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BERTRAND COMPETITION UNDER UNCERTAINTY*

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We look at a Bertrand model in which each firm may be inactive with a known probability, so the number of active firms is uncertain. The model has a mixed-strategy equilibrium, in which industry profits are positive and decline with the number of firms, the same features which make the Cournot model attractive. Unlike those in a Cournot model with similar uncertainty, Bertrand profits always increase in the probability that firms are inactive. Profits decline more sharply than in the Cournot model, the pattern found empirically in Bresnahan and Reiss [1991].

I. INTRODUCTION

Suppose a carpenter is asked by a homeowner to submit a tender for renovating a house. He considers it very likely that if the homeowner has asked for tenders from other carpenters then the lowest price will win the job. He also knows there is a chance that the homeowner has not found any other carpenter free to do the work this month and will give the job to him even if his tender is rather high. What price will he offer the homeowner? It will certainly be above marginal cost. We model this situation by having the carpenter know that with some probability he is a monopolist who can charge the monopoly price and that with some probability he does face price competition. We will show that there exists an equilibrium in mixed strategies and that expected profits are positive and rising with concentration. Moreover, the model does reasonably well in explaining the empirical results of Bresnahan and Reiss [1991] on *how* industry profits increase.

The paper is related to several different literatures. A variety of models, of which Salop and Stiglitz [1977] and Varian [1980] are early examples, have shown that competitive markets can have price dispersion even in equilibrium. Different firms charge different prices for an identical good because of heterogeneous consumer search, some consumers observing

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more prices than others. The closest of these to the present model is Burdett and Judd [1983], in which some consumers might observe one price, some two prices, some three, and so forth. The number of searches is endogenous, and in equilibrium a given consumer observes only one or two prices. Our firms, however, are strategic, not competitive, and a firm believes there is a fixed probability any of its competitors is active. Therefore, in our model, when the number of firms becomes larger, a seller knows that the probability that at least one other firm is actively competing with it has become closer to one, which drives prices closer to marginal cost, whereas in Burdett and Judd price dispersion and positive industry profits persist as the number of firms becomes infinite.[1]

II. THE MODEL

Let there be N firms and a homogeneous good. Before deciding on price, a firm does not know how many other firms are active in the market. The probability a given firm is active is α, where $0 \leq \alpha \leq 1$. If $\alpha = 1$, the market is described by the Bertrand model of price competition, and the equilibrium price equals marginal cost. If $\alpha = 0$ our firm is assured of being a monopolist and charges the monopoly price. For simplicity, we will assume that there is one consumer, who buys at most one unit, and his maximum willingness to pay for the good is v. Marginal cost is normalized to 0.

There is no symmetric Nash equilibrium with any firm putting positive probability on choosing any particular price. Suppose Firm 1 charges p' with positive probability θ, rather than mixing over a continuous range of prices and putting infinitesimal probability on each. Putting positive probability on $p' = 0$ is not profit maximizing, because if the firm charged the monopoly price of v instead on those occasions it would have an expected payoff of $(1-\alpha)^{N-1}v$, so let us focus on $p' > 0$.

If $p' > 0$, and two firms are putting positive probability θ on p', then with positive probability θ^2 they will both charge p' and will each have a contribution proportional to $(\theta^2/2)(p' - 0)$ towards their expected profits. Firm 1 could increase its expected profit, however, by deviating to putting zero weight on p' and positive weight on $p' - \epsilon$, for sufficiently small ϵ. This would replace the expected profit of $(\theta^2/2)(p' - 0)$ with the larger expected

[1] Also related are Spulber [1995] and Baye and Morgan [1999], which also address positive profits in oligopoly. Spulber analyzes Bertrand competition with private costs. Baye and Morgan [1999] show that if firms only choose prices to reach within epsilon of their maximal profit, then a mixed-strategy equilibrium exists in which profits are positive and large compared to the value of epsilon.

profit of $(\theta^2)(p' - \epsilon)$. Thus, it cannot be that both firms put positive probability on any p' in equilibrium.

