Oligopolistic competition in price and quality

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Abstract

We consider an oligopolistic market where firms compete in price and quality and where consumers have heterogeneous information: some consumers know both the prices, and quality of the products offered, some know only the prices, and some know neither. We show that if there are sufficiently many uninformed consumers, then there exists a unique equilibrium where price is a perfect indicator of quality. This equilibrium is characterized by dispersion and Pareto-inefficiency of the price/quality offers, where better price/quality combinations are associated with lower prices.

Key Words: oligopoly, competition, price, quality, imperfect information, signalling.

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1 Introduction

The very nature of competition implies that firms compete in as many ways as possible, not just in price. Firms always choose those combinations of strategic variables that serve their interests best. However, if consumers are homogeneous in their preferences among those variables and if they are fully informed about all relevant product characteristics, this multi-dimensional form of competition can be expressed in terms of a one dimensional competition model that is essentially identical in nature to that of price competition.

When consumers' preferences differ, or when consumers are better informed about some product characteristics than about others, the competitive process involving many dimensions does not have a single dimension analogue and should be analyzed in its own right. In this paper, we restrict our attention to markets where firms compete in two dimensions: price and quality. There are different approaches known in the literature dealing with endogenous price/quality competition. Klein and Leffler (1981) focus on repeated interaction between firms and consumers who are unaware of quality. Chan and Leland (1982), Cooper and Ross (1984), and Schwartz and Wilde (1985) emphasize the heterogeneity of information among consumers. Wolinsky (1983), Rogerson (1988), and Besancenot and Vranceanu (2004) additionally emphasize the heterogeneity of preferences. These models are similar in that they all consider either perfect or monopolistic competition as a form of market interaction.

Contrary to the aforementioned literature, we address the issue of price/quality competition in a *strategic* oligopoly model where price and quality are endogenously chosen and concentrate on the role of consumers having heterogeneous information (and therefore take consumer preferences to be identical). In a recent paper, Armstrong and Chen (2009) consider a similar framework where they focus on boundedly rational consumers who observe prices but who do not infer the corresponding quality, even if such inference is possible. We, on the other hand, focus on rational consumers, and the inference concerning quality that they make after observing prices plays a central role in our study.

There is a large literature, where firms are characterized by exogenous quality differences, that studies whether price choices of different types of firms can signal the quality produced (see, e.g., Bagwell and Riordan 1991). This literature uses the framework of signalling games to see whether separating equilibria exist in which different types choose different prices. The current paper differs from this literature in the sense that quality is an endogenous choice variable and the firms are ex ante identical.[1] To mark this difference, we will speak not of price signalling quality, but instead of price

[1] Another important difference is that in our model, as will become clear later, firms are indifferent between the different equilibrium price/quality offers whereas in a signalling equilibrium at least one type strictly prefers its equilibrium choice.

being an indicator of quality if consumers can infer the quality they buy upon observing the price.

We ask three questions. First, do firms differentiate themselves with different prices and/or quality choices, or do they make the same choices? Everyday experience suggests that price and/or quality dispersion is quite common in many markets.[2] Second, can consumers infer quality on the basis of price? Third, how should we characterize the outcomes in terms of Pareto-efficiency?

Stigler (1961) has pointed out that acquiring information about market prices is costly. As consumers can have different search costs, different groups of consumers can be present in a market: those who know all prices and those who do not. This idea is central in Varian (1980). The idea also readily extends to quality. In Cooper and Ross (1984) for example, some consumers know prices and are informed about quality, while the rest only know prices. We combine these two approaches in the following way. As quality is usually a more complex notion than price (especially under linear pricing, or with unit demand), it is often more costly for a consumer to learn the quality than to learn the price a firm charges. We therefore assume that there are three groups of consumers in our model: fully informed consumers who know the prices and quality of the products in a market, partially informed consumers who know the prices but not the quality, and fully uninformed consumers who know neither prices nor quality. We emphasize the role of partially informed consumers. When they are present in a market, price is not just an instrument of competition between firms, but also an indicator for quality. When they are absent, our model essentially reduces to a variation of Varian (1980), where price is replaced by a price/quality combination.

We analyze the consequences of this informational scenario in a model where two firms choose price and quality and consumers buy one good at most. Either firm is unaware of the quality choice of its competitor before it has to make its own choice over the price. The formal model is therefore one where firms choose prices and quality simultaneously. The case where firms have to choose prices while being unaware of the quality offered by their competitors is also at the heart of Daughety and Reinganum (2007, 2008), Janssen and Roy (2010), and Janssen and van Reeven (1998). The cited papers however treat quality as exogenous, while we consider endogenous quality.

In this model, with an endogenous quality choice, the main result shows that there exists an equilibrium characterized by a dispersion of prices and quality, and by price being a perfect indicator for quality if, and only if,

[2] How to explain price dispersion was first addressed by Stigler (1961). The role of imperfect information in explaining quality dispersion has been less emphasized (but receives some attention in, e.g., Chan and Leland 1982).

the fraction of uninformed consumers is above a critical number. Moreover, the equilibrium is unique in the class of perfect indicator equilibria. This kind of equilibrium correspondence between price and quality is formally described as a curve in a price-quality space with a one-dimensional distribution of price/quality offers over that curve. Firms are indifferent between selling to few consumers at higher margins and to many consumers at lower margins. In such an equilibrium, partially informed consumers correctly infer the true quality after observing prices. We also show that (i) higher prices indicate higher quality, that (ii) consumers' preferences over the resulting price/quality offers are monotone in price with consumers preferring the cheapest offer over the more expensive one, and that (iii) price/quality combinations offered in the equilibrium are Pareto-inefficient.

To understand the main qualitative features of the equilibrium that we derive it is useful to go back to Varian's model of sales with informed and uninformed consumers only, and to extend that analysis with a quality choice for firms.[3] With two instruments (price and quality) to compete for informed consumers a firm is best off to compete with the instrument that does not distort total welfare. In case of decreasing marginal returns to quality, which is what we have in our model, the only non-distorting instrument is price. Consequently, the firms compete in prices and the market equilibrium is characterized by an efficient level of quality and by a price dispersion as in Varian (1980). Notably, a firm charging a lower price has a larger expected market share. Our model introduces partially informed consumers who observe prices, but not qualities set by firms. Therefore, the firms have an incentive to distort quality downwards. Moreover, a firm charging a lower price has a greater incentive to do so because the associated cost reduction is larger due to the larger market share. Because firms reduce quality relative to its efficient level and because firms choosing lower prices reduce quality more, a positive price-quality relationship emerges in equilibrium and the equilibrium becomes inefficient.

The main reason as to why the equilibrium exists if, and only if, the number of uninformed consumers is large enough, is as follows. The firms have monopoly power over uninformed consumers, and this monopoly power sustains positive expected profits in the equilibrium. As the number of uninformed consumers diminishes, so do the expected equilibrium profits. This, in turn, creates stronger incentives to deviate from the equilibrium either by providing more attractive offers to the informed consumers or by lowering quality and exploiting partially informed consumers. Once the number of uninformed consumers falls below a critical level, these deviations become profitable and the equilibrium breaks down.

We show existence and uniqueness within a specific class of equilibria where price is a perfect indicator for quality. Other types of equilibria might

[3] We thank an anonymous referee for suggesting to think along the lines described here.

exist. In particular, when a perfect indicator equilibrium fails to exist due to a small number of uninformed consumers, strategies with randomized quality for a given price or prices (no perfect indication) are possible candidates for constructing another equilibrium. As we focus on perfect indication, we do not pursue this alternative venue further.

It is interesting to observe that the equilibrium price quality relationship we derive is similar in nature to the Wolinsky (1983) and Rogerson (1988) models in that along the equilibrium curve firms offer a higher quality together with a price difference that exceeds the marginal cost of producing that extra quality. In those models this result arises from consumers having different preferences over price/quality offerings. In our case the explanation for this result is purely information-based. In our case, firms are indifferent between offering any of the price quality bundles that are on the equilibrium curve, and those consumers that are informed about the bundles prefer to buy the ones associated with lower prices and quality. Therefore, for the firms to be indifferent, the margin on the lower price-quality bundles should be smaller.[4]

The paper is organized as follows. Section 2 formally introduces the model. Section 3 provides the equilibrium analysis, whilst section 4 illustrates the more complicated expressions of section 3 through an example. Section 5 concludes. All proofs are given in the appendix.

2 Model Setup

We consider a market with three players: two firms and a representative consumer. Each firm chooses the price and quality of the product it offers. Apart from price and quality the products are identical. The representative consumer can be of different types with respect to the information he has on firms' offers.[5]

The game has three stages. First, nature determines the type of the representative consumer. He can either be fully informed (H-type), partially informed (M-type), or uninformed (L-type), with respective probabilities λ_H, λ_M, and λ_L, and with $\lambda_H + \lambda_M + \lambda_L = 1$. Second, each firm i chooses a price p_i and a quality q_i for its product. Both firms choose simultaneously and are unaware of the consumer's type. Third, the consumer receives information on firms' offers according to his type. The H-type learns (p_i, q_i)

[4] In addition to the literature mentioned here, there is a large literature on price-quality where firms' quality choices are known before firms choose prices (see, e.g., Tirole 1988).

[5] By using a representative consumer of different types we account for the information heterogeneity among consumers without introducing technical complications. Alternatively, we could model population heterogeneity directly by having a finite population with certain fractions of different types of consumers. All our results except for the existence Theorem 2 will still hold in this alternative formulation. Theorem 2 will continue to hold if the population is large.

for $i \in \{1, 2\}$, the M-type learns only p_i, and the L-type learns nothing. The consumer then chooses whether to visit firm 1 or firm 2. Upon his choice, the game formally ends and the payoffs are realized. We next proceed with a description of these payoffs.

The production technology is such that producing higher quality comes at a higher cost. For simplicity, we assume a linear dependency so that the per-unit profits are given by

$$\Pi(p, q) = p - aq,$$

where the coefficient $a > 0$ characterizes the quality production technology. We take a to be the same across the two firms. The firms make their production decisions at the moment of sale, so that if the consumer buys from firm i the firm gets $\Pi(p_i, q_i)$ and otherwise the firm gets zero.

The consumer purchases at most one unit of the good from the chosen firm. Given quality q, he has maximum willingness to pay $V(q)$ for that good. Define $U(p, q) = V(q) - p$. We will refer to $U(p, q)$ as the utility the consumer gets from accepting offer (p, q). The subsequent analysis is based on the assumption that the function $V(q)$ is well-behaved.

