}{V_2} = \frac{7}{11/3} = \frac{21}{11} = 1.9090$$

and the Marxian profit rates $\pi_j = S_j/(K_j + V_j)$ are

$$\pi_1 = \frac{S_1}{K_1 + V_1} = \frac{7/3}{5 + 11/3} = \frac{7}{26} = 0.2692, \quad \pi_2 = \frac{S_2}{K_2 + V_2} = \frac{7/3}{7 + 11/3} = \frac{7}{32} = 0.2187$$

Note that the organic composition of capital is different in the two sectors, $q_1 \neq q_2$. The Marxian profit rates are also different, $\pi_1 \neq \pi_2$. This may be compared with the uniform profit rate $r = 1/4 = 0.25$ in the price system.

The total amounts of constant capital, variable capital and surplus value in sector j are given by the per capita amounts times x_j, i.e. by $K_j x_J$, $V_j x_j$, and $S_j x_j$; and the total (labor) value of the output is $v_j x_j$.
This gives, with $x = (x_1, x_2)^{\mathsf{T}} = (180, 105)^{\mathsf{T}}$:

$$v_1 x_1 = 11 \times 180 = 1980,\ K_1 x_1 = 5 \times 180 = 900,\ V_1 x_1 = \frac{11}{3} \times 180 = 660,\ S_1 x_1 = \frac{7}{3} \times 180 = 420$$

$$v_2 x_2 = 13 \times 105 = 1365,\ K_2 x_2 = 7 \times 105 = 735,\ V_2 x_2 = \frac{11}{3} \times 105 = 385,\ S_2 x_2 = \frac{7}{3} \times 105 = 245$$

One can check that

$$v_1 x_1 = 1980 = 900 + 660 + 420 = K_1 x_1 + V_1 x_1 + S_1 x_1$$

and

$$v_2 x_2 = 1365 = 735 + 385 + 245 = K_2 x_2 + V_2 x_2 + S_2 x_2$$

Clearly, these total values give the same organic compositions and profit rates as the per capita values, e.g.

$$\pi_2 = \frac{S_2 x_2}{K_2 x_2 + V_2 x_2} = \frac{245}{735 + 385} = \frac{245}{1120} = \frac{7 \times 35}{32 \times 35} = \frac{7}{32}$$

Finally, we compute the aggregate (economy - wide) values. The aggregegate constant capital is

$$\bar{K} = Kx = K_1 x_1 + K_2 x_2 = 900 + 735 = 1635$$

It is the same as the value of all inputs

$$vAx = (11, 13) \begin{pmatrix} 60 \\ 75 \end{pmatrix} = 660 + 975 = 1635$$

The aggregate variable capital is

$$\bar{V} = Vx = V_1 x_1 + V_2 x_2 = 660 + 385 = 1045$$

This is the same as the necessary labor:

$$L^a = vd^a = vcL = v_0 L = \frac{11}{18} \times 1710 = 1045$$

The aggregate surplus value (MEHRWERT) is

$$\bar{S} = Sx = S_1 x_1 + S_2 x_2 = 420 + 245 = 665$$

This is the same as the surplus labor:

$$L^s = vd^s = \varepsilon L^a = \frac{7}{11} \times 1045 = 665$$

The total value of the output is the sum of these terms:

$$\bar{K}+\bar{V}+\bar{S} = 1635+1045+665 = 3345 = 1980+1365 = v_1x_1+v_2x_2 = vx = vAx+vd = vAx+vd^a+vd^s$$

The aggregate (or average) composition of capital is

$$\bar{q} = \frac{\bar{K}}{\bar{V}} = \frac{165}{1045} = 1.56459$$

and the average rate of profit is

$$\bar{\pi} = \frac{\bar{S}}{\bar{K}+\bar{V}} = \frac{665}{1635+1045} = \frac{665}{2680} = 0.2481$$

Unlike the sectoral quantities q_j, π_j these aggregate quantities depend on the activity vector x.

