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RATIONALIZING FOCAL POINTS

ABSTRACT. Focal points seem to be important in helping players coordinate their strategies in coordination problems. Game theory lacks, however, a formal theory of focal points. This paper proposes a theory of focal points that is based on individual rationality considerations. The two principles upon which the theory rest are the Principle of Insufficient Reason (IR) and a Principle of Individual Team Member Rationality. The way IR is modelled combines the classic notion of description symmetry and a new notion of pay-off symmetry, which yields different predictions in a variety of games. The theory can explain why people do better than pure randomization in matching games.

KEY WORDS: Game theory, Focal points, Individual considerations

1. INTRODUCTION

One of the most fundamental ideas behind game theoretic solution concepts is that individual players choose their strategies on the basis of perceived differences in pay-offs. In line with the emphasis on pay-off differences as the criterion for choice is the view that rational players should not be concerned with the labels with which strategies are described. This view is most clearly expressed in Harsanyi and Selten (1988) who propose that strategies that do not differ in pay-offs be treated symmetrically by rational players. In matching games, this leads them to prescribe randomization over all available strategies.

Starting from Schelling (1960), it has been recognized that individuals who are confronted with a pure coordination game in everyday life frequently do much better than randomization. This observation has been confirmed in recent experiments (see, e.g., Bacharach and Bernasconi, 1997 and Mehta et al., 1994a,b). Apparently, individuals successfully use differences between strategy labels to coordinate their actions. Schelling has introduced the term ‘focal point’ to account for this strategic use of labels.

By showing that the use of focal points can be consistently incorporated into a framework of rational choice, this paper attempts

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to reconcile the view that rational players treat symmetric strategies in a symmetric way with the observation that in many coordination problems individuals do better than pure randomization. The key idea employed in this paper is similar to the one employed in Bacharach (1993), namely that the different labels that are attached to different strategies induce asymmetries between the strategies and that players can use these asymmetries in a rational way.

To illustrate this idea consider the following elementary example. Suppose individuals 1 and 2 have agreed to meet at restaurant chain Z in the city they live in. While being on their way, they suddenly realize that Z has three restaurants in the city. The restaurants are perceived to be identical to each other apart from their location: one restaurant is in the center and two are in suburban shopping malls. Both individuals are only interested in meeting each other. Can they make a rational choice where to go? Intuitively, the only sensible choice seems to be to go to the restaurant in the center. The two principles that are used in this paper to arrive at this solution are a Principle of Insufficient Reason (IR) and a Principle of Individual Team Member Rationality (TMR). IR basically says that a rational choice cannot discriminate between two strategies if they have the same characteristics. In this particular example, looking at the location of the restaurants divides the strategies into a set with one centrally located restaurant and a set with two suburban restaurants. A strategy that respects IR gives equal probability to the two suburban restaurants. On the other hand, TMR says that if there is a unique strategy combination that is Pareto-optimal, then individual players should do their part of that strategy combination. Here, the class of mixed strategy combinations that respect IR can be parameterized by $\{(1-2p_{1s}, p_{1s}, p_{1s}), (1-2p_{2s}, p_{2s}, p_{2s})\}$, where p_{1s} (p_{2s}) represents the probability with which individual 1 (2) goes to one of the suburban restaurants. It is easy to see that the unique Pareto-optimal strategy combination that satisfies IR is $\{(1, 0, 0), (1, 0, 0)\}$. Thus, the combination of the two principles explains the intuitive idea that rational individuals go to the restaurant in the center. The example illustrates the basic philosophy of the paper: TMR acts as an optimization principle and IR acts as a constraint on the set of feasible mixed strategies.

The present paper is most closely related to Variable Frame Theory as introduced by Bacharach (1991, 1993).[1] Bacharach introduces the idea that individual players conceptualize a game in a certain way. He argues that the way in which players 'cut up the world' is typically beyond their conscious control and shows how IR and a Principle of Coordination yield intuitively plausible solutions in a few examples. The present paper extends and modifies Bacharach's approach in the following way. First, and most importantly, the way IR is modelled in the present paper combines the notions of *description symmetry* and *pay-off symmetry*. Bacharach (1993) confines attention to asymmetries implied by the labels of the strategies; a notion I will call description symmetry. The notion of pay-off symmetry is inspired by Crawford and Haller (1990) and implies that strategies with different labels can be symmetric to each other as well. Second, Bacharach (1993) analyzes how his theory works out in a few selected examples, whereas I present results about uniqueness and optimality for a general class of descriptions. Third, Bacharach's (1993) Principle of Coordination *assumes* that players pick one of the Pareto-efficient equilibria.[2] The present paper, on the other hand, is not an equilibrium analysis. It is shown that a Nash equilibrium results when players are guided by IR and TMR. The example analyzed in the next section is chosen in such a way that the similarities and differences with Bacharach's analysis clearly come to the fore.

Sugden (1995) presents an alternative theory of focal points. He considers games in which a stochastic procedure determines the labels of different strategies. A player's decision rule picks a strategy for each possible private description. He explores the implications of different stochastic labeling procedures in specific games. As the present paper does not consider alternative labeling procedures, there is no direct connection between the present paper and Sugden's.

The principles of IR and TMR are not uncontroversial. A good discussion of IR and a justification for its use is provided by Sinn (1980) and Bacharach (1991). Bacharach (1991) takes as a starting point that players seek arguments for their choices. To have a valid argument to choose a particular option (and not another) implies that given the available information a player can make an inference that he should choose that option and that they cannot make a similar inference resulting in the conclusion that he should choose another option. If two options are symmetric, then any valid argument for one option is also a valid argument for the other. But then there cannot be any argument for any of the two options separately. Sinn (1980) shows that IR follows from two axioms commonly employed in the theory of expected utility, namely Completeness and Independence.

TMR is based on the idea that in situations in which there is a unique Pareto-optimal strategy combination, individual players can think of the set of players as a team. The best team solution to a problem is to choose a Pareto-optimal strategy combination. In any game each rational player can figure out what the set of Pareto-optimal strategy combinations is. If this set has only one member, all players, *reasoning individually*, know that there is only one way that they can reach this Pareto-optimal strategy combination and that is by doing their part of it. An alternative defense is given in Colman and Bacharach (1997). TMR is similar to the principle of coordination (Gauthier, 1975), pay-off dominance (Harsanyi and Selten, 1988) and collective rationality (Sugden, 1991). Gilbert (1990), among others, criticizes the principle on the ground that in certain situations other criteria as risk dominance (Harsanyi and Selten, 1988) may imply a different choice.

The rest of the paper is organized as follows. Section 2 presents an example showing the main differences with Bacharach's analysis. Section 3 provides formal definitions of the concepts of IR and TMR. Section 4 shows that IR and TMR assure that individual strategies are uniquely determined and that they together form a Nash equilibrium. Moreover, it is shown that by employing IR and TMR players can generally do better than pure randomization. The main part of the paper (Sections 2, 3 and 4) exclusively deals with matching games. Section 5 presents an example of a non-matching game that can be solved using ideas developed in Section 3. Section 6 considers an extension in which two components of a player's conceptual apparatus are distinguished: a player may think of a certain concept and not know how to use it (in a particular context) or a player may think of a concept and also know how to use it. Using an example, Section 6 discusses to what extent the results of Sections 3 and 4 generalize to this context. Section 7 concludes with a discussion and proofs are given in an appendix.

2. A COMPARISON WITH BACHARACH'S ANALYSIS: A SIMPLE EXAMPLE

This section presents an example highlighting the differences in implications of the alternative symmetry notions that Bacharach (1993) and I employ. For simplicity, and following Bacharach (1991), the examples in this section and the next are of the following form. There are two rooms and there are two players, one in each room. The players cannot communicate. Each of them is presented with the same tray with m objects. Inside each object there is a number (each object contains a different number) that helps to distinguish the different objects (and that facilitates the exposition below). The players cannot observe this number. They only observe some exterior characteristics of the objects, like shape, color, positions, and so on. Each of the players is asked to pick an object from the tray. The players receive a pay-off of 1 if they pick the same object (identified by the number inside the object). If they do not pick the same object their pay-off is equal to 0. The question is which object a rational player will choose?

EXAMPLE 1. *There are five objects and the players only observe their color (e.g., the positions are not clearly identifiable and of no use for coordinating, the objects have the same size and shape, and so on). The colors are as follows: green (no. 1), yellow (no. 2), red (no. 3), red (no. 4) and blue (no. 5).*

According to the Variable Frame Theory of Bacharach (1993) this example can be analyzed in the following way. There are four different colors; hence, there are five possible 'act descriptions': choose a block at random, choose the green, choose the yellow, choose a red at random and choose the blue. As there is no way to discriminate between the two red objects, they should receive the same probability. As there are clear and independent act descriptions for the other three objects, none of them is symmetric to one of the other objects.