Assumption 1. *$V(q)$ is strictly increasing, strictly concave, and twice continuously differentiable in q on the interior of its domain. Moreover, for any $M > 0$ there exists q_M such that $V'(q) > M$ for all $q < q_M$; and for any $\varepsilon > 0$ there exists q_ε such that $V'(q) < \varepsilon$ for all $q > q_\varepsilon$.*

Typical expressions for $V(q)$ such as $V(q) = \ln q$ or $V(q) = \sqrt{q}$ satisfy this assumption. To get a nontrivial analysis, there must be gains from trade:

Assumption 2. *The model is non-degenerate, i.e., there exists q such that $V(q) > aq$.*

The partially informed and fully uninformed types do not have information about quality. As a consequence, if a firm has zero chance of attracting the fully informed type, then the firm has an incentive to "cheat" – to set the lowest possible quality. Similar to Klein and Leffler (1981), we make an assumption that cheating is restricted in the sense that the consumer, after visiting the firm, learns the true quality of the product and can determine whether the offer gives him a non-negative utility. This assumption is valid, for example, when firms have a return policy, as is often the case for durable consumption goods, and when the consumer can learn the quality of the product he purchased within the return period. The assumption is also valid when the costs of assessing price and quality are negligible once the consumer is in the shop. For example, one often observes that firms have testers (for perfumes) or sample products (for food items, or electronics). Once in the shop, but before making his purchase, the consumer can assess the quality of the product he is about to buy. Either interpretation is consistent with our setup as long as comparative shopping is not allowed, i.e. as long as the consumer can not visit one firm, not buy its product and then visit another firm.

In summary, the payoffs are as follows: if the consumer chooses firm i and if $U(p_i, q_i) \geq 0$, then the consumer gets $U(p_i, q_i)$, firm i gets $\Pi(p_i, q_i)$ and the other firm gets 0; if the consumer chooses firm i and $U(p_i, q_i) < 0$, then all three players get 0.

Equilibria. We use the notion of subgame perfect equilibrium (SPE) to analyze this game. Additionally, we focus on what we call a perfect indicator equilibrium (PIE). This concept is introduced formally in the following paragraphs.

First, note that because the consumer can happen to be fully informed or fully uninformed, there do not exist symmetric or asymmetric pure strategy SPE in this model. Pure strategy equilibria with zero profits do not exist, because either firm can earn positive profits by exploiting the uninformed type. Symmetric pure strategy equilibria with positive profits do not exist as firms would have an incentive to increase quality slightly, which would attract the fully informed type at only a marginally higher cost. The same applies to asymmetric equilibria where firms offer different price-quality bundles that yield the same utility levels to the consumer. Other asymmetric pure strategy equilibria are also not possible as the firm offering the highest utility level has an incentive to lower the costs by slightly lowering the quality.

Thus, we continue with an analysis of mixed strategy equilibria. As firms have no information when making their choices, a firm's strategy is a distribution over all possible (p, q) bundles. Let P_i and Q_i be the random variables that stand for the price and quality offered by firm i and let $\mathbb{P}_i$ be the probability measure that defines the strategy of firm i.

In general, a certain price p may indicate a specific non-degenerate distribution of quality, when $\text{var}(Q_i|P_i = p) > 0$. We restrict attention, however, to symmetric equilibria where a certain price p is a perfect indicator of a certain quality $\hat{q}(p)$. We name the function $\hat{q}(p)$ an equilibrium curve. This restriction considerably simplifies the analysis, and we show that in this class of equilibria interesting price and quality choices can be made. Formally, we define perfect indicator equilibria as follows:

Definition 1. *A subgame perfect equilibrium is called a perfect indicator equilibrium if (i) firms' strategies are symmetric, i.e* $\mathbb{P}_1 \equiv \mathbb{P}_2$*, (ii) supp*$(\mathbb{P}) = \{(p, q) : p \in [p_l, p_h], q = \hat{q}(p)\}$*, (iii)* $\hat{q}(p)$ *is a function that is continuously differentiable in* p *over* $[p_l, p_h]$*, where* p_l *and* p_h *are some arbitrary bounds, (iv) the M-type chooses firm* i *whenever* $p_i \in [p_l, p_h]$ *and* $p_{-i} \notin [p_l, p_h]$*, and (v) the L-type plays a mixed strategy, with equal probabilities of choosing* *either firm.*

Conditions (iv) and (v) will be explained in the following paragraphs. Note that conditions (i) through (v) restrict the set of equilibrium strategies. We impose no restrictions on strategies per se, e.g. a firm can deviate and play any (p,q) bundle if it finds doing so profitable.

Let $\mu_H(p,q)$, $\mu_M(p)$ and μ_L denote the probabilities that in a PIE a firm gets the representative consumer of the respective type if the firm offers a bundle (p,q) with $V(q) \geq p$. In general, μ_H, μ_M, and μ_L depend upon the equilibrium in question, but useful expressions for μ_H, μ_M and μ_L can nevertheless be derived.

Given our information structure, proper subgames exist for the H-type consumer only. For any pair of alternative offers, (p_1,q_1) and (p_2,q_2), there is a subgame where the H-type has to choose which firm to visit. Trivially, subgame perfection implies that the H-type chooses the offer yielding the highest utility. If the offers give the same utility – which in a perfect indicator equilibrium happens with zero probability, we take that the H-type is equally likely to choose either firm.

Let $F(u) = \mathbb{P}(U(P,Q) < u)$. If a firm offers a bundle with utility level u, then $F(u)$ gives the probability that the rival offers a lower utility level when playing an equilibrium strategy. Additionally, for convenience, let $dF(u) = \mathbb{P}(U(P,Q) = u)$. Then subgame perfection implies

$$\mu_H(p,q) = F(U(p,q)) \cdot \lambda_H + dF(U(p,q)) \cdot \frac{\lambda_H}{2}.$$

Upon observing a price p and given the equilibrium strategies of firms, the M-type consumer has an expectation about the utility he can get from the offer. Denote this expected utility with $\hat{U}(p)$. Given that for any realization of P, there is a unique corresponding realization of $Q = \hat{q}(P)$, we obtain

$$\begin{aligned}\hat{U}(p) = \mathbb{E}(\max(U(P,Q),0) \mid P = p) = \\ \mathbb{E}(\max(U(P,\hat{q}(P)),0) \mid P = p) = \max(U(p,\hat{q}(p)),0)\end{aligned}$$

for all $p \in [p_l, p_h]$.

A strategy of the M-type is the best response to the equilibrium strategies of firms if, upon observing any two offers p_1 and p_2 from the support $[p_l, p_h]$, he chooses the one that yields him the highest expected utility $\hat{U}$.[6] Once again, if two offers have the same expected utility, we assume that the M-type makes a uniform random choice.

For $p \notin [p_l, p_h]$, the SPE concept does not give us any restrictions on the best response strategy of the M-type consumer. We assume that upon

[6]Strictly speaking, on a set of price realizations of measure zero (according to $\mathbb{P}$) the M-type is free to choose anything. The strategy still remains a best response. Clearly, all such best responses are equivalent.

observing a $p \notin [p_l, p_h]$, the M-type believes the corresponding quality to be so low that the offer is never competitive – thus, we have refinement (iv). Alternatively, perfect equilibria or sequential equilibria can be used to analyze off-the-equilibrium beliefs of the M-type consumer. However, those concepts are formally not defined for infinite games and, more importantly, our further investigation primarily concerns on-the-equilibrium beliefs.

Given the best response strategy of the M-type, we get

$$\mu_M(p) = F(\hat{U}(p)) \cdot \lambda_M + dF(\hat{U}(p)) \cdot \frac{\lambda_M}{2}$$

for $p \in [p_l, p_h]$ and $\mu_M(p) = 0$ for $p \notin [p_l, p_h]$.

Finally, as the L-type has no information to base his choice upon, we simply assume that he visits either firm with equal probability – refinement (v). Because the firms play the same strategy in an equilibrium, such a strategy of the L-type constitutes a best response. We have $\mu_L = \frac{\lambda_L}{2}$.

Define, for convenience, $\mu(p,q) = \mu_H(p,q) + \mu_M(p) + \mu_L$. Then, given the consumer plays a best response strategy, the expected firms' profits are given by

$$\pi(p,q) = \begin{cases} \mu(p,q) \cdot \Pi(p,q) & \text{if } V(q) \geq p, \\ 0 & \text{otherwise.} \end{cases}$$

For a firm, choosing a (p,q) bundle over an equilibrium curve $\hat{q}(p)$ is a best response strategy if, and only if, the profit function $\pi(p,q)$ attains its maximum along that equilibrium curve:

$$\operatorname{supp}(\mathbb{P}) \in \arg\max_{p,q} \pi(p,q). \tag{1}$$

Equation (1) gives necessary and sufficient conditions for there to be a perfect indicator equilibrium.

Since $\pi(p,q) \geq \mu_L \cdot \Pi(p,q)$ if $U(p,q) \geq 0$, a firm can always guarantee itself some positive profits given Assumption 2. So, in an equilibrium, no firm will offer (p,q) such that $U(p,q) < 0$. Consequently,

$$\hat{U}(p) = \max(U(p,\hat{q}(p)), 0) = U(p,\hat{q}(p)) \quad \forall p \in [p_l, p_h]. \tag{2}$$

In the next section, we show that in a PIE $U(p,\hat{q}(p))$ is strictly monotone in p. Therefore, the strategies of firms are fully characterized by the equilibrium curve $\hat{q}(p)$, by the boundary points p_l and p_h, and by the distribution of utilities along the equilibrium curve, namely by $F(u)$.

3 Analysis

In this section, we solve for a perfect indicator equilibrium, i.e., we solve for $F(u)$, $\hat{q}(p)$, p_l and p_h given $U(p,q)$ and given the other parameters of the model. At first, we assume that there exists a PIE and characterize its properties. Conditions under which such an equilibrium exists are discussed later.

We begin with stating some simple facts. First, $\hat{U}(p)$ is continuous in p because $U(p,q)$ is continuous in p and q, and $\hat{q}(p)$ is differentiable. Second, as a continuous function of one variable maps a closed interval into a closed interval, and in equilibrium firms choose prices from the set $[p_l, p_h]$, the corresponding equilibrium utility levels $\hat{U}(p)$ form a closed interval; we denote this interval by $[U_l, U_h]$. Third, it is a standard result that $F(u)$ does not have atoms:

Lemma 1. *$F(u)$ is continuous and $dF(u) \equiv 0$.*

In economic terms, the chance that the rival provides the same utility level is zero. This result is very similar in nature to the result that the price distribution in the "model of sales" (Varian, 1980) is atomless. The formal proof of this statement is therefore omitted. The reason for this result comes from a strictly positive chance that the consumer is either of the fully informed type, over which the firms compete, or of the fully uninformed type, over which the firms have market power. The next lemma argues that U_l must be equal to 0. The main reason is that, in a perfect indicator equilibrium, the firm offering the worst utility only gets the uninformed type, and if $U_l > 0$, it could make more profit by providing him a worse deal.

Lemma 2. $U_l = 0$.

Given Lemma 1 and equation (2), it is straightforward to simplify the expression for $\pi(p,q)$.

Lemma 3. *For $p \in [p_l, p_h]$, the profits are given by*

$$\pi(p,q) = \left(F(U(p,q)) \cdot \lambda_H + F(U(p,\hat{q}(p))) \cdot \lambda_M + \frac{\lambda_L}{2} \right) \cdot \Pi(p,q)$$

if $V(q) \geq p$ and 0 otherwise.

To find the functional form of $F(u)$, we need to be able to define equilibrium per-unit profits as a function of utility (Lemma 5 will clarify why this is necessary). The following lemma allows us to do so.