The rate of profit and the rate of exploitation.
We have chosen the bundle $c = (1/18, 0)^{\mathsf{T}}$ arbitrarily, under the constraint that $pc = 1$. We could also choose another bundle c' which satisfies $pc' = 1$. The profit rate $r = 1/4$ and the prices $p = (18, 22)$ would not be affected by this, but the rate of exploitation will change. For example, if we choose $c' = (0, c_2)^{\mathsf{T}}$, then $pc = (18, 22)(0, c_2)^{\mathsf{T}} = 1$ implies $c' = (0, 1/22)^{\mathsf{T}}$. The labor value of this is $v_0 = vc' = (11, 13)(0, 1/22)^{\mathsf{T}} = 13/22$. Therefore

$$\varepsilon' = (1 - vc')/vc' = 9/13 = 0.6923$$

Given the profit rate $r = 1/4$, a worker can choose to consume any combination $c = (c_1, c_2)^{\mathsf{T}}$ of the two goods which satisfies the budget constraint $pc = 18c_1 + 22c_2 = p_0 = 1$. If he consumes only good 1, the rate of ecploitation is $\varepsilon = 7/11 = 0.63$. If he consumes only good 2, the rate of exploitation is $\varepsilon' = 9/13 = 0.69$, and he seems to be "more exploited". The reason for this is that the price of good 2 rises faster than the price of good 1 when the profit factor increases. Thus there is no $(1 - 1)$-relationship between profit rates and rates of exploitation. The latter depend also on the composition of the workers' consumption bundle.

18 Constant Relative Prices

The Marxian belief that labor values were in some sense more fundamental than market prices has led Marxist economists to look for cases where the latter are equal or at least proportional to the former. One may say that the *Labor Theory of Value holds in the strict sense* when prices are proportional to labor values.

We have seen that this true when the rate of profit is zero, but not in general. As the rate of profit increases, the prices of all goods increase relative to the wage, but some prices increase faster, and others more slowly. It is even possible that a good is more expensive than another for low interest rates, becomes cheaper than the other as the interest rate increases, and becomes more expensive again if the interest rate increases further (reswitching). Thus there is no hope to explain the relative prices of goods in terms of labor values, when the profit rate is not zero, at least in general.

But there is a special type of technology in which the commoditiy prices are always proportional to their labor values, independently of the profit rate. We will now show that such technologies are characterized by the property that the organic composition of capital is the same in all sectors.

Let (a_0, A) be a productive technology for which labor is indispensable. Denote by $\alpha^* = \hat{\alpha}(A)$ the expansion factor and by $\lambda^* = 1/\alpha^*$ the dominant eigenvalue of A. Recall that two vectors x and y are proportional, written $x \sim y$, if there is a number $k \neq 0$ with $y = k.x$.

Consider the value equation

$$v = a_0 + vA \tag{18.1}$$

and the price equation

$$p = \rho(p_0 a_0 + pA) \qquad (0 < \rho < \alpha^*) \tag{18.2}$$

The value equation has a unique positive solution $v > 0$. For $\rho \in (0, \alpha^*)$, denote by $(p_0(\rho), p(\rho))$ a positive solution of the price equation. Such a solution exists by Th. 9.2, and is unique up to multiplication by a constant.

Case 1. Simple Commodity Production: $A = 0$.

Consider first the trivial case where all goods are produced by direct labor alone (Marx' "simple commodity production"). This means $a_0 > 0$ and $A = 0$. Clearly $\lambda^* = 0$, $\alpha^* = \infty$, and the price equation (18.2) becomes

$$p(\rho) = \rho\, p_0(\rho) a_0 \qquad \text{for all} \quad \rho > 0$$

Commodity prices are always proportional to direct labor inputs, $p(\rho) \sim a_0$ for all $\rho > 0$, these coincide with labor values, $v = p(1) = a_0$, and relative prices do not change with ρ.

