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Cycles and multiple equilibria in the market for durable lemons*

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Summary. The paper investigates the nature of market failure in a dynamic version of Akerlof (1970) where identical cohorts of a durable good enter the market over time. In the dynamic model, equilibria with qualitatively different properties emerge. Typically, in equilibria of the dynamic model, sellers with higher quality wait in order to sell and wait more than sellers of lower quality. The main result is that *for any distribution of quality* there exist an *infinite number* of cyclical equilibria where *all goods are traded* within a certain number of periods after entering the market.

Keywords and Phrases: Dynamic trading, Asymmetric information, Entry, Durable goods.

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1 Introduction

Since the pioneering work of Akerlof (1970) it is a commonplace to argue that asymmetric information is an important source of market failure in competitive markets. The standard model of adverse selection considers a static market with atomistic agents whose valuations depend on quality. A standard result is that only low quality goods are traded (if at all) even if the buyers are willing to pay

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more than the reservation price of sellers for each individual quality (see also, Wilson, 1979, 1980). This so-called *lemons problem* affects a large spectrum of markets, including the classic example of second-hand car markets, insurance markets, labor markets, financial markets and, apparently, even the market for thoroughbred yearlings (Chezum and Wimmer, 1996). In many of the above cases, the good under consideration is a durable good.

Durability introduces two complicating factors that are difficult to analyze in a static model. First, goods not traded in any period can be offered for sale in the future and, in addition, new cohorts of potential sellers may enter the market over time. Janssen and Roy (1999, 2001) have investigated some of the issues that arise when durability is explicitly taken into account in a dynamic model. Janssen and Roy (2001) address the issue whether a *given stock of goods* can be traded over time. They show that in any dynamic competitive equilibrium all goods eventually will be traded. That paper makes use of the essential idea that durable goods have a use value in every period the good is owned. The main intuition for their result is that given the same sequence of intertemporal prices low quality sellers have less incentives to wait before selling (due to a lower use value of the good) compared to high quality sellers. Once certain (low) qualities are sold, however, only relatively high qualities remain in the market. Consumers can predict that sellers of different qualities will sort themselves into different time periods and, hence, they are willing to pay higher prices in later periods. The equilibrium is thus one in which higher qualities are sold in later periods at higher prices.

Janssen and Roy (1999) address the same issue in the context of markets where identical cohorts of goods enter the market over time. They restrict their analysis, however, to the case where the quality of cars entering the market is uniformly distributed. In such markets, the infinite repetition of the static equilibrium under adverse selection is an equilibrium of the dynamic model. In fact, it is the unique stationary equilibrium and also the only equilibrium where prices and average quality traded are (weakly) monotonic over time. They show that there exists *at least one* other equilibrium, however, where *all* goods are traded within finite time after they have entered the market. This equilibrium is cyclical in prices and quantities in the sense that once all goods are traded, prices and quantities will fall. Up to the moment all goods are sold, prices and expected quality monotonically increase.

In the present paper we generalize the model of Janssen and Roy (1999) and derive a number of additional results. First, we relax the assumption that in every period a cohort of *uniformly distributed* qualities enters the market. Instead, we allow for almost any arbitrary distribution to enter the market over time. Second, our results are stronger in the sense that we show the existence of an *infinite* number of equilibria, where *all* goods are traded within finite time after they have entered the market. Hence, there is a strong sense in which a coordination problem is present in such dynamic markets with asymmetric information. Finally, we show the extent to which the uniform distribution is special. It turns out that for distributions that have relatively little probability mass in the neighborhood of the static equilibrium it is impossible (if the discount factor is relatively small) to construct a dynamic equilibrium with monotonically increasing prices and quantities up to the moment everything is sold. We provide an example where this is the case. Hence, the equilibrium construction that is used to prove the existence of a cyclical equilibrium for the uniform distribution does not extend naturally to the class of all distributions.

The main economic insight this paper delivers is to provide a different perspective on the adverse selection problem. In the static Akerlof-Wilson model, the adverse selection problem manifests itself in the fact that relatively high quality goods cannot be traded despite the potential gains from trade. In the dynamic market for durable goods, the *lemons problem* is not so much the impossibility of trading relatively high quality goods, but rather that sellers with relatively high quality goods need to wait in order to trade.[1] So, the cost of waiting becomes an important factor in the welfare loss arising due to asymmetric information. Also, as there exists an infinite number of equilibria, there is a serious coordination problem present in dynamic markets with adverse selection.

Our specific model is as follows. We consider a competitive market for a perfectly durable good where potential sellers are privately informed about the quality of the goods they own. Each period, a cohort of sellers of equal size and with an identical, but arbitrary, distribution of quality enters the market. The demand side is modeled in the following simple way. Buyers are identical and demand one unit of the good. Moreover, for any given quality a buyer's willingness to pay is larger than the seller's reservation price. As buyers do not know the quality, their willingness to pay in a period equals the expected valuation of goods traded in that period. Moreover, there are more buyers than sellers in each period so that, in equilibrium, price equals expected valuation. Once traded, goods are not re-sold in the same market.[2]

The Akerlof-Wilson model can be considered the static version of our model. Adverse selection implies that in any equilibrium only certain ranges of relatively low qualities can be traded. The infinitely repeated version of a static equilibrium outcome is also an equilibrium in our dynamic model. Hence, the issue of existence of dynamic equilibria is easily resolved. In this dynamic equilibrium high quality goods remain unsold forever.

We concentrate on the existence of other equilibria with more interesting properties where prices and average quality traded fluctuate over time. We provide a characterization result saying that in all such equilibria the range of quali-

[1] There are certain situations in which the fact that a seller has waited for a long time might indicate low rather than high quality. This would be true, for example, when the buyers can inspect quality – high valuation buyers are more likely to inspect and select the relatively high quality houses – leaving unsold goods of relatively low quality for later periods (Taylor, 1999). A paper with a similar spirit is that of Vettas (1997). As stated earlier, our model is designed to understand the nature of the lemons problem and so we do not allow for any technology, which can directly modify the information structure.

[2] Our analysis bears some resemblance to that by Sobel (1991) of a durable goods monopoly where new cohorts of consumers enter the market over time. Unlike our framework, there is no correlation between the valuations of buyers and sellers in his model.

ties that are eventually traded in the market exceeds that in the stationary (static) outcome. Moreover, sellers of different qualities within each cohort of entrants separate themselves out over time where owners of goods with lower quality trade earlier and owners of higher quality goods wait longer. In order to highlight the waiting aspect of the adverse selection problem in dynamic markets, the main part of the analysis is devoted to proving the existence of an equilibrium where *every* potential seller entering the market trades within a certain finite number of periods after entering the market. We show in fact that an infinite number of these equilibria exist.