What if only Firm 1 chooses p' with positive probability? Then prices in a neighborhood around p' are not chosen with a strictly positive probability mass. We distinguish two possibilities. First, there exists a neighborhood $[p', x)$ with $x > p'$ such that the probability that any firm charges a price in the neighborhood equals 0. This cannot be an equilibrium, as Firm 1 can increase p' without reducing its chance of winning the customer. Second, there exists a neighborhood (p', x) with $p' < x$ such that the probability that at least one other firm charges a price in the whole neighborhood is strictly positive. This can also not be part of an equilibrium, however, as one of the other firms has incentive to shift probability mass from prices just above p' to prices just below it. Hence, there cannot be any equilibrium in which only one firm puts strictly positive probability on any single price either, which is what we had to show.

Note, too, that the support over which a firm mixes in equilibrium is connected. Suppose Firm 1 randomized over an unconnected support including $[\beta_1, \gamma_1]$ and $[\beta_2, \gamma_2]$. The optimal response of Firm 2 (in mixed strategies) would not include prices in $[\gamma_1, \beta_2]$. There exists, however, an $\epsilon > 0$ such that Firm 1 will not be indifferent between setting a price of $\gamma_1 - \epsilon$ and setting a price of $\beta_2 + \epsilon$, which is a necessary condition for Firm 1 to mix over $[\beta_1, \gamma_1]$ and $[\beta_2, \gamma_2]$.

The equilibrium mixed strategies thus have a convex support. Let $F(p_i)$ be the probability that firm i charges a price smaller than p_i. Firm i's expected payoff from p_i when all other firms use $F(p_i)$ is

$$\pi_i(p_i, F_i(p)) = \Sigma_{k=0}^{N-1} \binom{N-1}{k} (1-\alpha)^k [\alpha(1-F(p_i))]^{N-k-1} p_i, \tag{1}$$

because the probability exactly $N-k-1$ of the other $N-1$ firms are active equals

$$\binom{N-1}{k} (1-\alpha)^k \alpha^{N-k-1}. \tag{2}$$

Firm i's expected payoff when $N-k-1$ firms are active and it charges p_i equals p_i times the probability each of other firm charges more than p_i: $(1-F(p_i))^{N-k-1} p_i$, leading to expression (2).

Applying a standard result of the Binomial Theorem, equation (1) can be rewritten as

$$\pi(p_i, F(p_i)) = [1 - \alpha F(p_i)]^{N-1} p_i. \tag{3}$$

In equilibrium, firm i must be indifferent between all pure strategies in the support of the mixed strategy. Hence, on some interval of prices the derivative of expression (3) with respect to p_i is zero, and

$$[1-\alpha F(p_i)]^{N-1} - (N-1)[1-\alpha F(p_i)]^{N-2}\alpha f(p_i)p_i = 0, \tag{4}$$

or

$$1-\alpha F(p_i) - \alpha(N-1)f(p_i)p_i = 0, \tag{5}$$

where f is the density associated with cumulative distribution function F.

It is straightforward to show that the solution to equation (5) is

$$F(p_i) = \frac{1-(1-\alpha)\left(\sqrt[N-1]{\frac{v}{p_i}}\right)}{\alpha}, \tag{6}$$

for $(1-\alpha)^{N-1}v \leq p_i \leq v$. Result (6) implies a unique symmetric equilibrium with compact support.

Proposition 1. The unique symmetric equilibrium of the Bertrand model with an uncertain number of competitors is in mixed strategies and the distribution function of a player's strategy is

$$F(p_i) = \begin{cases} 0 & \text{for } p_i \leq (1-\alpha)^{N-1}v \\ \dfrac{1-(1-\alpha)\left(\sqrt[N-1]{\frac{v}{p_i}}\right)}{\alpha} & \text{for } (1-\alpha)^{N-1}v \leq p_i \leq v \\ 1 & \text{for } p_i \geq v \end{cases} \tag{7}$$

It can be easily verified that as N increases, each firm chooses relatively low prices with higher probability. As N becomes large, the cumulative density function approaches 1 for all values of p that are strictly positive. Of course, the equilibrium price under perfect competition is also equal to 0. The perfectly competitive outcome can be regarded as the limit case of the present model when the number of firms becomes very large. The intuition is straightforward. As the number of potential competitors increases, the probability of at least one other firm actively producing the same product rises. With greater probability of competition, the firm reduces its prices. In the limit, a firm is extremely likely to have at least one active competitor. Standard Bertrand competition comes into effect and each firm charges a price equal to marginal cost.