The organic composition of capital $q_j = K_j/V_j$ is zero in all sectors, because there is no constant capital, $K_j = va^j = 0$ for all j. Note also that

$$a_0 A = \lambda^* a_0 = 0 \tag{18.3}$$

i.e. a_0 is a (left) eigenvector of A associated with the dominant eigenvalue (which is zero in this case).

Case 2. The General Case: $A \gneq 0$.

Consider next the general case where A is not zero. The organic composition of capital in sector j is $q_j = K_j/V_j$. It is the same in all sectors if $q_j = q$ for all j, i.e. if $K = qV$ for some number q, or, equivalently, if $K \sim V$.

We begin with a simple observation.

Lemma 18.1. *Let (a_0, A) be a productive technology with $A \gneq 0$ and $a_0 > 0$, and let $\lambda^* = 1/\alpha^* < 1$ be the dominant eigenvalue of A. Then the following conditions are equivalent:*

(a) the organic composition of capital is the same in all sectors, $K = qV$

(b) $vA \sim a_0$

(c) $a_0 A \sim a_0$

Moreover, in this case, $a_0 A = \lambda^* a_0$, $vA = \lambda^* v$, and $a_0 = (1 - \lambda^*)v$.

Proof of Lemma 18.1.

To prove $(a) \leftrightarrow (b)$, recall that $q_j = K_j/V_j$, where $K_j = va^j$ and $V_j = v_0 a_{0j}$ for some positive number v_0. Therefore $K = (K_1, \dots K_n) = vA$, and $V = (V_1, \dots V_n) = v_0 a_0$. If $q_j = q$ for all j, we have $K = qV$ or, equivalently, $vA = qv_0 a_0$, i.e. $vA \sim a_0$.

To prove $(b) \rightarrow (c)$, assume that $a_0 \sim vA$. The value equation $v = a_0 + vA$ implies then $v \sim a_0 \sim vA$. By Cor. 8.4 $v \sim vA$ and $v > 0$ imply $vA = \lambda^* v$. Since $a_0 \sim v$, this implies also $a_0 A = \lambda^* a_0$.

To prove $(c) \rightarrow (b)$, assume that $a_0 A = \lambda a_0$ for some λ. Define $w = \frac{1}{1-\lambda} a_0$. Then

$$a_0 + wA = a_0 + \frac{1}{1-\lambda} a_0 A = a_0 + \frac{1}{1-\lambda} \lambda a_0 = \frac{1}{1-\lambda} a_0 = w$$

i.e. w satisfies the value equation.Therefore $v = w = \frac{1}{1-\lambda} a_0$ or, equivalently, $a_0 = (1-\lambda)v$. The value equation now implies $v = (1 - \lambda)v + vA$ or $\lambda v = vA$. By Cor. 8.4, $\lambda = \lambda^*$. Since $a_0 \sim v$, this implies also $a_0 A = \lambda^* a_0$.

■

Lemma 18.1 says that the organic composition of capital is the same in all sectors if and only if the labor input vector a_0 is proportional to the vector $a_0 A$. This property of the technology (a_0, A) can be checked easily (simply by computing $a_0 A$), without recourse to labor values. The proof of Th. 18.2 below shows that in this case the commodity prices p are proportional to the direct labor inputs a_0, for all profit rates (in particular for $\rho = 1$, i.e. labor values v are also proportional to a_0).

In the Marxist literature, this simple result is usually presented in terms of labor values and the organic composition of capital, and we follow this tradition in the formulation of Th. 18.2 below.

Theorem 18.2. *Let (a_0, A) be a productive technology with $A \gneqq 0$ and assume that $a_0 > 0$. Let α^* be the expansion factor and $\lambda^* = 1/\alpha^*$ be the dominant eigenvalue of the matrix A. Denote by $(p_0(\rho), p(\rho))$ production prices associated with the profit factor ρ. The following conditions are equivalent:*

(a) $p(\rho) \sim v$ for $0 < \rho < \alpha^*$, i.e. commodity prices are proportional to labor values for all profit rates.