The main intuition for (and the driving force behind) the results is that buyers and sellers are interested in the use (consumption) value of the good as well as in its exchange value. We think this is true for durable goods like (used) cars and houses. Given a sequence of market prices (which are the same for every seller), sellers decide to sell at the moment when the market price at a particular moment is larger than the discounted sum of the use value of the good from that moment onwards. As the use value of low quality goods is lower than that of high quality goods, low quality sellers sell earlier (and at lower prices) than high quality sellers. Buyers are interested in buying the good as we maintain the assumption (typical of all adverse selection models in the spirit of Akerlof) that for any given quality, a buyer's use value exceeds the use value of the seller. Hence, market prices can be such that they are larger than the discounted sum of the seller's use value of the good, but lower than the discounted sum of the buyers' use value.

There are three important intertemporal factors in the market, which determine the market dynamics in all the non-stationary equilibria of our model. First, once a certain range of quality is traded, only sellers of higher quality goods are left in the market, which tends to improve the distribution of quality of potentially tradable goods in the future. Second, the entry of a new cohort of potential sellers with goods of all possible quality dilutes the average quality of potentially tradable goods, as they cannot be distinguished by buyers from higher quality sellers left over from the past. Finally, as time progresses and stocks of untraded goods accumulate from the past, the new cohort of traders entering the market in any period becomes increasingly less significant in determining the distribution of quality of tradable goods.[3]

Other recent literature[4] on adverse selection has focused on various processes (such as signaling and screening) through which the difficulties of trading under asymmetric information may be resolved and has emphasized the role of non-market institutions in this context (such as certification intermediaries and leasing). The present paper, in contrast, is motivated by a more basic issue which

[3] If there is no entry of sellers after the initial period, or equivalently, if buyers can distinguish the period of entry of sellers in the market, then only the first factor is relevant. In that case, it has been shown earlier for fairly general distributions of quality (see, Janssen and Roy (2001) that in *every* equilibrium all goods are traded in finite time. Vincent (1990) analyzes a dynamic auction game with similar features.

[4] See, for instance, Guha and Waldman (1997), Hendel and Lizzeri (1999a, b), Lizzeri (1999) and Waldman (1999).

also underlies the original Akerlof paper viz., the functioning of the price mechanism in a perfectly competitive market when traders have private information. We first want to understand the nature of market failures due to adverse selection before analyzing the role of institutions in mitigating these failures.

The paper is organized as follows. Section 2 sets out the model, the equilibrium concept and some preliminary results. Section 3 provides a characterization result. The main result of the paper relating to the existence of an infinite number of equilibria where all goods are traded within finite time after entry into the market are outlined in Section 4. Section 5 concludes with a short summary and a discussion of the use of some specific modeling assumptions. Proofs are contained in the Appendix.

2 The model

Consider a Walrasian market for a perfectly durable good whose quality, denoted by θ, varies between $\underline{\theta}$ and $\overline{\theta}$, where $0 < \underline{\theta} < \overline{\theta} < \infty$. Time is discrete and is indexed by $t = 1, 2, \ldots \infty$. Each time period t a set of sellers I_t enters the market and I is the set of all sellers, i.e., $I = \bigcup_{t=1}^{\infty} I_t$ and t_i is the period of entry of a seller $i \in I$. Each seller is endowed with one unit of the durable good of quality θ_i. Let the total Lebesgue measure of sellers from the set I_t who own a good of quality less than or equal to θ be a function $\mu\left(\{i \,|i \in I_t, \theta_i \leqslant \theta\}\right) \equiv \mu(\theta)$, which is independent of t. We assume that $\mu(\theta)$ is strictly increasing and absolutely continuous with respect to the Lebesgue measure.

The measure of all sellers who enter the market in each period is strictly positive, so $\mu\left(\overline{\theta}\right) > 0$. Each seller i knows the quality θ_i of the good he is endowed with and derives flow utility from ownership of the good until he sells it. Therefore, the seller's reservation price is the discounted sum of gross surplus due to ownership and we assume that it is exactly equal to θ_i. This implies that per period gross surplus is $(1 - \delta)\,\theta_i$.

Each time period t a set of buyers, with measure larger than $\mu\left(\overline{\theta}\right)$, enters the market. All buyers are identical and have unit demand. A buyer's valuation of quality θ is equal to $v\theta$, where $v > 1$. The buyers' valuation of the good is also based on the discounted sum of gross surplus due to ownership. As $v > 1$, under full information, a buyer's valuation exceeds the seller's. All buyers know the *ex ante* distribution of quality, but do not know the quality of the good offered by a particular seller. When a buyer buys a good he leaves the market forever. All players discount the future with common discount factor δ, $0 < \delta < 1$. They are risk neutral and rational agents.

We will denote the expected quality of the good from seller i conditional on the fact that he belongs to a certain subset $I' \subset I$ by $\eta\left(I'\right)$. This value is defined for all $I' \subset I$ such that $\mu\left(I'\right) > 0$ and it follows that

$$\eta\left(I'\right) \equiv \frac{1}{\mu(I')} \int_{i \in I'} \theta_i \, d\mu\left(I'\right).$$

In order to have an adverse selection problem we assume $vE(\theta) < \overline{\theta}$, where $E(\theta)$ is the unconditional expected quality of all goods, $E(\theta) = \eta(I_t) = \eta(I)$. This assumption implies that the static Akerlof-Wilson version of the model has a largest equilibrium quality, which we will denote by $\theta_S \in \left(\underline{\theta}, \overline{\theta}\right)$:

$$\theta_S = \max_{\theta} \left\{ \theta \left| v\eta\left(\{i \mid i \in I, \theta_i \in [\underline{\theta}, \theta]\}\right) = \theta \right.\right\}.$$

To simplify our analysis we introduce the following regularity assumption. Throughout this paper, we assume that this assumption holds. Basically, it assures that the distribution of quality is well-behaved for some left-neighborhood of θ_S.

Assumption 1. The measure function $\mu(\theta)$ is strictly increasing and absolutely continuous with respect to the Lebesgue measure on $[\theta_S - \varepsilon_\mu, \overline{\theta}]$ for some $\varepsilon_\mu > 0$; there exist numbers m_μ and M_μ such that $0 < m_\mu < \frac{\mu(\theta'')-\mu(\theta')}{\theta''-\theta'} < M_\mu$ for any θ' and θ''. Finally, the measure function $\mu(\theta)$ is differentiable at $\theta = \theta_S$ and $\mu'(\theta_S) = f \geqslant m_\mu > 0$.

Assumption 1 basically says that each quality from the range $[\underline{\theta}, \overline{\theta}]$ is represented in the market and guarantees that the measure function is sufficiently smooth above θ_S and stresses that the measure function is differentiable at θ_S. As it will be clear part of the proofs deals with the reciprocal of the measure density function at θ_S and Assumption 1 guarantees that it exists.

