Budget Constraints in Combinatorial Clock Auctions*

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April 16, 2015

1 Introduction

We consider Combinatorial clock auctions (CCAs) as multi-unit auctions with a clock phase followed by a supplementary phase where bidders can make bids on packages. The typical example is a telecom auction for frequencies in multiple bands. The supplementary phase is a Vickrey-Clark-Groves mechanism, where the bids are constrained by the bids in the clock phase. CCAs have been introduced by Ausubel, Cramton, and Milgrom (2006) and is subject to many recent investigations (see, e.g., Ausubel and Baranov (2014), Knapek and Wambach (2012), Janssen and Karamychev (2014), Levin and Skrzypacz (2014), and papers in this volume ....) One of the issues that is underexplored is to understand the properties of CCAs when bidders are budget-constrained .

It is difficult to obtain direct evidence of the fact that budget constraints play an important role in real-life CCAs. On the other hand, it is also difficult to believe bidders in recent telecom auctions do not consider some financial constraint on their bidding behavior. The amount of money typically paid is in the billions of euros and even though a firm may think it will earn that money back in the years after the auction, it is likely it has to borrow the money in one way or the other. Also, casual empiricism suggests that share prices of companies participating in a long-lasting auction decline after the auction (*cf.* share prices of KPN in the Netherlands in 2012 and A1 in Austria in 2013).

*Parts of this paper were first included in another paper, entitled Spiteful Bidding and Gaming in Combinatorial Clock Auctions. We have benefited from discussions with Martin Bichler, Jacob Goeree, Paul Klemperer, Jon Levin, Emiel Maasland, Tuomas Sandholm, David Salant, Andrzej Skrzypacz, and Achim Wambach. The first two authors have advised bidders in recent auctions in Europe. This research was supported by funds of the Oesterreichische Nationalbank (Oesterreichische Nationalbank, Anniversary Fund, project number: 15994).

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Budget constraints can be hard or soft. A hard budget constraint implies that bidders cannot pay more than a certain exogenously determined amount. Soft budget constraints are determined by senior management and inform the team that is bidding on the company's behalf is not allowed to spend more than a certain amount of money. That is, under soft budget constraints, senior management may set aside a certain amount of money to be invested in acquiring spectrum. Soft budget may be updated during the auction when it turns out that a certain minimum desired package cannot be obtained with the agreed upon budget. If soft budget constraints do affect bidding behavior, it is an interesting question to ask how these constraints can be optimally chosen. This paper considers, however, hard budget constraints only.

This paper points at different effects of budget constraints in a CCA under two alternative sets of preferences. First, we consider "standard" preferences, where bidders only care about the spectrum they win and the price they have to pay for that spectrum. Second, we consider bidders having alternative preferences and, in addition to their own surplus, also care about the price other bidders pay for the spectrum they win. Under these alternative preferences, bidders have a spite motive and, *ceteris paribus*, prefer outcomes where rivals pay more for their winning allocation. Janssen and Karamychev (2014) argue that comments by the Austrian regulator RTR can be interpreted as saying that in the supplementary round bidders have made many bids on very large packages the bidders knew they were unlikely to win, and that these bids were effective in raising the prices other bidders had to pay. Janssen and Karamychev (2014) provide two arguments why real-world bidders in telecom auctions are likely to engage in spiteful bidding, in addition to the evidence quoted from the Austrian regulator (see also Milgrom (2004) ....) .

We model the spite motive in a lexicographic way, i.e., a bidder always prefers outcomes with a larger intrinsic surplus (the value of the winning package minus the payment); rival payments only come into consideration to distinguish between outcomes with identical intrinsic surplus. This lexicographic preference ordering is an elegant way to select among the many equilibria of the supplementary round resulting in the same spectrum allocation, but different payments.

Paradoxically (maybe), having a hard budget constraint does not mean that in a CCA, a bidder should avoid bidding above budget. In particular, a bidder may insert such bids in the supplementary round that are not winning (and therefore do not affect the allocation). Bidders with a spite motive make these bids only to raise rivals' cost. Thus, we distinguish three ways in which a bidder can satisfy his budget constraint, depending on how much risk he accepts that he has to pay more than his budget: (i) a conservative, (ii) a neutral, or (iii) a risky way. A conservative bidder does not make any bid above budget. A neutral bidder only makes bids that are such that whatever bids the other bidders make in the supplementary phase (given that they are feasible given the constraints imposed by the cap rule), he will never pay more than his budget. A risky bidder has certain expectations of the bidding behavior of the rivals, given these expectations the bidder does not have to pay more than his budget, and these expectations are correct in equilibrium.

We obtain the following results for standard preferences. First, when considering the supplementary phase as a standard VCG mechanism, we show that the VCG mechanism does not have a weakly dominant strategy anymore. Bidders face the following trade-off: bidding full budget on all packages with a value larger than budget increases the chances of winning at least some package, but it may not be the most profitable package to win (given some bid strategy of others). This implies that even if bidders' behaviour in the clock phase is such that the constraints on supplementary round bids are not binding, optimal bidding in the supplementary round may be nontrivial and depends on the bidders' expectations of rival bids.