Bacharach's (1993: 266) Principle of Coordination says that a solution is one of the Pareto-efficient equilibria of the reformulated game. In the present game, there are three such equilibria, namely both players choosing the green, both players choosing the yellow *and both players choosing the blue object. The pay-off associated with each solution is equal to 1.*

Bacharach's account can be criticized on the grounds that it does not offer the players a particular reason to choose their part of any one of the three 'solutions' of the game: any reason a player may have to choose, say, the green object (apart from the fact that it happens to be green) is also a reason to choose say the yellow object. It may very well happen that one of the players chooses the yellow object, while the other chooses the green, i.e., the Principle of Coordination cannot guarantee that the players coordinate in case of multiple solutions. In fact, one may argue that from the perspective of the players themselves the theory does not offer a real solution at all, because there are multiple 'solutions' and the players know that the other player may act according to one of the other ones. The players' pay-off is only equal to the pay-off associated with the 'solution' if they somehow manage to coordinate on the same 'solution', but they do not have any means to do this. Moreover, as there are three 'solutions' and as there is an act description 'choosing a red', which prescribes to randomize over two objects only, choosing a 'non-solution' may, in a sense to be made precise, give a higher expected pay-off.

To resolve these difficulties, the present paper extends the notion of symmetry, it allows players to randomize over 'act descriptions' and it modifies the Principle of Coordination. The notion of symmetry used in this paper combines Bacharach's notion of description symmetry *with a notion of* pay-off symmetry. *Description symmetry says that players should treat objects they cannot distinguish from each other in the same way. In the present example a player cannot distinguish objects 3 and 4. Hence, a player i, i=1,2, who chooses a* $p_i = (p_{i1}, p_{i2}, p_{i3}, p_{i4}, p_{i5})$ *with* $p_{ij} > 0$, $\sum_{j=1}^{5} p_{ij} = 1$ *should set* $p_{i3} = p_{i4}$. *Note that this parameterization of the strategy space allows a player to choose according to any of the five act descriptions indicated above as well as any randomization between them.*

In the present example, the notion of pay-off symmetry can be introduced as follows.[3] *Write* $\pi(p)$ *for the pay-off to the players when they choose the strategy combination* $p = (p_1, p_2)$. *Consider two arbitrary mixed strategies* p *and* p'. *Two act descriptions* j *and* *k are called pay-off symmetric if for any p and p′ satisfying description symmetry the following holds: if* $p_{ij} = p'_{ik}$, $p'_{ij} = p_{ik}$ *and for* $h \neq j, k$ $p_{ih} = p'_{ih}$, $i = 1, 2$, *then* $\pi(p) = \pi(p')$. *Intuitively, two act descriptions are pay-off symmetric if the players receive the same expected pay-off by interchanging the probabilities assigned to these act descriptions and leaving the other probabilities unaffected.*

Given the above argument, it is easy to see that in the present example the three 'solutions' offered by Variable Frame Theory are pay-off symmetric to each other. Hence, they should be treated symmetrically by a permissible choice of rational players seeking reasons for their actions. Accordingly, a second constraint imposed on the set of permissible strategies is $p_{i1} = p_{i2} = p_{i5}$. *The two requirements taken together exhaust the implications of the notion of symmetry. The permissible strategies of player i can then be parameterized by* $(\frac{1}{3}x_i, \frac{1}{3}x_i, \frac{1}{2}y_i, \frac{1}{2}y_i, \frac{1}{3}x_i)$, *with* $x_i + y_i = 1$, *where* x_i *is the total probability given to the set of objects with different colors and* y_i *is the total probability given to the set of red objects.*

Given the above constraints on the permissible mixed strategies, it is easy to see that the expected pay-off to both players is equal to $\frac{1}{3}x_1x_2 + \frac{1}{2}y_1y_2$. *Hence, there is a unique Pareto-efficient outcome, which both players reasoning independently can easily calculate:* $y_1 = y_2 = 1$. *TMR tells the players to choose this strategy.*

Note that the solution that I arrive at tells the players to randomize over the two red objects. Hence, the implications of my symmetry notion are radically different from the implications of Bacharach's notion. □

It is interesting to note that doing one's part in an optimal strategy combination does not guarantee coordination. In particular, in the above example the chance of coordinating on the same object is equal to $\frac{1}{2}$.

3. CONCEPTS AND DEFINITIONS

In this Section, I will give a more formal analysis of focal points. Consider a two-player matching game with the following properties. The set of actions both players can choose is finite and given by $\Sigma = \{1, \ldots, m\}$. The actions play the role of the numbers of the objects in Section 2. There is an arbitrary number of different dimensions (concepts) that may be used to describe the actions: $d_1, \ldots, d_z$. Each of these dimensions induces a partition of Σ. Let us call a partition of Σ induced by a single dimension a *basic partition* and let this set of basic partitions of Σ be denoted by $\mathbb{B}$. A typical element of $\mathbb{B}$ will be denoted by β.

As some dimensions come more easily to the mind than others, a player may think of some dimensions of the actions, but not about others. Bacharach (1993) introduces the notion of *availability* to formalize the idea of conspicuousness (Lewis, 1969) or prominence (Schelling, 1960). The different dimensions a player thinks of may be thought of as the *frame* through which he looks at the coordination problem. In this section and the next I consider the situation in which if a player thinks of a certain dimension, he is also able to use it. In Section 6 I consider a more general notion of a frame and discuss the extent to which the present analysis generalizes. A player's frame will be denoted by F and F is an arbitrary subset of $\mathbb{B}$.

I will model the situation as a *kind* of simultaneous move game of incomplete information in which the frame F of a player is considered to be a player's type. A strategy for player i is a function from the set of possible frames to the set of (random) actions. Denote the randomization player i chooses by p_i and $p_i \in \Delta$, $i = 1, 2$, where $\Delta = \{x_i \in \mathbb{R}^m | x_i \geqslant 0, \Sigma_{i=1}^m x_i = 1\}$. For each pair $p = (p_1, p_2)$, let the expected pay-off of a player be denoted by $\pi(p)$. The actual pay-offs are as in a matching game. Our framework necessitates, however, to deviate in one important way from a standard game of incomplete information. As a player with frame F does not think of any dimension outside F, he has no idea about the existence of types with a frame that contains dimensions not contained in F. In fact, a player thinks he considers all the dimensions that are possible to distinguish. A consequence is that unlike the usual game of incomplete information, a player cannot optimize given the optimal actions of all other types, but instead only given the actions of types he thinks are possible.

The probability that all dimensions in F and no dimensions outside F come to the mind of player i is denoted by $V(F)$, the availability of F (which is the same for each player).[4] A player of type F has no idea about the existence of dimensions outside F. He has beliefs about the dimensions the other player can actively use. In particular, given that he is of type F, the conditional probability that the other player is of type G, $G \subseteq F$, is denoted by $V(G|F)$.

It is assumed that the availabilities of dimensions in F are independent of each other. Hence, the following expressions for the probabilities $V(F)$ and $V(G|F)$ can be derived if we write v_β for the availability of dimension β:

$$V(F) = \Pi_{\beta\in F} v_\beta \Pi_{\beta\in\mathbb{B}\setminus F}(1 - v_\beta);$$
$$V(G|F) = \begin{cases} \Pi_{\beta\in G} v_\beta \Pi_{\beta\in F\setminus G}(1 - v_\beta) & \text{for } G \subseteq F \\ \text{undefined} & \text{for } G \not\subset F. \end{cases}$$

Having introduced the above terminology, the expected pay-off to a player of type F is given by

$$\pi_i(p(\cdot)|F) = \Sigma_{G\subseteq F} V(G|F) \cdot \pi(p_i(F), p_{-i}(G)),$$

where $p_i(F)$ denotes the randomization chosen by type F, $p_{-i}(G)$ denotes the randomization chosen by type G and $\pi(p_i(F), p_{-i}(G))$ denotes the players' expected pay-off when these randomizations are chosen.

3.1. *Description symmetry*

Suppose $F = \{\beta_1 \ldots, \beta_k\}$ and define $\mathcal{C}(F) = \beta_1 \vee \beta_2 \vee .. \vee \beta_k$, where $\beta_j \vee \beta_k$ is the join of partitions β_j and β_k. A player of type F perceives all the sets of possible actions that are elements of $\mathcal{C}(F)$. A typical element of $\mathcal{C}(F)$ is a *perceived class* and will be denoted by C. It is clear that $\mathcal{C}(F)$ is a partition of Σ. In example 2.1 $\mathcal{C}(F)$ for a player who observes color consists of four cells, namely {1}, {2}, {3, 4} and {5}. A first constraint on the set of feasible randomization implied by IR for a player of type F reads

(a) For all $C \in \mathcal{C}(F)$ if $k, l \in C$, then $p_{ik}(F) = p_{il}(F)$, $\quad i = 1, 2$.