Given a sequence of market prices $\mathbf{p} = \{p_t\}_{t=1}^{\infty}$ each seller i chooses whether or not to sell and if he chooses to sell, the time period in which to sell. If he chooses not to sell his gross surplus is equal to θ_i and therefore his net surplus equals zero, while if he decides to sell in period $t \geqslant t_i$ his gross surplus is $\sum_{\tau=t_i}^{t-1}(1-\delta)\theta_i\delta^{\tau-t_i} + \delta^{t-t_i}p_t = \theta_i\left(1-\delta^{t-t_i}\right) + \delta^{t-t_i}p_t$ and, therefore, his net surplus equals to $s_i = \theta_i\left(1-\delta^{t-t_i}\right) + \delta^{t-t_i}p_t - \theta_i = (p_t - \theta_i)\delta^{t-t_i}$. The set of time periods in which it is optimal to sell for a seller i is given by

$$T_i(\mathbf{p}) \equiv \arg\max_{t \geqslant t_i} \{s_i \mid s_i \geqslant 0\} = \arg\max_{t \geqslant t_i} \left\{(p_t - \theta_i)\delta^{t-t_i} \mid (p_t - \theta_i) \geqslant 0\right\}.$$

If $p_t - \theta_i < 0$ for all $t \geqslant t_i$ then $T_i(\mathbf{p}) = \emptyset$.

Each potential seller i chooses a time period $\tau_i \in T_i$ in which to sell. Let $\boldsymbol{\tau} = \{\tau_i\}_{i \in I}$ be a set of all selling decisions. We will denote a set of the sellers who choose time period t for trade as J_t, and it follows that $J_t \equiv \{i \in I \mid \tau_i = t\}$. This generates a certain distribution of qualities over all time periods and the expected quality of the goods offered for sale in time period t is $\eta_t = \eta(J_t)$ when $\mu(J_t) > 0$.

In the sections that follow we will use the following additional notation. We denote by $\mu(x, y)$ the measure of sellers from I_t whose goods are of quality from the range $[x, y]$: $\mu(x, y) = \mu\left(\left\{i \mid i \in I_t, \theta_i \in [x, y]\right\}\right)$. It follows that $\mu(x, y) = \mu(y) - \mu(x)$ and $\mu(y, x) = -\mu(x, y)$. Then, $\eta(x, y)$ is used for the expected quality of goods from sellers who belong to I_t whose goods are of quality from the range $[x,y]$: $\eta(x,y) = \eta\left(\left\{i \,\middle|\, i \in I_t, \theta_i \in [x,y]\right\}\right)$. It follows that $\eta(x,y) = (\mu(x,y))^{-1} \int_x^y \theta \, d\mu$ for $\theta_S - \varepsilon_\mu \leqslant x < y \leqslant \overline{\theta}$, $\eta(x,x) = x$ and $\eta(y,x) = \eta(x,y)$.

The following lemma assures that $\eta(x,y)$ is continuous in its arguments.

Lemma 1. *For all $\theta_S - \varepsilon_\mu \leqslant x \leqslant y \leqslant \overline{\theta}$ the function $\eta(x,y)$ is a strictly increasing continuous function. Moreover, there exist numbers m_η and M_η such that $0 < m_\eta < \frac{\eta(x,y)-\eta(x,x)}{y-x} < M_\eta$ for $x < y$.*

A dynamic equilibrium is a sequence of prices and buying and selling decisions such that all players maximize their objectives, expectations are fulfilled and markets clear in every period. On the equilibrium path, buyers' expectations of quality in a period where a strictly positive measure of goods is offered for sale must equal the expected quality in that time period. As all buyers are identical, we assume that their expectations of quality in period t are symmetric and denoted by E_t.

Definition 1. *A dynamic equilibrium is described in terms of a sequence of prices $\mathbf{p} = \{p_t\}_{t=1}^{\infty}$, a set of selling decision $\boldsymbol{\tau} = \{\tau_i\}_{i \in I}$ and a sequence of buyers' quality expectations $\mathbf{E} = \{E_t\}_{t=1}^{\infty}$ such that:*

a) **Seller maximize:** $\tau_i \in T_i(\mathbf{p})$ *for all $i \in I$, i.e., seller i chooses time period τ_i to trade optimally.*
b) **Buyers maximize and market clear***: If $\mu(J_t) > 0$ then $p_t = vE_t$, i.e., if there is a strictly positive amount of trade in time period t, then each buyer earns zero net surplus so that he is indifferent between buying and not buying and market clears. If $\mu(J_t) = 0$ then $p_t \geqslant vE_t$, i.e., if zero measure of trade occurs in time period t then each buyer can earn at most zero net surplus and not buying is optimal for him.*
c) **Expectations are fulfilled when trade occurs***: If $\mu(J_t) > 0$ then $E_t = \eta_t$.*
d) **Expectations are reasonable even if no trade occurs:** *For all $t E_t \geqslant \underline{\theta}$.*

Given the set-up described above, conditions (a)–(c) are quite standard. Condition (d) is introduced for the formal reason that expected quality is not defined when no trade occurs. The condition says that even in periods in which (at most) zero measure of sellers intend to sell, buyers should believe that the expected quality is larger than the *a priori* lowest possible quality. This condition assures that autarky, i.e., no trade in any period, cannot be sustained in an equilibrium of the dynamic model. Given the condition, the willingness to pay, hence, the price in any period, is restricted from below by $v\underline{\theta}$ and sellers with low enough qualities prefer to sell against this price rather than not sell.

3 Characterization of equilibrium

We start the analysis characterizing the properties of any dynamic equilibrium. In part (a) of Proposition 1 we first argue that if a good of certain quality sells in period t, then all goods with lower qualities that have entered the market in and before period t will also sell in that period. This fact allows us to define for each period a marginal seller θ_t, which is the seller of the highest quality in period t, and the marginal surplus s_t, which is his surplus, i.e., $s_t = p_t - \theta_t$. This part of the Proposition 1 basically follows from the fact that the use value of low qualities is lower than that of high qualities.

Part (b) then argues that the marginal seller in any period makes non-negative net surplus, hence, all the other sellers in that period make strictly positive surplus.

Part (c) of Proposition 1 argues that the marginal seller in period t is indifferent between selling in period t and selling in the first future period in which a quality larger than his own quality is sold. Prices in that period will be higher, reflecting higher average quality, but the discounted surplus is such that the seller is indifferent.

The last part (d) of Proposition 1 says that the highest quality that will ever be sold in a dynamic equilibrium, if it exists, is either equal to $\overline{\theta}$ or it is such that the seller makes zero surplus. It is clear that if a seller makes zero net surplus, prices in all future periods cannot be higher as then this seller will have an incentive to wait and sell in that future period. This part also says that if the highest quality sold in a dynamic equilibrium makes strictly positive surplus, then it must be equal to $\overline{\theta}$.