We next analyze some aspects of clock phase bidding by means of two examples. A first example shows that the information during earlier clock rounds may be such that a bidder knows he can safely bid above budget without running the risk of winning that package and having to pay above budget. (IS THIS SO?) Bidding above budget may be beneficial as it relaxes the constraints on supplementary round bidding allowing the bidder to bid true marginal values in that round. In this case the (positive) role of the clock phase is to provide bidders with information of the possibility of making bids without the risk of winning them. This role of the clock phase has so far been neglected. A second example shows that

Finally, we show that in a CCA the spite motive interacts in a complicated way with budget constraints. We show that if all bidders are budget constrained in a supplementary round that is unconstrained by clock phase bidding (essentially the VCG mechanism) there cannot be an equilibrium where all bidders place all their bids at or below budget. Essentially, the idea is that at least one bidder has an incentive to raise the bid on a package above budget to increase a rival's payment. In the context of an example, we also show that conservative bidders (those bidders without bids above budget) may have to pay more for identical packages than their risk-taking competitors pay. Ironically, conservative bidding is associated with the risk of having to pay more than competitors! In an example, we show that budget constraints can lead to multiple equilibria with a Hawk-Dove type flavor: risky bidders perform well against conservative bidders, but their bidding leads to payments above budget if all bidders are risky.

Cramton (1995) and Salant (1997) highlight the importance of budget constraints in spectrum auction. However, most academic papers on multi-unit auctions ignore budget constraints despite their practical importance. Che and Gale (1998) and Benoit and Krishna (2001) are relatively early papers that discuss single and multi-unit auction with budget constrained bidders respectively. If bidders are budget-constrained, the single-unit second-price auction has a weakly dominant strategy (e.g. Krishna 2010). On the contrary, the multi-unit version of the VCG auction does no longer have an equilibrium in weakly dominant strategies (e.g. Ausubel and Milgrom 2006). This already indicates the problems a budget constraint causes for auction designers and bidders in auctions.

For the SMRA, Brusco and Lopomo (2008) show that private budget constraints lead to strategic demand reduction and therefore to inefficiencies. In a sequel paper, Brusco and Lopomo (2009) analyze a simple model (two bidders, two units) of the SMRA with complementaries and known budget constraints. Without budget constraints there exists an efficient non-collusive equilibrium. But with budget constraints, the exposure problem might arise. In equilibrium, the bundle can be assigned to the bidder with lower budget and lower valuation for the bundle. A positive use of bidders' budget constraints is exemplified in Bulow et al. (2009). The authors describe a way to forecast final auction revenue based on budget constraints in an SMRA relatively early in the auction. Moreover, they present a real-world example in which this information was successfully used in a high-stake spectrum auction. Ausubel and Milgrom (2002) introduce an ascending pay-as-bid auction. The pay-as-bid payment rule facilitates bidding under a budget constraint, since there is no uncertainty about the final price if a bid becomes winning.

Lawrence M. Ausubel (2004) puts forward a dynamic version of the sealed-bid VCG auction and illustrates that bidding under a budget constraint might be easier and more efficient than under the sealed-bid version. In an example much like in our Example 3.1, he shows that efficiency can be hard to obtain in the sealed-bid version, but relatively easy in his dynamic "clinching" auction. In the clinching auction, bidders learn the VCG price during auction. If at least aggregate demand is revealed in every round, bidders know at which prices they clinched some goods. Therefore, they know the price they have to pay for their current clinches and can calculate the difference between budget and the price for current clinches. If this difference is above the current price, bidders can keep demanding truthfully. However, Ausubel restricts attention to decreasing marginal values. In the case of complementaries, the exposure problem might arise. The CCA is another dynamic version of the sealed-bid VCG auction. Unlike Ausubel's (2004) auction, the CCA is a package auction and solves the exposure problem. In the CCA bidders do not directly learn parts of the allocation and final prices during the clock phase, but they can compute upper and lower bounds on final VCG prices. This information is provided through the activity rule and can be used to forecast that the final VCG price. If the forecast shows that the final price is never above budget, bidding above budget might be allowed.

Dobzinski et al. (2012) show if bidders have linear utility over a homogeneous good and budgets are public knowledge, there exists no deterministic, individually rational and dominant-strategy incentive-compatible auction that does not involve any positive transfers and yields Pareto optimal outcomes. Moreover, they show that Ausubel's (2004) clinching auction is Pareto optimal and incentive compatible. When there are exactly two bidders, this is the unique (up to tie-breaking) auction that satisfies the stated properties.

This might suggest that the Ausubel auction is the best choice if bidders have (known) budget constraints and if they have (weakly) decreasing marginal utilities. If one of the two requirements are not met, the analysis of Dobzinski et al. (2012) does no longer apply. In this paper, we show that the CCA is another dynamic variant of the sealed-bid VCG auction and that the dynamic nature might facilitate bidding under a budget constraint and restore efficiency. If bidders have complementaries in their valuation, it is better to use a package auction. However, Levin and Skrzypacz (2014) show that the CCA has many inefficient equilibria. If the clock ends with market clearing, the final cap activity rule implies that the final clock allocation is the final allocation. In the supplementary phase, the bids of bidder i have no affect on their own payoff, but only on the VCG prices of the other bidders. On the one hand, if bidders express the true marginal values in the supplementary phase (Levin and Skrzypacz (2014) call this *consistent bidding*), then truthful bidding is optimal in the clock and the final outcome is efficient. On the other hand, if they do not fully raise the bids, other bidders have an incentive for demand expansion in the clock phase, resulting in an inefficient outcome. Although the CCA is not fully dominance-solvable, there exist efficient equilibria (Janssen and Kasberger 2015), the exposure problem is not an issue and the activity rule might provide enough information to allow package and final VCG price discovery.

