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Auctions, aftermarket competition, and risk attitudes ☆

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ABSTRACT

With the experience of the sequence of UMTS auctions held worldwide in mind, we consider a situation where firms participate in license auctions to compete in an aftermarket. It is known that when a monopoly right is auctioned, auctions select the bidder that is least risk-averse. This firm will choose a higher value of the aftermarket strategic variable than any other firm will do, thereby implying a higher market price under price setting behavior and a lower price due to higher quantity under quantity-setting behavior. This paper extends the analysis to oligopoly aftermarkets and analyzes whether the monopoly result carries over to oligopoly settings. We argue that with multiple licenses and demand uncertainty auctions actually perform even worse from a welfare point of view than the monopoly case would suggest. A strategic effect strengthens the monopoly result with respect to prices, but weakens the result with respect to quantities.

1. Introduction

In many markets around the world, governments use auctions to allocate licenses to interested parties. A characteristic feature of these auctions is that after the auction is held, the winning firms compete with each other in an aftermarket where the revenues and profits depend not only on the market strategies that the licensed firms choose, but also on uncertainty concerning important market features, such as demand or cost. Winning a license is very much like winning a lottery ticket where the value of the ticket is uncertain. Important examples that share these features are the UMTS auctions that have been held in the past years all around the world. Other examples include the selling of radio frequencies for commercial radio and the selling of licenses to exploit gasoline stations on certain locations.

If firms were completely risk-neutral (an assumption often made for analytical tractability), this uncertainty would not affect firms' behavior in a crucial way. However, as we will briefly argue below, there are good reasons to believe that firms are not necessarily risk-neutral and, in fact, that firms differ in their risk attitudes. In this paper, we analyze how firms' risk attitudes affect their bidding behavior in an auction followed by an aftermarket.

Janssen and Karamychev (2007) ask a similar question in case governments decide to allocate only one license, thereby creating a monopoly in the aftermarket. They identify a risk attitude effect and show that the least risk-averse firm has the highest certainty equivalent for the aftermarket game (lottery) and, therefore, will win the license in any standard auction format. This firm chooses a higher value of its strategic variable, *e.g.*, price or quantity, than any other firm would choose. Thus, if the strategic variable is price, auctions select the firm that sets the highest aftermarket price, leading to a welfare loss relative to any other allocation of licenses. On the other hand, if the strategic variable is quantity, auctions select the firm that sets the highest quantity, resulting in a lower aftermarket price (and a welfare gain). We refer to this selection aspect of auctions as to the “monopoly result”.

A crucial assumption in the present paper and in Janssen and Karamychev (2007) is that firms are not necessarily risk-neutral. Empirical studies in finance indicate that firms may indeed be risk-averse, or that their behavior is as if they were risk-averse. Nance et al. (1993) and Geczy et al. (1997), among others, argue that firms hedge against different types of exogenous shocks such as exchange rate volatility. In a study on the gold mining industry, Tufano (1996) argues that delegation of control to a risk-averse manager, whose renumeration is linked to the firm's performance, may cause the firm to take actions in a risk-averse manner. As delegation of control differs between firms, and as the payment structure of managers differs from

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firm to firm, firms may very well act as if their attitudes towards risk differ significantly from one to the other. On the other hand, liquidity constraints and prospects of bankruptcy may force managers to undertake risky actions. Therefore, firms' managers can be both risk-averse and risk-seeking. In an experiment that they conducted with 224 managers, Laughhunn et al. (1980) showed that there are indeed large individual differences in risk attitudes between managers.

It is well-known that under a general form of risk aversion, the action price, which is sunk at the market stage, affects firms' aftermarket behavior. In particular, firms with identical utility functions make different choices in risky environments depending on how wealthy they are. McMillan (1994) argues that as auctions force firms to pay considerable amounts of money for licenses, auctions may force firms to behave differently in the marketplace. This wealth effect, however, disappears when firms are characterized by constant absolute risk aversion (CARA). As we want to concentrate on the selection aspect of auctions and not on the wealth effect, we assume that firms have CARA utility functions.

The main question we address in this paper is whether the monopoly result discussed above carries over to a situation where governments auction multiple licenses. This question is relevant in most applications (UMTS, radio frequencies, *etc.*) as governments often create oligopoly markets by auctioning off more than one license. We will say that the monopoly result generalizes to the oligopoly context if the auction has an equilibrium in which the most risk-seeking (least risk-averse) firms make the highest bids. We refer to such an equilibrium as a "risk-seekers' equilibrium", and we analyze the conditions under which it exists. To avoid complications arising from the possibility that players would like to signal their type through their bids, we analyze this question in a private information scenario, where firms are neither informed about each other bids, nor about the risk attitudes of each other in the aftermarket.[1]

This private information scenario arises in a sealed-bid uniform-price multi-unit auction where winning firms are the firms with the highest bids, and they pay a license fee which is equal to the highest non-winning bid. This is the auction format we adopt in this paper and the main features of this format were used, for example, in the Danish UMTS auction in 2001, see National Telecommunications Agency Denmark (2001). Qualitatively similar (although more complicated) results can, however, also be obtained for other formats, for example a pay-your-bid multi-unit auction, see, *e.g.*, Janssen and Karamychev (2005). Using this auction format, we consider both demand and cost uncertainty, and distinguish between differentiated Bertrand (strategic complements) and Cournot (strategic substitutes) oligopolies.

The main result we obtain is as follows. Under demand uncertainty, auctions actually perform worse than what could be expected on the basis of the monopoly result. Under differentiated Bertrand oligopoly, the aftermarket externality (due to strategic interactions between firms) reinforces the risk attitude effect. This implies that indeed the least risk-averse firms are selected and they choose the highest prices. Thus, like in the monopoly setting, auctioning multiple licenses results in higher market prices than if any other allocation mechanism is used. Under Cournot oligopoly, the aftermarket externality works in the opposite direction and weakens the risk attitude effect. When the externality is sufficiently strong, a risk-seekers' equilibrium fails to exist. This implies that the least risk-averse firms do not necessarily get licenses. Thus, and contrary to the monopoly setting, auctioning multiple licenses does not necessarily result in higher quantities and, therefore, lower market price than if the licenses were allocated differently. Hence, from a consumer welfare perspective the attractiveness of using auctions relative to using any other allocation mechanism is higher in the case of a single license than in the case of multiple licenses.[2]

The reason why the aftermarket externality works in the opposite direction in Cournot competition is that in a risk-seekers' equilibrium, the winning firms compete with the least risk-averse rivals. These firms choose the highest quantities and, therefore, are the most aggressive competitors. It remains true that for a given set of other players in the aftermarket, a relatively less risk-averse firm is willing to pay more for the license than a relatively more risk-averse firm is willing to pay. However, it may well be that a set of the least risk-averse firms make less profits and, therefore, have a smaller willingness to pay for the licenses than a set of more risk-averse firms do. That is why less risk-averse firms are not willing to pay high auction prices and outbid more risk-averse firms.

Our second result for Cournot competition is that the aftermarket externality and the *ex-ante* affiliation of firms' risk types together give rise to a strategic effect that does not only cause that a risk-seekers' equilibrium fails to exist, but that creates the conditions for another, "risk-averse players' equilibrium" to exist. In a risk-averse players' equilibrium, the most risk-averse firms (or the least risk-seeking firms—depending on the risk attitudes under consideration) submit the highest bids, win the licenses, and choose the lowest production levels, which again, as in the Bertrand case, results in the highest possible aftermarket price.

Another type of uncertainty that may exist and which we analyze as well is uncertainty about production costs. Here we show that if the uncertainty is about the level of fixed cost then the monopoly result does generalize to the oligopoly situation, whereas if the uncertainty is about marginal cost then a strategic effect appears, and the monopoly result for both price and quantity setting does not generalize.

The paper is organized as follows. An overview of related literature is provided in Section 2. As the model itself and the equilibrium analysis are fairly complicated, we start off in Section 3 with an example where we make some simplifying assumptions. The example makes it intuitively clear why under demand uncertainty the monopoly result carries over under Bertrand competition, but not under Cournot. Section 4 then presents the formal model where these simplifying assumptions are not made. The main results for the general model are provided in Section 5. Section 6 derives implications of the general results in case of demand uncertainty, while Section 7 delves into the case of cost uncertainty. Section 8 concludes and the Appendix A contains all proofs.

2. Literature review

There is a relatively large, recent literature on the possibility of inefficient allocation of licenses in auctions due to the presence of interdependencies. First, there is a literature where one license is auctioned and the auction winner competes in the aftermarket with non-winners.[3] Moldovanu and Sela (2003) analyze a situation where aftermarket competition is characterized by Bertrand competition and cost is private information at the auction stage.[4] When in such a situation a patent for a cost-reducing technology is auctioned amongst the competitors, they show that standard auction formats do not exhibit efficient equilibria where bids are increasing in the firms' efficiency parameter. Goeree (2003) and Das Varma (2003) analyze a similar setting

[1] Signaling plays an important role in a different scenario where the bids of the winning firms but not their types become public so that the auction stage has to be analyzed as an *N*-player signaling game. Another scenario, used in Section 3 for illustration purposes, where types of all winning firms become public, does not seem to be very realistic. Nevertheless, all these scenarios can be analyzed in a similar way to the private information scenario.

[2] We compare consumer welfare for different number of licenses indirectly, *i.e.*, through its comparison with welfare generated by, *e.g.*, a random allocation. In our analysis, the main effect of the number of licenses on consumer welfare is independent from firms' risk attitudes. Thus, a normative analysis of the optimal (from consumers' prospective) number of licenses does not depend on whether the licenses are auctioned or assigned randomly.

[3] Jehiel and Moldovanu (2006) provide an overview of existing work in this area and argue that in case firms' aftermarket profits depend on private information in the hands of other winning firms there is an informational externality. See, also, Jehiel et al. (1996) and Jehiel and Moldovanu (2000) for related papers where an (informational) externality may lead to inefficiency even in standard single-unit auctions.

[4] To avoid signaling issues, they consider the case where the true production costs of the bidding firms are revealed after the auction.

but allow for signaling private information through the auction bid. Das Varma (2003) shows that when there is Cournot competition (strategic substitutes) in the aftermarket an efficient equilibrium does exist as firms have an incentive to signal their (high) efficiency. Under Bertrand competition (strategic complements), this is not the case as other firms will adjust their market price downwards, the more efficient a firm is known to be. Katzman and Rhodes-Kropf (2008) show how different bid-announcement policies affect an auction's revenue and efficiency.

The above papers show that single-object auctions may result in inefficiencies, if values are strongly interdependent. We add to this literature by arguing that in multi-object auctions, inefficiencies may arise even when the aftermarket externality is weak provided the bidders' types are *ex-ante* affiliated. Moreover, external effects of firms' attitudes towards risk have not been analyzed before. In this paper we propose a technique that is capable of providing a full analysis of a multi-player signaling game for the imperfect information scenario, when firms' bids are made public after the auction stage.

There is also a literature (see, for example, Hoppe et al., 2006 and Grimm et al., 2003) on inefficiencies created by multiple license auctions if players have interdependent valuations. However, these papers usually address issues related to collusion and/or asymmetries between different participants.[5]

Apart from the auction literature, the present paper is intimately related to the early literature on price and quantity-setting behavior of a risk-averse monopolist (cf., Baron, 1970, 1971 and Leland, 1972). They show that the more risk-averse a price setting monopolist is the lower the price it sets. These results have recently been generalized to the case of market competition (see Asplund, 2001).