Expected profit for one firm can be found using the pure strategy profit from charging $p = v$. Since the firm is active with probability α, that profit is

$$\pi_i = \alpha(1-\alpha)^{N-1}v. \tag{8}$$

Note that individual profit is declining in N and its sum, industry profit, is equal to[2]

[2] Note that although the profits of the different firms are not independent, the expected profits are, so this summation is legitimate.

$$N\alpha(1-\alpha)^{N-1}v. \tag{9}$$

Let Π_b denote expected industry profit under Bertrand competition of this kind given that at least one firm is active. The profit in equation (9) can be written as

$$\sum_{i=1}^{N}\pi_i = N\alpha(1-\alpha)^{N-1}v = (1-\alpha)^N(0) + [1-(1-\alpha)^N]\Pi_b, \tag{10}$$

yielding

$$\Pi_b = \frac{N\alpha(1-\alpha)^{N-1}v}{1-(1-\alpha)^N}. \tag{11}$$

To see how industry profit changes with N, note that after some manipulation,

$$\frac{d\Pi_b}{dN} = \left[\frac{(1-(1-\alpha)^N) + N\log(1-\alpha)}{(1-(1-\alpha)^N)^2}\right]\left[\alpha(1-\alpha)^{N-1}v\right]. \tag{12}$$

Derivative (12) is well-defined, even though only integer values of N have an economic interpretation. Its sign is the same as the sign of

$$1-(1-\alpha)^N + N\log(1-\alpha). \tag{13}$$

For $N = 1$, expression (13) becomes $\alpha + \log(1-\alpha)$, which is negative because $\alpha < 1$. For larger N, expression (13) becomes even more negative, because its derivative with respect to N is $-(1-\alpha)^N\log(1-\alpha) + \log(1-\alpha) = \log(1-\alpha)[1-(1-\alpha)^N] < 0$. Thus,

$$\frac{d\Pi_b}{dN} < 0, \tag{14}$$

and profits fall as the number of firms increases.

We can show more: profits are convexly decreasing in the number of firms in the industry. The second derivative $d^2\Pi_b/dN^2$ is derived from (12), which can be rewritten as

$$\frac{d\Pi_b}{dN} = \alpha v\left\{\frac{(1-\alpha)^{N-1}}{1-(1-\alpha)^N} + \frac{(1-\alpha)^{N-1}N\log(1-\alpha)}{[1-(1-\alpha)^N]^2}\right\}. \tag{15}$$

The derivative of expression (15) is

(16) $d^2\Pi_b/dN^2$

$$= \alpha v \left\{ 2\frac{(1-\alpha)^{N-1}\log(1-\alpha)}{[1-(1-\alpha)^N]^2} + \frac{\begin{array}{c}(1-\alpha)^{N-1}N\log^2(1-\alpha)[1-2(1-\alpha)^N] \\ +(1-\alpha)^{2N}+2(1-\alpha)^N-2(1-\alpha)^{2N}\end{array}}{[1-(1-\alpha)^N]^4} \right\}$$

$$= \alpha v \left\{ 2\frac{(1-\alpha)^{N-1}\log(1-\alpha)}{[1-(1-\alpha)^N]^2} + \frac{(1-\alpha)^{N-1}N\log^2(1-\alpha)[1-(1-\alpha)^{2N}]}{[1-(1-\alpha)^N]^4} \right\}$$

$$= \frac{(1-\alpha)^{N-1}\log(1-\alpha)}{[1-(1-\alpha)^N]^2}\left\{2+\frac{N\log(1-\alpha)[1+(1-\alpha)^N]}{1-(1-\alpha)^N}\right\}\alpha v.$$

The first term of expression (16) is negative because $\log(1-\alpha)$ is negative. The second term has the same sign as

$$(17) \qquad 2-2(1-\alpha)^N + N\log(1-\alpha)[1+(1-\alpha)^N].$$

We will show that expression (17) is also negative for all N and all $\alpha \in (0,1)$. Suppose $N = 1$. We can define $f(\alpha) = 2\alpha + (2-\alpha)\log(1-\alpha)$. It is easy to see that $f(0) = f'(0) = 0$ and that $f''(\alpha) = -(\alpha/(1-\alpha)^2)$, which is strictly negative for all $\alpha > 0$. Hence, for all $\alpha \in (0,1), f(\alpha) < 0$.