The rest of the paper is organized as follows. Section 2 determines the set of strategies that are not dominated in a VCG mechanism where bidders are budget constrained and have standard preferences. Section 3 discusses the two examples illustrating the different optimal behaviours in the clock phase under a budget constraint. Section 4 discusses the complexities of combining spiteful bidding with a budget constraint. Section 5 concludes with a discussion. Proofs are in the Appendix.

2 Budget-Constrained Bidders in VCG

A well-known result for second-price auctions is that bidders have a weakly dominant strategy to bid their value. For one-unit auctions, this result has an analogue when bidders are budget-constrained: bidding the minimum of the value of the object and the budget is a weakly dominant strategy (see, *e.g.*, Krishna (2000, Proposition 4.2)). This Section shows that this result does not generalize to VCG auctions. Accordingly, bidding under a budget constraint is a non-trivial exercise in a multi-unit second-price auction.

To show which strategies are weakly dominated, and which cannot be eliminated as weakly dominated strategies, we use the following notation. Let there be K different types of objects to auction with n_k objects of type $k = 1, \ldots, K$. The set of all feasible packages is denoted by Π, and the aggregate supply is denoted by $\overline{\pi} = (\pi_1, \ldots, \pi_K)$. We use π to denote generic packages, $\pi \in \Pi$, and use the Greek letter superscripts to refer to specific packages, *e.g.*, π^α. The intrinsic valuation of bidder i for any package π^α is denoted by $v_i^\alpha = v_i(\pi^\alpha)$. Let $\Psi_i \subseteq \Pi$ be a subset of packages that bidder i bids on in a VCG mechanism. Accordingly, let $\Phi_i = \{b_i^\alpha : \pi^\alpha \in \Psi_i\} = \{b_i(\pi^\alpha) : \pi^\alpha \in \Psi_i\}$ be the set of bidder i's bids in the VCG mechanism, where $b_i^\alpha = b_i(\pi^\alpha)$ is the monetary amount b_i^α that bidder i bids on package π^α. Bidder i has a budget D_i, which is assumed to be a hard budget restriction. When no confusion is possible we drop subscript i.

In the following proposition, we state which strategies (set of bids) are weakly dominated in the VCG mechanism, and which set of bids are not.

Proposition 1. *Let all bids be potentially be pivotal and let $\pi^{\max}$ be the most valuable package of bidder i,* i.e., $v_i(\pi_i^{\max}) \geq v_i(\pi)$ *for all* $\pi \in \Pi$, *and* $v_i^{\max} = v_i(\pi^{\max})$ *be the corresponding value. Then, a collection of VCG bids Φ_i is weakly dominated if, for some package π^α:*

1. $b_i^\alpha > \min(v_i^\alpha, D_i)$, *or*
2. $b_i^\alpha < \max\{\min(v_i^{\max}, D_i) + (v_i^\alpha - v_i^{\max}), 0\}$.

The set of dominated bids consists of:

1. $b_i^{\max} = \min(v_i^{\max}, D_i)$ on the most valuable package $\pi_i^{\max}$, and
2. $b_i^\alpha \in [\max\{\min(v_i^{\max}, D_i) + (v_i^\alpha - v_i^{\max}), 0\}, \min(v_i^\alpha, D_i)]$ on all other packages π^α.

The proposition can be relatively easily understood. Under a hard budget constraint it is never optimal to bid above value or above budget. In an optimal strategy a bidder bids the full budget on his most valuable package, and the bid difference between this bid and the bids on all other packages will *not* be larger than the difference in valuations. For these other packages, a bidder faces the trade-off between winning at least some package (in which case they will bid full budget on less valuable packages as well), or winning the most profitable package (in which case they will bid full budget minus the value difference on less valuable packages as well).

The following example, which also will be used in the next Section on strategic bidding under a budget constraint in the clock phase, illustrates the Proposition.

Example 2. *Undominated strategies in the VCG with a budget constraint*
In this first example, there are three bidders competing for one band in which four units are for sale. The set of feasible packages is, for simplicity, $\Pi = \{1, 2, 3, 4\}$, *and bidders' realized values are the following four-tuples:*

$$\begin{aligned}\{v_1^\alpha\} &= \{5.9, 12, 12, 12\} \\ \{v_2^\alpha\} &= \{5, 9.5, 10, 10\} \\ \{v_3^\alpha\} &= \{5, 8, 8, 9\}\end{aligned}$$

Bidder $i = 1$ has a budget of $D_1 = 9$. Bidders do not know each other's valuation (private information scenario). In particular, bidder $i = 1$ knows that values of his rivals are either as stated above, or as follows:

$$\{\widehat{v}_2^\alpha\} = \{3.5, 9.9, 10, 10\}$$
$$\{\widehat{v}_3^\alpha\} = \{2.5, 9.9, 8, 9\}$$