Lemma 4. *Given $u \in [U_l, U_h]$, per-unit profits $\Pi(p, \hat{q}(p))$ are the same for all $p \in \hat{U}^{-1}(u)$.*

Consider a set of equilibrium $(p, \hat{q}(p))$ offers that give the same utility u. Given the same utility, the chances of attracting the representative consumer are also the same. Hence, per-unit profits are the same as otherwise expected profits differ and no firm will mix over those offers. So, per-unit profits do not depend on a particular offer from this set, only on the corresponding utility u. We use the notation $\hat{\Pi}(u)$ to denote this per-unit profits.

Formally, take an arbitrary $\tilde{p}(u)$ such that $\tilde{p}(u) \in \hat{U}^{-1}(u)$ for all $u \in [U_l, U_h]$. Then

$$\hat{\Pi}(u) = \Pi(\tilde{p}(u), \hat{q}(\tilde{p}(u))).$$

It is not possible to define $\hat{\Pi}(u)$ explicitly as it involves choosing a particular $\tilde{p}(u)$, and the functional form of $V(q)$ is not given. However, once a specific functional form of $V(q)$ is adopted, and once $\hat{q}(p)$ is known, it is possible to choose a particular $\tilde{p}(u)$ and hence solve for $\hat{\Pi}(u)$.

Now we can solve for the functional form of $F(u)$ using the fact that, in a mixed strategy equilibrium, firms have to be indifferent between any of the pure strategies in the support of the equilibrium mixtures.

Lemma 5.

$$F(u) = \frac{1}{2} \cdot \frac{\lambda_L}{\lambda_H + \lambda_M} \left(\frac{\hat{\Pi}(0)}{\hat{\Pi}(u)} - 1 \right) \quad \textit{for} \quad u \in [U_l, U_h].$$

It then follows that U_h is implicitly given by $F(U_h) = 1$ or, more explicitly, by

$$\hat{\Pi}(U_h) = \frac{1/2 \cdot \lambda_L \cdot \hat{\Pi}(0)}{\lambda_H + \lambda_M + 1/2 \cdot \lambda_L}.$$

The previous lemma shows that the distribution of utility levels over the equilibrium curve is such that the likelihood that the consumer is fully informed relative to the likelihood that he is not determines the spread of utility.[7] This is intuitive as the firms have market power over the uninformed type, and if there is a high chance for that, the price/quality offers concentrate around the offers that are cheapest to provide. We next give a description of the equilibrium curve $\hat{q}(p)$.

Lemma 6. *If there is a perfect indicator equilibrium, then $\hat{q}(p)$ has to satisfy*

$$\frac{d\hat{q}}{dp} = \frac{\lambda_H + \lambda_M}{\lambda_M} \cdot \frac{1}{V'(\hat{q}(p))} - \frac{\lambda_H}{a\lambda_M}.$$

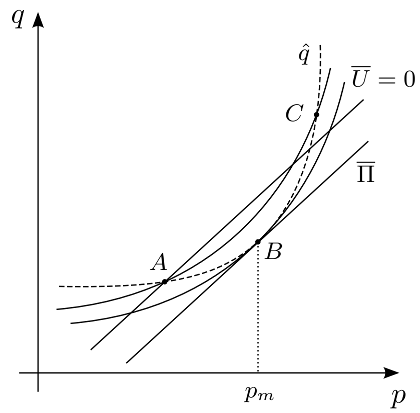
everywhere on (p_l, p_h).

To illustrate the impact of the lemma, Figure 1 depicts an equilibrium curve $\hat{q}(p)$ together with iso-utility curves and isolines of per-unit profits. To see why the equilibrium curve has the shape as in the figure, rewrite the differential equation for $\hat{q}(p)$ as follows:

$$\frac{d\hat{q}}{dp} = \frac{\lambda_H}{\lambda_M} \left(\frac{1}{V'(\hat{q}(p))} - \frac{1}{a} \right) + \frac{1}{V'(\hat{q}(p))}, \tag{3}$$

[7] This perspective is valid as long as $\hat{\Pi}(u)$ does not depend on λ_L. Further on we show that, given a fixed ratio of λ_H/λ_M, this is indeed the case.

Figure 1: Equilibrium Curve



Notation: $\overline{X}$ stands for $X(p,q) = \text{const}$, where constant is arbitrary; $\overline{X} = X_0$ stands for $X(p,q) = X_0$, where X_0 is some specific value.

and recall that the isoline of per-unit profits has a slope equal to $1/a$. The slope of the iso-utility curves is $\frac{1}{V'(\hat{q}(p))}$. Therefore it follows from (3) that, if the slope of an iso-utility curve is less than $1/a$ (point A for example), the slope of the equilibrium curve is even smaller at that point and vice versa. If the slope is exactly $1/a$ (point B), the iso-utility curve and the equilibrium curve are tangent to each other and they are also tangent to the isoline of per-unit profits at that point. Therefore, the equilibrium curve relative to the underlying iso-utility curves should look as depicted in the figure.

According to Lemma 2, the lowest attainable utility along an equilibrium curve equals 0. Thus the equilibrium curve in the figure "lies" on the iso-utility curve that corresponds to $U = 0$. Denote the corresponding point with (p_m, q_m):

Definition 2. *The point (p_m, q_m) is uniquely*[8] *defined by*

$$(p_m, q_m) = \arg\max_{(p,q):U(p,q)\geq 0} \Pi(p,q).$$

Now we are ready to state the following theorem:

Theorem 1. *If there exists a perfect indicator equilibrium, then $[p_l, p_h] = [p_l, p_m]$, $\hat{q}(p)$ is strictly increasing in p over this interval, and $U(p, \hat{q}(p))$ is strictly decreasing. Hence, in an equilibrium, higher prices indicate higher quality but lower utility.*

[8]The solution to the optimization problem exists by Assumption 1. Moreover, the solution is unique because $U(p,q)$ is strictly quasi-concave by the same assumption.

Theorem 1 is proved using several lemmas. First, it is proved that p_m is on the equilibrium curve. The reason is that there is one offer on the equilibrium curve that gives the lowest utility. At this offer only the uninformed type buys, and if he gets a utility that is strictly higher than his reservation utility, then the firms could profitably deviate by providing a worse offer. A technical lemma then shows that if $p_h > p_m$, the second-order conditions for profit maximization do not hold. So, the only remaining possibility is $[p_l, p_h] = [p_l, p_m]$. The fact that the equilibrium curve is monotonically increasing follows from the fact that the slope is positive at (p_m, q_m), that the equilibrium curve is twice differentiable and that the slope cannot be equal to 0 as that would imply that the whole equilibrium curve is horizontal. The fact that utility is decreasing over the equilibrium curve when prices and quality increase follows from the shape of the equilibrium curve as depicted in Figure 1 and from equation (3). At any intersection point with an iso-utility curve, the slope of the equilibrium curve is smaller then that of the iso-utility curve, so higher prices and qualities on the equilibrium curve lie on lower iso-utility curves.

The next theorem asserts that a PIE exists if and only if λ_L is large enough, and that when it exists it is unique.

Theorem 2. *Given any finite and strictly positive ratio $\frac{\lambda_H}{\lambda_M}$ there exists $\lambda_L^* \in (0,1)$ such that for all $\lambda_L < \lambda_L^*$ there are no perfect indicator equilibria and for all $\lambda_L > \lambda_L^*$ there exists a unique perfect indicator equilibrium.*

While we assume $(\lambda_L, \lambda_M, \lambda_H) > 0$, to obtain an intuition for this result let us consider $\lambda_L = 1$. Then $\lambda_H = \lambda_M = 0$, and both firms simply maximize their profits subject to $p \leq V(q)$. Thus, both firms offer (p_m, q_m), and there is a unique equilibrium in pure strategies.

Consider $\lambda_L = 0$. Suppose there exists a PIE equilibrium with zero expected profits. A firm offering (p, q) with $U(p, q) = U_h$ has strictly positive chances of attracting the M and H types. For its expected profits to be zero it should be that $\Pi(p, q) = 0$. But then, if $V'(q) \neq a$, there is a Pareto-improving deviation which yields higher Π, higher U and, consequently, positive profits from at least the H-type. And if $V'(q) = a$, then $U_h > U_R$ (due to Assumption 2) and thus the firm can lower its offer's quality and obtain positive profits from the M-type. So, no zero-profits equilibrium exists. Suppose $dF(U_l) = 0$, i.e. suppose there is no atom at the bottom of the utility distribution. Then any firm offering U_l earns zero profits, but that contradicts the earlier discussion. The remaining possibility is a PIE with positive expected profits and with $dF(U_l) > 0$. But in this case any firm offering U_l has an incentive to increase the quality by an infinitely small amount to fully attract the H-type consumer. Thus, no equilibrium exists for $\lambda_L = 0$.[9]

[9]The step from $dF(U_l) = 0$ to zero expected profits at U_l is based on the definition

The difficulty which Theorem 2 tackles is showing that the cut-off point in λ_L is strictly between 0 and 1, i.e. that existence and nonexistence regions are not of measure zero. Still, the incentives underlying the result of the theorem are precisely as outlined earlier. In particular, if $\lambda_L \to 0$, then the offers with lower utilities push the equilibrium profits towards zero. But as profits approach zero any firm offering high utility has an incentive to either lower its quality and gain from the M-type or to increase both its price and its quality in a Pareto-improving way and gain from the H-type. Figure 4 in the upcoming example suggests as much.

Pareto-efficiency. An allocation is Pareto-efficient in this model if the corresponding iso-utility curve is tangent to the corresponding isoline of per-unit profits. Thus, Pareto-efficient allocations are characterized by $V'(q) = a$. Considering Figure 1, one can see that for any $(p, \hat{q}(p))$ with $p \in [p_l, p_h)$ this is not the case. Therefore, equilibrium allocations are almost surely Pareto-inefficient. Another way to see this is to reconsider equation (3). The contract curve of Pareto-optimal points is a horizontal line where $V'(q) = a$. However, it follows from equation (3) that whenever $V'(q) = a$, the equilibrium curve has a positive slope. Thus, it cannot be the case that all the points on the equilibrium curve are Pareto-efficient.

This result marks a sharp contrast to Varian's model of sales, which is essentially this model (with prices being replaced by utilities) but excluding the partially informed type. In that model, or in our model without the M-type, all the equilibrium allocations will be Pareto-efficient. In our model, when $\lambda_M = 0$, the equilibrium curve will be a horizontal line coinciding with the contract curve. Thus, the presence of the partially informed type, and the incentives he creates for firms to distort quality downwards, is what brings Pareto-inefficiency. The fully uninformed type does not create Pareto-inefficiency on his own, he merely creates a redistribution in welfare.

4 An Example

In this section we give an example of a perfect indicator equilibrium. Take

$$V(q) = \frac{1}{2}\ln q + 2,\ \lambda_H = \lambda_M = \frac{1}{5},\ \lambda_L = \frac{3}{5},\ a = 1.$$

We begin by solving for (p_m, q_m). To do so we solve

$$\max_{p,q} \Pi(p,q) \quad \text{s.t.} \quad U(p,q) = V(q) - p \geq 0$$

of PIE, more precisely, on the existence of an equilibrium curve $\hat{q}(p)$. If firms mix over quality for a given price, then this step is not valid. So, intuitively, if there exists some other type of equilibrium for $\lambda_L = 0$ (or small λ_L), it has to involve mixing over quality (given the price or prices).

and obtain

$$p_m = \frac{1}{2}\ln\frac{1}{2} + 2, \quad q_m = \frac{1}{2}.$$

By Theorem 1, $p_h = p_m$.