This requirement says that a randomization is feasible only if all members of a perceived class C receive the same probability. Let $P_i^v(F) \subset \Delta$ denote the set of mixed strategies that satisfies requirement (a) for player i. I will say that a player of type F has the *vocabulary to distinguish* between actions j and k if there exists a $p_i(F) \in P_i^v(F)$ such that $p_{ij}(F) \neq p_{ik}(F)$.

3.2. *Pay-off symmetry*

Requirement (a) does not exhaust the implications of IR. Example 1 illustrates that certain elements of $\mathcal{C}(F)$ can be pay-off symmetric to each other. I will first define when two sets of actions are pay-off symmetric to each other. Subsequently, I define the notion of pay-off symmetry for a whole partition of Σ. The definition uses an inductive construction, because different sets of actions that consists of actions that are pay-off symmetric to each other may themselves be pay-off symmetric to each other.

Consider a $F \subseteq \mathbb{B}$ and assume a fixed family of mixed strategies $q(G), G \subseteq F$ in Δ. Two sets $A, B \in C$ are *pay-off-symmetric* relative to $(F, q(G))$ if $\pi_i(p(\cdot)|F) = \pi_i(p'(\cdot)|F)$ for any two families $p(\cdot) = (p_i(G), p_{-i}(G))$ and $p'(\cdot) = (p'_i(G), p'_{-i}(G)), G \subseteq F$ such that

(i) $p(G) = p'(G) = q(G)$, for all $G \subset F$.

(ii) For $i = 1, 2$ and all $j, k \in A$, $p_{ij}(F) = p_{ik}(F)$ and $p'_{ij}(F) = p'_{ik}(F)$ and for all $j, k \in B$, $p_{ij}(F) = p_{ik}(F)$ and $p'_{ij}(F) = p'_{ik}(F)$.

(iii) For $i = 1, 2$, $\Sigma_{k\in A} p_{ik}(F) = \Sigma_{k\in B} p'_{ik}(F)$ and $\Sigma_{k\in B} p_{ik}(F) = \Sigma_{k\in A} p'_{ik}(F)$.

(iv) For $i = 1, 2$, $p_{ik}(F) = p'_{ik}(F)$ for all $k \notin A \cup B$.

The definition basically says that two sets (that treat their own elements in a symmetric way – requirement (ii)) are pay-off symmetric if interchanging the probabilities given to these two sets (requirement (iii)) while leaving all the other probabilities unaffected (requirement (iv)) does not change the expected pay-off. As the expected pay-off depends on $p(G), G \subset F$, I require that the two families are identical for all strict subsets of F (requirement (i)). Note that this definition reduces to the definition given in Example 1 in case F consists of only one dimension and the sets A and B are singletons.

Consider now any partition $\mathcal{C}$ of Σ. Pay-off symmetry relative to $(F, q(G))$ defines an equivalence relation $\sim_C$ on $\mathcal{C}$. For any $\mathcal{C} \in C$, let $[C]_{\mathcal{C}}$ denote the $\sim_C$-equivalence class of C and set $\mathcal{F}_{\mathcal{C}}[C] \equiv \cup_{A\in[C]_{\mathcal{C}}} A$. In example 1 if $\mathcal{C}$ is the partition induced by the color dimension and $C = \{1\}$, an element of this partition (namely, the set of green objects), then $\mathcal{F}_{\mathcal{C}}[C] = \{1, 2, 5\}$. Let $\mathcal{C}^*$ denote the set of all partitions on Σ. Define the operator $\mathcal{F}^*_{F,q} : \mathcal{C}^* \to \mathcal{C}^*$ by

$$\mathcal{F}^*_{F,q}(\mathcal{C}) = \{\mathcal{F}_{\mathcal{C}}(C) : C \in \mathcal{C}\} \text{ for } \mathcal{C} \in \mathcal{C}^*.$$

It is clear that for any $\mathcal{C} \in \mathcal{C}^*$, $\mathcal{F}^*_{F,q}(\mathcal{C})$ defines a superset for each element of $\mathcal{C}$. Define $\mathcal{C}_{k+1} = \mathcal{F}^*_{F,q}(\mathcal{C}k)$. Since Σ is finite, a partition of Σ has only a finite number of elements and each element can grow only finitely many times. So, starting from any $\mathcal{C}_0$, there is a finite number K, depending on F and $q(G)$, such that $\mathcal{C}_K = \mathcal{F}^*(\mathcal{C}_K)$. Moreover, $\mathcal{C}_K$ is a partition.

3.3. *Symmetry*

Using the concepts defined above, the notion of (F, q^*)-symmetry can be defined by the following inductive construction:

(b) For any $F \subseteq \mathbb{B}$,

(i) Set $\mathcal{C}_0 = \mathcal{C}(F)$ and define $\mathcal{C}_{k+1} = \mathcal{F}^*_{F,q}(\mathcal{C}_k)$;
Let K be such that $\mathcal{C}_K = \mathcal{F}^*_{F,q}(\mathcal{C}_K)$ and define $\mathcal{C}_{(F,q)} = \mathcal{C}_K$.

(ii) Let $i = 1, 2$. For all $C \in \mathcal{C}_{(F,q)}$ if $j, k \in C$, then $p_{ij}(F) = p_{ik}(F)$.

The set of randomizations that satisfies this symmetry condition for type F is denoted by $P^r_i(F, q) \subset \Delta$. I will say that a player of type F has *reasons to distinguish* between actions j and k if there is a $p_i(F) \in P^r_i(F, q)$ such that $p_{ij}(F) \neq p_{ik}(F)$. IR implies that a player only considers (mixed) strategies $p_i(F) \in P^r_i(F, q)$ if F has been realized for him. Note that $P^r_i(F, q) \subseteq P^v_i(F)$.

3.4. *Individual team rationality and Nash equilibrium*

The notion of TMR serves two purposes. First, it helps to determine the expectation a player has about his opponents' strategy. In particular, it determines the strategy he expects her to choose when $G \subseteq F$ is realized. Second, it assures that players do their part of a strategy combination that is uniquely Pareto-superior given these expectations. The definition of TMR is recursive with respect to set inclusion. Let player i be of type F and suppose that for all $G \subset F$, there is a (mixed) strategy $q^*_{-i}(G)$ that is team member rational (TMR). Define $\pi_i(p(\cdot)|F, q^*) \equiv \Sigma_{G\subset F} V(G|F) \cdot \pi(p_i(F), q^*_{-i}(G)) + V(F|F) \cdot \pi(p_i(F), p_i(F))$, i.e., $\pi_i(p(\cdot)|F, q^*)$ is equal to $\pi_i(p(\cdot)|F)$ if we substitute (i) for all $G \subset F$, $q^*_{-i}(G)$ for $p^*_{-i}(G)$ and (ii) $p_i(F)$ for $p_{-i}(F)$. TMR can then be defined as in (c):

(c) For any $F \subset \mathbb{B}$, a strategy $p^*_i(F) \in P^r_i(F, q^*)$ is *team member rational* (TMR) if there does not exist a $p_i(F) \in P^r_i(F, q^*)$ with $\pi_i(p(\cdot)|F, q^*) \geqslant \pi_i(p^*(\cdot)|F, q^*)$.

Note that $p^*_i(F)$ is not defined in case of non-uniqueness for any type G, $G \subset F$. Example 2 illustrates the procedure defined by (a)–(c).

EXAMPLE 2. *This example is an extension of Example 1 in the sense that (at least one of) the players observe apart from their color also the shape of the objects. Objects 1 and 2 are pyramids, object 3 is a cube, 4 has a rectangular shape and 5 is a ball.*

Let the availability of shape and color be denoted by v_S, respectively, v_C. Suppose that player 1 observes both color and shape of the objects. This implies that player 1 thinks, for example, that the conditional probability $P(\{C\}|\{S, C\})$ of player 2 perceiving color but not shape (while he himself observes both color and shape) is equal to $v_C(1 - v_S)$ and that the conditional probability $P(\emptyset|\{S, C\})$ of player 2 perceiving neither color nor shape is equal to $(1 - v_C)(1 - v_S)$.