Actual values of the other bidders

Bids of Bidder 1	$b(2,2,0)$	$b(2,1,1)$	$b(1,2,1)$	$b(1,1,2)$	$b(2,0,2)$	$b(0,2,2)$
$b_1 = (5.9, 12, 12, 12)$	21.5	**22**	20.4	18.9	20	17.5
$b_1 = (5.9, 9, 9, 9)$	18.5	19	**20.4**	18.9	17	17.5
$b_1 = (2.9, 9, 9, 9)$	18.5	**19**	17.4	15.9	17	17.5

Alternative values of the other bidders

Bids of Bidder 1	$b(2,2,0)$	$b(2,1,1)$	$b(1,2,1)$	$b(1,1,2)$	$b(2,0,2)$	$b(0,2,2)$
$b_1 = (5.9, 12, 12, 12)$	21.9	**22**	20.8	20.8	21.9	19.8
$b_1 = (5.9, 9, 9, 9)$	18.9	19	**20.8**	**20.8**	18.9	19.8
$b_1 = (2.9, 9, 9, 9)$	18.9	19	17.8	17.8	18.9	**19.8**

Table 1: The impact of a budget constrained bidder on the final allocation in the VCG auction

We refer to the first set of valuations as "actual" and to the second set as "alternative". The winner determination problem is to find a feasible allocation that maximizes the function $b(x_1, x_2, x_3) = b_1(x_1) + b_2(x_2) + b_3(x_3)$ *such that* $x_1 + x_2 + x_3 \leq 4$.

Table 1 shows the values of the sum of all the bids for the two possible sets of valuations for different bids of the first bidder and truthful bidding of players 2 and 3, respectively. If all bidders bid truthfully in a VCG auction, then the final allocation is the efficient allocation $(2,1,1)$ *and bidder 1's final VCG price is*

$$p_1^{VCG} = 17.5 - 10 = 7.5.$$

In the world of alternative preferences, bidder 1 still wins 2 units in the efficient (and final) allocation, but now he has to pay 9.8, which is above the budget of 9. Bidder 1 does not have a weakly dominant strategy. With actual preferences, he wants to win 2 units because the VCG prices is below budget. Therefore, it is better to submit a bid that makes it more likely that he wins 2 units. Preserving the true increase in utility from 1 to 2 units makes it more likely to win 2 units. If he bids $b_1 = (2.9, 9, 9, 9)$*, the marginal increase from 1 unit to 2 units is higher, therefore it is more likely to win 2 units. On the other hand, he risks winning nothing. This can be seen in the lower panel of Table 1. Under actual preferences, he would win two units, but under the alternative preferences he would not win anything. A bid that makes it more likely that he wins 1 unit is* $b_1 = (5.9, 9, 9, 9)$*. If he submits this bid, the marginal bid on 1 is relatively large, therefore it is more likely that he will win 1 in the end. For both sets of preferences of rival bidders, he wins one unit, which is his optimal share as the VCG price for two units is above bidder 1's budget.* ///

Thus, the example confirms that bidding under uncertainty and a budget constraint is a non-trivial task and the optimal bid depends on the beliefs about the other bidder's valuation and play. The next Section shows that sometimes the clock phase may provide bidders with information that allow them to infer that the VCG price is below budget so that they may actually bid above budget (which may restore eficiency).

3 Budget-Constraints in the clock of the CCA

This section concentrates on some aspects of strategic bidding under a budget constraint that explicitly involve the clock phase of the CCA. First, we show that the outcome of the CCA can be efficient, whereas the outcome of the VCG auction is not. This may be the case because bidders are willing to bid above budget as long as they know that they will certainly not pay more than budget. The VCG mechanism does not provide bidders with the possible bids of the other bidders, and therefore it may be the case that bidders have to pay more than their budget if they make some bids above budget. The dynamic nature of the clock phase of the CCA, paired with the activity rule that links the clock phase to the supplementary phase, allows bidders to compute upper bounds on the final VCG price if they make certain bids. This is sometimes called a bidder's exposure. if the exposure is below budget, bidders may safely bid above budget without running the risk of having to pay more than their budget.

The second aspect of the clock phase ... EXAMPLE VLADIMIR

Add notation for CCA (not yet complete):

round prices p^t, round demand d_i^t, auction price p_i (π^α)

3.1 An efficiency restoring role of the clock phase

The activity rules of a CCA translate bidders' clock demand to constraints on the admissible bidding function in the supplementary phase. Bidders can use the information that is revealed in the clock phase to compute upper (and lower) bounds on the other bidders' supplementary bidding function. This information allows them to forecast the maximal VCG price. Thus, depending on the development of the clock phase and the monetary value of the budget, bidders may learn that even if they bid above budget in the clock phase, and subsequently in the supplementary phase, they never have to pay above budget. The applicability of this observation crucially depends on the activity rules and the information disclosure policy. The more information on the other bidder's clock demand is revealed, the easier it is for budget constrained bidders to forecast future final prices. The next example shows how this may work.