Proposition 1. *Any dynamic equilibrium has the following properties.*

a) For any t $\exists\theta_t \in [\underline{\theta}, \overline{\theta}]$ such that $J_t = \{i \,|\theta_i \in [\underline{\theta}, \theta_t], t_i \leqslant t\}$, i.e., in every period t in which trade occurs the set of qualities traded is a range $[\underline{\theta}, \theta_t]$.
b) $s_t \equiv p_t - \theta_t \geqslant 0$, i.e., in every period t the marginal surplus is non-negative.
c) Let $\tilde{t}(t) = \min_{\tau>t} \{\tau \,|\theta_\tau > \theta_t\}$, i.e., $\tilde{t}(t)$ is the first period τ after t where $\theta_\tau > \theta_t$. Then $p_t - \theta_t = \delta^{\tilde{t}(t)-t}\left(p_{\tilde{t}(t)} - \theta_t\right)$.
d) Let $\hat{t}(t) = \min_t \arg\max_{\tau>t} \theta_\tau$. Then $\left(p_{\hat{t}(t)} - \theta_{\hat{t}(t)}\right)\left(\overline{\theta} - \theta_{\hat{t}(t)}\right) = 0$.

It is easily seen that the infinitely repeated outcome of the static model is a dynamic equilibrium of our model. Hence, existence of equilibrium is not really an issue. In the next section, we will show that in the dynamic model there are infinitely many other equilibria, each one starting from a certain neighborhood of the largest static equilibrium quality.

4 Equilibria trading all goods

We will now show that for any measure function $\mu(\theta)$ which satisfies Assumption 1 and for all generic values of the $\overline{\theta}$ there exist an infinite number of dynamic equilibria covering all qualities up to $\overline{\theta}$. As we already know that our model has at least one equilibrium, a general existence proof is trivial. That is why we use a constructive proof showing how to find an equilibrium sequence of marginal qualities that is such that all qualities up to $\overline{\theta}$ are traded.

Before we will go into the details of the analysis, we first introduce an important parameter. Assumption 1 allows us to define a parameter a, which describes the relation between the distribution of quality over the range $[\underline{\theta}, \theta_S]$ and the marginal distribution at θ_S itself:

$$a \equiv \frac{1}{\mu(\theta_S)} (v-1)\,\theta_S \frac{d\mu(\theta)}{d\theta}(\theta_s) = \frac{(v-1)\,\theta_S f}{\mu(\theta_S)}.$$

Obviously, a is strictly positive. We will provide an economic interpretation of the parameter a and argue that generically, it must be that $a < 1$. To this end, consider the surplus of the marginal seller in the static model as a function of θ:

$$s(\theta) \equiv p(\theta) - \theta = v\eta(\theta) - \theta = \frac{v}{\mu(\theta)} \int_{\underline{\theta}}^{\theta} \theta d\mu(\theta) - \theta,$$

and $\frac{ds}{d\theta}(\theta_S) = a - 1$. Hence, $a - 1$ can be interpreted as the way in which the surplus of the marginal seller changes in the neighborhood of the largest static equilibrium quality. Suppose then that $a > 1$. This would imply that $s > 0$ in some right neighborhood of θ_S that contradicts the assumption that θ_S is the highest static equilibrium quality. Finally, a is defined in terms of exogenous parameters and the case $a = 1$ can be said to be non-generic.

In the uniform case, we have $s(\theta) = \frac{v}{2}\underline{\theta} - \left(1 - \frac{v}{2}\right)\theta$ and $a = \frac{1}{2}v$. As in the uniform case adverse selection implies that $1 < v < 2$, the uniform distribution is a special case of the case when $a \in \left(\frac{1}{2}, 1\right)$. In Subsection 4.1 we will start with this simplest case, which generalizes the analysis in Janssen and Roy (1999). We show that one can construct a "monotonic" sequence of marginal qualities θ_t that are strictly increasing over time until all goods are sold. The main reason why the case $a \in \left(\frac{1}{2}, 1\right)$ is to be distinguished from other cases can be seen by looking at the following example. If we choose $\theta_1 = \theta_S$ then in the second period the measure of qualities above θ_S that are not yet sold is two times as high as the original measure. If $a \in \left(\frac{1}{2}, 1\right)$, the distribution of qualities in the second period is such that a new "second-period static equilibrium" emerges, which is larger than θ_S. As in the second period we can write for any $\theta_2 > \theta_S$: $s_2(\theta_2) = (2a - 1)(\theta_2 - \theta_S) + o((\theta_2 - \theta_S))$, it becomes possible to find $\theta_2 > \theta_1 = \theta_S$ close enough to θ_S such that $s_2 > 0$.

If $a < \frac{1}{2}$, however, it may not be possible to construct such a "monotonic" equilibrium and we show this by example. In Subsection 4.2 we show that dynamic equilibria nevertheless exist if $a < \tilde{a}(\delta)$, where $\tilde{a}(\delta)$ is some decreasing function of δ. The kind of equilibrium we obtain has marginal qualities θ_t strictly decreasing for some initial time periods after which they strictly increase until all goods are sold. The general theorem covering all values of a and δ is provided in Subsection 4.3. As the equilibrium construction here becomes quite complicated, Subsections 4.1 and 4.2 are also provided for didactical reasons.

The construction of equilibrium uses an "equilibrium sequence" which is defined below.

Definition 2. *An equilibrium sequence* $\Theta_T(U)$ *is a finite sequence of marginal qualities* θ_t *as functions of* θ_1, *the latter being defined over some range* $U = \left(\underline{\theta_1}, \overline{\theta_1}\right)$, *i.e.,* $\Theta_T(U) = \{\theta_t(\theta_1)\}_{t=1}^{T}$, *such that all equilibrium conditions in Definition 1 hold for all* $t = 1, \ldots, T-1$. *Moreover, for all* $\theta_1 \in U$ *the functions* $\theta_t(\theta_1)$ *and* $p_t = \upsilon\eta_t = p_t(\theta_1)$ *are continuous for all* $t = 1, \ldots, T$, *and* $\theta_T(\theta_1) > \theta_t(\theta_1)$ *for all* $t = 1, \ldots, T-1$.

It easy to see that there exists at least one equilibrium sequence as defined above, namely $\Theta_1\left(\left(\theta_S - \varepsilon_\mu, \theta_S\right)\right) = \{\theta_1\}$, where all the stated above conditions are trivially satisfied.

The main property of an equilibrium sequence we use is that if there is a dynamic equilibrium with marginal qualities $\{\theta_\tau\}_{\tau=1}^{\infty}$ such that for $\tau = 1, \ldots, T$ it can be described by a certain equilibrium sequence $\Theta_T(U) = \{\theta_t(\theta_1)\}_{t=1}^{T}$, then there is only one indifference equation, namely

$$p_T - \theta_T = \delta^{\left(\tilde{t}(T)-T\right)}\left(p_{\tilde{t}(T)} - \theta_T\right), \quad (1)$$

which relates prices p_τ and marginal qualities θ_τ for $\tau = T+1, \ldots \infty$ to prices p_k and marginal qualities θ_k for $k = 1, \ldots, T$. Intuitively, θ_T summarizes all the relevant properties of the sequence of marginal qualities up to time period T. Our aim, therefore, is to find an equilibrium sequence such that $\theta_T(\theta_1) = \overline{\theta}$ for some T and θ_1.