For fixed α,

$$(18) \qquad g(N) = 2-2(1-\alpha)^N + N\log(1-\alpha)[1+(1-\alpha)^N].$$

It can be shown that $g'(N)$ has the sign of

$$(19) \qquad (1-\alpha)^N - 1 - (1-\alpha)^N N\log(1-\alpha)$$

and that $g''(N)$ has the sign of

$$(20) \qquad -N(1-\alpha)^N\log^2(1-\alpha).$$

As $g(1)$, $g'(1)$, and $g''(N)$ are strictly negative, we can conclude that expression (17) is negative, so that

$$(21) \qquad \frac{d^2\Pi_b}{dN^2} > 0.$$

It is not difficult to generalize the analysis and consider more general demand functions, which we denote by $D(p)$. In the working paper version of the present article (Janssen and Rasmusen [2000]) we show that if $pD(p)$ is increasing and continuously differentiable in p for $p < p_m$, where p_m is the monopoly price, then (for $N = 2$) the mixed strategy price distribution is given by

$$(22)\qquad F_2(p) = \begin{cases} 0 & \text{if } p \le \underline{p} \\ \dfrac{1}{\alpha}\left[1 - \dfrac{(1-\alpha)p_m D(p_m)}{pD(p)}\right] & \text{if } \underline{p} < p \le p_m \\ 1 & \text{if } p > p_m \end{cases}$$

for some $\underline{p}$. Moreover, industry profits may be calculated as above and equal (for general N)

$$(23)\qquad \Pi_b = \frac{N\alpha(1-\alpha)^{N-1} p_m D(p_m)}{1-(1-\alpha)^N}.$$

The working paper version also explores ways in which the number of firms in the industry may be made endogenous.

III. COMPARING BERTRAND AND COURNOT

We will now compare the Bertrand and Cournot models under uncertainty. We will assume that α, the probability of activity, is a constant, independent of N.[3] To make the comparison clearer, we will use linear demand, $p\left(\sum_{i=1}^N q_i\right) = a - b\sum_{i=1}^N q_i$. The monopoly price equals $a/2$ and the quantity demanded is $a/2b$ at that price. Define $q(p)$ as the demand facing a monopolist at a price of p, so $q(p) = (a/b) - (p/b)$.

Applying equation (22) to the case of linear demand, industry profits in the Bertrand model with uncertainty are

$$(24)\qquad \Pi_{bertrand} = \frac{N\alpha(1-\alpha)^{N-1} p_m D(p_m)}{1-(1-\alpha)^N} = \frac{N\alpha(1-\alpha)^{N-1}\frac{a^2}{4b}}{1-(1-\alpha)^N}.$$

Adding uncertainty eliminates the discontinuous behavior of the original Bertrand model. Profits are always positive, but they fall whenever the number of firms or the probability of more firms being active increases. Figure 1 shows this for a particular numerical example.

Firm i's expected profit in a Cournot equilibrium under uncertainty is, if each other firm chooses output q^*,

$$(25)\qquad \pi_i(q_i, q^*) = \sum_{j=0}^{N-1} \binom{N-1}{j} (1-\alpha)^j \alpha^{N-1-j} [p(q_i + (N-1-j)q^*)]q_i.$$

Substituting the demand function, differentiating with respect to q_i, setting $q^* = q_i$, and solving for q^* yields the equilibrium expected Cournot industry profit conditional upon one firm being active, equation (26).

[3] This is an assumption some people question. If a firm becomes inactive because it sells out its capacity, for example, it may be that many small firms each have a bigger chance of running out of capacity for given demand than two large firms would.