Example 3 (The possibility of an efficiency restoring role of the clock phase)**.** *Consider again the example of the previous Section, but now introduce a clock phase. For now assume that bidders learn the individual demands after each clock round. Table 2 summarizes the demands at the given price. The clock started at a price of 1. Due to excess demand the price increased up to 4. At the price* $p = 4$, *bidder 1 sees that bidder 3 dropped demand to 1. Since there is still excess demand, the price will be increased to 5 in the next clock round.*

p	$D_1(p)$	$D_2(p)$	$D_3(p)$
1	2	2	2
2	2	2	2
3	2	2	2
4	2	2	1

Table 2: Observed bids in the clock phase

Under truthful bidding, bidder 1 would demand 2 units at this price. In this case he has to bid at least 10, which is above budget, for 2 units in the supplementary phase. He wants to bid above budget only if he knows that the final VCG price is below budget. Bidder 1 can infer from the observed history that in all admissible continuations of the clock, the final VCG price is below budget.

If bidder 1 demands two units at a price of 5, then the clock can end at $p = 5$ *with five possible final clock allocations that are consistent with the observed history. Either the clock ends with market clearing, in which case the final clock allocation is* $(2, 2, 0)$ *or* $(2, 1, 1)$*. Or it ends with excess supply, where* $(2, 1, 0), (2, 0, 1)$ *or* $(2, 0, 0)$ *are possible. The clock continues if* $(2, 2, 1)$ *is demanded.*

Table 3 summarizes the constraints on the supplementary bidding function for all possible clock demands that are consistent with the observed demands in Table 2 and the clock ending at $p = 5$*. The constraint on* $b_i(3)$ *and* $b_i(4)$ *are the same for all bidders. In the supplementary phase, it must be true that* $b_i(3) \leq b_i(2) + 1$ *and* $b_i(4) \leq b_i(2) + 2$*.*

Bidder 1 can now compute the possible final VCG prices if the clock ends at $p = 5$ *by him demanding two units. Similarly, bidder 1 can compute possible final VCG prices if he demands two units at* $p = 5$ *and the clock continues and he bids such that he respects his budget constraints at* $p > 5$*. In this case it must be that the demand equals* $(2, 2, 1)$ *at a price of 5. The detailed calculations in the Appendix show that in either case, bidder 1 will not have to pay more than his budget. Intuitively, if the final allocation is* $(2, 1, 1)$*, then the other bidders*

Possible clock demand histories					Constraints on $b_i(1)$	Constraints on $b_i(2)$
$D_i(1)$	$D_i(2)$	$D_i(3)$	$D_i(4)$	$D_i(5)$		
2	2	2	2	2	$b_i(1) \leq b_i(2) - 5$	$10 \leq b_i(2)$
2	2	2	2	1	$5 \leq b_i(1)$	$8 \leq b_i(2) \leq b_i(1) + 5$
2	2	2	2	0	$b_i(1) \leq 5$	$8 \leq b_i(2) \leq 10$
2	2	2	1	1	$5 \leq b_i(1)$	$6 \leq b_i(2) \leq b_i(1) + 4$
2	2	2	1	0	$4 \leq b_i(1) \leq 5$	$6 \leq b_i(2) \leq b_i(1) + 4$

Table 3: Transformation of clock demands into constraints on the supplementary bidding function

can make him pay at most 9, since bidders 2 and 3 cannot raise their bids on 2 more than 5 and 4 respectively.

If the clock finishes at $p = 5$, by bidding for two units at the final clock price bidder 1 is able to bid his true marginal values for all packages in the supplementary round. Thus, using the VCG pricing rule he will always acquire the bundle with the highest intrinsic value, and he certainly not want to deviate from this strategy.

The example depends on the activity rule and the informational policy. In this example, the limits of the relative cap are sufficient. With more than one band, the final cap sometimes determines lower upper bounds on the supplementary bidding function. The example suggests that under budget constraints tighter constraints can be efficiency enhancing. Consequently, a hybrid activity rule consisting of the final and the relative cap seem to be a good choice.

The analysis of the example was done under perfect information. Bidders learn the other bidders' past demand after every clock round. In the present example, the revelation of aggregate demand would suffice for bidder 1 to compute the maximal VCG price at round 5. However, in more complicated situations more demand revelation certainly helps in computing more accurate VCG prices. If no demand was revealed, bidders cannot compute upper bounds on the VCG price.

The auction designer faces a trade-off in the choice of informational policy. The revelation of more demand can foster collusion or spiteful behavior, but it can also enable the efficient outcome under budget constraints.

Bulow et al. (2009) define exposure as the maximal amount a bidder has to pay if all bids become winning. In the SMRA the exposure is the sum of the bidder's bids due to the pay-as-bid pricing rule. In the CCA, however, the exposure is the VCG price of the currently demanded package. The example shows that the exposure can be sufficiently different from the bid.