Let us now find $\hat{q}(p)$. Plugging our utility and the parameters into the differential equation for $\hat{q}(p)$ (Lemma 6) gives

$$\frac{d\hat{q}(p)}{dp} = 4\hat{q}(p) - 1.$$

Solving it, and using the boundary condition $\hat{q}(p_m) = q_m$ gives:

$$\hat{q}(p) = e^{4p-8} + \frac{1}{4}. \tag{4}$$

To find a utility distribution $F(u)$, we need to know $\hat{\Pi}(u)$. For that we need to find $\tilde{p}(u)$ such that $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$. Writing down this latter expression gives

$$\frac{1}{2}\ln\left(e^{4\tilde{p}(u)-8} + \frac{1}{4}\right) + 2 - \tilde{p}(u) = u.$$

After some algebraic manipulations we arrive at the solution:

$$\tilde{p}(u) = \frac{1}{2}\ln\left(\frac{1}{2}\left(e^{2u} - \sqrt{e^{4u}-1}\right)\right) + 2. \tag{5}$$

Having (4) and (5) we therefore also have

$$\hat{\Pi}(u) = \tilde{p}(u) - a\hat{q}(\tilde{p}(u)) = \tilde{p}(u) - \hat{q}(\tilde{p}(u))$$

and

$$F(u) = \frac{1}{2}\cdot\frac{\lambda_L}{\lambda_H + \lambda_M}\left(\frac{\hat{\Pi}(0)}{\hat{\Pi}(u)} - 1\right) = \frac{3}{4}\left(\frac{\hat{\Pi}(0)}{\hat{\Pi}(u)} - 1\right).$$

Given $F(u)$ we can find U_h from $F(U_h) = 1$. Define

$$z = \frac{1}{2}\left(e^{2U_h} - \sqrt{e^{4U_h}-1}\right).$$

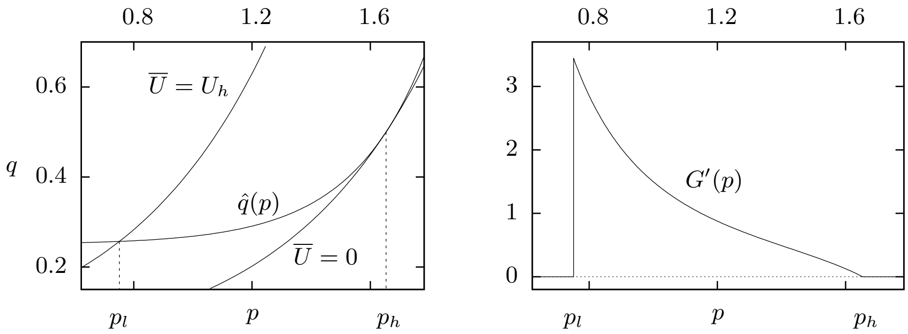
Thus, $z \leq \frac{1}{2}$, and $F(U_h) = 1$ can be rewritten as

$$\frac{1}{2}\ln z - z^2 + \frac{7}{4} - \frac{3}{7}\hat{\Pi}(0) = 0.$$

This equation can not be solved analytically but a numerical solution is easily obtainable: $z \approx 0.08226$. Then, from the definition of z,

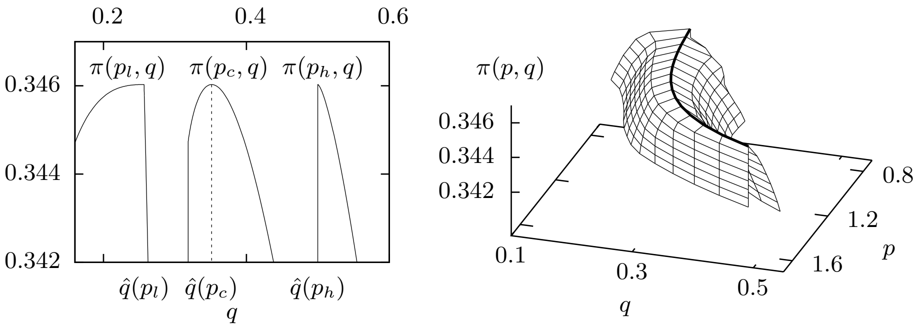
$$U_h = \frac{1}{2}\ln\left(\frac{4z^2+1}{4z}\right) \approx 0.56911$$

Figure 2: Equilibrium Characteristic Functions



Notation: $\overline{U} = u$ stands for $U(p, q) = u$.

Figure 3: Equilibrium Profits



Comments: $p_c = \frac{1}{4}p_l + \frac{3}{4}p_h$ (the left plot); the bold line depicts $\pi(p, \hat{q}(p))$, i.e. the profits along the equilibrium curve (the right plot); also, for convenience, only a summit of $\pi(p, q)$ is shown on the right plot.

Finally, $p_l = \tilde{p}(U_h) \approx 0.75109$.

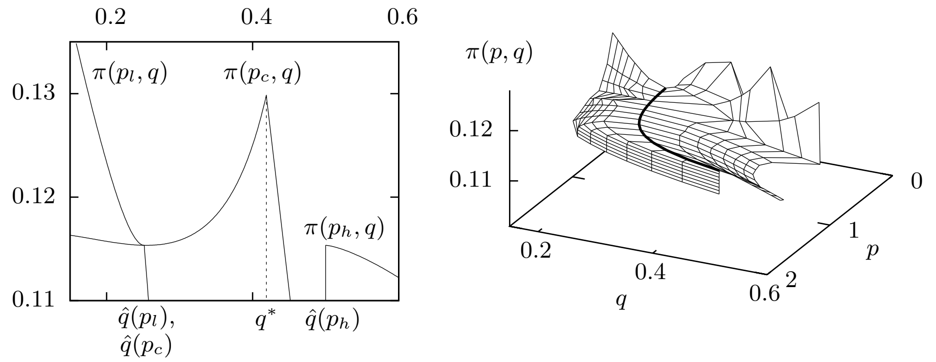
Figure 2 plots a few important functions of our equilibrium candidate. The left plot shows $\hat{q}(p)$ together with the iso-utility curves that correspond to $U_l = 0$ and U_h. It is easy to see from the plot that higher prices indicate lower utility in the equilibrium. The right plot shows the density function of the price distribution, namely $G'(p)$. Earlier we exclusively worked with the utility distribution $F(u)$, but there is an easy transformation as

$$G(p) = \mathbb{P}(P < p) = \mathbb{P}(U(P, \hat{q}(P)) > U(p, \hat{q}(p))) = \\ 1 - \mathbb{P}(U(P, \hat{q}(P)) \leq U(p, \hat{q}(p))) = 1 - F(U(p, \hat{q}(p))).$$

From the plot we can see that lower prices, combined with lower quality, are chosen more often than higher prices combined with higher quality.

Recollect that $\pi(p, q)$ gives the expected profits of one firm when the other firm is playing the equilibrium strategy. For there to be an equilibrium,

Figure 4: Disequilibrium Profits



Comments: $p_c = \frac{4}{5}p_l + \frac{1}{5}p_h$ and q^* is such that $U(p_c, q^*) = U_h$ (the left plot); the bold line depicts $\pi(p, \hat{q}(p))$, i.e. the profits along the equilibrium curve (the right plot); also, for convenience, only a summit of $\pi(p, q)$ is shown on the right plot; the sharp spokes to the back and right of the 3D picture are rendering artifacts – should be one steep "hill".

it should be that $\pi(p, q)$ attains its maximum along the equilibrium curve $\hat{q}(p)$. Figure 3 plots the function $\pi(p, q)$. The left plot gives 2D slices of $\pi(p, q)$ for various values of p, the right plot attempts to provide a 3D presentation. One can readily see that the condition in question is satisfied indeed and so we have a perfect indicator equilibrium.

However, Theorem 2 states that given a fixed ratio $\frac{\lambda_H}{\lambda_M}$, which is 1 in our case, the equilibrium fails to exist if λ_L is sufficiently small. So, consider the same example but with $\lambda_L = \frac{1}{5}$ and $\lambda_H = \lambda_M = \frac{2}{5}$. It can be solved in the same way as before. Figure 4 gives the same plots as before but for the new parameter values. We know that if there is an equilibrium we should have that $\pi(p, q)$ attains its maximum along the equilibrium curve. As this is not the case, we can conclude that no perfect indicator equilibria exist.

5 Conclusions

We have considered a market where oligopolistic firms compete for consumers by varying prices and quality of their products, and where consumers are heterogeneous in their knowledge of the prices and quality of the products offered: some consumers know both the quality and the prices, some know only the prices, and some consumers know neither.[10] We have derived a perfect indicator equilibrium for this setting that is characterized by firms playing a mixed strategy over a curve in a price-quality space. The dispersion of prices and quality in this equilibrium is such that *lower* prices indicate lower quality, but higher utility. The equilibrium characterization is

[10]Technically, we modelled heterogeneous consumers using a representative consumer who could be of different types.

different from that in the standard signalling literature, as here firms are indifferent between any of the price/quality offers in the equilibrium support. We have shown that a perfect indicator equilibrium certainly exists if there are sufficiently many uninformed consumers in the population. Finally, we have shown that the equilibrium is inefficient due to the fact that there are partially informed consumers who infer quality on the basis of price. Firms therefore distort their price quality offers in comparison with Pareto-efficient outcomes: in the equilibrium, the marginal cost of producing better quality is lower than the marginal valuation of better quality.

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Appendix

Lemma 2. $U_l = 0$.

Proof. Consider $p \in [p_l, p_h]$ such that $U(p, \hat{q}(p)) = U_l$. Such p exists because U_l belongs by definition to the support of $F(u)$. Also by definition, $F(U_l) = 0$. Therefore

$$\pi(p, \hat{q}(p)) = \frac{\lambda_L}{2} \cdot \Pi(p, \hat{q}(p)).$$

Clearly, $U_l \geq 0$. Suppose that $U_l = U(p, \hat{q}(p)) > 0$. $U(p, q)$ is continuous in q, so it is possible to choose $\varepsilon > 0$ such that

$$U(p, \hat{q}(p) - \varepsilon) > 0.$$

Additionally, $U(p, q)$ is strictly increasing in q, so $U(p, \hat{q}(p) - \varepsilon) < U(p, \hat{q}(p))$ and, consequently, $F(U(p, \hat{q}(p) - \varepsilon)) = 0$. Given that $\Pi(p, q)$ is strictly decreasing in q, we obtain:

$$\pi(p, \hat{q}(p) - \varepsilon) = \frac{\lambda_L}{2} \cdot \Pi(p, \hat{q}(p) - \varepsilon) > \frac{\lambda_L}{2} \cdot \Pi(p, \hat{q}(p)) = \pi(p, \hat{q}(p)).$$

This contradicts

$$(p, \hat{q}(p)) \in \arg\max_{(\tilde{p}, \tilde{q})} \pi(\tilde{p}, \tilde{q}).$$

So, $U_l = 0$. □

Lemma 4. *Given $u \in [U_l, U_h]$, per-unit profits $\Pi(p, \hat{q}(p))$ are the same for all $p \in \hat{U}^{-1}(u)$.*