(b) The organic composition of capital is the same in all sectors.

Moreover, in this case, a solution of the price equation is given by

$$p_0 = p_0(\rho) = 1 - \rho\lambda^*, \quad p = p(\rho) = \rho a_0 \tag{18.4}$$

Proof of Th. 18.2.

To show that (a) implies (b), let (p_0, p) be a positive solution of the price equation, for some $\rho \neq 1$. and assume that $p \sim v$. Then $p = kv$ for some $k > 0$ and the value and price equations imply

$$ka_0 + kvA = kv = \rho(p_0 a_0 + kvA)$$

Therefore $(k - \rho p_0)a_0 = k(\rho - 1)vA$, with $\rho - 1 \neq 0$, so that $a_0 \sim vA$. By Lemma 18.1 this implies (b).

Conversely, assume that the organic composition is the same in all sectors. By Lemma 18.1 this implies that $a_0 A = \lambda A$ where $\lambda = \lambda^*$ is the dominant eigenvalue of A. We have $0 < \lambda < 1$ because A is productive ($\alpha^* > 1$). Now define (p_0, p) by

$$p_0 = p_0(\rho) = 1 - \rho\lambda, \quad p = p(\rho) = \rho a_0$$

We claim that these prices satisfy the price equation (18.2). Indeed, substituting it in (18.2) gives

$$\rho a_0 = \rho[(1 - \rho\lambda)a_0 + \rho a_0 A] = \rho[(1 - \rho\lambda)a_0 + \rho\lambda a_0] = \rho a_0$$

so that $(p_0(\rho), p(\rho)) = (1 - \rho\lambda, \rho a_0)$ is a solution of (18.2), for all $\rho \in (0, \alpha^*)$. Clearly $p(\rho) = \rho a_0 \sim a_0 \sim v$. Since the prices are unique up to a positive multiple, every solution of the price equation satisfies $p \sim v$, i.e. (a) holds.

■

Th. 18.2 and its proof show that the following conditions are all equivalent:

$$p \sim v \;\Leftrightarrow\; v \sim a_0 \;\Leftrightarrow\; v \sim vA \;\Leftrightarrow\; a_0 \sim a_0 A \;\Leftrightarrow\; p \sim pA \;\Leftrightarrow\; a_0 \sim vA \tag{18.5}$$

The first two conditions say that commodity prices p are proportional to labor values v, which are in turn proportional to direct labor inputs a_0. The next three conditions say that v, a_0 and p are left eigenvectors of the matrix A; and the last condition says that the organic composition of capital is the same in all sectors.

Moreover, if any one of these conditions holds, the eigenvectors a_0, v and p are associated with the dominant eigenvalue $\lambda = 1/\alpha^*$ of A (by Cor. 8.4):

$$a_0 A = \lambda a_0, \quad vA = \lambda v, \quad pA = \lambda p \tag{18.6}$$

(in particular, (18.3) holds also in the general case). The value equation then gives $v = a_0 + \lambda v$ or

$$a_0 = (1 - \lambda)v \tag{18.7}$$

The prices given in (18.4) are normalized such that they take a particularly simple form: As ρ increases from 0 to $\alpha^* = 1/\lambda$, the wage $p_0(\rho) = 1 - \rho\lambda$ falls from 1 to zero, and the commodity prices $p_j(\rho) = \rho a_{0j}$ increase linearly with ρ, for all $j \in N$.

The commodity prices relative to the wage (i.e. the prices if labor is the numéraire) are

$$\frac{p(\rho)}{p_0(\rho)} = \frac{\rho}{1 - \rho\lambda} a_0 = \frac{\rho}{1 - \rho/\alpha^*} a_0 \tag{18.8}$$

As ρ increases from 0 to α^*, these prices increase from 0 to infinity. For $\rho = 1$ they coincide with the labor values:

$$\frac{p(1)}{p_0(1)} = \frac{1}{1 - \lambda} a_0 = v$$

Connection with dated labor inputs.