WHAT IF THE CLOCK DOES NOT STOP AT P=5. THE STORY BELOW IS NOT SO CONVINCING: I WANT TO KNOW SOMETHING WHAT BIDDER ! WILL DO AT EACH POSSIBLE CLOCK DEVELOPMENT TO ASSURE HE WILL NOT PAY MORE THAN BUDGET: Under alternative preferences, bidders 2 and 3 demand two units until the price reaches $p = 5$, at which they drop demand to one unit. The clock ends with excess supply. Bidder 1 cannot forecast a VCG price lower than his budget before bidding at $p = 5$. Therefore he must drop demand to 1. This drop limits his choice of the supplementary bid function. In particular, it rules out bidding true marginal values on 2 units. If bidder 1 believes that bidders 2 and 3 demanded truthfully in the clock, then he can interpret their demand as a signal that both value 2 units relatively highly. On the one hand, he can reason that if he gets two units, it might be that he has to pay more than budget. On the other hand, if his bid on 1 is too low, then it might be that he does not win anything. Therefore, the clock has shown (and partially forced) bidder 1 to submit the bidding function $b_1 = (5.9, 9, 9, 9)$. If the other two bidders bid truthfully, then, by Table 1 the final allocation is either $(1, 2, 1)$ or $(1, 1, 2)$. The clock "forces" bidder 1 to bid at least 5 for 1. Under the present preferences, the final allocation would not change, if bidder 1 bid $b_1 = (5, 10, 10, 10)$. I THINK THE STORY ABOUT SIGNALING SHOULD GO OUT: ///

In the above example, bidder 1 uses information from the clock phase to realize that some bids above his budget are safe to make. This can restore the efficiency of the auction (which, as we saw in the previous section, is not necessarily guaranteed by the VCG mechanism). The example assumed that bidders are informed about the demands of each and every bidder, but by carefully reviewing the example it is clear that the same argument would go through if bidders would only be informed about total demand in each round. In that case, it may be that the exposure of bidders is larger and that the argument becomes less tight, but the same principles apply.

If bidders cannot infer that they can safely bid above budget during the clock phase, budget-constrained bidders have to adjust their clock phase bidding, as the following example demonstrates. WHAT IS KEY MESSAGE OF THIS EXAMPLE?

Example 4. *Example Vladimir, $D_1 = 23$*
In this example, there are three bidders competing for two bands with supply $\overline{\pi} = (6, 3)$. The eligibility points equal 1 for the first band and 2 for the second. The set of feasible packages is $\Pi = \{(1,1), (2,1), (1,2), (3,1), (3,2)\}$. Let bidders have the following (symmetric) values:

$$v(1,1) = 30.5,\ v(2,1) = 45,\ v(1,2) = 39.5,\ v(3,1) = 54.5,\ v(3,2) = 55.5$$

We assume that the CCA begins with reserve prices $p^1 = (10, 1)$, and price increments in both bands are equal to one. The following table represents the clock phase development. If bidders bid truthfully in the clock phase their behaviour is given by this table. It is clear that round $t = 5$ is the final clock round.

Round, t	*Prices, p^t*	d_1^t	d_2^t	d_3^t
1	$(10, 1)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
2	$(10, 2)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
3	$(10, 3)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
4	$(10, 4)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
5	$(10, 5)$	$(2, 1)$	$(2, 1)$	$(2, 1)$

If bidders also bid truthfully in the supplementary round, they bid their values on all packages:

$$b(1,1) = 30.5,\ b(2,1) = 45,\ b(1,2) = 39.5,\ b(3,1) = 54.5,\ b(3,2) = 55.5.$$

Applying the winner determination tool, results in each bidder winning package $(2, 1)$ at a price

$$p(2,1) = (b(3,2) + b(3,1)) - (b(2,1) + b(2,1)) = 20$$

so that surplus equals 25.

Suppose now that bidder 1 has a budget of $D_1 = 23$. *In this example, the exposure of bidding* $d_i^5 = (2,1)$ *in round* $t = 5$ *is larger than the available budget. if the auction would end at that round, the others can maximally raise their bid* $b(3,2)$ *to* $b(2,1)+15$ *and their bid* $b(3,1)$ *to* $b(2,1)+10$ *so that together the other bidders can raise the price bidder 1 has to pay for obtaining package* $(2,1)$ *to* 25. *As a result, bidding* $d_i^5 = (2,1)$ *in round* $t = 5$ *is not feasible for him. Among the (still feasible) packages,* $(1,1)$ *and* $(1,2)$, *package* $(1,2)$ *is the most profitable one at prices* $p^5 = (10,5)$ *so that one may assume bidder 1 bids* $d_i^5 = (1,2)$. *If this happens, the price for band 2 keeps increasing until round* $t = 7$, *when bidder 1 switches to a package* $(1,1)$ *and the clock stops. The following table represents the clock phase development for rounds* $t = 5, \ldots, 7$ *if bidder 1 has a budget constraint of* $D_1 = 23$.