Proof. Take $p \in [p_l, p_h]$. It follows from Lemma 3 that

$$\pi(p, \hat{q}(p)) = \left(F(\hat{U}(p)) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2} \right) \cdot \Pi(p, \hat{q}(p)). \tag{6}$$

If there are different p_1, p_2 such that

$$\hat{U}(p_1) = \hat{U}(p_2) = u,$$

then $\Pi(p_1, \hat{q}(p_1)) = \Pi(p_2, \hat{q}(p_2))$. Indeed, if this is not the case, then equilibrium profits, i.e. the profits along an equilibrium curve, will differ between p_1 and p_2 as readily seen from (6). But profits have to attain their maximum along the equilibrium curve and hence they have to be constant along it as well. $\square$

Lemma 5.

$$F(u) = \frac{1}{2} \cdot \frac{\lambda_L}{\lambda_H + \lambda_M} \left(\frac{\hat{\Pi}(0)}{\hat{\Pi}(u)} - 1 \right) \quad \textit{for} \quad u \in [U_l, U_h].$$

Proof. It follows from Lemma 3 that

$$\pi(p, \hat{q}(p)) = \left(F(U(p, \hat{q}(p))) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2} \right) \cdot \Pi(p, \hat{q}(p)). \tag{7}$$

Evaluating (7) at $\tilde{p}(u)$, noticing that $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$ and using the definition of $\hat{\Pi}(u)$ gives

$$\pi(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = \left(F(u) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2} \right) \cdot \hat{\Pi}(u).$$

For there to be an equilibrium the strategies of the firms should provide maximal profits along the equilibrium curve. Therefore $\pi(p, \hat{q}(p))$ is constant over this interval, and we denote its value by $\hat{\pi}$. Since $\tilde{p}(u) \in [p_l, p_h]$ we get

$$\hat{\pi} = \left(F(u) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2} \right) \cdot \hat{\Pi}(u).$$

By definition, $F(U_l) = 0$. Also, $U_l = 0$. So,

$$\hat{\pi} = \frac{\lambda_L}{2} \cdot \hat{\Pi}(0).$$

Plugging it back and solving for $F(u)$ gives the result. $\square$

Lemma 6. *If there is a perfect indicator equilibrium, then $\hat{q}(p)$ has to satisfy*

$$\frac{d\hat{q}}{dp} = \frac{\lambda_H + \lambda_M}{\lambda_M} \cdot \frac{1}{V'(\hat{q}(p))} - \frac{\lambda_H}{a\lambda_M}.$$

everywhere on (p_l, p_h).

Proof. It should be that

$$\left.\frac{\partial \pi(p, q)}{\partial p}\right|_{(\tilde{p}, \hat{q}(\tilde{p}))} = 0, \quad \left.\frac{\partial \pi(p, q)}{\partial q}\right|_{(\tilde{p}, \hat{q}(\tilde{p}))} = 0,$$

because otherwise a firm may get higher profits by deviating along the gradient vector. Using Lemma 3 we get

$$\begin{aligned}\frac{\partial \pi(p,q)}{\partial p}\Big|_{(\tilde p,\hat q(\tilde p))} &= \Big(-F'(U(p,q))\cdot\lambda_H + \\ &\quad F'(U(p,\hat q(p)))\cdot\big(V'(\hat q(p))\cdot\hat q'(p)-1\big)\cdot\lambda_M\Big)\cdot(p-aq)+ \\ &\quad \Big(F(U(p,q))\cdot\lambda_H + F(U(p,\hat q(p)))\cdot\lambda_M + \frac{\lambda_L}{2}\Big)\cdot 1\Big|_{(\tilde p,\hat q(\tilde p))} = \\ &\quad F'(U(\tilde p,\hat q(\tilde p)))\Big(V'(\hat q(\tilde p))\cdot\hat q'(\tilde p)\cdot\lambda_M-(\lambda_H+\lambda_M)\Big)\cdot \\ &\quad (\tilde p - a\hat q(\tilde p)) + \Big(F(U(\tilde p,\hat q(\tilde p))\cdot(\lambda_H+\lambda_M)+\frac{\lambda_L}{2}\Big) = 0 \qquad (8)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \pi(p,q)}{\partial q}\Big|_{(\tilde p,\hat q(\tilde p))} &= \Big(F'(U(p,q))\cdot V'(q)\cdot\lambda_H\Big)\cdot(p-aq)+ \\ &\quad \Big(F(U(p,q))\cdot\lambda_H + F(U(p,\hat q(p)))\cdot\lambda_M + \frac{\lambda_L}{2}\Big)\cdot(-a)\Big|_{(\tilde p,\hat q(\tilde p))} = \\ &\quad \Big(F'(U(\tilde p,\hat q(\tilde p)))\cdot V'(\hat q(\tilde p))\cdot\lambda_H\Big)\cdot(\tilde p - a\hat q(\tilde p))+ \\ &\quad \Big(F(U(\tilde p,\hat q(\tilde p))\cdot(\lambda_H+\lambda_M)+\frac{\lambda_L}{2}\Big)\cdot(-a) = 0. \qquad (9)\end{aligned}$$

From (8) and (9) it follows after some algebra that

$$\frac{d\hat q}{dp} = \frac{\lambda_H+\lambda_M}{\lambda_M}\frac{1}{V'(\hat q(p))} - \frac{\lambda_H}{a\lambda_M}.$$

□

Theorem 1. *If there exists a perfect indicator equilibrium, then* $[p_l, p_h] = [p_l, p_m]$*,* $\hat q(p)$ *is strictly increasing in* p *over this interval, and* $U(p,\hat q(p))$ *is strictly decreasing. Hence, in an equilibrium, higher prices indicate higher quality but lower utility.*

Proof. The formal proof is fully contained in the following lemmas A.1-A.5. Here we only give an overview of the structure.

For there to be an equilibrium, the profit function $\pi(p,q)$ should attain its maximum along the equilibrium curve $\hat q(p)$. The idea of the proof is to apply second order necessary conditions to check whether that can be the case, given different choices of p_l and p_h. For the purpose of this proof it suffices to check concavity in q only, i.e., we examine the following second order necessary condition:

$$\frac{\partial^2\pi(p,q)}{\partial^2 q}\Big|_{(p,\hat q(p))} \le 0 \quad \text{for} \quad p\in[p_l,p_h].$$

Lemma A.4 provides the expression for $\frac{\partial^2\pi(p,q)}{\partial^2 q}\Big|_{(p,\hat q(p))}$. The expression is complicated and it is hard to evaluate its sign for an arbitrary p from $[p_l,p_h]$. However, by considering a limiting case with $p\to p_m$, Lemma A.5 shows that $[p_l,p_h]=[p_l,p_m]$. The results on the monotonicity of $\hat q(p)$ and $U(p,\hat q(p))$ follow from lemmas A.2 and A.3 respectively. □

Lemma A.1. *$p_m \in [p_l, p_h]$ and $q_m = \hat{q}(p_m)$, i.e. point (p_m, q_m) belongs to an equilibrium curve.*

Proof. Pick an arbitrary $p_0 \in [p_l, p_h]$ such that $U(p, \hat{q}(p)) = U_l$ (existence of such a point follows from the definition of U_l). Let $q_0 = \hat{q}(p_0)$. Since $U(p_0, q_0) = U_l$ we have that $F(U(p_0, q_0)) = 0$ and therefore

$$\pi(p_0, q_0) = \frac{\lambda_L}{2}\Pi(p_0, q_0).$$

As (p_0, q_0) belongs to the equilibrium curve, it maximizes the profits. Hence $\pi(p_0, q_0) \geq \pi(p, q)$ for any (p, q). But $\pi(p, q) \geq \frac{\lambda_L}{2}\Pi(p, q)$ if $U(p, q) \geq 0$. Therefore

$$\frac{\lambda_L}{2}\Pi(p_0, q_0) \geq \frac{\lambda_L}{2}\Pi(p, q)$$

for any (p, q) such that $U(p, q) \geq 0$. Hence, (p_0, q_0) is a solution to the following optimization problem:

$$\max_{p,q} \Pi(p, q) \quad \text{s.t.} \quad U(p, q) \geq 0. \tag{10}$$

But (10) is precisely the definition of (p_m, q_m). Thus $(p_0, q_0) = (p_m, q_m)$. □

Lemma A.2. *$\frac{d\hat{q}}{dp} > 0$.*

Proof. We have

$$\frac{d\hat{q}}{dp} = \frac{\lambda_H + \lambda_M}{\lambda_M}\frac{1}{V'(\hat{q}(p))} - \frac{\lambda_H}{a\lambda_M}. \tag{11}$$

The boundary condition is provided by Lemma A.1: $\hat{q}(p_m) = q_m$. By definition,

$$(p_m, q_m) = \arg\max_{(p,q):U(p,q)\geq 0} \Pi(p, q).$$

The first order condition is $V'(q_m) = a$. Thus, $\hat{q}'(p_m) = \frac{1}{a} > 0$.

As is readily seen from (11), $\hat{q}'(p)$ is continuous in $\hat{q}(p)$. In turn, $\hat{q}(p)$ is itself continuous in p as a solution to a differential equation. So, $\hat{q}'(p)$ is continuous in p.

Suppose there exists $\tilde{p}$ such that $\hat{q}'(\tilde{p}) \leq 0$. Then, given the continuity of $\hat{q}'(p)$ and given that $\hat{q}'(p_m) > 0$, there exists p_0 in between of $\tilde{p}$ and p_m such that $\hat{q}'(p_0) = 0$.

Define q_0 as a solution to $V'(q_0) = a\frac{\lambda_H + \lambda_M}{\lambda_H}$. Then $\hat{q}(p_0) = q_0$.

Define $\tilde{q}(p) = q_0$. Clearly, $\tilde{q}(p)$ is also a solution to (11), and it also passes through the point (p_0, q_0). However, it is distinct from $\hat{q}(p)$ as $\tilde{q}'(p) = 0$ for all p. As two distinct solutions to (11) can not pass through the same point, we get that there is no $\tilde{p}$ such that $\hat{q}'(\tilde{p}) \leq 0$.

□

Lemma A.3. *$\frac{d}{dp}U(p, \hat{q}(p)) > 0$ for $p > p_m$ and $\frac{d}{dp}U(p, \hat{q}(p)) < 0$ for $p < p_m$.*

Proof. For convenience let $\hat{U}$ stand for $U(p, \hat{q}(p))$ and let the same be for the derivatives, e.g. $\hat{U}_p$ stands for $U_p(p, \hat{q}(p)) = \frac{\partial U(p,q)}{\partial p}\Big|_{(p,\hat{q}(p))}$. We have:

$$\frac{d}{dp}U(p, \hat{q}(p)) = \hat{U}_p + \hat{U}_q\frac{d\hat{q}}{dp} = \frac{\lambda_H}{\lambda_M}\left(1 - \frac{V'(\hat{q}(p))}{a}\right)$$

and

$$\frac{d^2}{dp^2}U(p,\hat{q}(p)) = -\frac{V''(\hat{q}(p))}{a}\frac{d\hat{q}}{dp}.$$

$V'' < 0$ by assumption and $\hat{q}' > 0$ by Lemma A.2. Thus $\frac{d^2}{dp^2}U(p,\hat{q}(p)) > 0$.