We have seen in Sec. 11 that the "reason" for the irregular behavior of commodity prices as the profit rate increases lies in the uneven distribution of the dated labor inputs for different goods. But if the proportionaliy condition (18.5) holds, this cannot happen: if $a_0 A = \lambda a_0$, the dated labor inputs $\ell_t(d) = a_0 A^t d$ take a particularly simple form. To see this, we note first that (18.3) implies

$$a_0 A = \lambda a_0, \; a_0 A^2 = \lambda a_0 A = \lambda^2 a_0, \; \ldots a_0 A^t = \lambda^t a_0, \; \ldots$$

so that $\ell_t(d) = \lambda^t a_0 d$. The sequence of dated labor inputs

$$(\ell_0(d), \ell_1(d), \ell_2(d), \dots) \;=\; (1, \lambda, \lambda^2, \dots)\, a_0 d$$

is proportional to the sequence $(1, \lambda, \lambda^2, \dots)$ for every commodity bundle d. Therefore the dated labor inputs for all periods and for all commodities are always proportional to each other, and a change in the profit factor ρ affects the price of every commodity bundle in the same way. Relative commodity prices do not change with ρ. In terms of dated labor inputs, the production conditions for all commodities are the same. Such a technology is of course a very special case, and one cannot expect to find it in reality. It is mainly of theoretical interest.

Example 18.3.

Example 17.1 had the property that the relative price of the two commodities changes with the profit rate. For $\rho = 1$ we had $p_1/p_2 = 11/13 = 0.8461$, and for $\rho = 5/4$ we had $p_1/p_2 = 18/22 = 0.8181$. The reason for this is that the organic composition of capital was different in the two sectors.

Let us now change the technology such that the organic composition is the same in both sectors. To this end, consider again the familiar matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

but assume that the labor input vector is now

$$a_0 = (6, 8)$$

(instead of $(6, 6)$ as before). Then the matrices $I - A$ and $(I - A)^{-1}$ and the expansion factor $\alpha^* = \hat{\alpha}(A) = 2$ and the dominant eigenvalue $\lambda^* = 1/\alpha^* = 1/2$ remain the same as in Ex. 17.1. But labor values and prices change.

Labor values $v = (v_1, v_2)$ are now given by

$$v = a_0(I - A)^{-1} = (6, 8).\frac{1}{6} \begin{pmatrix} 8 & 4 \\ 3 & 9 \end{pmatrix} = (12, 16) = \frac{1}{1 - \lambda^*} a_0 = 2 \times (6, 8)$$

This implies that

$$vA = (12, 16)\frac{1}{10} \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \frac{1}{10}(12 + 48, 48 + 32) = (6, 8) \sim a_0 = (6, 8)$$

i.e. the organic composition is the same in both sectors.

By (18.8), and if we put $p_0 = 1$, the prices $p = (p_1, p_2)$ are given by

$$p = (p_1, p_2) = \frac{\rho}{1 - \rho\lambda^*} a_0 = \frac{\rho}{1 - \rho/2}(6, 8) \qquad (0 < \rho < 2 = \alpha^*)$$

For $\rho = 1$ this gives $p = \frac{1}{1-1/2}(6, 8) = (12, 16)$, i.e. the labor values. For $\rho = 5/4$ this gives

$$p = (p_1, p_2) = \frac{5/4}{1 - 5/8}(6, 8) = \frac{10}{3}(6, 8) = (\frac{60}{3}, \frac{80}{3}) = (20, 26.66)$$

We can check this by substituting it in the price equation $p = \rho(p_0 a_0 + pA)$ (with $p_0 = 1$ and $\rho = 5/4$):

$$\rho(p_0 a_0 + pA) = \frac{5}{4}[(6, 8) + (\frac{60}{3}, \frac{80}{3})\frac{1}{10}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}] = \frac{5}{4}[(6, 8) + \frac{1}{3}(30, 40)] = (20, 80/3) = p$$

So commodity prices are proportional to labor values (and to a_0).