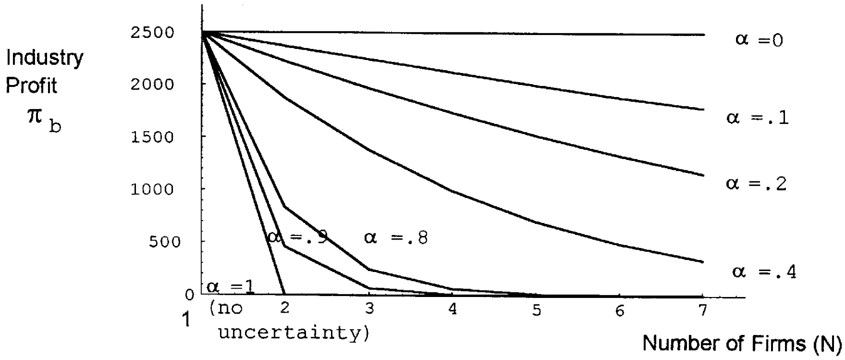

Figure 1
Bertrand Profits For Different Probabilities of Activity, α, and Numbers of Firms, N

(from equation (24), $a = 100$, $b = 1$, conditional on at least one firm being active; $\alpha = 0$ is the event that one firm is active and the expected number of other firms is zero)

Figure 2 shows profit for different degrees of activity and concentration using the same numerical parameters as in Figure 1.

$$(26) \qquad \Pi_{Cournot} = \frac{a^2 \alpha N}{b[1-(1-\alpha)^N][2+\alpha(N-1)]^2}.$$

Figure 2 shows that depending on the number of firms in the industry, uncertainty can either increase or reduce Cournot profits, but it does not radically change the equilibrium. Under Cournot competition, a firm expands its output when it expects fewer rivals to be helping push down the price and the net effect on expected industry output is unclear.

Using profit equations (24) and (26), the ratio of industry profits under Bertrand and Cournot competition is decreasing in both N and α:

$$(27) \qquad \frac{\Pi_{Bertrand}}{\Pi_{Cournot}} = (1-\alpha)^{N-1}\left[1+\frac{\alpha}{2}(N-1)\right]^2,$$

which can be shown to be both decreasing in N and α. Figure 3 shows the outcomes for different degrees of concentration under Cournot and Bertrand behavior when $\alpha = 0.8$.

All the curves in Figures 1 through 3 have convex shapes, but the curvatures differ. Figure 3 show that profits decline much more rapidly in Bertrand than in Cournot. Bertrand profits fall from the monopoly level of 2,500 to duopoly profits of 833, triopoly profits of 242, and low levels thereafter, whereas Cournot profits decline at a steadier rate. This is reminiscent of the empirical finding of Bresnahan and Reiss [1991], who analyzed markets in small towns. An example will illustrate their method. If a town is very small—say, 500 people—it will have no dentist since even a monopolist dentist incurs a fixed cost and could not make a profit. If it

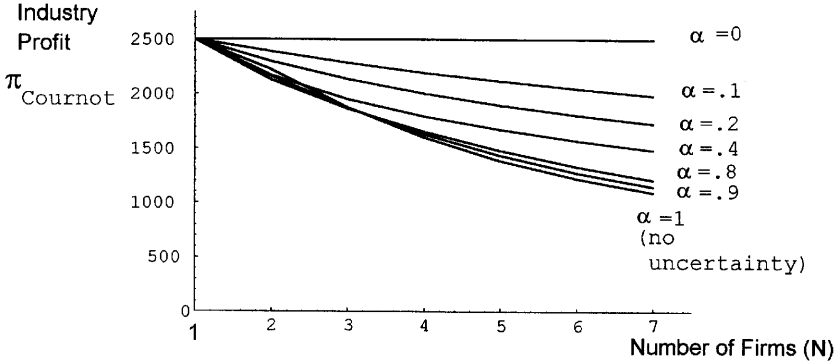

Figure 2
Cournot Profits For Different Probabilities of Activity, α, and Numbers of Firms, N

(from equation (25), $a = 100$, $b = 1$, conditional on at least one firm being active and on Nq^* not being so large as to drive the price to zero)