We have that $V'(\hat{q}(p_m)) = a$ (see the proof of Lemma A.2), thus $\frac{d}{dp}U(p,\hat{q}(p)) = 0$ at $p = p_m$. As $\frac{d}{dp}U(p,\hat{q}(p))$ is strictly increasing, we immediately get $\frac{d}{dp}U(p,\hat{q}(p)) > 0$ for $p > p_m$ and $\frac{d}{dp}U(p,\hat{q}(p)) < 0$ for $p < p_m$. □

Lemma A.4. *For any* $p \neq p_m$

$$\frac{\partial^2 \pi(p,q)}{\partial q^2}\Big|_{(p,\hat{q}(p))} = \frac{a^2\lambda_L\lambda_M}{\lambda_H} \cdot \frac{\hat{\Pi}(0)}{\Pi(p,\hat{q}(p))}\left(\frac{V''(\hat{q}(p))}{2V'(\hat{q}(p))} \cdot \frac{1}{V'(\hat{q}(p)) - a} + \frac{1}{\Pi(p,\hat{q}(p))}\right).$$

Proof. We prove this lemma in a straightforward way. Recollect from Lemma 3 that

$$\pi(p,q) = \left(F(U(p,q)) \cdot \lambda_H + F(U(p,\hat{q}(p))) \cdot \lambda_M + \frac{\lambda_L}{2}\right) \cdot \Pi(p,q),$$

where $\Pi(p,q) = p - aq$ and $F(u)$ is given by Lemma 5. If we are to differentiate $\pi(p,q)$ we need to know $\hat{q}'(p)$ and $\hat{\Pi}'(u)$. The former derivative we take from Lemma 6:

$$\frac{d\hat{q}}{dp} = \frac{\lambda_H + \lambda_M}{\lambda_M} \cdot \frac{1}{V'(\hat{q}(p))} - \frac{\lambda_H}{a\,\lambda_M}. \tag{12}$$

As for the latter derivative, recollect that

$$\hat{\Pi}(u) = \Pi(\tilde{p}(u), \hat{q}(\tilde{p}(u))), \tag{13}$$

where $\tilde{p}(u)$ could be any function such that $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$. We'll be looking at the second order derivative of $\pi(p,q)$ at point $(p_0, \hat{q}(p_0))$ of an equilibrium curve with $p_0 \neq p_m$. For this point we can be more precise about $\tilde{p}(u)$. Indeed, from Lemma A.3 we know that

$$\frac{d}{dp}U(p,\hat{q}(p)) \neq 0 \quad \text{for} \quad p \neq p_m.$$

Also $U(p,\hat{q}(p))$ is twice differentiable because for $U(p,q)$ it was assumed and $\hat{q}(p)$ is itself defined by a differential equation that involves only differentiable functions. So, by an inverse function theorem there is a unique continuously differentiable $\tilde{p}(u)$ defined in the neighbourhood of $u_0 = U(p_0, \hat{q}(p_0))$ by $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$, with its derivative given by

$$\frac{d\tilde{p}(u)}{du} = \frac{1}{U_p(\tilde{p}(u), \hat{q}(\tilde{p}(u)) + U_q(\tilde{p}(u), \hat{q}(\tilde{p}(u)) \cdot \hat{q}'(\tilde{p}(u))} = -\frac{a\lambda_M}{\lambda_H}\frac{1}{V'(\hat{q}(\tilde{p}(u))) - a}. \tag{14}$$

Expressions (12), (13) and (14) allow us to calculate the second order derivative of $\pi(p,q)$ in q in a straightforward way. Let Π stand for $\Pi(p,q)$, $\hat{\Pi}$ – for $\hat{\Pi}(U(p,q))$,

$\hat{\Pi}_0$ – for $\hat{\Pi}(0)$, V_q – for $V_q(q)$, $\tilde{V}_q$ – for $V_q(\hat{q}(\tilde{p}(U(p,q))))$, and similarly for V_{qq} and $\tilde{V}_{qq}$. Then

$$\frac{\partial^2\pi(p,q)}{\partial q^2} = \frac{a\lambda_L\hat{\Pi}_0}{2\lambda_H\hat{\Pi}^3\tilde{V}_q^3}\Bigg(\frac{a\lambda_M\Pi\hat{\Pi}V_q^2\tilde{V}_{qq}}{\tilde{V}_q - a} + \lambda_H\Pi\hat{\Pi}\left(V_{qq}\tilde{V}_q^2 - V_q^2\tilde{V}_{qq}\right) + 2a\lambda_H\left(\Pi V_q^2\tilde{V}_q - \hat{\Pi}V_q\tilde{V}_q^2\right) + 2a\lambda_M\Pi V_q^2\tilde{V}_q\Bigg) \quad (15)$$

in the neighbourhood of point $(p_0, \hat{q}(p_0))$.

Evaluating (15) at $(p_0, \hat{q}(p_0))$, noticing that

$$\tilde{p}(U(p_0, \hat{q}(p_0))) = p_0$$

and that, consequently, $\Pi = \hat{\Pi}$, $V_q = \tilde{V}_q$, $V_{qq} = \tilde{V}_{qq}$, and noticing that p_0 was chosen arbitrary just not to equal p_m, we immediately obtain the result of the lemma. □

Lemma A.5. $p_h = p_m$.

Proof. Suppose $p_h > p_m$. As $p_m \in [p_l, p_h]$ (Lemma A.1), we can consider the limit of $\frac{\partial^2\pi(p,q)}{\partial q^2}\Big|_{(p,\hat{q}(p))}$ as p approaches p_m from the right. To do so let us first consider the limit of $V'(\hat{q}(p))$. We have

$$\frac{d}{dp}U(p,\hat{q}(p)) = \hat{U}_p + \hat{U}_q\frac{d\hat{q}}{dp} = \frac{\lambda_H}{\lambda_M}\left(1 - \frac{V'(\hat{q}(p))}{a}\right).$$

By Lemma A.3, $\frac{d}{dp}U(p,\hat{q}(p)) > 0$ for $p > p_m$. Also, $V'(\hat{q}(p_m)) = a$. Thus

$$V'(\hat{q}(p)) \uparrow a \quad \text{as} \quad p \downarrow p_m$$

and so

$$\lim_{p\downarrow p_m}\frac{\partial^2\pi(p,q)}{\partial q^2}\Big|_{(p,\hat{q}(p))} = \lim_{p\downarrow p_m}\frac{a^2\lambda_L\lambda_M}{\lambda_H}\cdot\frac{\hat{\Pi}(0)}{\Pi(p,\hat{q}(p))}\left(\frac{V''(\hat{q}(p))}{2V'(\hat{q}(p))}\cdot\frac{1}{V'(\hat{q}(p)) - a} + \frac{1}{\Pi(p,\hat{q}(p))}\right) = +\infty \quad (16)$$

(also using $\hat{\Pi}(0) > 0$, $\Pi(p,\hat{q}(p)) > 0$, $V' > 0$ and $V'' < 0$).

However, (16) contradicts the necessary condition that $\frac{\partial^2\pi(p,q)}{\partial q^2}\Big|_{(p,\hat{q}(p))} \leq 0$ for all $p \in [p_l, p_h]$. Therefore if there is a perfect indicator equilibrium it should be that $p_h \leq p_m$ (in this case we can not consider a limit from the right and the contradiction does not hold). But $p_m \in [p_l, p_h]$, hence $p_h = p_m$. □

The following two lemmas – A.6 and A.7 – will be required in the proof of Theorem 2.

Lemma A.6. *$\frac{d}{du}\tilde{p}(u) < 0$ and $\frac{d}{du}\hat{\Pi}(u) < 0$ for $u > 0$.*

Proof. By Lemma A.5 $p \leq p_m$. As $\frac{d}{dp}U(p, \hat{q}(p)) < 0$ for $p < p_m$ (Lemma A.3), equation

$$U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$$

uniquely defines $\tilde{p}(u)$. Also, $\tilde{p}(u) < p_m$ for $u > U_l = 0$. Differentiating this equation with respect to u gives

$$\big(V'(\hat{q}(\tilde{p}(u))) \cdot \hat{q}'(\tilde{p}(u)) - 1\big)\tilde{p}'(u) = 1$$

or, using Lemma 6,

$$\frac{\lambda_H}{\lambda_M}\left(1 - \frac{V'(\hat{q}(\tilde{p}(u)))}{a}\right)\tilde{p}'(u) = 1.$$

For $p < p_m$ we have that $\hat{q}(p) < q_m$ (lemmas A.1, A.2). So, as $V'(q_m) = a$ and $V'(q)$ is strictly decreasing in q, we get that $V'(\hat{q}(\tilde{p}(u))) > a$. Consequently, $\tilde{p}'(u) < 0$.

Next,

$$\hat{\Pi}(u) = \tilde{p}(u) - a\hat{q}(\tilde{p}(u)).$$

So,

$$\hat{\Pi}'(u) = (1 - a\hat{q}'(\tilde{p}(u))) \cdot \tilde{p}'(u) = \frac{\lambda_M + \lambda_H}{\lambda_M}\left(1 - \frac{a}{V'(\hat{q}(\tilde{p}(u)))}\right)\tilde{p}'(u) < 0.$$

□

Lemma A.7. *For $p \leq p_m$ there exists a unique solution to the initial value problem defined by*

$$\frac{d\hat{q}}{dp} = \frac{\lambda_H + \lambda_M}{\lambda_M}\frac{1}{V'(\hat{q})} - \frac{\lambda_H}{a\lambda_M}$$

and $\hat{q}(p_m) = q_m$.

Proof. Let

$$f(q) = \frac{\lambda_H + \lambda_M}{\lambda_M}\frac{1}{V'(q)} - \frac{\lambda_H}{a\lambda_M}$$

and let $N(x, \varepsilon)$ denote the ε-neighbourhood of point x.

Let $\tilde{q}$ be the unique solution to $f(\tilde{q}) = 0$. Choose $\varepsilon > 0$ such that $f(q)$ is Lipschitz continuous over $N(\tilde{q}, \varepsilon)$. Both of these operations can be performed given our assumptions on V. Moreover, given the assumptions on V, $f(q)$ will be Lipschitz continuous over $N(q, \varepsilon)$ for any $q > \tilde{q}$.

Let $p_0 = p_m$, $q_0 = q_m$ and $S_0 = \sup_{q \in N(q_0, \varepsilon)} |f(q)|$. We have that $q_0 > \tilde{q}$, thus $f(q)$ is Lipschitz continuous over $N(q_0, \varepsilon)$. We further have

$$S_0 = \max\left(\frac{\beta}{a}, f(q_0 + \varepsilon)\right).$$

By Picard's theorem there exists a unique solution to the stated boundary problem, defined over $N(p_0, \varepsilon/S_0)$. We are interested in the left side of this solution. Define $p_1 = p_0 - \varepsilon/S_0$, $q_1 = \hat{q}(p_1)$, and $S_1 = \sup_{q \in N(q_1, \varepsilon)} |f(q)|$. According to

Lemma A.2, $f(q_1) > 0$.[11] Therefore $q_1 > \tilde{q}$ and so $f(q)$ is Lipschitz continuous over $N(q_1, \varepsilon)$. Also due to Lemma A.2, $q_1 < q_0$. Therefore

$$S_1 = \max\left(\frac{\beta}{a}, f(q_1 + \varepsilon)\right) \leq S_0$$

and $N(p_1, \varepsilon/S_1) \supset N(p_1, \varepsilon/S_0)$.