In Example 1 I have argued that a player who only observes color should randomize over the red objects, i.e., $p^*_i(C) = (0, 0, \frac{1}{2}, \frac{1}{2}.0)$. Similarly, it is easy to see that a player who only observes shapes should randomize over the pyramids, i.e., $p^*_i(S) = (\frac{1}{2}, \frac{1}{2}.0, 0, 0)$. Player 1 expects player 2 to choose the above probabilities when player 2 is of type C, respectively, S. Let us then consider the choice made by player 1. It is clear that $C(F_1) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$. Moreover, $\{1\}$ and $\{2\}$ are pay-off symmetric relative to $(F, q^*(G))$ for all v_C and v_S. the same holds true for $\{3\}$ and $\{4\}$. Using the definition of pay-off symmetry, I will explain why 1 and 2 are pay-off symmetric. For any family $p(\cdot) = (p_1(G), p_2(G))$, $G \subseteq F$, such that $p_2(G) = q^*(G)$ for $G \subset F$, $\pi_1(p(\cdot)|F)$ is equal to

$$(1 - v_c)(1 - v_s)/5 + \tfrac{1}{2}(1 - v_C)v_s(p_{11} + p_{12}) + \tfrac{1}{2}v_c(1 - v_s)$$
$$(p_{13} + p_{14}) + v_s v_c(p_{11}p_{21} + p_{12}p_{22} + p_{13}p_{23} + p_{14}p_{24} + p_{15}p_{25}).$$

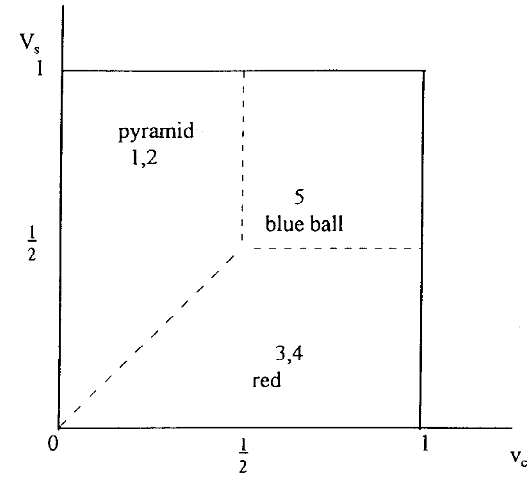

Figure 1.

It is clear from this expression that $\pi_1(p(\cdot)|F)$ *remains unaffected if both players interchange the probabilities given to the first two objects, while leaving all the other probabilities unchanged. Hence, objects* $\{1\}$ *and* $\{2\}$ *are pay-off symmetric relative to* $(F, q^*(G))$ *for all* v_C *and* v_S. *A similar argument applies to* $\{3\}$ *and* $\{4\}$.

It follows that player 1 may choose any p_1 *such that* $p_1 = (\frac{1}{2}x, \frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}y, z)$, $x + y + z = 1$, *where* x *is the total probability given to the set of pyramids and* y *is the total probability given to the set of red objects. For any* $p_i(F) \in P_i^r(F)$, $\pi_i(p(\cdot)|F, q^*)$ *is then given by*

$$(1 - v_c)(1 - v_s)/5 + \tfrac{1}{2}(1 - v_c)v_s x + \tfrac{1}{2}v_c(1 - v_s)y + v_s v_c (\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + z^2).$$

If $v_s > v_c$ *and* $v_c < \frac{1}{2}$, *this expression reaches a unique maximum at* $x = 1$.

If $v_s < v_c$ *and* $v_s < \frac{1}{2}$, *this expression reaches a unique maximum at* $y = 1$.

If $v_c > \frac{1}{2}$ *and* $v_s > \frac{1}{2}$, *this expression reaches a unique maximum at* $z = 1$.

It follows that the choice implied by IR and TMR depends on the availabilities v_c *and* v_s *and coincides with the above three cases. Figure 1 illustrates the dependence of the optimal choice on* v_c *and*

v_s. *The cases in which one of inequalities holds as an equality are discussed in the next section.* □

Finally, I define the concept of *Nash equilibrium* for the game considered here. TMR implies that the proposed strategy combination forms a Nash equilibrium when the strategy space is restricted to $P_i^r(F, q^*)$. However, the players have the vocabulary that enables them to think of a larger strategy set, namely $P_i^v(F)$. Proposition 1 in the next section assures that it is not in the interest of individual players to deviate from the proposed strategy combination even if the strategies in the larger strategy set are considered.

To this end, let us consider for $F \subseteq \mathbb{B}$ a family of strategy profiles $p(G) = (p_1(G), p_2(G)), G \subseteq F$. I first define for $i = 1, 2$ and $q_i(F) \in \Delta$, $p(\cdot|q_i(F))$ as the family of strategy profiles that coincide with $p(\cdot)$ except that $p(F) = (p_i(F), p_{-i}(F))$ is replaced by $(q_i(F), p_{-i}(F))$. That is, $p(\cdot|q_i(F))$ is a family of mixed strategy profiles generated by F, whose ith component is $q_i(F)$. The notion of Nash equilibrium can then be formulated as:

(d) A family of strategy profiles $p(F) \in P^v(F), F \subseteq \mathbb{B}$, one strategy profile for each type F, is a *Nash equilibrium* if

$$\pi_i(p(\cdot)|F) \geqslant \pi_i(p(\cdot|q_i(F))|F) \text{ for all } F \subseteq \mathbb{B}, i = 1, 2$$
$$\text{and all } q_i(F) \in P_i^v(F).$$

This completes the description of the concepts used in the next section.

4. EQUILIBRIUM, UNIQUENESS AND OPTIMALITY

This section analyzes the implications of IR and TMR for generic parameter values. Before stating the results, I first briefly discuss the notion of genericity that is employed. Suppose F contains r different dimensions. The availabilities can be represented by a point in the r-dimensional space. A result is said to hold generically if it holds for all availabilities except those of a null set.

The proofs of the two propositions that follow partly rely on the issue when generically two sets are (F, q^*)-symmetric to each other. This issue is addressed in lemma 1. In the lemma and the proofs of the propositions the following notation turns out to be useful. Write $\#C$ for the number of elements in C and define $\pi(C, p_{-i}(G)) = \pi(p_i(F), p_{-i}(G))$ for $p_{ij}(F) = 1/\#C$ if $j \in C$ and $p_{ij}(F) = 0$ if $j \notin C$; $\pi(C|F)$ and $\pi(C|F, q^*)$ are defined in a similar way.

LEMMA 1. *Two sets $C_j, C_k \in C(F)$ are generically (F, q^*)-symmetric if, and only if,*

(i) $\#C_j = \#C_k$ and
(ii) $\pi(C_j, p_{-i}(G)) = \pi(C_k, p_{-i}(G))$ for all $G \subset F$ and $p_{-i}(G) = q^*(G)$.

The first proposition shows that in all generic cases the procedure outlined in the previous section yields a unique solution and selects one of the (mixed) strategy Nash equilibria of the game. This is important, because if this is, so individual players have no incentive to deviate from the optimal strategy combination that satisfies the symmetry requirements.

PROPOSITION 1. [5] *Consider a matching game and a set of basic partitions $\mathbb{B}$. In all generic cases, the following holds:*

(a) For every type F there is a unique optimal strategy $p_i^*(F)$, $i = 1, 2$.
(b) The family of strategy profiles $p^*(F)$, $F \subseteq \mathbb{B}$ is a Nash equilibrium.

Next, I consider the question whether by employing IR and TMR players generally do better than pure randomization. The next proposition shows that in all generic cases this is indeed the case.

PROPOSITION 2. *Consider a type F, $F \subseteq \mathbb{B}$ and suppose there is a dimension $\beta \in F$ such that $\mathcal{C}_\beta \notin \Sigma$. In all generic cases, $\pi_i(p^*(\cdot)|F, q^*) > 1/m$.*

The two theorems taken together constitute this paper's explanation of the observation that in matching games people actually coordinate more frequently than could be expected if they would purely randomize over their options. The analysis of Example 2 below illustrates why the above propositions apply for all generic cases, but not for all values of availabilities. In particular, I show that for particular values of the availabilities the strategy combination p^*(F) is not unique or not better than pure randomization.