3. An Example

Consider an example where $N=3$ risk-averse firms compete for $n=2$ licenses. We first consider the case where firms winning the auction *compete in prices* in a differentiated product aftermarket and they face uncertain demand that is represented by the following demand functions:

$$q_1(p_1,p_2)=1+\theta-p_1+p_2 \text{ and } q_2(p_1,p_2)=1+\theta-p_2+p_1,$$

where θ is a demand shock that is normally distributed with zero mean and variance $D>0$. In this example, we allow demand to be negative which happens when $\theta<-1$. Moreover, we assume in this Section that risk attitudes of the two winning bidders are revealed after the auction. With these simplifying assumptions, a closed-form solution can be obtained for most of the analysis, so that we are better able to explain the main intuition for the results we get. In the main body of the paper, we consider a private information case, where risk attitudes of the winners are not revealed after the auction so that the bidders remain uncertain about risk attitudes of each other in the market stage, and we do not restrict the analysis to specific demand functions or specific forms of demand uncertainty.

Normalizing firms' production costs to zero, the profit of firm 1 is given by

$$\pi^1\equiv\pi(p_1,p_2,\theta)=p_1\cdot q_1(p_1,p_2)=p_1(1+\theta-p_1+p_2).$$

As explained in the Introduction, we assume firms having CARA utility functions throughout the paper so as to focus on the selection aspect of auction. Thus, the utility function $U^i(\pi^i)$ of firm i is given by

$$U^i\left(\pi^i\right)\equiv U\left(\pi^i,r_i\right)=\begin{cases}(1-\exp(-\pi^i r_i))/r_i & ,r_i\neq 0\\ \pi^i & ,r_i=0\end{cases}, \tag{1}$$

where r_i measures the absolute risk aversion of firm i. We assume in this example that firms' risk attitudes are positive, *i.e.*, $r_i\geq 0$, and are weakly affiliated.[6] This, together with the normally distributed demand shock θ, implies that the expected utility W^1 (of receiving an uncertain aftermarket profit) of firm 1 can be written as follows:

$$\begin{aligned}W^1\equiv W(p_1,p_2,w,r_1)&=\frac{1}{(2\pi D)^{\frac{1}{2}}}\int_{-\infty}^{\infty}U^1(\pi^1-w)\exp\left(-\frac{1}{2D}\theta^2\right)d\theta\\&=\frac{1}{r_1}\left\{1-\exp\left(-r_1\left(p_1(1-p_1+p_2)-w-\frac{1}{2}Dr_1p_1^2\right)\right)\right\}\\&=U^1\left(\pi(p_1,p_2,0)-w-\frac{1}{2}Dr_1p_1^2\right)\end{aligned}$$

where w is the price to be paid for the license. In other words, the certainty equivalent of the uncertain aftermarket profit is the net profit $\pi(p_1,p_2,0)-w$ adjusted by the amount $-Dr_1p_1^2/2$ due to risk aversion. When firms get to know each others' risk types after the auction, they maximize W^i w.r.t. p_i taking the price of its competitor as given. Maximizing W^1 w.r.t. p_1 yields the first-order condition:

$$0=1-(2+Dr_1)p_1+p_2,$$

and the following (linear) reaction function for firm 1:

$$p_1(p_2)=(1+p_2)/(Dr_1+2). \tag{2}$$

Nash equilibrium prices are then easily calculated to be

$$p_1^*=\frac{Dr_2+3}{(Dr_1+2)(Dr_2+2)-1}\text{ and }p_2^*=\frac{Dr_1+3}{(Dr_1+2)(Dr_2+2)-1}. \tag{3}$$

Here, one can see that these prices are decreasing in both firms' risk attitudes. As a player becomes more risk-averse, his reaction curve shifts downwards (this is because the bad states of the world, in which the firm optimally sets low prices, receive more weight in the utility evaluation) and, because of strategic complementarity, both equilibrium prices decrease. The reduced-form certainty equivalent $\tilde{\pi}$ of winning and getting the uncertain aftermarket profit for firm 1 is

$$\begin{aligned}\tilde{\pi}(r_1,r_2,w)&\equiv\pi\left(p_1^*,p_2^*,0\right)-w-\frac{Dr_1\left(p_1^*\right)^2}{2}\\&=\frac{(Dr_2+3)^2(Dr_1+2)}{2((Dr_1+2)(Dr_2+2)-1)^2}-w=v^1-w,\end{aligned}$$

where we define

$$v^1\equiv v(r_1,r_2)=\frac{(Dr_2+3)^2(Dr_1+2)}{2((Dr_1+2)(Dr_2+2)-1)^2}. \tag{4}$$

The amount $v(r_1,r_2)$ is firm 1's valuation of the license in the auction stage, had the firm known the risk attitude of the other winning firm. The signs of the partial derivatives are:

$$\frac{\partial v}{\partial r_1}=-\frac{D(Dr_2+3)^2((Dr_1+2)(Dr_2+2)+1)}{2((Dr_1+2)(Dr_2+2)-1)^3}<0,\text{ and} \tag{5}$$

$$\frac{\partial v}{\partial r_2}=-\frac{D(Dr_1+2)(Dr_2+3)(Dr_1+3)}{((Dr_1+2)(Dr_2+2)-1)^3}<0. \tag{6}$$

First, firms' valuations for licenses monotonically decrease with their own risk attitude. This is the *risk attitude* effect we noted in the Introduction, and it is always negative. Second, firms' valuations monotonically decrease with risk attitudes of their rivals. This is the aftermarket externality that affects firms' bidding behavior during the auction. With an increase in a firm's risk attitude, its reaction function

[5] Krishna (2002), Example 6.4, shows that it might happen that the bidder with the highest type is the bidder with the lowest valuation and that standard auction formats do not have efficient equilibria.

[6] Weak affiliation includes the case of statistical independence; see, *e.g.*, Krishna (2002).

(2) moves downwards and, due to the strategic complementarity, firms' Nash equilibrium prices (3) decrease. This has a negative impact on the firm's profit and, consequently, on its valuation for the license.

In the auction stage of the game, firms' risk attitudes are drawn from a certain joint distribution, and are private information to them. A bidding strategy in the auction is, therefore, a monetary bid conditional on a firm's risk attitude, which we denote by $b(r)$. In order to calculate the optimal (equilibrium) bid, we make in this example two additional simplifying assumptions. First, we formally assume that the variance D is so small that we can approximate firms' reaction functions (2), the Nash equilibrium prices (3), and the valuation functions (4) by their linear approximations around $D=0$:

$$p_1 = \frac{1+p_2}{2}\left(1-\frac{1}{2}r_1 D\right),\; p_1^* = 1-\left(\frac{2}{3}r_1+\frac{1}{3}r_2\right)D,$$
and $v(r_1, r_2) = 1-\frac{1}{6}(5r_1+4r_2)D.$

Second, we assume that firms' risk attitudes are independently and uniformly distributed over the interval [0,1].

Given these additional simplifications, we can construct a risk-seekers' equilibrium, where the most risk-seeking firms will win the licenses, *i.e.*, where the equilibrium bidding function $b(r)$ is a decreasing function. From the perspective of firm 1 in such an equilibrium it is the most risk-averse of the other two firms (let us call him firm 3) that determines the auction price, *i.e.*, $w=b(r_3)$. The aftermarket rival, firms 2, has risk attitude $r_2 \le r_3$. Hence, firm 1′s expected utility V^1 of winning the auction given type r_3 of firm 3 is:

$$\begin{aligned} V^1 \equiv V(r_1, r_3) &= E\left(U^1(v(r_1, r_2)-b(r_3))|r_2<r_3\right) \\ &= \frac{1}{r_3}\int_0^{r_3} \frac{1-\exp(-r_1(v(r_1, r_2)-b(r_3)))}{r_1}dr_2 \\ &= \frac{1}{r_1 r_3}\left(r_3-\exp\left(-r_1\left(1-\frac{5}{6}r_1 D-b(r_3)\right)\right)\frac{\exp(2r_1 r_3 D/3)-1}{2r_1 D/3}\right) \end{aligned}$$

For small values of D we can approximate the latter exponent as follows:

$$\exp(2r_1 r_3 D/3) \approx 1+2r_1 r_3 D/3.$$

This leads to the final expression

$$V(r_1, r_3) = \frac{1-\exp(-r_1(1-5r_1 D/6-b(r_3)))}{r_1} = U^1\left(1-\frac{5}{6}Dr_1-b(r_3)\right). \tag{7}$$

This implies that the certainty equivalent of the auction game is equal to the *ex-ante* value for the license $(1-5Dr_1/6)$ minus the auction price $b(r_3)$.

In a uniform-price auction, which is a generalization of a second-price auction, winning firms pay the highest losing bid and a bidder's Nash equilibrium strategy is to bid their own value. Hence, the bidding function

$$b(r) = 1-5Dr/6$$

constitutes the unique monotone symmetric Nash equilibrium.[7] This illustrates that the monopoly result generalizes to oligopoly competition settings with uncertain demand and strategic complements.

We now explain the more general intuition behind this result. The result depends on two forces that are at work. First, due to the risk attitude effect, a firm that is more risk-averse bids less because it has a lower valuation *for given types of its competitors*. This is reflected in the first derivative in Eq. (5). Second, even if firms' risk attitudes are *ex-ante* statistically independent, they are positively correlated amongst the firms that win the auction *ex-post*. Thus, when bidding for a license, a firm takes into account that if it wins the auction the rival firm is not just a firm with an average risk attitude. In particular, in a monotone decreasing equilibrium, a firm knows that if it wins the auction, it cannot be the case that the other two firms have lower risk attitudes than it has itself. This implies that a firm that is more risk-averse expects to compete (upon winning) with a firm that is more likely to be more risk-averse.

As this point is one of the crucial points made in this paper, we illustrate it graphically. In Fig. 1, the square $(r_2,r_3) \in [0,1]^2$ represents the *ex-ante* support of the joint distribution of (r_2,r_3). Let risk attitude of firm 1 take a value of r_1^*. If firm 1 wins the auction, it realizes that the risk attitudes of firms 2 and 3 cannot be both below r_1^*:

$$\Pr\left(r_2<r_1^*, r_3<r_1^* | \text{Firm 1 wins a license}\right) = 0.$$

The reason for this is that if it were that $r_2<r_1^*$ and $r_3<r_1^*$, firm 1 would not have won the license because firms 2 and 3 would have bid higher than firm 1. Hence, the expected risk attitude of the aftermarket rival of firm 1 is the lowest of the two risk attitudes r_2 and r_3, *i.e.*, $\min(r_2,r_3)$ calculated over the remaining (shaded) area.