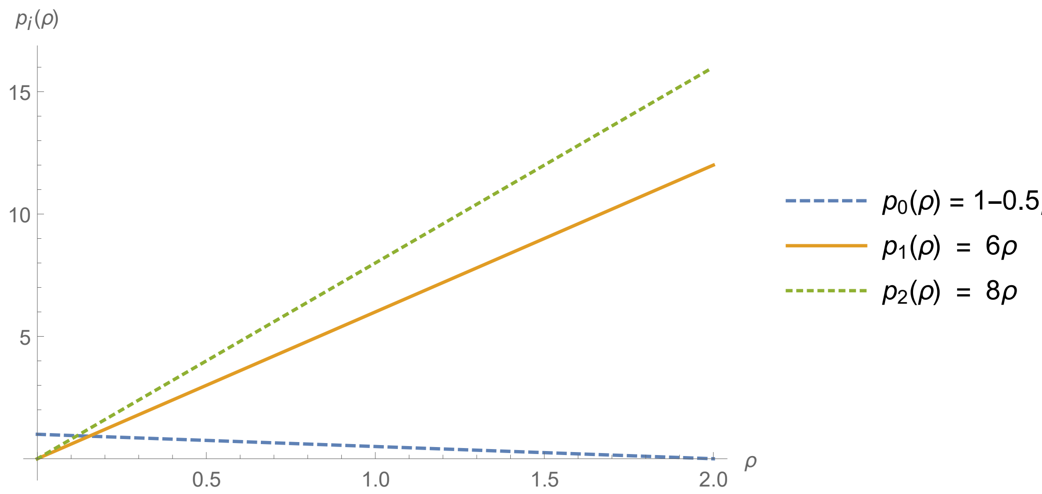

Figure 8: Prices $p_i(\rho)$ in Example 18.3 (constant organic composition of capital), normalized as in (18.4).

19 Appendix

Lemma 19.1. (Geometric Series) *Let A be a nonnegative square matrix. Then the following statements are equivalent:*

(a) the matrix $(I - A)$ is invertible and $(I - A)^{-1}$ is nonnegative

(b) the infinite sum $\sum_{t=0}^{\infty} A^t = I + A + A^2 + A^3 + \cdots$ is convergent

(c) $\lim_{t \to \infty} A^t = 0$

Moreover, if any of the above conditions is satisfied, then

$$(I - A)^{-1} = \sum_{t=0}^{\infty} A^t \tag{19.1}$$

Formula (19.1) generalizes the well-known formula for the sum of the geometric series

$$1 + a + a^2 + a^3 + \cdots = 1/(1-a) = (1-a)^{-1} \quad \text{for } 0 \leqq a < 1 \tag{19.2}$$

to nonnegative matrices.

Proof of Lemma 19.1.
Define the partial sums $M_s = \sum_{t=0}^{s} A^t$ for $s = 0, 1, 2, \ldots$. This implies

$$(I - A)M_s = M_s(I - A) = I - A^{s+1} \leqq I \quad \forall s \tag{$*$}$$

Assume (a), i.e. $(I-A)^{-1}$ is nonnegative. Multiplying $(*)$ by $(I-A)^{-1}$ gives $M_s \leqq (I-A)^{-1}$ for all s, i.e. the sequence (M_s) is bounded above. It is also increasing, hence convergent, $M_s \to M = \sum_{t=0}^{\infty} A^t$. Therefore $(a) \to (b)$.
The implication $(b) \to (c)$ is trivial.
By passing to the limit in $(*)$ we get $M = (I - A)^{-1}$. This proves the formula (19.1).
Now assume (c), i.e. $A^t \to 0$. Then, for all s sufficiently large, the $(n \times n)$-matrix $I - A^{s+1}$ is invertible. Therefore, by $(*)$, the matrices $I - A$ and M_s are also invertible, and $(*)$ implies