4.1 The case where $a > 1/2$

In this subsection we prove the existence of an increasing sequence $\{\theta_t\}_{t=1}^{T}$, where $\theta_T = \overline{\theta}$ when $a > \frac{1}{2}$. As the uniform distribution is a special case, the result obtained here shows to what extent the results obtained in Janssen and Roy (1999) can be generalized to allow for other types of distribution functions. The following theorem contains a statement of the formal result.

Theorem 1. *For any* $a \in \left(\frac{1}{2}, 1\right)$ *and for any generic value of* $\overline{\theta}$, *there exist an infinite number of dynamic equilibria such that all goods are sold within T periods after entering the market. The sequence* $\{\theta_t\}_{t=1}^{T}$ *is monotonically increasing.*

The proof consists of three steps. In Proposition 2 we prove that it is possible to construct an equilibrium sequence of an arbitrary length where marginal qualities $\{\theta_t\}$ are strictly increasing and very close to the static equilibrium quality θ_S. Under these circumstances the main indifference equation (1) takes the following form:

$$p_t - \theta_t = \delta\left(p_{t+1} - \theta_t\right). \quad (2)$$

In other words, the marginal seller in period t is just indifferent between selling in that period and in the next period. We will denote such monotonic equilibrium sequences as $\Theta^1(U)$ and call a dynamic equilibrium, which is based on them, a “dynamic equilibrium of type I”.

Proposition 2. *If $a \in \left(\frac{1}{2}, 1\right)$, then there exist an infinite number of $\Theta^1(U)$. Moreover, $\exists T_0$ such that for all $t > T_0$ $\exists U_t^0 = \left(\theta_1^0(t), \theta_S\right)$ and $\exists \Theta_t^1\left(U_t^0\right)$ such that:*

a) for all $\tau = 1, \ldots, t$ $\theta_\tau(\theta_1)$ is differentiable at $\theta_1 = \theta_S$ and $\theta_\tau(\theta_S) = \theta_S$;
b) for all $\theta_1 \in U_t^0$ $0 < \theta_t(\theta_1) - \theta_S < \varepsilon_\gamma \frac{1}{\delta} s_t(\theta_1)$, where $\varepsilon_\gamma = \frac{\delta}{2a-1}$.

Proposition 2 implies that if $a \in \left(\frac{1}{2}, 1\right)$, we can construct an equilibrium sequence of an arbitrarily long length t such that in period $t+1$ there will be more sellers with high quality ($\theta_i > \theta_S$) goods than the number of sellers with low quality ($\theta_i < \theta_S$). This allows us to expand the equilibrium sequence Θ_t for some more periods.

Next, in Proposition 3, we prove that when we are able to construct an equilibrium sequence of an arbitrary length where all marginal qualities belong to a certain neighborhood of θ_S, then we can expand it in such a way that the surplus of the last marginal quality θ_t could be made any value between 0 and $(\upsilon - 1)\theta_t$. More precisely, given any equilibrium sequence Θ_t with $0 < \theta_t(\theta_1) - \theta_S < \varepsilon_\gamma \frac{1}{\delta} s_t(\theta_1)$ we can construct another sequence Θ'_{t}, where $t' > t$, such that $\Theta_t \subset \Theta'_t$ and $p_t(\theta_1)$ covers the whole interval $(\theta_t(\theta_1), \upsilon\theta_t(\theta_1))$. The conditions under which the Proposition 3 holds are the same as the conclusion reached in Proposition 2. These conclusions are replicated here, as in later subsections we will also make use of it.

Proposition 3. *If there exist $\varepsilon_\gamma > 0$ and T_0 such that for all $t > T_0$ $\exists U_t^0 = \left(\theta_1^0(t), \theta_S\right)$ and $\exists \Theta_t\left(U_t^0\right)$ such that for all $\theta_1 \in U_t^0 \theta_t(\theta_1) - \theta_S < \varepsilon_\gamma \frac{1}{\delta} s_t(\theta_1)$, then for any $\varepsilon_S > 0$ and $\varepsilon_\theta > 0$ $\exists T_S$ such that for all $t \geqslant T_S$ $\exists U_t^S = \left(\theta_1^S(t), \theta_S\right) \subset U_t^0$ and $\exists \Theta_t\left(U_t^S\right)$ such that:*

a) for any $\theta_1 \in U_t^S$ $|\theta_t(\theta_1) - \theta_S| < \varepsilon_\theta$;
b) $s_t(\theta_S) = 0$ and $s_t\left(\theta_1^S\right) > (\upsilon - 1)\theta_t\left(\theta_1^S\right) - \varepsilon_S$.

Proposition 3 tells us that if we could trade goods for many time periods and, therefore, accumulate “high quality sellers”, then we can organize trade in such a way that in the last time period of the equilibrium sequence “almost” all sellers who prefer to sell in that period will have goods of quality very close to θ_S.

Finally, in Proposition 4 we prove that if we are able to trade goods along an equilibrium path from a certain range of qualities such that the price in the last period of the equilibrium sequence can be made any value between the marginal quality and buyer’s valuations of the marginal quality, then we can expand that equilibrium sequence in such a way that wider range of qualities could be traded with the same properties. Doing so, after a finite number of iterations we generically can construct an equilibrium sequence where $\theta_T = \overline{\theta}$, i.e., all goods are traded by period T.

Proposition 4. *If $\exists \theta^{(k)} \in [\theta_S, \overline{\theta})$ such that for any $\varepsilon_S > 0$ and $\varepsilon_\theta > 0$ $\exists T_S^{(k)}$ such that for all $t > T_S^{(k)}$ $\exists U_t^{(k)} = \left(\underline{\theta}_1(t,k), \overline{\theta}_1(t,k)\right)^5$ and $\exists \Theta_t\left(U_t^{(k)}\right)$ such*

[5] Here we don’t make a distinction between $\underline{\theta}_1 < \overline{\theta}_1$ and $\underline{\theta}_1 > \overline{\theta}_1$. All we need is $U_t^{(k)}$ to be a nonempty open set while $\underline{\theta}_1$ and $\overline{\theta}_1$ are its boundary points.