To see that there is a positive correlation between risk attitudes of firms that have won licenses, we calculate the expected risk attitude of a rival winning firm conditional on the risk attitude of firm 1 if the latter has won a license. This expected value equals to

$$E\left(\min(r_2, r_3)|\max(r_2, r_3)>r_1^*\right) = E\left(r_2|r_3>r_1^*, r_3>r_2\right) = \frac{1}{3}\left(1+\frac{r_1^{*2}}{1+r_1^*}\right),$$

and it is easy to see that it increases for all $r_1^* \in (0,1)$. Hence, a firm's own risk attitude has a second, indirect effect on its bid through (*i*) the *ex-post* positive correlation, and (*ii*) the aftermarket externality $\partial v/\partial r_2<0$. This indirect effect is what we call the *strategic* effect.

In the case of demand uncertainty and price competition, the strategic effect is negative (because the externality is negative) and, therefore, reinforces the risk attitude effect: a more risk-averse firm expects (while bidding in the auction) to compete in the aftermarket with a more risk-averse rival if it wins a license. Because a firm's risk attitude has a negative impact on the other firm's valuation, this further reduces the firm's incentive to bid high. Consequently, the monopoly result in the present example carries over to the oligopoly case.

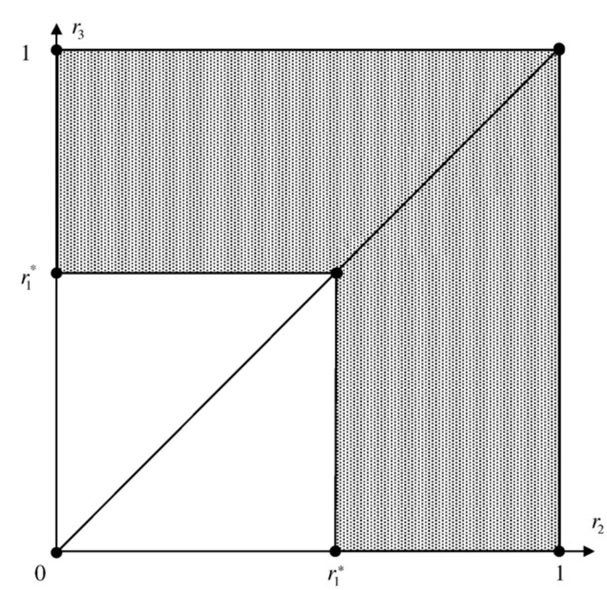

Fig. 1. Support (shaded) of the joint distribution of (r_2,r_3) conditional on $r_1=r_1^*$.

[7] Note that this is indeed a decreasing function so that we have a consistent solution. In addition, it can be shown that the second-order condition is satisfied in this example. The general treatment of the second-order condition is in Proposition 1 in Section 5.

When firms compete in quantities a la Cournot the situation is different. This can be illustrated by modifying our example so that firms compete in quantities and the profit of firm 1 is given by

$$\pi^1 \equiv \pi(q_1, q_2, \theta) = q_1(1 + \theta - q_1 - q_2).$$

Under the CARA specification (1), expected utility W^1 of firm 1 can be written as follows:

$$W^1 \equiv W(q_1, q_2, w, r_1) = U^1\left(\pi^1 - w - Dr_1 q_1^2/2\right).$$

Maximizing W^1 with respect to q_1 yields the following reaction function of firm 1:

$$q_1(q_2) = (1 - q_2)/(Dr_1 + 2).$$

Nash equilibrium quantities are then given by

$$q_1^* = \frac{Dr_2 + 1}{(Dr_1 + 2)(Dr_2 + 2) - 1} \text{ and } q_2^* = \frac{Dr_1 + 1}{(Dr_1 + 2)(Dr_2 + 2) - 1}.$$

Cournot equilibrium quantities are decreasing in a firm's own risk attitude, but – more importantly – are increasing in the risk attitude of the other winning firm with whom they are competing in the aftermarket. As a player becomes more risk-averse, his reaction curve shifts downwards and, due to the fact that quantities are strategic substitutes, a player's own quantity choice decreases, whereas the quantity of his rival increases.

The valuation of firm 1 for the license in the auction stage can easily be calculated to be:

$$v^1 \equiv v(r_1, r_2) = \frac{(Dr_2 + 1)^2 (Dr_1 + 2)}{2((Dr_1 + 2)(Dr_2 + 2) - 1)^2}.$$

This function has the following monotonicity properties:

$$\frac{\partial v}{\partial r_1} = -\frac{D(Dr_2 + 1)^2(1 + (Dr_1 + 2)(Dr_2 + 2))}{2((Dr_1 + 2)(Dr_2 + 2) - 1)^3} < 0,$$

$$\text{and } \frac{\partial v}{\partial r_2} = \frac{D(Dr_1 + 2)(Dr_2 + 1)(Dr_1 + 1)}{((Dr_1 + 2)(Dr_2 + 2) - 1)^3} > 0.$$

This implies that the risk attitude of a firm's rival has a positive effect on the firm's valuation, *i.e.*, the aftermarket externality now is positive. Consequently, in a risk-seekers' equilibrium, as a firm that is more risk-averse expects (due to *ex-post* positive correlation) to compete with another firm that is also more risk-averse, and because firms under Cournot competition prefer to compete with more risk-averse rivals, the indirect strategic effect is now positive, and it works against the risk attitude effect.

The relation between these two effects determines the condition for the existence of a monotone symmetric bidding equilibrium. When the strategic effect is small, the unique equilibrium is a risk-seekers' equilibrium in which players who are less risk-averse bid higher, and the equilibrium bidding function is decreasing in a firm's risk attitude. When the strategic effect is larger than the risk attitude effect *for some* values of risk attitudes, a risk-seekers' equilibrium fails to exist. Finally, when the strategic effect is larger than the risk attitude effect *for all* values of risk attitudes, not only the risk-seekers' equilibrium fails to exist, but another, risk-averse players' equilibrium appears. In such an equilibrium, the most risk-averse firms (or the least risk-seeking firms) submit the highest bids, and the bidding function monotonically increases in firms' risk attitude.

We now proceed to describe the general model that we consider in the main body of the paper, and show that the above properties do hold more generally. The main difference with the example presented here is that when uncertainty is not normally distributed and information about risk attitudes is not revealed after the auction, closed-form solutions cannot be obtained. We therefore will resort to the technique of Taylor expansions to arrive at solutions in the general model.

4. The model

We consider a market where $n \geq 2$ firms compete by simultaneously choosing a value of the strategic variable s. Depending on the market, we interpret s as either a price (strategic complements) or a quantity (strategic substitutes). The profit π^i of firm i is determined by the level of s that firm i chooses, and on the levels of s chosen by all other $(n-1)$ firms. In addition to this, we assume that the firms' market profitability is affected by a random shock θ, which can represent either uncertainty concerning market demand or industry-wide production cost. Thus, we write the market profit of firm i as $\pi^i = \pi^i(s_1, s_2, \ldots s_n, \theta)$.

We assume that π^i is symmetric in all s_j for $j \neq i$, and identical for all i. Hence, π^i can be written as $\pi^i = \pi(s_i, s_{-i}, \theta)$, where s_{-i} stands for choices of s of the firms others than i. To shorten notation, we denote the partial derivatives of π as follows:

$$\pi_i \equiv \partial\pi/\partial s_i,\ \pi_j \equiv \partial\pi/\partial s_j \text{ for } j \neq i,\ \pi_\theta \equiv \partial\pi/\partial\theta,\ \pi_{i,j} \equiv \partial^2\pi/\partial s_i \partial s_j,\ etc.$$

We use similar notations for partial derivatives of other functions.

The random shock θ is assumed to be uniformly distributed over the interval $[-\alpha, \alpha]$, where $\alpha \in (0,1)$ is a parameter.[8] If θ is a demand shock and firms compete in prices then $s = p$ and firms fulfill their random demands $q_i = q(p_i, p_{-i}, \theta)$. If, to the contrary, firms compete in quantities then $s = q$, prices that consumers pay for these quantities are $p_i = p(q_i, q_{-i}, \theta)$, and firms accept to sell their pre-determined quantities q_i at these random prices. Finally, if θ is an industry cost's shock then firms' profit function π can be written as $\pi(s_i, s_{-i}, \theta) = R_i(s_i, s_{-i}) - C(q_i(s_i, s_{-i}), \theta)$, where firms' revenues $R_i(s_i, s_{-i})$ and outputs $q_i(s_i, s_{-i})$ are determined by the nature of market competition (prices or quantities).

With respect to the uncertainty parameter θ we assume that the single-crossing property holds, *i.e.*, $\pi_{i,\theta}$ keeps its sign. This single-crossing property, which is sometimes referred to in this context as to the Principle of Increased Uncertainty (PIU) due to Leland (1972), states that the marginal profit $\pi_i \equiv \partial\pi/\partial s_i$ is monotonically increasing in θ.

Access to the market is limited to the firms that have obtained licenses to operate in the market. There are $N \geq (n+1)$ firms, and the government allocates n licenses in the multi-unit uniform-price auction, where n highest bids win licenses, and the winners pay a license fee equal to the highest non-winning bid. This uniform-price auction allows us to simplify the exposition of results while keeping the market stage competition quite general.

Firms differ in their attitude towards risk: some firms are more risk-averse than others. We assume that firm i has CARA utility function $U^i(\pi^i)$, given by Eq. (1). Risk types r_i, $i = 1, \ldots, N$, are identically distributed over the finite support $[\underline{r}, \bar{r}]$ in accordance with the joint distribution function $F(r_1, \ldots, r_N)$. Positive values of r_i correspond to risk-averse firms whereas negative values correspond to risk-seeking firms. A player's risk attitude is his type, and is private information in both the auction stage and the aftermarket stage of the game.

The expected utility of a firm i that gets a license to operate in the market at the auction price w, has a risk type r_i, sets a level s_i of the strategic variable, and competes with firms that set levels s_{-i} of the strategic variable, is denoted by W^i. It follows that

$$W^i \equiv W(s_i, s_{-i}, w, r_i) = \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} U(\pi(s_i, s_{-i}, \theta) - w, r_i) d\theta.$$

In order to ensure the existence, uniqueness and stability of the (Bayes–) Nash equilibrium in the aftermarket, the expected utility

[8] This formulation does not affect the generality of the model as any change in the distribution of θ can be modeled as a corresponding change of the profit function $\pi(s_i, s_{-i}, \theta)$.

function W must satisfy a stability requirement. We follow Bulow et al. (1985) and assume that $W_{i,i}+\sum_{j\neq i}W_{i,j}<0$ if $W_{i,j}>0$, and that $W_{i,i}-W_{i,j}<0$ for all $j\neq i$ if $W_{i,j}<0$.