$$M_s = (I - A)^{-1}(I - A^{s+1})$$

For $s \to \infty$ the right-hand side converges to $(I - A)^{-1}$, hence $(I - A)^{-1} = \lim_{s \to \infty} M_s$ is nonnegative. Therefore $(c) \to (a)$.

■

Corollary 19.2. *Let $A \geqq B \geqq 0$ be two productive matrices. Then*

$$(I - A)^{-1} \geqq (I - B)^{-1}$$

Proof of Cor. 19.2.
Immediate from (19.1): assume $A \geqq B$. Then $A^t \geqq B^t$ for $t = 0, 1, 2, \ldots$ and

$$(I - A)^{-1} = I + A + A^2 + \cdots \geqq I + B + B^2 + \cdots = (I - B)^{-1}$$

■

Lemma 19.3. (Inversion Lemma.) *Let $M = [m_{ij}]$ be a square matrix, with $m_{ij} \leqq 0$ for $i \neq j$. Then M is nonnegatively invertible if and only if $M\bar{x} > 0$ for some nonnegative vector $\bar{x} \geqq 0$.*

Proof of Lemma 19.3.
If M is nonnegatively invertible, $M^{-1} \geqq 0$, choose some $d > 0$ and put $\bar{x} = M^{-1}d$. Then $\bar{x} \geqq 0$ and $M\bar{x} = d > 0$.

To show the converse, we proceed in three steps.

Step 1. First we show that if M satisfies the hypothesis of the Theorem, then $Mx \geqq 0$ implies $x \geqq 0$. Let $\bar{x} \geqq 0$ and $M\bar{x} > 0$. For each i, we have

$$[M\bar{x}]_i = m_{ii}\bar{x}_i + \sum_{j \neq i} m_{ij}\bar{x}_j > 0, \text{ hence } m_{ii} > 0 \text{ and } \bar{x}_i > 0, \text{ hence } \bar{x} > 0 \qquad (*)$$

where the first implication follows from the assumption that $m_{ij} \leqq 0$ for $j \neq i$. Now assume that $Mx \geqq 0$, but $x_i < 0$ for some i. Choose the smallest number ϑ such that $x' = x + \vartheta\bar{x} \geqq 0$. Then $\vartheta > 0$ and $x'_k = 0$ for some k. Moreover, $Mx' = Mx + \vartheta M\bar{x} > 0$, hence $x' > 0$ by $(*)$, a contradiction. Therefore $Mx \geqq 0$ implies $x \geqq 0$. This proves Step 1.

Step 2. Next we show that M is invertible. It suffices to show that $Mx = 0$ implies $x = 0$. By Step 1, $Mx = 0$ implies $x \geqq 0$. It also implies $-Mx = 0$, hence, again by Step 1, $-x \geqq 0$. Therefore $x = 0$ and M^{-1} exists.

Step 3. It remains to show that M^{-1} is nonnegative. Choose an arbitrary $y \geqq 0$ and put $x = M^{-1}y$. Then $Mx = y \geqq 0$, hence $x \geqq 0$, again by Step 1. Thus $y \geqq 0$ implies $M^{-1}y \geqq 0$. Choosing for y the unit vectors, $y = e^j$, shows that $M^{-1} \geqq 0$.

■

Theorem 19.4. (Perron-Frobenius) *Let $A \geqq 0$ be a square matrix with expansion factor $\alpha^* = \hat{\alpha}(A)$, and let $\lambda^* = 1/\alpha^*$. Then*

(a) The number λ^* is a nonnegative eigenvalue of A.