EXAMPLE 2 (continued). *In example 2 above I only considered generic values of v_c and v_s. Here, we consider some non-generic cases. I first consider the case $v_c = \frac{1}{2}$ and $v_s > \frac{1}{2}$. Given these availabilities, it is clear that $\{1, 2\}$ and $\{3, 4\}$ are not symmetric to each other and, also, that it cannot be optimal to give any probability mass to $\{3, 4\}$ as $v_S > v_C$. I next ask the question whether $\{1, 2\}$ and $\{5\}$ can be symmetric to each other. Let us consider two parameterizations:*

$$p_i(F) = (\tfrac{1}{2}x_i, \tfrac{1}{2}x_i, \tfrac{1}{2}y_i, \tfrac{1}{2}y_i, z_i), \quad x_i + y_i + z_i = 1 \textit{ and}$$
$$p_i'(F) = (\tfrac{1}{2}x +' i, \tfrac{1}{2}x_i, \tfrac{1}{2}y_i, \tfrac{1}{2}y_i, z_i), \quad x_i' + y_i + z_i = 1.$$

The sets $\{1, 2\}$ and $\{5\}$ are pay-off symmetric relative to (F, q^*) if for all p_i and p_i' such that $x_i = z_i'$, $z_i = x_i$ and $y_i = y_i'$: $\pi_i(p(\cdot)|F) = \pi_i(p'(\cdot)|F)$ for $q_2(G) = q_2'(G) = q^*(G)$, $G = S$ or C. It is easy to see that in this case

$$\pi_1(p(\cdot)F) = (1 - v_s)/10 + \tfrac{1}{4}v_s x_1 + \tfrac{1}{4}(1 - v_s)y_1 + \tfrac{1}{2}v_s(\tfrac{1}{2}x_1x_2 + \tfrac{1}{2}y_1y_2 + z_1z_2)$$

and

$$\pi_1(p'(\cdot)|F) = (1 - v_s)/10 + \tfrac{1}{4}v_s x_1' + \tfrac{1}{4}(1 - v_s)y_1' + \tfrac{1}{2}v_s(\tfrac{1}{2}x_1'x_2' + \tfrac{1}{2}y_1'y_2' + z_1'z_2').$$

Using the conditions $x_i = z_i'$ and $z_i = x_i$, the two expressions are equal to each other if $\frac{1}{4}v_s(x_1 - z_1) = \frac{1}{2}v_s(\frac{1}{2}x_1x_2 - \frac{1}{2}z_1z_2)$. This is the case if, and only if, $x_1(1 - x_2) = z_1(1 - z_2)$. Hence, the sets $\{1, 2\}$ and $\{5\}$ are not pay-off symmetric relative to (F, q^*). It is clear, however, that $\pi_1(p(\cdot)|F, q^*) = (1 - v_s)/10 + \frac{1}{4}v_s x_1 + \frac{1}{4}(1 - v_s)y_1 + \frac{1}{2}v_s(\frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 + z_1^2)$ is maximal at $x_1 = 1$ and at $z_1 = 1$. Accordingly, the optimal strategy for player 1 is not unique, TMR is not defined and Proposition 1(a) does not hold.

Let us then look at the case $v_c = v_s = v$. If $p_2(G) = q^*(F)$, $G \subset F$, $\pi_1(p(\cdot)|F)$ is equal to $(1 - v)^2/5 + \frac{1}{2}(1 - v)vx_1 + \frac{1}{2}v(1 - v)y_1 + v^2(\frac{1}{2}x_1x_2 + \frac{1}{2}y_1y_2 + z_1z_2)$. It follows that in this case the sets $\{1, 2\}$ and $\{3, 4\}$ are pay-off symmetric relative to (F, q^*). Hence, all $p_1(F) \in P_1^r(F)$ can be parameterized by $p_1 = (\frac{1}{4}x_1, \frac{1}{4}x_1, \frac{1}{4}x_1, \frac{1}{4}x_1,$

z_1) *and the expected pay-off* $\pi_1(p(\cdot)|F, q^*)$ *is equal to*

$$(1-v)^2/5 + \frac{1}{2}(1-v)vx_1 + v^2(\frac{1}{4}x_1 + z_1).$$

Setting $x_1 = 1$ *gives an expected pay-off of* $\frac{1}{2}v - \frac{1}{4}v^2$ *and setting* $z_1 = 1$ *gives an expected pay-off of* v^2*. Simple calculations shows that for* $0 < v < 0.4$ *there is a unique optimal strategy combination, namely to randomize over the objects 1,2,3 and 4. Similarly, for* $0.4 < v < 1$ *there is a unique optimal strategy combination, namely to set* z_1*=1. For* v*=0.4, one can show that there are two maximal solutions for* $\pi_1(p(\cdot)|F, q^*)$ *and -again- Proposition 1(a) does not hold. It is easy to see that for* $v = 1$*, all five objects are symmetric to each other and the only feasible option for the players is to randomize with probability 1/5 over the five objects, i.e., Proposition 2 does not hold.* □

5. AN EXAMPLE OF A NON-MATCHING GAME

In this section I briefly consider how the approach described in the previous sections can be applied to games in which pay-off differences are the only useful information players have.

Following Crawford and Haller (1990) differences in pay-offs can be incorporated in the procedure outlined in Section 3 in the following way. A pay-off can be considered an attribute of a combination of strategies. In case pay-off differences are the only useful information players have and if we are outside the framework of matching games, pay-off symmetry can be defined as follows. Two actions j and k are said to be *pay-off symmetric for player* 1 if $\pi_1(p) = \pi_1(p')$ for all p and p' such that for all i and $h \neq j, k$ $p_{ih} = p'_{ih}$, $p_{ij} = p_{ik}$ and $p_{ik} = p_{ij}$. A similar definition holds true for player 2. This definition basically says that if both players interchange the probabilities given to actions j and k while keeping the other probabilities constant and if a player gets the same pay-off in both cases, then actions j and k are pay-off *symmetric for that player*. Pay-off symmetry induces a partition of Σ. It is natural to suppose that players observe pay-off differences and the availability of the partition induced in this way is set equal to 1.

EXAMPLE 3. *Battle of the Sexes Consider a version of the Battle of the Sexes with three actions. As in the classic 2×2 action Battle of the Sexes game, one player prefers to go to a ballet, while the other player prefers to go to a football match; they both prefer going together to going alone. Unlike the classic Battle of the Sexes game, there are two football matches scheduled for the same evening. The pay-off matrix is given below.*

		Player 2		
		ballet	football match #1	football match #2
	ballet	3, 2	0, 0	0, 0
Player 1	football match #1	0, 0	2, 3	0, 0
	football match #2	0, 0	0, 0	2, 3

The only discriminating partition is to divide the strategy space into a set with the ballet and a set with the football matches. Note that this partition is consistent with pay-off symmetry considerations. The class of mixed strategy combinations respecting IR can be parameterized by $(x_i, \frac{1}{2}y_i, \frac{1}{2}y_i)$, *with* $0 \leqslant x_i, y_i \leqslant 1$ *and* $x_i + y_i = 1$, $i = 1, 2$, *where* $x_i(y_i)$ *represents the probability with which player* i *goes to the ballet (a football match).*

The expected pay-off $\pi_1(p(\cdot)|F, q^*) = 3x_1x_2 + 1y_1y_2$ *and* $\pi_2(p(\cdot)|F, q^*) = 2x_1x_2 + 1\frac{1}{2}y_1y_2$. *It is easily seen that* $x_1 = x_2 = 1$ *is the unique Pareto-optimal strategy combination that satisfies IR. The main reason for this conclusion is that* reasoning individually, *the players cannot find a reason to go to football match 1 instead of going to football match 2 and they know that the other does not have a reason to discriminate between the two football matches either. Hence, the expected pay-off of going to the ballet are larger for both players than the expected pay-off of going to a football match.* □

6. POSSESSING A CONCEPT, BUT NOT KNOWING HOW TO USE IT

So far, I have considered situations in which players with a certain frame could not think of dimensions the other player may be able to use even though they themselves are not able to use them. In this section we relax this assumption. From now on, I distinguish two components of a frame. First, a player may think of a certain dimension (concept), but does not know how to use it (in a particular context). Second, a player may think of a certain dimension and also know how to use it.[6] The difference between the two components may be clarified by the following example. Suppose two players are asked to taste a certain number of different wines and choose one of them after tasting. They win a prize if, and only if, they choose the same wine. A player may observe that the wines are made of different grape varieties and use this knowledge in one way or the other to 'solve' the coordination problem. Another player may have heard of the fact that wines can be made of different grape varieties even though she herself cannot distinguish wines on this basis. In this Section I will discuss by means of examples the way in which our framework can be extended to situations like this.

In line with the above, two components of the notion of availability now need to be distinguished. Let v_β be the probability that dimension β comes to the mind of a player and that he is also able to use it, i.e., he knows which partition is induced by that dimension. Let v_{β^-} be the probability that dimension β comes to the mind of a player, but that he is not able to use it, i.e., he does not know which partition is induced by that dimension. The probability that a dimension β does not come at all to the mind of a player is then given by $1 - v_\beta - v_{\beta^-}$. As before let F denote the set of dimensions that comes to the mind of a player and that he is able to use.

The notion of a player's type has to be extended in two ways. In the previous sections, a player's type could be identified by the dimensions in his frame F. Now, of course, a player's type description should include both the dimensions that he himself is able to use *and* the dimensions a player thinks the other player may be able to use even though he himself is not able to use them. In addition, however, a player's type description should also include the particular partition that is induced by the dimensions that he is able to use (as the other player may be of a type that is not able to use some dimensions). Moreover, the type of players are correlated with each other in the sense that if a player is able to use a certain dimension, then he knows that the other player observes the same partition if the other player is also able to use the same dimension. Example 4 illustrates this point.