In the auction stage, a firm i submits a bid b_i, which is based on its risk type r_i, and we denote a monotone symmetric equilibrium bidding function by $b(x)$, so that $b_i=b(r_i)$. We define Z to be the risk type of the firm that submits the nth highest bid amongst all $(n-1)$ firms other than firm i. Hence, if the bidding function is an increasing function $b^{(+)}(x)$, Z is the nth highest-order statistics amongst r_j, $j\neq i$; we denote the distribution of Z conditional on $r_i=x$ by $G^{(+)}(z|x)$. If, on the other hand, the bidding function $b^{(-)}(x)$ decreases, Z is the nth lowest-order statistics; its conditional distribution is denoted by $G^{(-)}(z|x)$. Thus,

$$G^{(\pm)}(z|x)=\Pr(Z<z|r_i=x).$$

The corresponding distribution density function is denoted by $g^{(\pm)}(z|x)$:

$$g^{(\pm)}(z|x)=dG^{(\pm)}/dz.$$

In the market stage, firm i maximizes its expected utility W with respect to its strategy s_i conditional on the fact that all its competitors (we index them by j) won the auction and all other firms (we index them by k) lost the auction. Suppose that Z takes a value z. Thus, if a firm i of type $r_i=x$ wins a license and pays an auction price $w=b(z)$, it chooses its Nash equilibrium strategy $s_i^*=s(x,z)$ where

$$s(x,z)=\arg\max_{s_i}E\left(W(s_i,s_{-i},w,r_i)|r_i=x,b(r_j)>b(z),b(r_k)\leq b(z)\right).$$

We denote by $V(x,z)$ the reduced-form expected utility of a winning firm i:

$$V(x,z)=E\left(W\left(s_i^*,s_{-i}^*,w,r_i\right)|r_i=x,b(r_j)>b(z),b(r_k)\leq b(z)\right).$$

This expression has its counterpart in the previous section in Eq. (7).

Suppose firm i of a risk type $r_i=x$ submits a bid $b_i=b(y)$ in the auction stage. If $b(y)<b(z)$, firm i looses the auction and gets its utility from the outside option which we normalize to zero. If, on the other hand, $b(y)>b(z)$, firm i gets the license at the auction price $w=b(z)$ and gets the expected utility $V(x,z)$. The *ex-ante* expected utility of a firm with risk type x which bids $b(y)$ is

$$\overline{V}(x,y)=\int_{b(z)<b(y)}V(x,z)dG^{(\pm)}(z|x).$$

The firm maximizes $\overline{V}(x,y)$ with respect to y, and the condition $x=\arg\max_y\overline{V}(x,y)$ implicitly defines the equilibrium bidding function $b(x)$, which we will also denote by $b^{(+)}(x)$ or $b^{(-)}(x)$ depending on whether it is increasing or decreasing.

We derive the exact solution to the model as an expansion power series with respect to the parameter α, *i.e.*, we solve the model for

$$s(x,z)=\sum_{t=0}^{\infty}\alpha^t s^{(t)}(x,z),\text{ and }b(x)=\sum_{t=0}^{\infty}\alpha^t b^{(t)}(x),$$

and analyze conditions for a risk-seekers' (with decreasing bidding function) and a risk-averse players' (increasing bidding function) equilibrium to exist or not to exist.[9] The first non-trivial term in those expansions will capture the main effects of firms' risk attitudes on their equilibrium bidding behavior and aftermarket strategy. It turns out that the first such term is of the second order, and in the next Section we analyze the second-order Taylor expansions of firms' exact strategies $s(x,z)=s^{(0)}(x,z)+\alpha^2s^{(2)}(x,z)$, and $b(x)=b^{(0)}(x)+\alpha^2b^{(2)}(x)$.

[9] We implicitly, therefore, make the standard assumption that the exact solution exists and can be represented by its Tailor expansion with positive radius of convergence.

5. Analysis

Without uncertainty, the aftermarket game has a unique symmetric Nash equilibrium $s(x,z)=s^{(0)}$, in which neither firms' risk attitudes nor auction price w play a role. For the sake of notational simplicity, we drop the arguments of the profit function π and its derivatives, and implicitly assume that they are evaluated at the point $(s_i,s_{-i},\theta)=(s^{(0)},s_{-i}^{(0)},0)$. The stability assumption requires that in case of strategic complements $\pi_{i,i}<-(n-1)\pi_{i,j}<0$, and in case of strategic substitutes $\pi_{i,i}<\pi_{i,j}<0$.

Let firm i have type x and let it win a license at auction price $w=b(z)$. By $H^{(\pm)}(x,z)$ we denote the expected risk attitude of an aftermarket competitor in case of an increasing and a decreasing bidding function $b^{(\pm)}(x)$, respectively:

$$H^{(-)}(x,z)\equiv E\left(r_j|r_i=x,r_j<z,r_k\geq z\right),\text{ and }H^{(+)}(x,z)\equiv E\left(r_j|r_i=x,r_j>z,r_k\leq z\right).$$

In other words, $H^{(-)}$ ($H^{(+)}$) is an expectation of risk attitudes of any of the $(n-1)$ winning firms j conditional on these risk types being below (above) z, risk types of all the other $(N-n)$ loosing firms k are above (below) z, and the risk type of firm i itself being equal to x. When risk types are affiliated (including the case of statistical independence), both partials of $H^{(\pm)}(x,z)$ are non-negative (see, *e.g.*, Krishna, 2002, pp. 272), *i.e.*, $H_x^{(\pm)}(x,z)\geq 0$ and $H_z^{(\pm)}(x,z)\geq 0$. Finally, by continuity we have $H^{(-)}(x,\underline{r})=\underline{r}$, and $H^{(+)}(x,\overline{r})=\overline{r}$.

In the following proposition, we derive bidding functions and existence conditions for a risk-seekers' equilibrium (*i.e.*, an equilibrium where players who are less risk-averse bid more and the equilibrium bidding function is decreasing in a firm's risk attitude), and a risk-averse players' equilibrium (*i.e.*, an equilibrium where players who are more risk-averse bid more and the equilibrium bidding function is increasing in a firm's risk attitude).

Proposition 1. *Let coefficients* $B^{(0)}$, $B^{(1)}$, $B^{(2)}$, *and function* $v^{(\pm)}(x,z)$ *be defined as follows:*

$$B^{(0)}\equiv\frac{1}{6}\left(\pi_{\theta,\theta}-\frac{(n-1)\pi_j\pi_{i,\theta,\theta}}{(\pi_{i,i}+\pi_{i,j}(n-1))}\right),\qquad B^{(1)}\equiv\frac{(n-1)\pi_j\pi_{i,j}\pi_{i,\theta}\pi_\theta}{3(\pi_{i,i}+(n-1)\pi_{i,j})(\pi_{i,i}-\pi_{i,j})}+\frac{1}{6}(\pi_\theta)^2,$$

$$B^{(2)}\equiv\frac{(n-1)\pi_j\pi_{i,i}\pi_{i,\theta}\pi_\theta}{3(\pi_{i,i}+(n-1)\pi_{i,j})(\pi_{i,i}-\pi_{i,j})},\text{ and }\quad v^{(\pm)}(x,z)\equiv\pi+\alpha^2\left(B^{(0)}-B^{(1)}x+B^{(2)}H^{(\pm)}(x,z)\right).$$

The auction stage has at most one risk-seekers' equilibrium and at most one risk-averse players' equilibrium, such that:

a) A risk-averse players' (+) or risk-seekers' (−) equilibrium exists if $0<\pm v_x^{(\pm)}$ and $0<\pm(v_x^{(\pm)}+v_z^{(\pm)})$ (sufficient conditions).
b) If a risk-averse players' or risk-seekers' equilibrium does exist, it is unique and given by $b^{(\pm)}(x)=v^{(\pm)}(x,x)$.
c) A risk-averse players' (+) or risk-seekers' (−) equilibrium exists *only* if $0\leq\pm v_x^{(\pm)}(x,x)$ and $0\leq\pm(v_x^{(\pm)}(x,x)+v_z^{(\pm)}(x,x))$ (necessary conditions).

In accordance with Proposition 1, there is no first-order effect of firms' risk attitudes on their bidding behavior. The reason for this is that the concept of attitude towards risk is essentially of the second-order, *i.e.*, it is determined by the second-order derivative of the utility function. In the first-order approximation, all firms behave as if they were all risk-neutral.

When $\alpha=0$, all firms get identical market profits π and, therefore, bid $b_i=\pi$ in the auction. The other part $\alpha^2(B^{(0)}-B^{(1)}x+B^{(2)}H^{(\pm)}(x,x))$ of the bidding function is the second-order effect of firms' risk attitudes. In accordance with Proposition 1, $v^{(\pm)}(x,z)$ appears to be a valuation function, *i.e.*, a certainty equivalent of the market-stage game, of a firm i, which has a risk type x and pays an auction price determined by a firm of type z. When $z=x$, *i.e.*, when both firms i and j have the same risk type and compete for only one remaining license, they bid their values $v^{(\pm)}(x,x)$. Hence, the auction price w must be equal to $v^{(\pm)}(x,x)$, which is, essentially, the first-order condition $b^{(\pm)}(x)=v^{(\pm)}(x,x)$.

The first term $B^{(0)}$ is of least importance as it captures the effect of aftermarket uncertainty on firms' valuations that is independent of firms' risk attitudes. This effect disappears when firms' profits are linear in uncertainty θ, *i.e.*, when $\pi_{\theta,\theta}=\pi_{i,\theta,\theta}=0$.

The second term $B^{(1)}x$ represents the effect of the firm's risk attitude on its valuation for the license for given types of the firm's competitors. Its derivative with respect to x, *i.e.*, $B^{(1)}$, is the direct, or risk attitude effect. This corresponds to inequality (5) in the example of Section 3. If $B^{(1)}>0$, a less risk-averse firm is willing to pay more for the license in order to compete in the aftermarket with a given set of competitors. If only one license were auctioned, *i.e.*, in a monopolistic aftermarket, the direct effect is the only effect that is present and the only equilibrium that exists is a risk-seekers' equilibrium.

The third term $B^{(2)}H^{(\pm)}(x,z)$ represents the aftermarket externality, *i.e.*, the effect of the expectation of competitors' risk attitudes r_j on the firm's valuation for a given type x of the firm. When $B^{(2)}<0$, the externality is negative, and it reinforces the risk attitude effect so that only a risk-seekers' bidding equilibrium can exist and, in fact, does exist. This is the case in the Bertrand example in Section 3; see inequality (6). When, on the other hand, $B^{(2)}>0$, the externality is positive and works against the direct risk attitude effect so that a risk-seekers' equilibrium may fail to exist. The expression for $B^{(2)}$ makes it possible to analyze special cases where the strategic effects are absent so that $\pi_j=0$, or the uncertainty does not effect marginal profit (as is the case when the only uncertainty is with respect to a fixed cost parameter) so that $\pi_{i,\theta}=0$. In these two special cases, $B^{(2)}=0$ so that the monopoly result generalizes and the equilibrium is always a risk-seekers' equilibrium.

The sensitivity of the externality $B^{(2)}H^{(\pm)}(x,z)$ with respect to its own type x, *i.e.*, $B^{(2)}H_x^{(\pm)}(x,z)$ is what we call the indirect, or strategic effect. When the externality is positive and sufficiently strong, and $H_x^{(\pm)}(x,z)$ is sufficiently large, not only does a risk-seekers' equilibrium fail to exist but also another, risk-averse equilibrium may appear (see Section 6). The strategic effect is absent if $B^{(2)}=0$, *i.e.*, when there is no externality. Alternatively, the strategic effect is absent if firm types are statistically independent, *i.e.*, when $H^{(\pm)}(x,z)$ is independent of x and $H_x^{(\pm)}(x,z)=0$. Hence, risk-averse players' equilibria may only exist when multiple licenses are auctioned and when firms' risk types are *ex-ante* correlated.