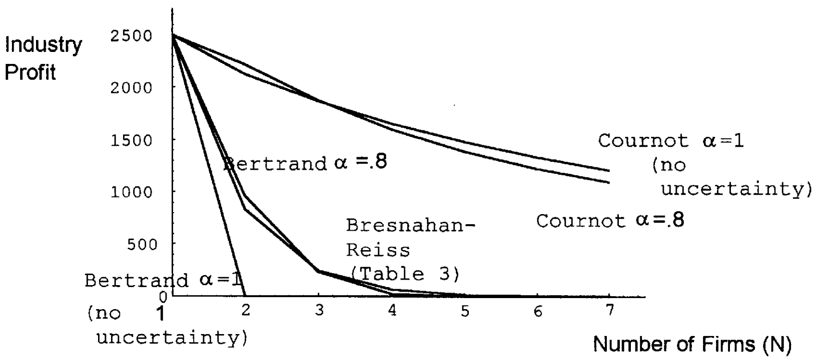

Figure 3
Bertrand and Cournot Profits

grows to 800 people, it will have one dentist, since monopoly revenues exceed the fixed cost. If the town grows to 1,600 people, it will still have only one dentist, because entry by the second dentist would not just split the industry profits, but reduce them. Bresnahan and Reiss used this approach to estimate the thresholds s_i for entry in small markets for a number of industries. Table 1 shows these thresholds in thousands of inhabitants per firm.[4] Table 2 rescales the same numbers to be very roughly comparable with the numerical example used earlier in this

[4] Table 1 is calculated from Table 5A of Bresnahan and Reiss [1991]. The entry of 0.79 in the second row of their original paper is a mistake and should be 1.09, and their Figure 3 illustrates s_i/s_5, not the s_5/s_i in the legend.

TABLE I
BRESNAHAN-REISS ENTRY THRESHOLDS s_i: ORIGINAL (1,000S OF INHABITANTS)

Number of Firms, N	1	2	3	4	5
Doctors	0.88	1.75	1.93	1.93	1.83
Dentists	0.71	1.27	1.39	1.36	1.28
Druggists	0.53	1.06	1.68	1.92	1.88
Plumbers	1.43	1.51	1.51	1.55	1.49
Tire Dealers	0.49	0.89	1.14	1.19	1.22

TABLE II
BRESNAHAN-REISS ENTRY THRESHOLDS: RESCALED AND ROUNDED $(25(s_m - s_i))/(s_m - s_1)$

Number of Firms, N	1	2	3	4	5
Doctors	2500	430	0	0	0
Dentists	2500	440	0	0	0
Druggists	2500	1550	430	0	0
Plumbers	2500	830	830	0	0
Tire Dealers	2500	1130	270	100	0
Average	2500	960	230	20	0

paper.[5] The rescaling is somewhat arbitrary, since the theory of Bresnahan and Reiss is that some quasi-rents remain to cover fixed cost even when the minimum scale for entry flattens out, but it creates a comparison measure for how the intensity of competition changes with the number of firms.

What is significant is how profits flatten out. Full-fledged competition kicks in quickly, so going from one firm to two is much more important than from two to three. This matches the Bertrand model with uncertainty very well but is inconsistent with the Cournot model.[6]

IV. CONCLUDING REMARKS

The Bertrand model with uncertainty about the number of competitors is simple, but its properties are interesting and useful. The extreme transition in standard Bertrand from monopoly to competition disappears. Expected profits are positive, but decline with the number of firms in the industry,

[5] Table 2's rescaling uses the following procedure. Define the monopoly level of profits in an industry to be 2,500, and the competitive level to be 0. Assume that when s_i reaches its maximum level s_m over $[1, 5]$, the competitive level of profits is reached and any further changes are measurement error. Apply the conversion formula $s_i^* = (25(s_m - s_i))/(s_m - s_1)$, and Table 2 results.

[6] We do not argue that the Bertrand model with uncertainty is the only model that may explain the data presented. The exponential decline of industry profits is nonetheless nicely captured by it.

and decline in a way that empirical work suggest is realistic. The model is useful both as a simple description of oligopoly and as a building block for more complex models, as in Gwin [1997] and Janssen and Van Reeven [1998].

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