So, we can continue applying Picard's theorem step by step, without those steps getting smaller. Thus for $p \leq p_m$ there is a unique solution to the stated boundary problem. □

Theorem 2. *Given any finite and strictly positive ratio $\frac{\lambda_H}{\lambda_M}$ there exists $\lambda_L^* \in (0,1)$ such that for all $\lambda_L < \lambda_L^*$ there are no perfect indicator equilibria and for all $\lambda_L > \lambda_L^*$ there exists a unique perfect indicator equilibrium.*

Proof. Given a fixed ratio $\frac{\lambda_H}{\lambda_M}$, we can vary λ_L from 0 to 1, while maintaining the normalization $\lambda_H + \lambda_M + \lambda_L = 1$. We prove the theorem in four steps: 1) we construct an equilibrium candidate, which is necessarily unique; 2) we show that if this candidate is not an equilibrium for some λ_L^1, then it is not an equilibrium for any $\lambda_L < \lambda_L^1$; 3) we show there exists λ_L sufficiently close to 0 such that the candidate is not an equilibrium; 4) we finally show that there exists λ_L sufficiently close to 1 such the candidate is an equilibrium. Step 1) delivers uniqueness, step 2) guarantees a unique cut-off point between the non-existence and existence regions, steps 3) and 4) guarantee that λ_L^* belongs to the interior of $[0,1]$, i.e. that the regions where the equilibrium exists and where it does not are not empty.

This proof uses some of the earlier lemmas. The earlier lemmas can be divided into two groups: 1) those lemmas (1, 2, 4, 5, 6, A.1, A.5) that derive certain properties of a perfect indicator equilibrium given it exists; and 2) those lemmas (3, A.2, A.3, A.4, A.6) that do not directly require the existence of an equilibrium in their proofs, but that might require it indirectly through the reliance on the lemmas of the first group. Below, we carefully use these lemmas so as not to prove existence by supposing it.

Step 1. Let $\beta = \frac{\lambda_H}{\lambda_M}$. If there is a perfect indicator equilibrium, then, according to Lemma 6, its equilibrium curve $\hat{q}(p)$ has to satisfy

$$\frac{d\hat{q}}{dp} = \frac{\lambda_H + \lambda_M}{\lambda_M} \frac{1}{V'(\hat{q}(p))} - \frac{\lambda_H}{a\lambda_M} = \frac{1+\beta}{V'(\hat{q}(p))} - \frac{\beta}{a}. \tag{17}$$

Additionally, due to Lemma A.1, $\hat{q}(p)$ has to pass through the point (p_m, q_m), which is uniquely defined. By Lemma A.7, this boundary problem has a unique solution for $p \leq p_m$; we denote this solution with $\hat{q}(p)$.

Given that $\hat{q}(p)$ was constructed based on lemmas 6 and A.1, Lemma A.3 holds: $U(p, \hat{q}(p))$ is strictly decreasing in p for $p < p_m$. Consequently, $\tilde{p}(u)$ is uniquely defined as a solution to $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$, $\tilde{p}(u) \leq p_m$. Let

$$\hat{\Pi}(u) = \Pi(\tilde{p}(u), \hat{q}(\tilde{p}(u))).$$

[11] Originally, Lemma A.2 is based on Lemma A.1, which itself is based on the condition that an equilibrium exists. In this proof we have the result of Lemma A.1 simply as a condition for the current lemma, so Lemma A.2 is used without requiring Lemma A.1 and thus without requiring the existence of an equilibrium.

In correspondence with Lemma 5, let

$$F(u) = \frac{1}{2} \cdot \frac{\lambda_L}{\lambda_H + \lambda_M} \left(\frac{\hat{\Pi}(0)}{\hat{\Pi}(u)} - 1 \right) = \frac{1}{2} \cdot \frac{\lambda_L}{1 - \lambda_L} \left(\frac{\hat{\Pi}(0)}{\hat{\Pi}(u)} - 1 \right).$$

Consequently, Lemma 1 holds by construction.

In correspondence with lemmas 2 and A.5, let $U_l = 0$ and $p_h = p_m$. Further, let $U_h = F^{-1}(1)$ and $p_l = \tilde{p}(U_h)$.

The equilibrium strategies of the firms are now uniquely defined. The equilibrium strategy of the consumer, subject to the refinements (iv) and (v) of the PIE definition, is also unique. So, step 1) is finalized.

It remains to note that all the lemmas of group one hold by construction now, and, consequently, all the lemmas of group two hold as well.

Step 2. For a fixed β we vary λ_L. The point (p_m, q_m) is constant. Given (17), $\hat{q}(p)$ depends on β only, so $\hat{q}(p)$ does not vary when we vary λ_L and neither does $\tilde{p}(u)$ or $\hat{\Pi}(u)$. However, the border points U_h and p_l do depend on λ_L, we will write $U_h(\lambda_L)$ and $p_l(\lambda_L)$ therefore.

Define

$$B(\lambda_L) = \{(p, q) \,|\, p_l(\lambda_L) \le p \le p_m,\, 0 \le U(p, q) \le U_h(\lambda_L),\, q \le q_m\}.$$

For $(p, q) \in B(\lambda_L)$ we have

$$\pi(p, q) = \left(F(U(p, q)) \cdot \lambda_H + F(U(p, \hat{q}(p))) \cdot \lambda_M + \frac{\lambda_L}{2} \right) \Pi(p, q) =$$
$$\frac{\hat{\Pi}(0)\lambda_L}{2} \left(\frac{\beta}{1+\beta} \cdot \frac{1}{\hat{\Pi}(U(p, q))} + \frac{1}{1+\beta} \cdot \frac{1}{\hat{\Pi}(U(p, \hat{q}(p)))} \right) \Pi(p, q). \quad (18)$$

Consider $\pi(p, q)$ as a function of λ_L. Given (18) we have

$$\pi(p, q; \alpha\lambda_L) = \alpha\pi(p, q; \lambda_L) \quad (19)$$

as long as $(p, q) \in B(\alpha\lambda_L)$ and $(p, q) \in B(\lambda_L)$. The profit function changes in scale with respect to λ_L, but not in shape.

Next, we prove two statements about $B(\lambda_L)$: B1) $\pi(p, q) \le \max_{(p,q)\in B(\lambda_L)} \pi(p, q)$, i.e. the profits outside the $B(\lambda_L)$ region are never larger than the profits inside that region, and B2) $B(\lambda_L^1) \supset B(\lambda_L^2)$ if $\lambda_L^1 < \lambda_L^2$, i.e. as λ_L decreases, the region $B(\lambda_L)$ expands.

Consider a point $(p_0, q_0) \notin B$. Suppose $0 \le U(p_0, q_0) \le U_h$, $p_0 \in [p_l, p_h]$ and $q_0 > q_m$. Let $\tilde{p}(q) = V(q) - U(p_0, q_0)$, thus $\tilde{p}(q)$ defines the iso-utility curve going through the point (p_0, q_0). Let $p_1 = \tilde{p}(q_m)$. We get $U(p_1, q_m) = V(q_m) - p_1 = U(p_0, q_0)$ and, consequently, $F(U(p_1, q_m)) = F(U(p_0, q_0))$. As $q_m < q_0$, $p_1 < p_0$. By Lemma A.3 $U(p_1, \hat{q}(p_1)) > U(p_0, \hat{q}(p_0))$. Therefore $F(U(p_1, \hat{q}(p_1))) > F(U(p_0, \hat{q}(p_0)))$. Finally, consider $\Pi(\tilde{p}(q), q)$. We have $\frac{d}{dq}\Pi(\tilde{p}(q), q)) = V'(q) - a < 0$ for $q > q_m$ (because $V'(q_m) = a$ and $V''(q) < 0$). Thus, $\Pi(p_1, q_m)) = \Pi(\tilde{p}(q_m), q_m)) > \Pi(\tilde{p}(q_0), q_0)) = \Pi(p_0, q_0)$. Summing up these results and using Lemma 3 gives $\pi(p_1, q_m) > \pi(p_0, q_0)$. It only remains to note that $(p_1, q_m) \in B(\lambda_L)$.

Suppose $0 \le U(p_0, q_0) \le U_h$ and $p_0 > p_h$. Let $\tilde{p}(q)$ and p_1 be defined as before. Then $F(U(p_1, q_m)) = F(U(p_0, q_0))$ and $\Pi(p_1, q_m)) > \Pi(p_0, q_0)$ still hold. As the partially informed consumer never buys at $p_0 > p_h$ we immediately obtain that $\pi(p_1, q_m) > \pi(p_0, q_0)$.

Suppose $0 \le U(p_0, q_0) \le U_h$ and $p_0 < p_l$. Let $\tilde{q}(p)$ be such that $V(\tilde{q}(p)) - p = U(p_0, q_0)$. Given the assumption on $V(q)$ such $\tilde{q}(p)$ is uniquely defined. Let $q_1 = \tilde{q}(p_l)$. We get $U(p_l, q_1) = U(p_0, q_0)$ and $F(U(p_l, q_1)) = F(U(p_0, q_0))$. Further, $\frac{d}{dp}\Pi(p, \tilde{q}(p)) = 1 - a/V'(\tilde{q}(p)) > 0$ for $p < p_l$ (because $V'(q_m) = a$, $V''(q) < 0$ and $\tilde{q}(p) < q_m$ for $p < p_l$). As the partially informed consumer never buys at $p_0 < p_l$ we obtain that $\pi(p_l, q_1) > \pi(p_0, q_0)$.

So, for $(p_0, q_0) \notin B$ and such that $0 \le U(p_0, q_0) \le U_h$ we have that $\pi(p_0, q_0) < \max_{(p,q)\in B} \pi(p, q)$.

Suppose $U(p_0, q_0) > U_h$. Let $\tilde{q}$ be such that $U(p_0, \tilde{q}) = U_h$. Then $\tilde{q} < q$, $\Pi(p_0, \tilde{q}) > \Pi(p_0, q_0)$ and $F(U(p_0, \tilde{q})) = F(U(p_0, q_0)) = 1$. Thus, $\pi(p_0, \tilde{q}) > \pi(p_0, q_0)$. If $(p_0, \tilde{q}) \in B$ we immediately get that $\pi(p_0, q_0) < \max_{(p,q)\in B} \pi(p, q)$. Otherwise we come to the same conclusion but using the result of the previous paragraph.

Finally, suppose that $U(p_0, q_0) < 0$. Then $\pi(p_0, q_0) = 0 < \frac{\lambda_L}{2}\Pi(p_m, q_m) = \pi(p_m, q_m)$, where $(p_m, q_m) \in B$. This finalizes the proof of B1.

To prove B2 it suffices to show that $p_l(\lambda_L)$ strictly increases in λ_L and $U_h(\lambda_L)$ strictly decreases in λ_L.