In light of the direct and strategic effects, the necessary ($0\le\pm v_x^{(\pm)}(x,x)$) and sufficient ($0<\pm v_x^{(\pm)}$) conditions in Proposition 1 can be restated as follows. If the sum of the direct and strategic effects is always strictly positive (negative) then a risk-averse players' (risk-seekers') equilibrium exists. If a risk-averse players' (risk-seekers') equilibrium does exist then the sum of the direct and strategic effects in equilibrium is not negative (positive).[10]

6. Cournot competition under demand uncertainty

In this Section and the next, we provide illustrations of Proposition 1 for regular oligopoly models when there is either demand or cost uncertainty. Section 3 has already provided an illustration for the case of differentiated Bertrand competition with three firms competing for two licenses and linear uncertain demand. Applying Proposition 1 to a more general setting with an arbitrary number of firms and licenses, and more general demand and cost functions, gives a similar result, namely that only a risk-seekers' equilibrium (an equilibrium where players who are less risk-averse bid more, and the equilibrium bidding function is decreasing in a firm's risk attitude) exists, resulting in the highest possible aftermarket prices being set.

This section proceeds with an analysis of Cournot competition as this analysis is richer than the analysis of the Bertrand model. We first show that even with linear demand and statistically independent risk attitudes, a risk-seekers' equilibrium may fail to exist, providing evidence that the positive welfare result for the quantity-setting monopoly case does not automatically generalize. Next, we show by means of an example that under Cournot competition, one actually may get a completely reverse result compared to the quantity-setting monopoly result.

Let firms compete in quantities, *i.e.*, $s_i=q_i$, with market (inverse) demand being given by $p=1+\theta-q_i-\sum_{j\neq i}q_j$. Normalizing production costs to zero, we write firms' profit function as $\pi=pq_i$. We denote Nash equilibrium outputs at $\theta=0$ by q_0: $q_0=1/(n+1)$. In this set-up one can derive the following expressions for the partial derivatives:

$$\pi_\theta = q_0,\ \pi_{\theta,\theta} = \pi_{i,\theta,\theta} = 0,\ \pi_{i,\theta} = 1,\ \pi_{i,i} = -2,\ \pi_j = -q_0,\ \pi_{i,j} = -1.$$

Applying Proposition 1 leads to the following expressions:

$$B^{(0)} = 0,\ B^{(1)} = \frac{3n-1}{6(n+1)^3}>0,\ B^{(2)} = \frac{2(n-1)}{3(n+1)^3}>0,\ \text{and}$$
$$v^{(\pm)}(x,z) = \pi + \frac{\alpha^2}{6(n+1)^3}\left(-(3n-1)x + 4(n-1)H^{(\pm)}(x,z)\right).$$

As $B^{(2)}>0$, the externality is positive and works against the direct effect. Intuitively, in a risk-seekers' equilibrium, firms suffer from competing with more risk-seeking firms, and this effect may actually be so strong that a risk-seekers' equilibrium does not exist even when firms' types are statistically independent. In order to show this, we take the risk types to be independently distributed over an interval $[\underline{r},\bar{r}]$ with a distribution function $F(r) = (r-\underline{r})^\gamma/(\bar{r}-\underline{r})^\gamma,\quad \gamma>0$. The conditional expectation function $H^{(-)}(x,z)$ for this distribution is:

$$H^{(-)}(x,z) = \int_{\underline{r}}^{z} r dF(r) / \int_{\underline{r}}^{z} dF(r) = \frac{\gamma z + \underline{r}}{\gamma + 1}.$$

This implies that a risk-seekers' equilibrium does not exist if the function

$$b^{(-)}(x) = v^{(-)}(x,x) = \pi + \frac{\alpha^2}{6(n+1)^3}\left(-(3n-1)x + 4(n-1)\frac{\gamma x + \underline{r}}{\gamma + 1}\right)$$

increases in x, *i.e.*, if $4(n-1)\gamma>(3n-1)(\gamma+1)$. This condition is satisfied when $n\ge4$ and $\gamma>(3n-1)/(n-3)$.

Hence, the monopoly result does not always generalize to the oligopoly case if firms compete in quantities, even with linear demand and independent risk types. The effect, however, appears in the current example only with more than four licenses and, therefore, with at least five competing firms in the auction. The positive (for welfare) selection effect of auctioning monopoly rights when firms choose quantities thus weakens and sometimes disappears when multiple licenses are auctioned.

6.1. When a risk-averse equilibrium exists

Until now, we focused on the question whether or not a risk-seekers' equilibrium exists. We now show that when the externality is sufficiently strong, and at the same time, firms' types are strongly correlated, a risk-averse players' equilibrium may emerge under Cournot competition. In such an equilibrium, we have the worst possible outcome, namely that the most risk-averse firms win the auctions and they choose the lowest possible production level, resulting in the highest possible market price.

In the example we take the following distribution F^* of firms' risk attitudes. Let there be $N=n+1$ firms, and let a macroeconomic fundamental such as an interest rate, oil price, the growth rate of the economy, a democracy index, *etc.*, β be distributed over the interval [0, 1] in accordance with an arbitrary twice differentiable distribution function $F_\beta(t)\equiv\Pr(\beta<t)$, $F_\beta\in C^2$ (the support [0, 1] has been chosen for exposition purposes only; the linearity of firms' equilibrium strategies guarantees that any other interval leads to the very same equilibrium conditions). Then, for any given realization of

[10] The other necessary ($0\le\pm v_x^{(\pm)}(x,x)\pm v_z^{(\pm)}(x,x)$) and sufficient ($0<\pm(v_x^{(\pm)}+v_z^{(\pm)})$) conditions in Proposition 1 are just necessary and sufficient conditions for the bidding function $b^{(\pm)}(x)=v^{(\pm)}(x,x)$ to be monotonically increasing (decreasing).

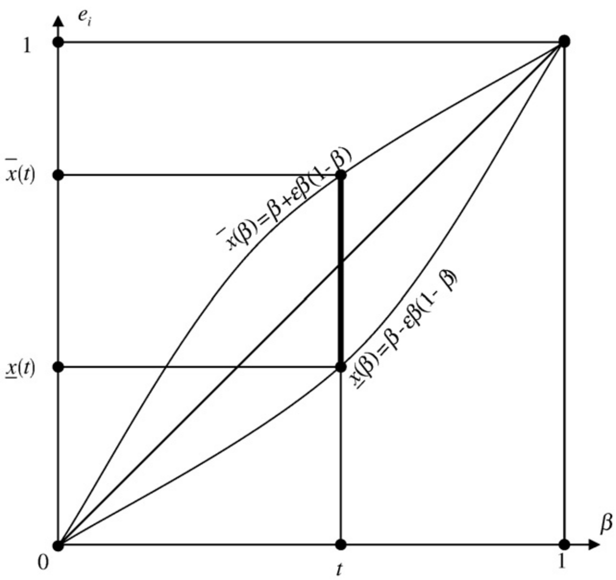

Fig. 2. Support of the conditional distribution $F_r(x|t)=\Pr(r_j<x|\beta=t)$.

β, let all N firms' risk types r_j be independently and uniformly distributed over the interval $[\underline{x}(\beta),\bar{x}(\beta)]$, where $\underline{x}(\beta)\equiv\beta-\varepsilon\beta(1-\beta)$, $\bar{x}(\beta)\equiv\beta+\varepsilon\beta(1-\beta)$, and $\varepsilon\in(0,1)$ is a parameter, *i.e.*, let the conditional distribution $F_r(x|t)\equiv\Pr(r_j<x|\beta=t)$ be

$$F_r(x|t)=\frac{x-\underline{x}(t)}{\bar{x}(t)-\underline{x}(t)}\ \text{for}\ x\in[\underline{x}(t),\bar{x}(t)].$$

The reason why we consider this specific distribution is that for small values of ε, if a firm i has a type x, the distribution of types of all other firms conditional on x is concentrated on a small neighborhood of x. Therefore, all firms that are competing in the aftermarket have approximately the same type, the Nash equilibrium is almost symmetric, and an equilibrium increasing bidding function can be analytically calculated in the limit when ε converges to zero. Fig. 2 shows the support (bold line) of the conditional distribution $F_r(x|t)$.

In the following lemma, we derive the conditional expectation function $H^{(\pm)}(x,z)$.

Lemma 1. *Let r_i be distributed in accordance with the distribution function F^*. Then, $H^{(\pm)}(x,z)$ for small ε can be written as follows:*

$$H^{(\pm)}(x,z)=\frac{nx+(n+2)z}{2(n+1)}\pm\frac{nx(1-x)}{(n+1)}\varepsilon+o(\varepsilon).$$

Let firms compete in the market with stochastic demand $p=\theta+Q^{-1/e}$ so that the stability requirement $e>1/n$ holds. Let firms marginal costs be constant and equal to $c>0$. Firms' profit function then is $\pi=(Q^{-1/e}+\theta-c)q_i$. The aftermarket Nash equilibrium output q_0 is then given by

$$q_0=((ne-1)/(nec))^e/n,$$

and the partials of the profit function at this output are:

$$\pi_\theta=q_0,\ \pi_{\theta,\theta}=\pi_{i,\theta,\theta}=0,\ \pi_{i,\theta}=1,\ \pi_j=-c/(ne-1),$$
$$\pi_{i,i}=-\frac{(2ne-(1+e))c}{neq_0(ne-1)},\ \text{and}\ \pi_{i,j}=-\frac{(ne-(1+e))c}{neq_0(ne-1)}.$$

This lead to the following expressions:

$$B^{(0)}=0,\ B^{(1)}=\frac{2(n-1)(ne-(1+e))+n(ne-1)}{6n(ne-1)}q_0^2,\ \text{and}\ B^{(2)}=\frac{(n-1)(2ne-(1+e))}{3n(ne-1)}q_0^2.$$

Using Lemma 1 and considering it in the limit when ε converges to zero yields the following expression for the valuation function:

$$v^{(\pm)}=\pi+\frac{\alpha^2q_0^2\left(-(2(n-1)(ne-(1+e))+n(ne-1))x+2(n-1)(2ne-(1+e))\frac{nx+(n+2)z}{2(n+1)}\right)}{6n(ne-1)}.$$

It is easy to see that $v_z^{(\pm)}(x,z)>0$. Thus, the only sufficient condition to be satisfied for the risk-averse players' equilibrium to exist is $v_x^{(\pm)}(x,z)>0$, that is:

$$0<-(2(n-1)(ne-(1+e))+n(ne-1))+2(n-1)(2ne-(1+e))\frac{n}{2(n+1)},\ \text{or}$$
$$e<\bar{e}(n)\equiv\frac{1}{n}\left(1+\frac{(n-1)(n^2+n+2)}{(n-1)(n^2+n-2)+2n^2}\right).$$

Hence, if $e\in(1/n,\bar{e}(n))$, then there exists an $\varepsilon>0$ and $\alpha>0$ so that the only equilibrium in the auction stage of the game is the risk-averse players' equilibrium. In such an equilibrium, the most risk-averse firms submit the highest bids, get the licenses, and compete in the aftermarket. They choose lower outputs (due to $\pi_{i,\theta}\pi_\theta>0$) and, therefore, market price is higher than if licenses were assigned by any other mechanism.

7. Cost uncertainty

We now briefly consider the implications of the general results obtained in Section 5 for the case where marginal costs depend on a common uncertain component. We show that in this case the monopoly results do not generalize for both price and quantity competition settings.