that $\left|\theta_t - \theta^{(k)}\right| < \varepsilon_\theta$, *for all* $\theta_1 \in U_t^{(k)}$, $0 \leqslant s_t\left(\underline{\theta}_1\right) < \varepsilon_S$ *and* $s_t\left(\overline{\theta}_1\right) > (v-1)\,\theta_t\left(\overline{\theta}_1\right) - \varepsilon_S$, *then either*

a) for any $\overline{\varepsilon_S} > 0$ *and* $\overline{\varepsilon_\theta} > 0$ $\exists\theta^{(k+1)} \in (v\theta^{(k)}, \overline{\theta}]$ *and* $\exists T_S^{(k+1)}$ *such that for all* $t > T_S^{(k+1)}$ $\exists U_t^{(k+1)} = \left[\underline{\theta}_1(t,k+1), \overline{\theta}_1(t,k+1)\right] \subset U_t^{(k)}$ *and* $\exists\Theta_t\left(U_t^{(k+1)}\right)$ *such that* $\left|\theta_t(\theta_1) - \theta^{(k+1)}\right| < \overline{\varepsilon_\theta}$ *for all* $\theta_1 \in U_t^{(k+1)}$, $0 \leqslant s_t\left(\underline{\theta}_1\right) < \overline{\varepsilon_S}$ *and* $s_t\left(\overline{\theta}_1\right) > (v-1)\,\theta_t\left(\overline{\theta}_1\right) - \overline{\varepsilon_S}$*; or*

b) $\exists\overline{\varepsilon_S} > 0$ *such that for any* T $\exists\overline{t} > T$, $\exists\tilde{\theta}_1\left(\overline{t}\right)$ *and* $\exists\Theta_{\overline{t}}\left(\tilde{\theta}_1\right)$ *such that* $\theta_{\overline{t}}\left(\tilde{\theta}_1\right) = \overline{\theta}$ *and* $s_{\overline{t}}\left(\tilde{\theta}_1\right) > \overline{\varepsilon_S}$.

Proposition 4 basically says that if we have constructed an equilibrium sequence for a sufficiently large number of periods, then we can either make sure that after some more time periods the next marginal quality can be chosen relatively far from the present marginal quality and such that all desirable properties are kept (case (a)), or we can reach $\overline{\theta}$ (case (b)). The last three propositions taken together give us a large part of the proof of Theorem 1.

4.2 The case of small a and δ

In this section, we construct an equilibrium sequence for the case when a and δ are small. We first provide an example showing why the analysis of the previous subsection does not continue to be valid. The example shows that when a and δ are small, there does not exist a $\theta_2 > \theta_1$ such that $s_2 \geqslant 0$ and such that θ_1 is indifferent between selling in period 1 and selling in period 2. The basic intuition exploited in the example is that when a is small, there are not enough sellers above the static equilibrium quality in period 2 to support high enough prices that allow these sellers to get positive surplus.

Example 1. Let us take $v = 1.2$, $\delta = 0.1$, $\underline{\theta} = 10$, $\overline{\theta} = 13$ and a measure function $\mu(\theta)$ such that $\mu(10) = 0$, $\mu'(\theta) = 101$ for $\underline{\theta} < \theta < 10.1$ and $\mu'(\theta) = 1$ for $10,1 < \theta < \overline{\theta}$. The static equilibrium quality for this case is unique and equals to $\theta_S = \sqrt{151.5} \approx 12.31$.

In any dynamic equilibrium we must have $\theta_1 \in [\underline{\theta}, \theta_S]$ (otherwise we would have had $s_1 < 0$). Figure 1 shows the graph of functions $X2 = \theta_2(X1)$ and $S2 = s_2(X1)$ where $X1 = \theta_1$.

We will now prove that if a is relatively small, particularly if $a < (1-\delta)^2$, then we are still able to construct infinitely many dynamic equilibria such that all goods from the range $[\underline{\theta}, \overline{\theta}]$ are traded. The equilibrium sequence is non-monotonic. Note that the parameter configuration analyzed here partially overlaps with the parameter configuration analyzed in the previous subsection. The result is formally stated in Theorem 2 below.

Theorem 2. *For any* $a < (1-\delta)^2$ *and for any generic value of* $\overline{\theta}$ *there exist an infinite number of dynamic equilibria such that all goods are sold in finite time after entering the market.*

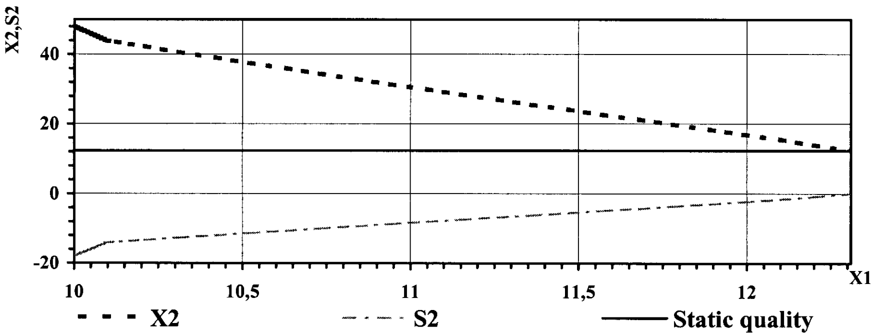

Figure 1. It is easy to see (and formally proof) that for any value of θ_1 we get $\theta_2(\theta_1)$ above θ_S and the surplus in the second period is negative

In order to prove this theorem we only need to show that when $a < (1-\delta)^2$ it is also possible to construct an equilibrium sequence of an arbitrary large length $t$where marginal qualities $\{\theta_t\}$ are very close to the static equilibrium quality θ_S. We will construct a sequence that is strictly decreasing for some time, $\theta_{\tau+1} < \theta_\tau$, and only the last marginal quality θ_T exceeds all previous ones. We denote such a sequence as "equilibrium sequence of type II" and write $\Theta^2(U)$.

In this case our indifference equation (1) becomes the following system:

$$p_\tau - \theta_\tau = \delta^{T-\tau}(p_t - \theta_\tau), \tau = 1, \ldots, T. \quad (3)$$

Proposition 5. *If $a < (1-\delta)^2$, then there exist an infinite number of $\Theta^2(U)$. Moreover, for $\forall \varepsilon_\gamma > 0$ $\exists T_0$ such that for all $t > T_0$ $\exists U_t^0 = \left(\theta_1^0(t), \theta_S\right)$ and $\exists \Theta_t^2\left(U_t^0\right)$ such that:*

a) for all $\tau = 1, \ldots, t$ $\theta_\tau(\theta_1)$ is differentiable at $\theta_1 = \theta_S$ and $\theta_\tau(\theta_S) = \theta_S$;
b) for all $\theta_1 \in U_t^0$ $0 < \theta_t(\theta_1) - \theta_S < \varepsilon_\gamma \frac{1}{\delta} s_t(\theta_1)$.

Note that the conclusions reached in Proposition 5 are identical to the conclusions reached in Proposition 2 so that we can make use of Proposition 3 and Proposition 4 to get the proof of Theorem 2.

4.3 The general case

Finally, we prove that for any value of a, we are able to construct infinitely many dynamic equilibria such that all goods in $[\underline{\theta}, \overline{\theta}]$ are traded. The structure of the corresponding equilibrium sequences becomes a mixture of the equilibrium sequences of type I and II.