Round, t	*Prices,* p^t	d_1^t	d_2^t	d_3^t
5	$(10,5)$	$(1,2)$	$(2,1)$	$(2,1)$
6	$(10,6)$	$(1,2)$	$(2,1)$	$(2,1)$
7	$(10,7)$	$(1,1)$	$(2,1)$	$(2,1)$

Relative cap rule puts the following restrictions on the supplementary round bids of bidder $i = 1$:

$$\begin{aligned} b_1(2,1) &\leq b_1(1,1)+10,\ b_1(1,2) \leq b_1(1,1)+7, \\ b_1(3,1) &\leq b_1(1,1)+20,\ b_1(3,2) \leq b_1(1,2)+20 \end{aligned}$$

If bidders 2 and 3 bid truthfully in the supplementary round, and bidder $i = 1$ *bids according to his budget:*

$$b_1(1,1) \in [17,23],\ \text{and } b_1(2,1) = 23,$$

he wins $(1,1)$ *at price* $p_1(1,1) = 10.5$ *and surplus 20, bidders* $i = 2,3$ *win* $(2,1)$ *and* $(3,1)$ *at prices* $p(2,1) = 24 - b_1(1,1)$ *and* $p(3,1) = 33.5 - b_1(1,1)$, *with surplus* $21 + b_1(1,1)$ *from both packages.*

With this clock phase, bidder i can only win package $(2,1)$ *for sure if he bids at least* $b_1(2,1) = 27$ *and* $b_1(1,1) = 17$. *In this case, the maximal price* $p_1(2,1)$ *that bidder* $i = 1$ *has to pay is* $p_1(2,1) = 25$. *This price realizes if,* e.g., *bidders 2 and 3 bid*

$$b(1,1) = 35,\ b(2,1) = 45,\ b(1,2) = 40,\ b(3,1) = 55,\ b(3,2) = 60$$

(these bids are feasible, according to the relative cap rule). Therefore, bidder $i = 1$ *cannot win* $(2,1)$ *with price surely below the budget.*

Alternatively, having observed that bidders 2 and 3 have switched to $(2,1)$ *in round* $t = 5$, *bidder* $i = 1$ *can drop to* $(0,0)$ *in round* $t = 6$. *The following table represents the clock phase development for rounds* $t = 5, 6$.

Round, t	*Prices,* p^t	d_1^t	d_2^t	d_3^t
5	$(10,5)$	$(1,2)$	$(2,1)$	$(2,1)$
6	$(10,6)$	$(0,0)$	$(2,1)$	$(2,1)$

Relative cap rule puts the following restrictions on the supplementary round bids of bidder $i = 1$:

$$b_1(2,1) \le 26,\ b_1(1,2) \le 22,\ b_1(3,1) \le 36,\ b_1(3,2) \le b_1(1,2) + 20$$

If bidders 2 and 3 bid truthfully in the supplementary round, and bidder $i = 1$ bids $b_1(2,1) = 23$, he wins $(2,1)$ at price

$$p(2,1) = (b(3,2) + b(3,1)) - (b(2,1) + b(2,1)) = 20$$

Therefore, bidder $i = 1$ has a chance to win $(2,1)$ by dropping out of the clock phase.

Discussion.

Example 5. *Example Vladimir, $D_1 = 19$*
Suppose that instead of the budget $D_1 = 23$ in the previous example, budget of bidder $i = 1$ is $D_1 = 19$. In this case, the truthful clock phase looks like as is shown in the following table.

Round, t	*Prices, p^t*	d_1^t	d_2^t	d_3^t
1	$(10,1)$	$(1,2)$	$(1,2)$	$(1,2)$
2	$(10,2)$	$(1,2)$	$(1,2)$	$(1,2)$
3	$(10,3)$	$(1,2)$	$(1,2)$	$(1,2)$
4	$(10,4)$	$(1,2)$	$(1,2)$	$(1,2)$
5	$(10,5)$	$(1,1)$	$(2,1)$	$(2,1)$

If bidders 2 and 3 bid truthfully in the supplementary round, and bidder $i = 1$ bids

$$b_1(1,1) \in [15,19],\ \text{and } b_1(2,1) = 19,$$

he wins $(1,1)$ at price $p_1(1,1) = 10.5$ and surplus 20, bidders $i = 2,3$ win $(2,1)$ and $(3,1)$ at prices $p(2,1) = 20 - b_1(1,1)$ and $p(3,1) = 29.5 - b_1(1,1)$, with surplus $25 + b_1(1,1)$ from both packages.

Dropping out seems to be a very risky option for bidder $i = 1$ in round $t = 5$.

4 Budget-Constrained Bidders Under Spite Motive

Proposition 6. *Let each bidder i be active in the final clock round T and have a budget D_i that is lower than his value $v_i\left(\pi_i^f\right)$ for his last clock round package $\pi_i^f = d_i^T$. Let also bidders have a spite motive. Then, in any undominated equilibrium of the supplementary round, either $b_i\left(\pi_i^f\right) < D_i$ or $b_i(\pi^\alpha) > D_i$ for some package π^α.*

5 Appendix

Proof of proposition 1.
We omit the subscript i. Any bid above the value, $b^\alpha > v^\alpha$, is dominated by $b^\alpha = v^\alpha$. Next, any bid above budget, $b^\alpha > D$, is dominated by $b^\alpha = D$. This proves part (1) of the proposition. In odrder to prove part (2), we define $z^\alpha = \min\left(v^{\max}, D\right) + \left(v^\alpha - v^{\max}\right) - b^\alpha$, and $Z = \left\{\pi^\alpha : \forall \pi^\beta \in \Psi : z^\alpha \geq z^\beta\right\}$. In other word, Z is a subset of packages that generate the largest surplus had the bidder win them at VCG prices equal to their bids b^α. Consider an alternative set of bids $\widetilde{\Phi} = \left\{\widetilde{b}^\alpha : \pi^\alpha \in \Psi\right\}$ where $\widetilde{b}^\alpha = b^\alpha$ if $\pi^\alpha \notin Z$ and $\widetilde{b}^\alpha = b^\alpha + \varepsilon$ if $\pi^\alpha \in Z$. In other words, we rase bids on all packages from Z by a small amount $\varepsilon > 0$. It is easy to verify that $\widetilde{\Phi}$ dominates Φ.