7.1. Differentiated Bertrand competition

Let firms compete in prices, *i.e.*, $s_i=p_i$, in differentiated Bertrand oligopoly with demand given by $q_i(p_i,p_{-i})=1-p_i+(\sum_{j\neq i}p_j)/(n-1)$. This case represents consumers with unit demand who only decide on from which firm to buy. Let the stochastic cost function be given by $(c-\theta)q_i$. In this case, a firms' profit function is given by $\pi=(p_i-c+\theta)q_i$. Denoting Nash equilibrium outputs at $\theta=0$ by q_0, we get the following expressions for the partials:

$$\pi_\theta=q_0,\ \pi_{\theta,\theta}=\pi_{i,\theta,\theta}=0,\ \pi_{i,\theta}=-1,\ \pi_{i,i}=-2,\ \pi_j=q_0/(n-1),\ \pi_{i,j}=1/(n-1).$$

This leads to the following expressions for $B^{(1)}$ and $B^{(2)}$:

$$B^{(1)}=\frac{2n-3}{6(2n-1)}q_0^2,\quad B^{(2)}=\frac{2(n-1)}{3(2n-1)}q_0^2.$$

Here the externality is positive and works against the direct effect. As in Section 6 under Cournot oligopoly, we show that the risk-seekers' equilibrium may fail to exist even when firms' types are statistically independent. Assuming that risk types are independently distributed over some interval $[\underline{r},\bar{r}]$ with the distribution function $F(r)=(r-\underline{r})^\gamma/(\bar{r}-\underline{r})^\gamma$, $\gamma>0$, we get $H^{(-)}(x,z)=(\gamma z+\underline{r})/(\gamma+1)$ and, consequently,

$$b^{(-)}(x)=v^{(-)}(x,x)=\pi+\frac{\alpha^2q_0^2}{6(2n-1)(\gamma+1)}(4(n-1)(\gamma x+\underline{r})-(2n-3)(\gamma+1)x).$$

The risk-seekers' equilibrium does not exist when the above function $b^{(-)}(x)$ does not decrease in x, *i.e.*, $\gamma>(2n-3)/(2n-1)$.

Hence, the monopoly result does not generalize to the oligopoly case if firms compete in prices and have uncertain production costs. The risk-seekers' equilibrium fails to exists even for $n=2$ licenses and uniform (and independent) types' distribution where $\gamma=1$.

7.2. Cournot competition

Let us go back to the case of Cournot competition with market (inverse) demand given by $p=1-q_i-\sum_{j\neq i}q_j$ and stochastic costs $(c-\theta)q_i$, as in the previous example. Firms' profit function in this case is given by $\pi=(p-c+\theta)q_i$. Denoting Nash equilibrium outputs at $\theta=0$ by q_0, we get $q_0=(1-c)/(n+1)$ and the following expressions for the partials:

$$\pi_\theta = q_0,\ \pi_{\theta,\theta} = \pi_{i,\theta,\theta} = 0,\ \pi_{i,\theta} = 1,\ \pi_{i,i} = -2,\ \pi_j = -q_0,\ \pi_{i,j} = -1.$$

One may see that these expressions exactly coincide with those from the Cournot example analyzed in the previous Section. Therefore, under quantity competition setting, whether there is a demand or cost uncertainty, the monopoly result does not generalize.

Comparing the results obtained for demand and cost uncertainty one may notice an asymmetry between different oligopoly settings. Under differentiated Bertrand competition, the sign of the aftermarket externality is negative with demand uncertainty, and is positive with cost uncertainty. Consequently, firms prefer competing with less risk-averse competitors if market demands are uncertain, whereas they prefer more risk-averse competitors if production costs are uncertain. To the contrary, under Cournot competition, the sign of the aftermarket externality is always negative. Consequently, firms always prefer competing with more risk-averse competitors irrespective of the source of uncertainty.

The reason for this asymmetry goes back to the single-crossing property of the effect of uncertainty on firms' profits. Under Cournot competition, if the uncertainty positively affects firms' profits, it also affects firms' *marginal* profits positively, in other words, $\pi_\theta\pi_{i,\theta}>0$ irrespective of the source of uncertainty. Consequently, the externality is always positive, and it always works against the risk attitude effect. Under differentiated Bertrand competition, a positive demand shock *increases* firms' marginal profits, *i.e.*, $\pi_\theta\pi_{i,\theta}>0$, whereas a positive cost shock *decreases* marginal profits of the firms, *i.e.*, $\pi_\theta\pi_{i,\theta}<0$. That is why the sign of the externality, the direction of the strategic effect, and consequently, whether the monopoly result generalizes or not in differentiated Bertrand competition, depends on whether the aftermarket uncertainty affects demand or cost.

8. Conclusion

In this paper, we have analyzed auctions where the winning firms get licenses to operate in an aftermarket, and play an oligopoly game. Profits in the marketplace are uncertain as there is uncertainty concerning future demand and/or future cost, and firms differ in their risk attitude. The difference between this model and a standard auction model with interdependent valuations (see, *e.g.*, Milgrom and Weber, 1982) is that in the current model firms' valuations only depend on risk attitudes of the winning firms; risk types of loosing firms only affect the distribution of risk attitudes of the firms that *do* play the market stage.

We have provided necessary and sufficient condition for a monotonic equilibrium to exist in a very general set-up. We have applied this set-up to the case of demand and cost uncertainty. We have shown that in oligopoly markets with demand uncertainty auctions perform worse compared to the monopoly result. The inefficiency of auctions for monopoly rights due to the selection of the least risk-averse firms (compared to a random allocation of licenses) carries over when the firms play a differentiated Bertrand game. The efficiency of auctions for monopoly rights when firms choose production levels does not generalize to an oligopoly setting (Cournot competition). The main reason for this is a strategic effect that works against the risk attitude effect: firms prefer to compete with competitors who are more risk-averse. We have also shown that when a firm's choice of output has a sufficient impact on price and when firms' risk types are *ex-ante* affiliated, the strategic effect becomes so large that the more risk-averse firms win licenses and set lower quantities (leading again to higher market prices) than when licenses were allocated by any other mechanism.

Appendix A

Proof of Proposition 1. The proof consists of two parts. In part 1, we derive firm's aftermarket strategies $s(x,z)$. In part 2, we derive firms' bidding functions and equilibrium necessary and sufficient existence conditions.

Part 1. For $\alpha=0$, the aftermarket equilibrium $s_i=s_i^{(0)}$ is symmetric, and firms' expected utility function W is

$$W(s_i, s_{-i}, w, r_i) = U(\pi(s_i, s_{-i}, 0)-w, r_i).$$

Maximizing this expression w.r.t. s_i is equivalent to maximizing the aftermarket profit $\pi(s_i,s_{-i},0)$. The first-order condition $\pi_i=0$ defines the unique symmetric Nash equilibrium strategy $s_i=s_i^{(0)}$ and the realized aftermarket profit π.

In the first-order approximation, $s_i=s_i^{(0)}+\alpha s_i^{(1)}$ and (taking into account $\pi_i=0$):

$$\pi(s_i, s_{-i}, \theta) = \pi + \alpha\pi_j \sum_{j\neq i} s_j^{(1)} + \theta\pi_\theta,\ \pi_i(s_i, s_{-i}, \theta) = \alpha\left(\pi_{i,i}s_i^{(1)} + \pi_{i,j}\sum_{j\neq i} s_j^{(1)}\right) + \theta\pi_{i,\theta},\ \text{and}$$

$$U_\pi(\pi(s_i, s_{-i}, \theta)-w, x) = U_\pi((\pi-w), x)\left(1-\left(\alpha\pi_j\sum_{j\neq i} s_j^{(1)} + \theta\pi_\theta\right)x\right).$$

Hence, the first-order condition $0=W_i(s_i^*,s_{-i}^*,w)$ which defines s_i^*, in the first-order approximation becomes:

$$\begin{aligned}0 = \overline{W}_i(x,z) &= \frac{\partial}{\partial s_i}E\left(\frac{1}{2\alpha}\int_{-\alpha}^{\alpha} U\left(\pi\left(s_i^*, s_{-i}^*, \theta\right)-w, r_i\right)d\theta | r_i = x, b(r_j)>b(z), b(r_k)\le b(z)\right)\\ &= \frac{1}{2\alpha}E\left(\int_{-\alpha}^{\alpha} \pi_i\left(s_i^*, s_{-i}^*, \theta\right)U_\pi\left(\pi\left(s_i^*, s_{-i}^*, \theta\right)-w, r_i\right)d\theta | r_i = x, b(r_j)>b(z), b(r_k)\le b(z)\right)\\ &= \alpha U_\pi((\pi-w), x)\left(\pi_{i,i}s_i^{(1)} + \pi_{i,j}E\left(\textstyle\sum_{j\neq i} s_j^{(1)} | r_i = x, b(r_j)>b(z), b(r_k)\le b(z)\right)\right).\end{aligned}$$

Thus, the following first-order condition must hold:

$$\pi_{i,i}s_i^{(1)} = -\pi_{i,j}(n-1)E\left(s_j^{(1)} | r_i = x, b(r_j)>b(z), b(r_k)\le b(z)\right). \tag{A.1}$$

In order to solve Eq. (A.1) for $s_i^{(1)}$, we first rewrite it for another firm j:

$$\begin{aligned}\pi_{i,i}s_j^{(1)} &= -\pi_{i,j}(n-1)E\left(s_l^{(1)} | l\neq j, r_j, \ldots\right)\\ &= -\pi_{i,j}\left((n-1)E\left(s_l^{(1)} | l\neq i, r_j, \ldots\right) + E\left(s_i^{(1)} | r_j, \ldots\right)-s_j^{(1)}\right)\end{aligned}$$

and, consequently,

$$(\pi_{i,i}-\pi_{i,j})s_j^{(1)} = -\pi_{i,j}(n-1)E\left(s_l^{(1)} | l\neq i, r_j, \ldots\right)-\pi_{i,j}E\left(s_i^{(1)} | r_j, \ldots\right).$$

Then, taking the expectation conditional on information that firm i has yields:

$$\begin{aligned}&(\pi_{i,i}-\pi_{i,j})E\left(s_j^{(1)} | j\neq i, r_i, \ldots\right)\\ &\quad = -\pi_{i,j}(n-1)E\left(E\left(s_l^{(1)} | l\neq i, r_j, \ldots\right) | j\neq i, r_i, \ldots\right)-\pi_{i,j}E\left(E\left(s_i^{(1)} | r_j, \ldots\right) | j\neq i, r_i, \ldots\right).\end{aligned} \tag{A.2}$$

By the law of iterative expectations:

$$\begin{aligned}&E\left(E\left(s_l^{(1)} | l\neq i, r_j, \ldots\right) | j\neq i, r_i, \ldots\right)\\ &\quad = E\left(s_j^{(1)} | j\neq i, r_i, \ldots\right) \text{ and } E\left(E\left(s_i^{(1)} | r_j, \ldots\right) | j\neq i, r_i, \ldots\right) = s_i^{(1)},\end{aligned}$$

so that Eq. (A.2) can be written as follows

$$(\pi_{i,i}-\pi_{i,j})E\left(s_j^{(1)} | j\neq i, r_i, \ldots\right) = -\pi_{i,j}(n-1)E\left(s_j^{(1)} | j\neq i, r_i, \ldots\right)-\pi_{i,j}s_i^{(1)}.$$

This equation together with Eq. (A.1) implies that

$$s_i^{(1)} = E\left(s_j^{(1)}|r_i = x, b(r_j)>b(z), b(r_k)\le b(z)\right) = 0.$$

Hence, risk attitudes have no first-order effect of on the aftermarket behavior.