I next consider a player i's expectations about the randomization $p_{-i}(G)$ chosen if the other player is able to use dimensions he himself is not able to use. For any such G, there is a unique $H \subseteq F$ such that H does not contain any dimensions that are not in G and for all $H' \subseteq F$, H does not share fewer dimensions with G than H'. Let $G = \{d_1, .., d_g\}$ and $H = \{d_1, .., d_h\}$, $h < g$. The player under consideration knows the partition induced by H, namely $\beta_1 \vee .. \vee \beta_h$, but does not know the partition induced by the dimensions $G \backslash H = \{d_{h+1}, .., d_g\}$. Recall that $\mathcal{C}^*$ denotes the set of all partitions on Σ. It is clear that from the perspective of the player under consideration the partition induced by $G \backslash H$ can be any $\mathcal{C} \in \mathcal{C}^*$. Hence, the resulting partition induced by G itself is $\beta_1 \vee .. \vee \beta_h \vee \mathcal{C}$, where $\mathcal{C}$ can be any partition in $\mathcal{C}^*$. For each of these partitions a player can carry out requirement (a) and (b) of Section 3. An additional implication of IR is that the player under consideration must consider any partition induced by a dimension that he himself cannot use equally likely.

The first example below illustrates that the general philosophy of the previous sections with the above modifications can be used to analyze a class of cases in which v_{β^-} is small enough, i.e., TMR is defined and propositions 1 and 2 hold true. The example also shows that the extension considered in this section is substantive in the sense that a player who thinks the other player may use a dimension he himself is not able to use may make a different choice than a player who does not consider this possibility.

EXAMPLE 4. *Consider an extension of Example 2. There are 5 wooden objects of which three dimensions may be distinguished, namely color (C), shape (S) and the wood (W) of which the object is made. Player 1 is able to use all three dimensions, while player 2 is only able to use the color and shape dimensions, while he thinks player 1 may also be able to use some other dimension. The availabilities are the following:* $v_S = .6$, $v_C = .6 - \epsilon$, $v_C - = v_{s-} = 0$, $v_w = 1 - 2\epsilon$, $v_{w-} = \epsilon$, *where* ϵ *is a small positive number. The color and shape dimensions induce the partitions mentioned in Example 3.1, while it turns out that player 1 who observes the wood* *dimension sees one oak object (no. 5) and four birch objects (no. 1, 2, 3 and 4).*

From example 2 it is clear that the types[7] *$\emptyset$, C, S and CS will respectively choose to randomize over all objects, to randomize over* $\{1, 2\}$*, to randomize over* $\{3, 4\}$ *and choose* $\{5\}$*. Similar calculations as below show that the types W_{-}, CW_{-} and SW_{-} also respectively choose to randomize over all objects, to randomize over* $\{1, 2\}$*, to randomize over* $\{3, 4\}$*, i.e., types who observe and know how to use color or shape or nothing make the same choices as those types who in addition think that their opponent may use another dimension, but are not able themselves to use it. I will show that type CSW_{-} makes a different choice than type CS. To this end, note that there are 52 possible partitions of five objects (see appendix B). In addition to the types mentioned above player 2 (who is of type CSW-) considers the possibility that player 1 is of some type W_k, CW_k, SW_k or CSW_k with* $k = 1, .., 52$*. In the table in Appendix B I mention the choice that each type W_k, CW_k, SW_k or CSW_k makes. While determining these choices, I use the fact that e is so small that the choices that the types $\emptyset$, C, S, CS, W_{-}, CW_{-}, SW_{-} and CSW_{-} make, are of secondary importance for the pay-off of a type who is able to use the dimension induced by W.*

Let us first concentrate on the symmetry implications for player 2 (of type CSW_{-}). It is clear that he has the same vocabulary as the CS type and I will argue that he also has no reasons to distinguish between actions $\{1\}$ *and* $\{2\}$ *or between* $\{3\}$ *and* $\{4\}$*. I will concentrate on the symmetry of* $\{1\}$ *and* $\{2\}$*. The expected pay-off of* $\pi_2(p(\cdot)|CSW_{-})$ *is a weighted average of* $\pi(p(\cdot)|CS)$ *and the expected pay-off of interacting with types W_k, CW_k, SW_k and CSW_k,* $k = 1, .., 52$*. From the table in the appendix it becomes clear that the expected pay-off of interacting with a type W_k is the same for actions* $\{1\}$ *and* $\{2\}$*. The same holds true for interacting with a type CW_k, SW_k or CSW_k. In Example 3.1. I have argued that type CS also treats* $\{1\}$ *and* $\{2\}$ *symmetrically. Hence, player 2 has no reasons to distinguish between actions* $\{1\}$ *and* $\{2\}$.

Finally, I will argue that type CSW_{-} is better off choosing to randomize over the first two objects than to choose object 5. For ϵ small enough, $\pi(p(\cdot)1CSW_{-})$ *is approximately equal to the expected pay-off of interacting with types W_k, CW_k, SW_k and CSW_k,* $k =$

1, .., 52. *From Table B.1 it follows that*

$$\pi_2(p(\cdot)|CSW_-, q^*) \approx \frac{1}{2} \cdot \frac{0.16 \cdot 20 + 0.24 \cdot 45 + 0.36 \cdot 26}{52} + \frac{0.16 \cdot 2}{5 \cdot 52} \textit{ for } p_2 = (\frac{1}{2} \cdot \frac{1}{2}, 0, 0, 0)$$

and

$$\pi_2(p(\cdot)|CSW_-, q^*) \approx \frac{0.16 \cdot 10 + 0.24 \cdot 14 + 0.36 \cdot 12}{52} + \frac{0.16 \cdot 2}{5 \cdot 52} \textit{ for } p_2 = (0.0, 0, 0, 1).$$

As it is clear that the first expression is larger than the second, player 2 will randomize over the first two objects and choose differently from type CS. □

It is possible to show that when v_{β^-} is *not* small, there is a generic non-uniqueness problem and TMR may not be defined. An example is given in the working paper version (Janssen, 2000). Intuitively, the reason is the following. In the previous example we have seen that the recursive definition of TMR does not *strictly* hold true any-more as some types will not only think about the actions chosen by ‘lower’ types, but also about the actions chosen by ‘higher’ types who can use dimensions they themselves cannot use. When all v_{β^-}’s are small enough, however, the actions chosen by these ‘higher’ types do not depend on the actions chosen by the types who think about these ‘higher’ types. This implies that we may still act as if the recursive procedure holds true. This is not true when some $v_{\beta-}$ are relatively large and the choices of ‘lower’ and ‘higher’ types have to be determined simultaneously. This may lead to the existence of multiple equilibria that are not Pareto-dominated (by each other).

7. CONCLUSION

This paper discusses a framework in which it is possible to explain why rational players are able to coordinate their actions in coordination games without communication. At the same time the paper suggests an explanation why players may not be extremely successful in coordinating their actions. The reason is that they might think of different clues to solve the problem. This reason does not defy, however, the general argument of the paper. The general argument is that in coordination games players try to use the information they have in such a way that they have a reason to choose one particular strategy and not another. For the approach to yield a coordinated outcome players have to start from some common background: a common set of dimensions and a shared understanding of the likelihood that the other player considers other sets of these dimensions. The paper then basically says that players can use their common background in a rational way.

The theory outlined here extends the analysis in Bacharach (1993) in a number of testable ways. This paper introduces the notion of pay-off symmetry and Example 1 shows that the use of this notion may yield predictions that differ from what Bacharach's theory predicts. The relative predictive power of the two theories may be easily tested in an experimental setting. Also, Example 4 shows that a player who thinks the other player may use a dimension he himself is not able to use may make a different choice than a player who does not consider this possibility. This prediction may also be tested in an experimental setting where apart from their actual choices, participants are asked to write down the way they have thought about the problem in order to get a hold on the dimensions they have considered and the likelihood they attach to the other player thinking about the same dimensions.