Theorem 3. *For any generic value of $\overline{\theta}$, there exist an infinite number of dynamic equilibria such that all goods are sold in finite time after entering the market.*

Again, like in Subsection 4.2, the only thing we need to prove is that it is possible to construct an equilibrium sequence of an arbitrary large length where marginal qualities $\{\theta_t\}$ are very close to θ_S. This is the content of Proposition 6.

Proposition 6. *There exist an infinite number of $\Theta(U)$. Moreover, for $\forall\varepsilon_\gamma > 0$ $\exists T_0$ and $\exists k_m \geqslant 1$ such that for all $t > T_0$ $\exists U^0_{tk_m+1} = \left(\theta^0_1(tk_m+1), \theta_S\right)$ and $\exists\Theta_{tk_m+1}\left(U^0_{tk_m+1}\right)$ such that:*

a) for all $\tau = 1, \ldots, tk_m + 1$ $\theta_\tau(\theta_1)$ is differentiable at $\theta_1 = \theta_S$ and $\theta_\tau(\theta_S) = \theta_S$;
b) for all $\theta_1 \in U^0_{tk_m+1}$ $0 < \theta_{tk_m+1}(\theta_1) - \theta_S < \varepsilon_\gamma \frac{1}{\delta} s_{tk_m+1}(\theta_1)$.

The main difference with the related Proposition 2 and Proposition 5 is that here the equilibrium sequence constructed around θ_S is partly composed of increasing subsequences and partly composed of decreasing subsequences. Therefore, we need two indices (t and k_m) to keep track of the whole equilibrium sequence.

Note that the conclusions reached in Proposition 6 are again identical to the conclusions reached in Proposition 2 so that we can make use of Proposition 3 and Proposition 4 to get the proof of Theorem 3.

5 Discussion and conclusion

In this paper we have provided a different perspective on the way the adverse selection problem manifests itself in durable good markets, where entry takes place in the same market. In the static Akerlof-Wilson model, adverse selection results in high quality goods not being able to trade despite the potential gains from trade. The infinite repetition of this static equilibrium is also an equilibrium in the dynamic model where a durable good is traded in a competitive market. Our main result in this paper is that there are infinitely many other equilibria where all goods are sold within finite time after entering the market. In each of these dynamic equilibria, the marginal quality that is sold in the first period lies in a small neighborhood of the static equilibrium. This result holds true for all generic values of the parameters governing the behavior of buyers and sellers and the distribution of qualities in the population of sellers.

There are a few general principles that are important to obtain most of the result, namely that owners of a durable good enjoy the use and exchange value of the good, that the use value is increasing in the quality of the good and that owners, but not buyers, know the quality. The generality of these principles suggests that the results we obtain must hold in settings where some of the specific assumptions employed in the model are not satisfied. We discuss some of these assumptions below. The assumption about more buyers than sellers is used only to get a simple characterization of market prices and the resulting buyers' behavior. It assures that buyers are indifferent between buying in any period and not buying at all so that we are able to concentrate on the issue whether time can separate out sellers to sell in different periods. As buyers' surplus is zero in any period, the discount factor is also of no importance and we may allow buyers and sellers to have different discount factors.

In an accompanying paper, Janssen and Karamychev (2000) we study a continuous time version of the present model. As some of the technical complications do not arise when markets operate in continuous time, we are also able to relax the assumption about the buyers' risk neutrality and homogeneity, and perfect durability of the goods. When buyers are risk averse, have different valuations for a certain quality and the good is almost perfectly durable, a similar type of market dynamics occurs as we discovered in the present paper.

The assumption about a constant distribution of qualities entering the market over time is used for the technical reason that the equilibrium construction we employ has the market operating in a neighborhood of the static equilibrium for quite a few periods. In this way, the market is able to build up enough "stock" of high quality goods to eventually trade all goods. As the initial changes in market prices and marginal qualities are very small, introducing small changes in the distribution from period to period would complicate the analysis considerably.

Finally, the assumption that buyers leave the market after buying the good is used for the following reason. If buyers can re-sell the good they bought in the same market, future supply would come from different sources: low valuation sellers with high quality goods who have never bought in this market before and high valuation sellers with low quality goods that have previously bought the good in the same market. This would ask for an analysis where the interaction between different (primary and secondary) markets is studied. Stationary equilibria of such a model are analyzed in Hendel and Lizzeri (1999). Analyzing dynamic non-stationary equilibria is an interesting topic for further research.

Appendix

In the appendix we will use the following additional notation.

a) $y_\tau = \theta_\tau - \theta_{\tau-1}, \quad z_\tau = y_\tau - y_{\tau-1}, \quad \gamma_\tau = \frac{1}{\delta} s_\tau$ and $\varphi_{t-1} = \theta_{t-1}(v-1) - \gamma_{t-1}$; (A.1)

b) $F_\tau = \frac{\mu(\theta_{\tau-1}, \theta_\tau)}{y_\tau}$ and $K_\tau = \frac{\eta(\theta_{\tau-1}, \theta_\tau) - \eta(\theta_{\tau-1}, \theta_{\tau-1})}{y_\tau}$; (A.2)

c) $g(\theta) = \frac{\theta - v\eta(\underline{\theta}, \theta)}{\theta - \theta_S}$ for $\theta \neq \theta_S$, $g(\theta_S) = \lim_{\theta \to \theta_S} g(\theta) = 1 - a$ and $\overline{g} = \max_{\theta \in [\theta_S, \overline{\theta}]} |g(\theta)|$; (A.3)

d) $\dot{\theta}_\tau = \frac{d\theta_\tau}{d\theta_1}(\theta_S)$, $\dot{s}_\tau = \frac{ds_\tau}{d\theta_1}(\theta_S)$, $\dot{y}_\tau = \frac{dy_\tau}{d\theta_1}(\theta_S)$, $\dot{z}_\tau = \frac{dz_\tau}{d\theta_1}(\theta_S)$ and $\dot{p}_\tau = \frac{dp_\tau}{d\theta_1}(\theta_S)$.