In the next, second-order approximation, $s_i = s_i^{(0)} + \alpha^2 s_i^{(2)}$ and:

$$\pi(s_i, s_{-i}, \theta) = \pi + \theta\pi_\theta + \alpha^2\pi_j \sum_{j\ne i} s_j^{(2)} + \frac{1}{2}\theta^2\pi_{\theta,\theta},$$

$$\pi_i(s_i, s_{-i}, \theta) = \theta\pi_{i,\theta} + \alpha^2\left(\pi_{i,i}s_i^{(2)} + \pi_{i,j}\sum_{j\ne i} s_j^{(2)}\right) + \frac{1}{2}\theta^2\pi_{i,\theta,\theta}, \text{ and}$$

$$U_\pi(\pi(s_i, s_{-i}, \theta) - w, x) = U_\pi(\pi - w, x)(1 - \theta\pi_\theta x).$$

Hence, the first-order condition $0 = W_i^i(s_i^*, s_{-i}^*, w)$ in the second-order approximation becomes:

$$0 = E\left(\frac{1}{2\alpha}\int_{-\alpha}^{\alpha} \pi_i\left(s_i^*, s_{-i}^*, \theta\right) U_\pi\left(\pi\left(s_i^*, s_{-i}^*, \theta\right) - w, r_i\right) d\theta | r_i, \ldots\right)$$
$$= \alpha^2 U_\pi(\pi - w, x)\left(\pi_{i,i}s_i^{(2)} + \pi_{i,j}E\left(\sum_{j\ne i} s_j^{(2)}|r_i, \ldots\right) + \frac{1}{6}\pi_{i,\theta,\theta} - \frac{1}{3}\pi_{i,\theta}\pi_\theta x\right)$$

This can be written as

$$6\pi_{i,i}s_i^{(2)} = 2\pi_{i,\theta}\pi_\theta x - 6\pi_{i,j}(n-1)E\left(s_j^{(2)}|j\ne i, r_i \ldots\right) - \pi_{i,\theta,\theta} \quad \text{(A.3)}$$

In order to solve Eq. (A.3) for $s_i^{(2)}$, we first rewrite it for another firm j:

$$6\pi_{i,i}s_j^{(2)} = 2\pi_{i,\theta}\pi_\theta r_j - 6\pi_{i,j}(n-1)E\left(s_l^{(2)}|l\ne j, r_j \ldots\right) - \pi_{i,\theta,\theta}$$
$$= 2\pi_{i,\theta}\pi_\theta r_j - 6\pi_{i,j}\left((n-1)E\left(s_l^{(2)}|l\ne i, r_j \ldots\right) + E\left(s_i^{(2)}|r_j \ldots\right) - s_j^{(2)}\right) - \pi_{i,\theta,\theta}$$

so that

$$6(\pi_{i,i} - \pi_{i,j})s_j^{(2)} = 2\pi_{i,\theta}\pi_\theta r_j - 6\pi_{i,j}\left((n-1)E\left(s_l^{(2)}|l\ne i, r_j \ldots\right) + E\left(s_i^{(2)}|r_j \ldots\right)\right) - \pi_{i,\theta,\theta}.$$

In a similar way as in the first-order approximation, we write:

$$6(\pi_{i,i} - \pi_{i,j})E\left(s_j^{(2)}|j\ne i, r_i \ldots\right) = 2\pi_{i,\theta}\pi_\theta E(r_j|j\ne i, r_i \ldots) - 6\pi_{i,j}E\left(E\left(s_i^{(2)}|r_j \ldots\right)|j\ne i, r_i \ldots\right)$$
$$-6\pi_{i,j}(n-1)E\left(E\left(s_l^{(2)}|l\ne i, r_j \ldots\right)|j\ne i, r_i \ldots\right) - \pi_{i,\theta,\theta}$$

$$6(\pi_{i,i} - \pi_{i,j})E\left(s_j^{(2)}|j\ne i, r_i \ldots\right) = 2\pi_{i,\theta}\pi_\theta E(r_j|j\ne i, r_i \ldots) - 6\pi_{i,j}s_i^{(2)}$$
$$-6\pi_{i,j}(n-1)E\left(s_j^{(2)}|j\ne i, r_i \ldots\right) - \pi_{i,\theta,\theta}.$$

$$6(\pi_{i,i} + \pi_{i,j}(n-2))E\left(s_j^{(2)}|j\ne i, r_i \ldots\right) = 2\pi_{i,\theta}\pi_\theta E(r_j|j\ne i, r_i \ldots) - 6\pi_{i,j}s_i^{(2)} - \pi_{i,\theta,\theta}$$

This equation together with Eq. (A.3) yields the following expressions:

$$\begin{cases} s_i^{(2)} = \dfrac{((\pi_{i,i} + \pi_{i,j}(n-2))x - \pi_{i,j}(n-1)E(r_j|j\ne i, r_i \ldots))\pi_{i,\theta}\pi_\theta}{3(\pi_{i,i} - \pi_{i,j})(\pi_{i,i} + \pi_{i,j}(n-1))} - \dfrac{\pi_{i,\theta,\theta}}{6(\pi_{i,i} + \pi_{i,j}(n-1))} \\ E\left(s_j^{(2)}|j\ne i, r_i \ldots\right) = \dfrac{\pi_{i,\theta}\pi_\theta(\pi_{i,i}E(r_j|j\ne i, r_i \ldots) - \pi_{i,j}x)}{3(\pi_{i,i} - \pi_{i,j})(\pi_{i,i} + \pi_{i,j}(n-1))} - \dfrac{\pi_{i,\theta,\theta}}{6(\pi_{i,i} + \pi_{i,j}(n-1))} \end{cases} \quad \text{(A.4)}$$

This ends Part 1 of the proof.

Part 2. In the second-order approximation, firms' equilibrium profits $\pi_i^* \equiv \pi(s_i^*, s_{-i}^*, \theta)$, utilities $U(\pi_i^*, b(z), r_i)$ and expected utilities $V(x,z)$ are

$$\pi_i^* = \pi + \theta\pi_\theta + \alpha^2\pi_j\sum_{j\ne i} s_j^{(2)} + \frac{1}{2}\theta^2\pi_{\theta,\theta},$$

$$U\left(\pi_i^* - b(z), r_i\right) = \theta\pi_\theta + \alpha^2\left(\pi_j\sum_{j\ne i} s_j^{(2)} + \frac{1}{2}\theta^2\left(\pi_{\theta,\theta} - r_i(\pi_\theta)^2\right) - b^{(2)}(z)\right),$$

$$V(x,z) = E\left(W\left(s_i^*, s_{-i}^*, b(z)\right)|r_i = x, b(r_j)>b(z), b(r_k)\le b(z)\right)$$
$$= \alpha^2\left((n-1)\pi_j E\left(s_j^{(2)}|r_i = x, b(r_j)>b(z), b(r_k)\le b(z)\right) + \frac{1}{6}\left(\pi_{\theta,\theta} - r_i(\pi_\theta)^2\right) - b^{(2)}(z)\right)$$

Using Eq. (A.4), this can be written as follows:

$$V(x,z) = \alpha^2\left(B^{(0)} - B^{(1)}x + B^{(2)}H^{(\pm)}(x,z) - b^{(2)}(z)\right),$$

where $B^{(0)}$, $B^{(1)}$, and $B^{(2)}$ are defined as in Proposition 1.

Suppose that the bidding function is decreasing, *i.e.*, $db^{(2)}/dx < 0$. The *ex-ante* expected utility of a firm with risk type x which bids $b(y)$ is

$$\overline{V}(x,y) = \int_{y<z} V(x,z)dG^{(-)}(z|x).$$

Maximizing $\overline{V}(x,y)$ w.r.t. y yields the following first-order condition:

$$0 = \frac{\partial\overline{V}}{\partial y}(x,x) = -V(x,x)g^{(-)}(x|x).$$

This implies that $V(x,x) = 0$, *i.e.*,

$$b^{(2)}(x) = B^{(0)} - B^{(1)}x + B^{(2)}H^{(-)}(x,x) \text{ and } b^{(-)}(x) = \pi + \alpha^2 b^{(2)}(x)$$
$$= v^{(-)}(x,x).$$

This bidding function is a unique decreasing equilibrium bidding function if $b^{(2)}(x)$ is indeed decreasing. Hence, the condition $0 \ge v_x^{(-)}(x,x) + v_z^{(-)}(x,x)$ is the first necessary condition for the risk-seeking equilibrium to exist. The second necessary condition is the second-order condition:

$$0 \ge \frac{\partial^2\overline{V}}{\partial y^2}(x,x) = \frac{d}{dx}\left(\frac{\partial\overline{V}}{\partial y}(x,x)\right) - \left(\frac{\partial}{\partial x}\frac{\partial\overline{V}}{\partial y}\right)(x,x) = -\frac{\partial^2\overline{V}}{\partial x\partial y}(x,x)$$
$$= V_x(x,x)g^{(-)}(x|x),$$

which can be written as $0 \ge v_x^{(-)}(x,x)$.

Suppose now that, first $0 > v_x^{(-)} + v_z^{(-)}$ so that $b^{(-)}(x)$ is a strictly decreasing function, and second $0 > v_x^{(-)}$. In equilibrium, firm i of type x bids $b^{(-)}(x)$ and gets expected utility

$$\overline{V}(x,x) = \int_{x<z} V(x,z)dG^{(-)}(z|x) = \int_{x<z}\left(v^{(-)}(x,z) - v^{(-)}(z,z)\right)dG^{(-)}(z|x).$$

If, on the other hand, firm i of bids $b^{(-)}(y)$, it gets

$$\overline{V}(x,y) = \int_{y<z} V(x,z)dG^{(-)}(z|x) = \int_{y<z}\left(v^{(-)}(x,z) - v^{(-)}(z,z)\right)dG^{(-)}(z|x).$$

This deviation yields strictly lower utility level because

$$\overline{V}(x,y) - \overline{V}(x,x) = \int_y^x \left(v^{(-)}(x,z) - v^{(-)}(z,z)\right)dG^{(-)}(z|x)$$
$$= \int_y^x\left(\int_z^x v_x^{(-)}(t,z)dt\right)dG^{(-)}(z|x) < 0.$$

Hence, $b^{(-)}(x) = v^{(-)}(x,x)$ is indeed an equilibrium bidding function.