(b) There exist nonnegative right and left eigenvectors of A, associated with λ^*.

(c) λ^* is the largest eigenvalue in absolute value: $\lambda^* \geqq |\mu|$ for every eigenvalue μ of A.

Proof of Th. 19.4.

If $A = 0$ is a zero matrix, then $\hat{\alpha}(A) = \infty$ and $Ax = 0$ for every x. The only eigenvalue is $\hat{\lambda} = 0$ and every nonzero x is an eigenvector. Assume now $A \gneqq 0$.

Put $\alpha^* = \hat{\alpha}(A)$ and $\lambda^* = 1/\alpha^* = \hat{\lambda}(A)$. To prove (a) and (b), choose some $c > 0$. For every positive $\alpha < \alpha^*$, there is $x(\alpha) > 0$ such that

$$(I - \alpha A)x(\alpha) = c \tag{$*$}$$

because αA is productive for $\alpha < \alpha^*$. By Cor. 19.2, every component of $x(\alpha) = (I - \alpha A)^{-1}c$ is (weakly) monotonically increasing in α, and so is the sum $\sigma(\alpha) = \sum_{i=1}^{n} x_i(\alpha)$. If α increases towards α^*, therefore either $x(\alpha) \to x^*$ converges or $\sigma(\alpha) \to \infty$. Assume first that $\alpha^* < \infty$. If $x(\alpha)$ converged, passing to the limit in $(*)$ would give $(I - \alpha^* A)x^* = c > 0$, i.e. $\alpha^* A$ would be productive, contradicting the definition of α^*. Therefore $\sigma(\alpha) \to \infty$. Define $y(\alpha) = x(\alpha)/\sigma(\alpha)$. Then $y(\alpha) \in \Delta$ for all α and by passing to a subsequence if necessary we may assume that $y(\alpha)$ converges to some $\hat{x} \in \Delta$ for $\alpha \to \alpha^*$. Dividing $(*)$ by $\sigma(\alpha)$ gives

$$(I - \alpha A)y(\alpha) = c/\sigma(\alpha)$$

and passing to the limit for $\alpha \to \alpha^*$ gives $(I - \alpha^* A)\hat{x} = 0$, or $A\hat{x} = \lambda^* \hat{x}$.

Assume now $\alpha^* = \infty$ (i.e. $\lambda^* = 0$). Then certainly $\alpha\sigma(\alpha) \to \infty$. Dividing $(*)$ by this gives

$$\left(\frac{1}{\alpha} I - A\right)y(\alpha) = \frac{c}{\sigma(\alpha)}$$

Passing to the limit for $\alpha \to \alpha^*$ gives $A\hat{x} = 0 = \lambda^* \hat{x}$. Applying the same reasoning to A^{T} proves the assertion for a left eigenvector $\hat{p}$. This proves (a) and (b).

To prove (c), let $Ax = \mu x$. If $\mu = 0$ then $|\mu| \leqq \lambda^*$ trivially. If $\mu \neq 0$, this implies $x = \beta A x$ with $\beta = 1/\mu$. By Cor. 8.2, this implies $|\beta| \geqq \alpha^*$ or, equivalently, $|\mu| = 1/|\beta| \leqq 1/\alpha^* = \lambda^*$.

■

References

Bliss, C. J. (1975): *Capital Theory and the Distribution of Income*. North-Holland, Amsterdam - Oxford.

Morishima, M. (1973): *Marx's Economics*. Cambridge University Press.

——— (1974): "Marx in the Light of Modern Economic Theory," *Econometrica*, 42(4), 611–632.

Samuelson, P. (1971): "Understanding the Marxian Notion of Exploitation: A Summary of the So-Called Transformation Problem Between Marxian Values and Competitive Prices," *Journal of Economic Literature*, 9(2), 399–431.

Sraffa, P. (1960): *Production of commodities by means of commodities. Prelude to a critique of Economic Theory*. Cambridge Universty Press.