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APPENDIX A: PROOFS

Proof of Lemma 1. For any F and $p_i(F) \in P_i^v(F)$, there are $C_1, \ldots,$ $C_K \in \mathcal{C}(F)$ and $0 < a_{i1}, .., \alpha_{iK} < 1$ such that $\Sigma_{h \in C_k} p_{ih}(F) = \alpha_{ik}$

for all $i, k = 1, .., K$. Consider w.l.o.g. the (F, q^*)-symmetry of C_1 and C_2. Write

$$\begin{aligned}\pi(p(\cdot)|F) = \; & \Sigma_{k\in K}\alpha_{ik}[\Sigma_{G\subset F}V(G|F)\cdot\pi(C_k, p_{-i}(G)) \\ & + V(F|F)\cdot\pi(C_k, p_{-i}(F))] \\ = \; & \alpha_{i1}[\Sigma_{G\subset F}V(_{G\subset F}v(G|F)\cdot\pi(C_1, p_{-i}(G)) \\ & + V(F|F)\cdot\pi(C_1, p_{-i}(F))]+ \\ & \alpha_{i2}[\Sigma_{G\subset F}V(G|F)\cdot\pi(C_2, p_{-i}(G)) \\ & + V(F|F)\cdot\pi(C_2, p_{-i}(F))]+ \\ & \Sigma_{k\neq 1,2}\alpha_{ik}[\Sigma_{G\subset F}V(G|F)\cdot\pi(C_k, p_{-i}(G)) \\ & + V(F|F)\cdot\pi(C_k, p_{-i}(F))].\end{aligned}$$

Substitute for all $G \subset Fp_{-i}(G)byq^*(G)$. As for all $j, kC_j \cap C + k = \emptyset, \pi(C_k, p_{-i}(F)) = \alpha_{-i,k}/\#C_k$. So that the above expression can be rewritten as

$$\begin{aligned}\pi(p(\cdot)|F) = \; & \alpha_{i1}[\Sigma_{G\subset F} \\ & V(G|F)\cdot\pi(C_1, q^*(G)) + \alpha_{-i,1}V(F|F)/\#C_1]+ \\ & \alpha_{i2}[\Sigma_{G\subset F}V(G|F)\cdot\pi(C_2, q^*(G)) + \alpha_{-i,1} \\ & V(F|F)/\#C_1]+ \\ & \Sigma_{k\neq 1,2}\alpha_{ik}[\Sigma_{G\subset F} \\ & V(G|F)\cdot\pi(C_k, q^*(G)) + \alpha_{-i,k}V(F|F)/\#C_k].\end{aligned} \tag{1}$$

As $V(F|F) \neq 0$, it follows immediately that C_1 and C_2 cannot be (F, q^*)-symmetric if $\#C_1 \neq \#C_2$. It is also clear from (1) that C_1 and C_2 are (F, q^*)-symmetric for all values of the conditional probabilities (and, hence, for all values of availabilities) if for all $G \subset F$ $\pi(C_1, q^*(G)) = \pi(C_2, q^*(G))$ and $\#C_1 = \#C_2$.

Let us then consider the reverse claim. Suppose that C_1 and C_2 are (F, q^*)-symmetric for generic parameter values (hence $\#C_1 = \#C_2$), but that for some $G \subset F$ $\pi(C_1, q^*(G)) \neq \pi(C_2, q^*(G))$. In this case there is a set of values of the availabilities $\{\bar{v}_1, \ldots, \bar{v}_z\}$ and $\epsilon > 0$ such that for all $v_j \in (\bar{v}_j - \epsilon, \bar{v}_{j+\epsilon})$, $j = 1, \ldots, z$, $\pi(p(\cdot)|F) = \pi(p'(\cdot)|F)$ for families $p(\cdot)$ and $p'(\cdot)$ satisfying $(i) - (iv)$ of the definition of pay-off symmetry. It is clear that if v_j takes on values in $(\bar{v}_j - \epsilon, \bar{v}_j + \epsilon)$, there are values of the conditional probabilities $\bar{V}(G|F)$, $G \subset F$, and an $\epsilon > 0$ such that all $V(G|F) \in (\bar{V}(G|F) - \epsilon', \bar{V}(G|F) + \epsilon')$ are reached. From (1) it follows that $\pi_i(p(\cdot)|F) = \pi_i(p'(\cdot)|F)$, if and only if, the conditional probabilities $V(G|F)$, $G \subset F$, satisfy a linearity condition. Hence, there do not exist $\bar{V}(G|F)$, $G \subset F$, such that $\pi_i(p(\cdot)|F) = \pi_i(p'(\cdot)|F)$ holds true in an ϵ' neighborhood. □

Proof of Proposition 1. It is clear that $p_1^*(F) = p_2^*(F)$. First, I will show that for all generic matching games $p_i^*(F)$ is such that $p_{ij}(F)$ is either 0 *or* $\alpha > 0$. Suppose $p_i^*(F)$ satisfies requirement (*b*) with respect to $q^*(G)$ for all $G \subset F$, but is not of this form. Then there are $0 < \alpha_1, \ldots, \alpha_K < 1$ and $C_1, \ldots, C_K \in \mathcal{C}_{(F,q^*)}$, $K \geqslant 2$, such that $\Sigma_{h \in C_k} p_{ih}^* = \alpha_k$ for all $i, k = 1, \ldots, K$ and all α_k differ from each other. Player i's expected pay-off of choosing $p_i^*(F)$ is given by

$$\begin{aligned}\pi(p^*(\cdot)|F, q^*) = \ &\Sigma_{k \in K}[\alpha_k \cdot \pi(C|F, q^*) \\ &+ V(F|F)\alpha_k(\alpha_k - 1)/\#C_k].\end{aligned} \tag{2}$$

There is at least one C_k, denoted by C^* such that $\pi(C^*|F, q^*) \geqslant \pi(C_k|F, q^*)$ for all k. As the second part of the RHS of (2) is negative, shifting all probability mass to strategies in C^* yields a pay-off of $\pi(C^*|F, q^*)$ which is strictly larger than $\pi(p^*(\cdot)|F, q^*)$. This contradicts the hypothesis that $p^*(F)$ is of the form described above.

The next part is to show that C^* is unique in all generic cases. Suppose it is not unique. Then generically there are at least two sets, denoted by C_1^* and C_2^*, that yield maximum pay-off if all probability mass is shifted to these groups. I will show that this implies that C_1^* and C_2^* are (F, q^*)-symmetric, which implies that $C_1^*, C_2^* \notin \mathcal{C}_{(F,q^*)}$.

Write $\pi(C_1^*|F, q^*) = \Sigma_{G \subset F} V(G|F) \cdot \pi(C_1^*, q^*(G)) + V(F|F)/\#C_1$ and $\pi(C_2^*|F, q^*) = \Sigma_{G \subset F} V(G|F) \cdot \pi(C_2^*, q^*(G)) + V(F|F)/\#C_2$. Hence, $\pi(C_1^*|F, q^*) - \pi(C_2^*|F, q^*) =$

$$\begin{aligned}\Sigma_{G \subset F} V(G|F) \cdot \ &[\pi(C_1^*(G)) - \pi(C_2^*, q^*(G))] \\ &+ V(F|F)[1/\#C_1^* - 1/\#C_2^*].\end{aligned} \tag{3}$$

We are looking for a case in which (3) is equal to 0 for generic parameter values. This can, however, only be the case if for all $G \subset F \pi(C_1^*, q^*(G)) = \pi(C_2^*, q^*(G))]$ and $\#C_1^* = \#C_2^*$. From lemma 1 it then follows that C_1^* and C_2^* are (F, q^*)-symmetric.

The last part of the proof consists of showing that the family of strategy profiles $p^*(G), G \subseteq F$ forms a Nash equilibrium. This part proceeds in two steps. First, it will be shown that $p_i^*(F)$ is a best response if all $p_i(F) \in P_i^r(F, q^*)$ are considered. Second, the argument will be expanded to all $p_i(F) \in P_i^V(F)$.

From the above analysis and definition (c) it follows that in all generic cases $\pi(p^*(\cdot)|F,q^*) > \pi(p(\cdot)|F,q^*) = \Sigma_{G\subset F} V(G|F) \cdot \pi(p_i(F), q^*_{-i}(G)) + V(F|F) \cdot \pi(p_i(F), p_i(F))$ for all $p_i(F) \in \cdot P^r_i(F,q^*)$. As $\pi(p_i(F), p_i(F)) \geqslant \pi(p_i(F), q^*_{-i}(F))$, it is easy to see that $\pi(p(\cdot)|F,q^*)$ in turn is larger than $\pi(p^*(\cdot|p_i(F))|F) = \Sigma_{G\subseteq F} V(G|F) \cdot \pi(p_i(F), q^*_{-i}(G))$. As the above argument is independent of F, it also holds for all $G \subset F$.