In exactly the same way, one may show that if a risk-averse players' equilibrium exists, it is unique and given by $b^{(+)}(x) = v^{(+)}(x,x)$. The necessary conditions for $b^{(+)}(x)$ to be a unique risk-averse players' equilibrium are $0 \le v_x^{(+)}(x,x) + v_z^{(+)}(x,x)$ and $0 \le v_x^{(+)}(x,x)$, and the sufficient conditions are $0 < v_x^{(+)} + v_z^{(+)}$ and $0 < v_x^{(+)}$. This ends the proof. ■

Proof of Lemma 1. We denote distribution functions as follows:

$$F_\beta(t) \equiv \Pr(\beta < t), \quad F_r(x,z) \equiv \Pr(r_i < x, r_k > z), \quad \text{and} \quad F_r(x,z|t) \equiv \Pr(r_i < x, r_k > z|\beta = t),$$

and the corresponding densities are

$$f_\beta(t) \equiv dF_\beta(t)/dx = 1, \quad f_r(x,z) \equiv dF_r(x,z)/dx, \quad \text{and} \quad f_r(x,z|t) \equiv dF_r(x,z|t)/dx.$$

Next, we define inverse functions

$$\underline{t}(x)\equiv\left((1+\varepsilon)-\sqrt{(1+\varepsilon)^2-4\varepsilon x}\right)/(2\varepsilon)\quad\text{and}\quad \bar{t}(x)\equiv\left(\sqrt{(1-\varepsilon)^2+4\varepsilon x}-(1-\varepsilon)\right)/(2\varepsilon),$$

so that $x\equiv\underline{x}(\bar{t}(x))$ and $x\equiv\bar{x}(\underline{t}(x))$ for any $x\in[0,1]$.

As $r_i\sim U(\underline{x}(\beta),\bar{x}(\beta))$, it follows that for all $x\in[\underline{x}(t),\bar{x}(t)]$:

$$F_r(x,z|t)=\begin{cases}(x-\underline{x}(t))(\bar{x}(t)-z)^{n-1}(\bar{x}(t)-\underline{x}(t))^{-n}, & \text{if } z\in(\underline{x}(t),\bar{x}(t))\\ (x-\underline{x}(t))/(\bar{x}(t)-\underline{x}(t)), & \text{if } z\in(0,\underline{x}(t))\end{cases}$$

and, therefore,

$$f_r(x,z|t)=\begin{cases}(\bar{x}(t)-z)^{n-1}(\bar{x}(t)-\underline{x}(t))^{-n}, & \text{if } z\in(t-\varepsilon t(1-t),\bar{x}(t))\\ 1/(\bar{x}(t)-\underline{x}(t)), & \text{if } z\in(0,t-\varepsilon t(1-t))\end{cases}$$

Hence,

$$f_\beta(t|x,z)\equiv\frac{f_r(x,z|t)f_\beta(t)}{f_r(x,z)} =\begin{cases}\dfrac{(\bar{x}(t)-z)^{n-1}f_\beta(t)}{(\bar{x}(t)-\underline{x}(t))^n f_r(x,z)}, & \text{if } t\in(\max(\underline{t}(x),\underline{t}(z)),\min(\bar{t}(x),\bar{t}(z)))\\ \dfrac{f_\beta(t)}{(\bar{x}(t)-\underline{x}(t))f_r(x,z)}, & \text{if } t\in(\min(\bar{t}(x),\bar{t}(z)),\bar{t}(x))\end{cases}$$

We define $\tilde{H}^{(+)}(x,z,t)\equiv E(r_k|r_i=x,r_l=z<r_k,\beta=t)$ and consider two cases.

a) When $z\geq x,\tilde{H}^{(+)}(x,z,t)=\frac{1}{2}(\bar{x}(t)+z)$ for $t\in(\underline{t}(z),\bar{t}(x))$. Hence, $H^{(+)}(x,z)$ can be written as

$$H^{(+)}(x,z)=E_\beta\left(\tilde{H}^{(+)}(x,z,\beta)|r_i=x,r_l=z<r_k\right)=P_1(x,z)/Q_1(x,z),\ \text{where}$$

$$P_1(x,z)\equiv f_r(x,z)\int_{\underline{t}(z)}^{\bar{t}(x)}\frac{1}{2}(\bar{x}(t)+z)f_\beta(t|x,z)dt=\int_{\underline{t}(z)}^{\bar{t}(x)}\frac{(\bar{x}(t)+z)(\bar{x}(t)-z)^{n-1}f_\beta(t)}{2(\bar{x}(t)-\underline{x}(t))^n}dt,$$

and

$$Q_1(x,z)\equiv f_r(x,z)\int_{\underline{t}(z)}^{\bar{t}(x)}f_\beta(t|x,z)dt=\int_{\underline{t}(z)}^{\bar{t}(x)}\frac{(\bar{x}(t)-z)^{n-1}f_\beta(t)}{(\bar{x}(t)-\underline{x}(t))^n}dt.$$

b) When $z\leq x,\tilde{H}^{(+)}(x,z,t)=\frac{1}{2}(\bar{x}(t)+z)$ for $t\in(\underline{t}(x),\bar{t}(z))$ and $\tilde{H}^{(+)}(x,z,t)=t$ for $t\in(\bar{t}(z),\bar{t}(x))$. Hence, $H^{(+)}(x,z)$ can be written as

$$H^{(+)}(x,z)=E_\beta\left(\tilde{H}^{(+)}(x,z,\beta)|r_i=x,r_l=z<r_k\right)=P_2(x,z)/Q_2(x,z),\ \text{where}$$

$$P_2(x,z)\equiv\int_{\underline{t}(x)}^{\bar{t}(z)}\frac{(\bar{x}(t)+z)(\bar{x}(t)-z)^{n-1}f_\beta(t)}{2(\bar{x}(t)-\underline{x}(t))^n}dt+\int_{\bar{t}(z)}^{\bar{t}(x)}\frac{tf_\beta(t)}{\bar{x}(t)-\underline{x}(t)}dt$$

and

$$Q_2(x,z)=\int_{\underline{t}(x)}^{\bar{t}(z)}\frac{(\bar{x}(t)-z)^{n-1}f_\beta(t)}{(\bar{x}(t)-\underline{x}(t))^n}dt+\int_{\bar{t}(z)}^{\bar{t}(x)}\frac{f_\beta(t)}{\bar{x}(t)-\underline{x}(t)}dt.$$

In order to evaluate $H^{(+)}(x,z)$ and its partials for small values of ε we use the 3rd-order approximation $\sqrt{1+\varepsilon}=1+\frac{1}{2}\varepsilon-\frac{1}{8}\varepsilon^2+\frac{1}{16}\varepsilon^3+o(\varepsilon^3)$, so that

$$\sqrt{(1-\varepsilon)^2+4\varepsilon x}=1+(2x-1)\varepsilon+2x(1-x)\varepsilon^2+2x(1-x)(1-2x)\varepsilon^3+o(\varepsilon^3),\ \text{and}$$

$$\sqrt{(1+\varepsilon)^2-4\varepsilon x}=1-(2x-1)\varepsilon+2x(1-x)\varepsilon^2-2x(1-x)(1-2x)\varepsilon^3+o(\varepsilon^3).$$

Hence,

$$\bar{t}(x)=x+x(1-x)\varepsilon+x(1-x)(1-2x)\varepsilon^2+o(\varepsilon^2),\quad \bar{t}'(x)=1+o(1),$$

$$\underline{t}(x)=x-x(1-x)\varepsilon+x(1-x)(1-2x)\varepsilon^2+o(\varepsilon^2),\quad\text{and } \underline{t}'(x)=1+o(1),$$

For an arbitrary $\bar{\varepsilon}\in(0,1)$, the uniform convergence with respect to $\varepsilon\in(0,\bar{\varepsilon})$ of the limits

$$\lim_{x\to0}\frac{\bar{t}(x)}{x}=\lim_{x\to1}\frac{1-\underline{t}(x)}{1-x}=\frac{1}{(1-\varepsilon)},\quad\text{and}\quad \lim_{x\to0}\frac{\underline{t}(x)}{x}=\lim_{x\to1}\frac{1-\bar{t}(x)}{1-x}=\frac{1}{(1+\varepsilon)},$$

implies the following expressions for $\underline{t}$ and $\bar{t}$:

$$\bar{t}(x)=x+x(1-x)\varepsilon(1+(1-2x)\varepsilon+o(\varepsilon)),\quad\text{and}\quad \underline{t}(x)=x-x(1-x)\varepsilon(1-(1-2x)\varepsilon+o(\varepsilon)).$$

We consider cases $z\geq x$ and $z\leq x$ separately.

a) Let $z\geq x$. Using the above Tailor expansions yields

$$Q_1(x,z)=\frac{1}{n}\left(f_\beta(x)+\frac{n-1}{n+1}(f_\beta(x)(1-2x)+f'_\beta(x)x(1-x))\varepsilon\right),$$

$$\frac{\partial Q_1}{\partial x}(x,x)=\frac{\bar{t}'(x)(f_\beta(x)+(f'_\beta(x)x(1-x)-f_\beta(x)(1-2x))\varepsilon)}{2x(1-x)\varepsilon},$$

$$\frac{\partial Q_1}{\partial z}(x,x)=-\frac{1}{2x(1-x)\varepsilon}\left(f_\beta(x)+\frac{n-2}{n}f'_\beta(x)x(1-x)\varepsilon\right),$$

$$P_1(x,x)=xQ_1(x,x)+\frac{f_\beta(x)x(1-x)\varepsilon}{n+1},$$

$$\frac{\partial P_1}{\partial x}(x,x)=\frac{\bar{t}'(x)(f_\beta(x)+x(f_\beta(x)+f'_\beta(x)(1-x))\varepsilon)}{2(1-x)\varepsilon},\ \text{and}$$

$$\frac{\partial P_1}{\partial z}(x,x)=\frac{Q_1(x,x)}{2}-\frac{1}{2(1-x)\varepsilon}\left(f_\beta(x)+\frac{(1-x)}{n}((n-1)f_\beta(x)+(n-2)f'_\beta(x)x)\varepsilon\right).$$

Hence,

$$H^{(+)}(x,x+0)=x+\frac{n}{n+1}x(1-x)\varepsilon,$$

$$H_x^{(+)}(x,x+0)=\frac{n}{2(n+1)},\quad\text{and } H_z^{(+)}(x,x+0)=\frac{(n+2)}{2(n+1)}.$$

b) Let $z\leq x$. As $P_2(x,x)=P_1(x,x)$ and $Q_2(x,x)=Q_1(x,x)$, it follows that $H^{(+)}(x,x-0)=H^{(+)}(x,x+0)$. Similarly, as $\partial P_2/\partial x=\partial P_1/\partial x$, $\partial P_2/\partial z=\partial P_1/\partial z$, $\partial Q_2/\partial x=\partial Q_1/\partial x$, and $\partial Q_2/\partial z=\partial Q_1/\partial z$ at (x,x), it follows that $H_x^{(+)}(x,x-0)=H_x^{(+)}(x,x+0)$ and $H_z^{(+)}(x,x-0)=H_z^{(+)}(x,x+0)$.

Thus, $H^{(+)}(x,z)$ is continuously differentiable it at (x,x) function with the partial derivatives $H_x^{(+)}(x,x)=n/(2(n+1))+o(1)$ and $H_z^{(+)}(x,x)=(n+2)/(2(n+1))+o(1)$, and, therefore, can be written as

$$H^{(+)}(x,z)=\frac{nx+(n+2)z}{2(n+1)}+\frac{nx(1-x)}{(n+1)}\varepsilon+o(\varepsilon).$$

The expression for $H^{(-)}$ immediately follows from $H^{(-)}(x,z)=1-H^{(+)}(1-x,1-z)$. ■

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