We have

$$U_h = F^{-1}(1) = \hat{\Pi}^{-1}\left(\frac{\hat{\Pi}(0)}{2/\lambda_L - 1}\right). \tag{20}$$

As $\hat{\Pi}(u)$ does not depend on λ_L and is strictly decreasing in u (Lemma A.6), we immediately obtain that $U_h(\lambda_L)$ is strictly decreasing in λ_L. Further, $p_l(\lambda_L) = \tilde{p}(U_h(\lambda_L))$, but $\tilde{p}(u)$ is strictly decreasing in u (same Lemma A.6), thus $p_l(\lambda_L)$ is strictly increasing in λ_L. This completes the proof of statement B2.

Suppose the equilibrium candidate is not an equilibrium for a certain λ_L^2. Then there exists (p_1, q_1) such that $\pi(p1, q1; \lambda_L^2) > \pi(p_m, q_m; \lambda_L^2)$. According to B1, $(p_1, q_1) \in B(\lambda_L^2)$. Choose any $\lambda_L^1 < \lambda_L^2$. According to B2, $B(\lambda_L^1) \supset B(\lambda_L^2)$, thus $(p_1, q_1) \in B(\lambda_L^1)$, formula (19) is valid and

$$\pi(p1, q1; \lambda_L^1) = \frac{\lambda_L^1}{\lambda_L^2} \cdot \pi(p1, q1; \lambda_L^2) > \frac{\lambda_L^1}{\lambda_L^2} \cdot \pi(p_m, q_m; \lambda_L^2) = \pi(p_m, q_m; \lambda_L^1).$$

So, the candidate is not an equilibrium for λ_L^1.

Step 3. Consider $\frac{\partial^2 \pi(p,q)}{\partial q^2}$ at a point $(p, q) = \big(p_l(\lambda_L), \hat{q}(p_l(\lambda_L))\big)$. Using Lemma A.4 and using $\Pi(p_l, \hat{q}(p_l)) = \Pi\big(\tilde{p}(U_h), \hat{q}(\tilde{p}(U_h))\big) = \hat{\Pi}(U_h)$ we obtain

$$\frac{\partial^2 \pi(p,q)}{\partial q^2}\Big|_{(p_l(\lambda_L), \hat{q}(p_l(\lambda_L)))} = \frac{a^2 \lambda_L}{\beta} \cdot \frac{\hat{\Pi}(0)}{\hat{\Pi}(U_h(\lambda_L))} \left(\frac{V''(\hat{q}(p_l(\lambda_L)))}{2V'(\hat{q}(p_l(\lambda_L)))} \cdot \frac{1}{V'(\hat{q}(p_l(\lambda_L))) - a} + \frac{1}{\hat{\Pi}(U_h(\lambda_L))}\right).$$

We next consider the limit of this expression as $\lambda_L \to 0$. Similar to the proof of Lemma A.2, define q_0 by $V'(q_0) = a\left(1 + \frac{1}{\beta}\right)$. Clearly, $q_0 < q_m$. Moreover, $\hat{q}(p)$ never reaches q_0 as p decreases, because if it did for some p_0, then it would be that $\hat{q}'(p_0) = 0$, but that contradicts Lemma A.2. Thus $q_0 < \hat{q}(p) \le q_m$ for any $p \in [p_l(\lambda_L), p_h]$. As $V'(q) > 0$, $V''(q) < 0$ and both are continuous, there exists a finite M such that

$$M \le \frac{V''(\hat{q}(p_l(\lambda_L)))}{2V'(\hat{q}(p_l(\lambda_L)))} < 0,$$

where $M < 0$ does not depend on λ_L, because neither q_0 nor q_m depend on λ_L.

As $p_l(\lambda_L)$ strictly increases in λ_L, $\hat{q}(p)$ strictly increases in p, and $V'(q)$ strictly decreases in q, we get that $V'(\hat{q}(p_l(\lambda_L)))$ is strictly decreasing in λ_L. As we look at $\lambda_L \to 0$, we can limit our analysis to $\lambda_L \le \frac{1}{2}$. In this case

$$0 < \frac{1}{V'(\hat{q}(p_l(\lambda_L))) - a} < \frac{1}{V'(\hat{q}(p_l(1/2))) - a}.$$

Finally, using $F(U_h) = 1$, we obtain

$$\lim_{\lambda_L \to 0} \frac{1}{\hat{\Pi}(U_h(\lambda_L))} = \lim_{\lambda_L \to 0} \left(\frac{2}{\lambda_L} - 1\right) \hat{\Pi}(0) = +\infty$$

and

$$\lim_{\lambda_L \to 0} \frac{\lambda_L}{\hat{\Pi}(U_h(\lambda_L))} = \lim_{\lambda_L \to 0} (2 - \lambda_L)\hat{\Pi}(0) = 2\hat{\Pi}(0).$$

Summing up yields

$$\lim_{\lambda_L \to 0} \frac{\partial^2 \pi(p,q)}{\partial q^2}\Big|_{(p_l(\lambda_L), \hat{q}(p_l(\lambda_L)))} = +\infty.$$

Thus, for a sufficiently small λ_L there is a profitable deviation from the equilibrium curve in the neighbourhood of the point $(p_l, \hat{q}(p_l))$. For a sufficiently small λ_L the equilibrium candidate is not an equilibrium.

Step 4. Define

$$D(\lambda_L) = B(\lambda_L) \cap \{(p,q) \,|\, U(p,q) > 0\} = \\ \{(p,q) \,|\, p_l(\lambda_L) \le p \le p_m,\ 0 < U(p,q) \le U_h(\lambda_L),\ q \le q_m\}.$$

Because $U(p,q) > 0$ and $q \le q_m$, $p_m \notin D(\lambda_L)$, i.e. $p < p_m$.

Consider $(p,q) \in D(\lambda_L)$. We have $\frac{d}{dp} U(p, \hat{q}(p)) < 0$, because $p < p_m$. Then, given that $U(p_l, \hat{q}(p_l)) = U_h$ and $U(p_m, \hat{q}(p_m)) = U_l$,

$$U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u \tag{21}$$

uniquely defines $\tilde{p}(u)$. Differentiating (21) with respect to u gives

$$\tilde{p}'(u) = -\frac{a\lambda_M}{\lambda_H} \frac{1}{V'(\hat{q}(\tilde{p}(u))) - a}$$

for all $u \in (0, U_h]$. Given that $\tilde{p}'(u)$ is defined for all u, we can repeat the steps of Lemma A.4 and obtain that

$$\frac{\partial^2 \pi(p,q)}{\partial q^2} = \frac{a\lambda_L \hat{\Pi}_0}{2\beta\hat{\Pi}^3 \tilde{V}_q^3} \left(\frac{a\Pi\hat{\Pi} V_q^2 \tilde{V}_{qq}}{\tilde{V}_q - a} + \right. \\ \left. \beta\Pi\hat{\Pi}\left(V_{qq}\tilde{V}_q^2 - V_q^2\tilde{V}_{qq}\right) + 2a\beta\left(\Pi V_q^2 \tilde{V}_q - \hat{\Pi} V_q \tilde{V}_q^2\right) + 2a\Pi V_q^2 \tilde{V}_q \right)$$

for all $(p,q) \in D(\lambda_L)$, where Π stands for $\Pi(p,q)$, $\hat{\Pi}$ – for $\hat{\Pi}(U(p,q))$, $\hat{\Pi}_0$ – for $\hat{\Pi}(0)$, V_q – for $V_q(q)$, $\tilde{V}_q$ – for $V_q(\hat{q}(\tilde{p}(U(p,q))))$, and similarly with V_{qq} and $\tilde{V}_{qq}$.

Consider $\lambda_L \to 1$ and consider an arbitrary sequence $(p(\lambda_L), q(\lambda_L)) \in D(\lambda_L)$. Given (20) and given that $\hat{\Pi}(u)$ is continuous and does not depend on λ_L, we obtain that $U_h(\lambda_L) \to 0$ as $\lambda_L \to 1$. By definition of $\tilde{p}$ we have $\tilde{p}(0) = p_m$. Moreover, $\tilde{p}(u)$ is continuous. Therefore $p_l(\lambda_L) = \tilde{p}(U_h(\lambda_L)) \to p_m$ as $\lambda_L \to 1$. So,

$$(p(\lambda_L), q(\lambda_L)) \to (p_m, q_m) \quad \text{as} \quad \lambda_L \to 1.$$

By assumption, $\Pi(p,q)$, $V(q)$, $V_q(q)$ and $V_{qq}(q)$ are continuous. Moreover, as defined, $\hat{q}(p)$ and $\tilde{p}(u)$ are continuous as well. Thus $\hat{\Pi}(U(p,q))$, $V_q(\hat{q}(\tilde{p}(U(p,q)))$ and $V_{qq}(\hat{q}(\tilde{p}(U(p,q)))$ are also continuous. Consequently, as $\lambda_L \to 1$,

$$\begin{gathered}
\Pi(p(\lambda_L), q(\lambda_L)) \to \Pi(p_m, q_m) > 0, \\
\hat{\Pi}(U(p(\lambda_L), q(\lambda_L))) \to \hat{\Pi}(0) = \Pi(p_m, q_m) > 0, \\
V_{qq}(q(\lambda_L)) \to V_{qq}(q_m) < 0, \\
V_{qq}(\hat{q}(\tilde{p}(U(p(\lambda_L), q(\lambda_L)))) \to V_{qq}(q_m) < 0, \\
V_q(q(\lambda_L)) \to V_q(q_m) = a > 0, \\
V_q(\hat{q}(\tilde{p}(U(p(\lambda_L), q(\lambda_L)))) \downarrow a,
\end{gathered}$$

where the direction of the last limit follows from $V_q(\hat{q}(\tilde{p}(U(p,q))) > a$.

Summing up yields

$$\left.\frac{\partial^2 \pi(p,q)}{\partial q^2}\right|_{(p(\lambda_L), q(\lambda_L))} \to -\infty \quad \text{as} \quad \lambda_L \to 1. \tag{22}$$

The sequence $(p(\lambda_L), q(\lambda_L))$ was chosen arbitrary, therefore there exists $\lambda_L^\varepsilon < 1$ such that $\frac{\partial^2 \pi(p,q)}{\partial q^2} < 0$ for all $(p,q) \in D(\lambda_L)$ for all $\lambda_L > \lambda_L^\varepsilon$. Indeed, if this is not the case, then there exists a sequence $(p(\lambda_L), q(\lambda_L))$ such that $\frac{\partial^2 \pi(p,q)}{\partial q^2} \geq 0$ for each element of the sequence, what contradicts (22).

Given $\lambda_L > \lambda_L^\varepsilon$ and given p, $\pi(p,q)$ is thus strictly concave in q. As $\left.\frac{\partial \pi(p,q)}{\partial q}\right|_{(p,\hat{q}(p))} = 0$, we obtain $\pi(p,q) \leq \pi(p, \hat{q}(p))$ for all $(p,q) \in D(\lambda_L)$ given that $\lambda_L > \lambda_L^\varepsilon$.

The profit function $\pi(p,q)$ is continuous. Therefore $\pi(p,q) \leq \pi(p,\hat{q}(p))$ for all $(p,q) \in B(\lambda_L)$ given that $\lambda_L > \lambda_L^\varepsilon$. Further, using B1, we obtain $\pi(p,q) \leq \pi(p,\hat{q}(p))$ for all (p,q) given that $\lambda_L > \lambda_L^\varepsilon$. So, for a λ_L sufficiently close to 1 the equilibrium candidate is indeed an equilibrium. □