Let us then consider strategies in $P^v_i(F)$. First, note that strategies that are elements of $P_i(F)\backslash P^r_i(F,q^*)$ are based on sets that are (F,q^*)-symmetric to each other. Consider an arbitrary $C \in \mathcal{C}_{(F,q^*}$ and suppose that $C \notin \mathcal{C}(F)$. Then there are $C_1, \cdots, C_m \in \mathcal{C}(F)$ with $C_j \cap C_k = \emptyset$ and $\cup_{k=1,\dots,m} C_k = C$. For all C_j, $j \in \{1,\dots,m\}$ there is a (union of) C_k, $k \in \{1,\dots,m\}$ such that C_j and a (union) of C_k are (F,q^*)-symmetric to each other. As the case in which C_j is symmetric to a union of C_k is similar to the case in which it is symmetric to a single C_k, I will only deal with the latter case. Consider arbitrary sets C_j and C_k such that $C_j, C_k \in C$. As C_j and C_k are symmetric to each other $\pi(C_j|F,q^*) = \pi(C_k|F,q^*)$. Write $\pi(C_j|F,q^*)$ as $\Sigma_{G\subset F} V(G|F) \cdot \pi(C_j, q^*(G)) + V(F|F)/\#C_j$. A similar expression holds for $\pi(C_k|F,q^*)$. As in all generic cases $\#C_j = \#C_k$, it follows that $\Sigma_{G\subset F} V(G|F) \cdot \pi(C_j, q(G)) = \Sigma_{G\subset F} V(G|F) \cdot \pi(C_k, q^*(G))$. Moreover, as $C_j, C_k \in C$ and as $p^*(F)$ gives the same probability to each element of $C, \pi(C_j|F) = \pi(C_k|F)$. It follows that it is not beneficial to deviate from p^*_i. □

Proof of Proposition 2. Consider $C_1, .., C_K \in \mathcal{C}_{(F,q^*)}$ such that $\cup_{k=1,..,K} C_k = \Sigma$ and $C_j \cap C_k = \emptyset$. I will show that there is a set C_k such that $\pi(C_k|F,q^*) > 1/m$.

One way to represent pure randomization is to give each set C_k, $k = 1, .., K$ a total probability of $\#C_k/m$. The expected pay-off of pure randomization is $1/m$. Thus, using (2) and setting $\alpha_k = \#C_k/m$ we get

$$\frac{1}{m} = \Sigma_{k\in K} \left[\frac{\#C_k}{m} \cdot \pi(C_k|F,q^*) + \left(\frac{\#C_k}{m} - 1 \right) \frac{V(F|F)}{m} \right].$$

If $\mathcal{C}_{(F,q^*)}$ has more than one element, then $\#C_k < m$ for all k and $\Sigma_{k\in K} \frac{\#C_k}{m} \cdot \pi(C_k|F,q^*) > \frac{1}{m}$. Also, from $\cup_{k=1,..,K} C_k = \Sigma$, it follows that $\Sigma_{k\in K} \#C_k = m$. Therefore, there must be a C^* such that $\pi(C^*|F,q^*) > 1/m$.

The only thing we have to check is that $\mathcal{C}_{(F,q^*)}$ contains more than one element if there is a dimension $\beta_1 \in F$ such that $C_{\beta_1} \neq \Sigma$. Write $C^*(\beta_1)$ for the set with the smallest number of actions in $C_(\beta_1)$. It is clear that the number of actions in $C^*(\beta_1)$ is smaller than m. The only reason why the above argument might not work in this case is that $C^*(\beta_1) \notin \mathcal{C}_{(F,q^*)}$. Two cases need to be distinguished: (i) $C^*(\beta_1) \in \mathcal{C}(F)$ and (ii) $C^*(\beta_1) \notin C(F)$.

(i) As there is a positive probability that the other player only considers the partition induced by β_1 there is a $G \subset F$ such that $\pi(C^*(\beta_1), q^*(G)) \neq \pi(C_j, q^*(G))$ for all $C_j \in C(F)$. It follows from lemma 1 that in all generic cases $C^*(\beta_1) \in \mathcal{C}_{(F,q^*)}$.

(ii) In this case, there are $C_1, .., C_K \in \mathcal{C}(F)$ such that $C_j \cap C_k = \emptyset$, $j, k \in \{1, .., K\}$ and $\bigcup_{j=1}^{K} C_j = C^*(\beta_1)$. A similar argument as in (i) shows that in all generic cases there is a set $C \subseteq C^*(\beta_1)$ such that $C \in \mathcal{C}_{(F,q^*)}$. □

APPENDIX B: DETAILS OF EXAMPLE 4

In this appendix I present a table indicating the choices made by types W_k, CW_k, SW_k and CSW_k, $k = 1, .., 52$. The 52 possible partitions induced by W alone are grouped in 16 columns. The partition that is indicated by 1,23,45, for example, divides the five objects in one one-element set $\{1\}$ and two two-element sets $\{2, 3\}$ and $\{4, 5\}$. The partitions mentioned in the same column induce the different types W_k, CW_k, SW_k and CSW_k to make the same choices. The rows mention the approximate probabilities (when $\epsilon \approx 0$) that a type W_k, CW_k, SW_k and CSW_k is drawn for a given partition induced by W.[8] In the cells of the matrix the choice that a particular type makes is mentioned, where 12, for example, means that that particular type randomizes with probability $\frac{1}{2}$ over objects 1 and 2.

Instead of presenting detailed calculations, I specify the type of considerations that lead to the table below. First, if on the basis of W alone a player cannot discriminate between any of the actions, i.e., column 1, then a type CW_k, for example, chooses the same action as type C. Second, if there is a single one-element set in the partition induced by W alone, i.e., Columns 2–6, then this set dominates all the calculations as the availability of W is the highest. Third, if there is a single two-element set in the partition induced by W alone or if

Table B.1.

Prob	Partitions	12345 1,2,3,4,5	1,23,45 1,24,35 1,25,34 1,2345	2,13,45 2,14,35 2,15,34 2,1345	3,12,45 3,14,25 3,15,45 3,1245	4,12,35 4,13,25 4,15,23 4,1235	5,12,34 5,13,24 5,14,23 5,1234	1,2,345 12,3,4,5 12,345	13,245 13,2,4,5 13,235	1,4,235 14,2,3,5 14,235	1,5,234 15,2,3,4 15,234	2,3,145 23,1,4,5 23,145	2,4,135 24,1,3,5 24,135	2,5,135 25,1,3,5 25,135	3,4,125 34,1,2,5 34,125	4,5,123 45,1,2,3 45,123	3,5,124 35,1,2,4 35,124
0.16	W_k	12345	1	2	3	4	5	12	13	14	15	23	24	25	34	45	35
0.24	SW_k	12	1	2	3	4	5	12	1	1	1	2	2	2	5	3	4
0.24	CW_k	34	1	2	3	4	5	5	3	4	2	3	4	1	34	4	3
0.36	CSW_k	5	1	2	3	4	5	5	1	1	1	2	2	2	5	4	3

there are exactly two one-element sets, i.e., columns 7–16, then the following holds:

– a W_k type randomizes over this set (these two sets),
– a CW_k type chooses the object that is also among the chosen objects for types C and W_k if there is such an object;
– a CW_k type chooses to randomize over the two objects that are also chosen by types C and W_k if there is such an object;
– a CW_k type chooses the object that is not chosen by types C and W_k if there is exactly one such an object (as the availabilities are larger than $\frac{1}{2}$);
– the same considerations as above for CW_k apply to SW_k; a CSW_k type chooses (in almost all cases) the object that is also chosen by type SW_k as this type has the largest chance of occurring (ϵ more than type CW_k); only in column 7 when type CSW_k cannot distinguish between objects 1 and 2, and therefore cannot choose one or the other, the fact that object 5 is a single object dominates the calculations.

NOTES

1. Janssen (1998) discusses the focal points literature in more detail.
2. Goyal and Janssen (1996) have criticized this implicit use of a coordination assumption in an explanation of how players (learn to) coordinate.
3. In general, considerations of pay-off symmetry may involve complex higher order symmetry requirements. A general account is given in the next section.
4. It is implicit that once one member of a dimension (say, green) comes to mind, all members (red, white of the other objects) come to mind, i.e., no color is more focal than the others. It is possible to relax this assumption by using similar xconcepts as the ones introduced here.
5. I thank André Casajus for pointing at a mistake in an earlier proof.
6. This distinction was first brought to my attention by Robin Cubitt. The previous sections followed Bacharach (1993) by only considering the second component of a player's frame. As a player cannot use a concept if he does not possess it, there are no other possibilities than the two considered here.
7. I use the following notation. A type CSW_{-} observes color and shape and knows how to use them and also thinks about the wood structure of the objects, but cannot use this attribute. Similarly, a type C observes color and knows how to use it and does not think about other attributes of the objects at all and a type W_k observes the wood structure and sees that it induces partition k.

8. Note that in example 6.2. these probabilities have to be divided by two. In that example, the same approximate probabilities apply to types W_-, CW_-, SW_- and CSW_- being drawn, namely 0.08, 0.12 and 0.18, respectively.

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