Detailed calculations for Example 3.1.

Suppose first that the clock phase ends at $p = 5$ and bidder 1 demanding two units. In the determination of the final allocation and the VCG prices, no bidder's bid on 4 can ever play a role. To see this, note that

$$b_i(4) \leq b_i(2) + 2 < b_i(2) + 6 \leq b_i(2) + b_j(2),$$

that is, the bid on 4 is always smaller than the bid on 2 by the same bidder and the bid on 2 by another bidder. If the clock ends with demands $(2, 2, 0)$, then this is the final allocation by the final cap rule. Since the bid on 4 plays no role in the determination of VCG prices, there are in principle three possible ways to construct the VCG price for bidder 1. However, only the cases

$$p_1^{VCG} = b_2(3) + b_3(1) - b_2(2)$$
$$p_1^{VCG} = b_2(2) + b_3(2) - b_2(2)$$

are relevant, since

$$b_2(2) + b_3(2) \geq b_2(2) - 5 + b_3(2) + 1 \geq b_2(1) + b_3(3).$$

In the first case, the VCG price is at most $p_1^{VCG} \leq 6$. This constraint comes from the fact that the bid on 3 can be at most 1 larger than the bid on 2 and from the last line in Table 3. In the second case, $p_1^{VCG} = b_3(2) \leq 9$, which is equal to the budget.

If the clock ends with $(2, 1, 1)$, then the final clock allocation is the final allocation. The possible VCG prices are

$$p_1^{VCG} = b_2(3) + b_3(1) - b_2(1) - b_3(1) \leq b_2(2) + 1 - b_2(1) \leq 6$$
$$p_1^{VCG} = b_2(2) + b_3(2) - b_2(1) - b_3(1) \leq b_2(1) + 5 - b_2(1) + b_3(1) + 4 - b_3(1) = 9.$$

If the clock ends with excess supply and if bidder 1 bids $b_1 = (5.9, 10, 10, 10)$, then in any final allocation in which bidder 1 gets 2 units, he does not pay more than budget. If the final allocation is $(2, 2, 0)$ or $(2, 1, 1)$, the same considerations as above apply. If the final allocation is $(2, 0, 2)$, then

$$p_1^{VCG} = b_2(2) + b_3(2) - b_3(2) = b_2(2) \leq 10.$$

But if $b_2(2) \geq \omega_1$, then $(2,0,2)$ is no longer implemented by the auctioneer since

$$b_1(1) + b_2(2) + b_3(1) \geq 5.9 + 9 + b_3(1) \geq 10 + b_3(1) + 4 \geq b_1(2) + b_2(2)$$

is true. Moreover, bidder 1 never gets 3 units since

$$b_1(3) + b_j(1) \leq b_1(2) + b_j(1) \leq b_1(2) + b_j(1) + b_{5-j}(1),$$

for $j = 2, 3$. Even if $b_2(2) = 0$, it is true that $b(1,2,1) > b(3,0,1)$.

Consider then the case where bidder 1 demands two units at $p = 5$ and the clock continues, so that the demand at $p = 5$ must be $(2,2,1)$. If the clock ends at some higher price $p > 5$, bidder 1 also never has to pay more than budget. Bidder 1 submits the supplementary bidding function $b_1 = (5.9, 10, 10, 10)$. Since he never has to pay more than his bid, the only case in which he has to pay more than budget is if he wins two units. But in no final allocation in which he wins two units, he has to pay more than budget. First, note that $(2,2,0)$ is never the final allocation since

$$b_1(1) + b_2(2) + b_3(1) \geq 5.9 + b_2(2) + 5 > b_1(2) + b_2(2) = 10 + b_2(2).$$

Second, also $(2,0,2)$ is never winning since

$$b_1(1) + b_2(2)b_3(1) \geq 5.9 + 10 + b_3(1) \geq 10 + 4 + b_3(1) \geq b_1(2) + b_3(2).$$

Third, $(2,1,1)$ is winning if

$$\begin{aligned} b_1(1) + b_2(2) + b_3(1) &\leq b_1(2) + b_2(1) + b_3(1) \\ b_2(2) &\leq 4.1 + b_2(1). \end{aligned}$$

In this case, the VCG price is

$$\begin{aligned} p_1^{VCG} &= b_2(2) + b_3(2) - b_2(1) - b_3(1) \leq 4 + 4.1 < 9 \\ p_1^{VCG} &= b_2(3) + b_3(1) - b_2(1) - b_3(1) \leq 5.1 < 9 \end{aligned}$$

less than budget in any case.

As a result, bidder 1 can safely demand two units at a price of 5 in the clock phase. He knows that the final VCG price is never above budget.

Proof of proposition 6.
To be done.

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