Proof of Proposition 1. We prove all statements of the proposition sequentially.

a) Let us take any period t of positive amount of trade J_t, so that $\mu(J_t) > 0$, and take any $i \in J_t$. By Definition 1 we can write:

$$t \in \arg\max_{\tau \geqslant t_i} \left\{ (p_\tau - \theta_i)\, \delta^{\tau - t_i} \,|(p_\tau - \theta_i) \geqslant 0 \right\}.$$

This implies $(p_t - \theta_i)\, \delta^{t-t_i} \geqslant (p_\tau - \theta_i)\, \delta^{\tau - t_i}$ for all $\tau > t \geqslant t_i$. Now we take any $\theta < \theta_i$:

$$(p_t - \theta)\,\delta^{t-t_i} - (p_\tau - \theta)\,\delta^{\tau-t_i} \geqslant (\theta_i - \theta)\left(1 - \delta^{\tau-t}\right)\delta^{t-t_i} > 0$$

Hence, for all sellers with a good of quality less then θ_i who are still in the market in a certain period and have not yet traded it is optimal to trade in that period. Thus, we can define θ_t as $\theta_t = \sup_i \{\theta_i \,|i \in J_t\}$ and then it is easy to see that $J_t = \{i\,|\theta_i \in [\underline{\theta}, \theta_t]\,, t_i \leqslant t\}$. Finally, if $\mu(J_t) = 0$ for some t, then we set $\theta_t = \underline{\theta}$.

b) By Definition 1, $E_t \geqslant \underline{\theta}$ and $p_t \geqslant vE_t$ for all t so that $p_t \geqslant v\underline{\theta}$. Thus, if $\mu(J_t) = 0$ then $p_t \geqslant v\underline{\theta} > \underline{\theta} \equiv \theta_t$. If $\mu(J_t) > 0$, it is optimal for the marginal seller θ_t to trade in period t and a necessary condition is $p_t - \theta_t \geqslant 0$. So, $p_t \geqslant \theta_t$ for all t.

c) Suppose $p_t - \theta_t - \delta^{(\tilde{t}-t)}(p_{\tilde{t}} - \theta_t) = \sigma > 0$. Then, there exists a seller i of quality $\theta_i = \theta_t + \frac{1}{2}\sigma$ such that $\theta_t < \theta_i < \theta_{\tilde{t}}$ and $t_i \leqslant t$, i.e., he is in the market by period t. By definition of $\{\theta_t\}$, he will trade in period $\tilde{t}$. But this is not optimal as

$$\begin{aligned}&\delta^{(\tilde{t}-t_i)}\left(p_{\tilde{t}} - \left(\theta_t + \frac{1}{2}\sigma\right)\right) - \delta^{(t-t_i)}\left(p_t - \left(\theta_t + \frac{1}{2}\sigma\right)\right)\\ &= \delta^{t-t_i}\left(-\sigma + \frac{1}{2}\sigma\left(1 - \delta^{(\tilde{t}-t)}\right)\right) < 0.\end{aligned}$$

So, it is not possible that $p_t - \theta_t > \delta^{(\tilde{t}(t)-t)}\left(p_{\tilde{t}(t)} - \theta_t\right)$. A similar argument shows that it is not possible to have $p_t - \theta_t < \delta^{(\tilde{t}(t)-t)}\left(p_{\tilde{t}(t)} - \theta_t\right)$ either.

d) Suppose $\overline{\theta} - \theta_{\hat{t}} = \sigma > 0$. We will show that in this case $p_{\hat{t}} - \theta_{\hat{t}} = 0$. Suppose not, then it must be $p_{\hat{t}} - \theta_{\hat{t}} = \varepsilon > 0$. Let us take a seller i of quality $\theta_i = \theta_{\hat{t}} + \frac{1}{2}\min\{\varepsilon, \sigma\}$, so that $\theta_{\hat{t}} < \theta_i < \overline{\theta}$, and $t_i = \hat{t}$. By definition of $\{\theta_t\}$, he will never trade because $\theta_t \leqslant \max_{t \geqslant \hat{t}}\{\theta_t\} = \theta_{\hat{t}} < \theta_i$ for all $t \geqslant t_i$. If he had, however, traded in period $\hat{t}$ he would have got $p_{\hat{t}} - \theta_i = \varepsilon - \frac{1}{2}\min\{\varepsilon, \sigma\} \geqslant \frac{1}{2}\varepsilon > 0$. Hence, $p_{\hat{t}(t)} - \theta_{\hat{t}(t)} = 0$. □

Proof of Proposition 2. Using the fact that $\theta_\tau > \theta_{\tau-1}$ we express the expected quality sold in period τ in terms of $\mu\left(\theta_{\tau-1}, \theta_\tau\right)$ and $\eta\left(\theta_{\tau-1}, \theta_\tau\right)$:

$$\begin{aligned}\eta_\tau\left(\theta_{\tau-1}, \theta_\tau\right) &= \frac{\tau\eta\left(\theta_{\tau-1}, \theta_\tau\right)\mu\left(\theta_{\tau-1}, \theta_\tau\right) + \eta\left(\underline{\theta}, \theta_{\tau-1}\right)\mu\left(\underline{\theta}, \theta_{\tau-1}\right)}{\tau\mu\left(\theta_{\tau-1}, \theta_\tau\right) + \mu\left(\underline{\theta}, \theta_{\tau-1}\right)}, \\ &\quad \tau \geqslant 1,\ \theta_0 \equiv \underline{\theta}.\end{aligned} \tag{A.4}$$

Now we consider the indifference condition (2) with $p_\tau = v\eta_\tau$. It can be written as

$$\eta_\tau\left(\theta_{\tau-1}, \theta_\tau\right) - \eta_{\tau-1}\left(\theta_{\tau-2}, \theta_{\tau-1}\right) = \frac{1-\delta}{v\delta}s_{\tau-1}. \tag{A.5}$$

The main part of the proof is by induction. In the full version of the paper we first prove that if all the conditions to be proved, except $\theta_t > \theta_S$, are true for some $t > 2$, i.e., if $\exists U^0_{t-1} = \left(\theta^0_1(t-1), \theta_S\right)$ and $\exists\{\theta_\tau\}^{t-1}_{\tau=1}$ such that for all $\theta_1 \in U^0_{t-1}$:

a) θ_τ and s_τ are differentiable at $\theta_1 = \theta_S$ and $\theta_\tau > \theta_{\tau-1}$, $s_\tau > 0$; $\theta_\tau(\theta_S) = \theta_S$, $s_\tau(\theta_S) = 0$;

b) $p_\tau - \theta_\tau = \delta(p_{\tau+1} - \theta_\tau)$ for all $\tau = 1, \ldots, t-2$ and $\dot{s}_\tau < 0$, $\dot{y}_\tau < 0$, $\dot{z}_\tau < 0$;

c) $\dot{s}_\tau - \beta\dot{s}_{\tau-1} \leqslant 0$ for all $\tau = 1, \ldots, t-2$, where $\beta = \frac{2a-(1-\delta)}{2a\delta}$;

then $\exists U_t^0 \subset U_{t-1}^0$ such that (A.5) determines θ_t on $\theta_1 \in U_t^0$ and those conditions are also true for $t+1$:

a) θ_t and s_t are differentiable at $\theta_1 = \theta_S$ and $\theta_t > \theta_{t-1}$, $s_t > 0$, $\theta_t(\theta_S) = \theta_S$, $s_t(\theta_S) = 0$;

b) $p_{t-1} - \theta_{t-1} = \delta\left(p_t - \theta_{t-1}\right)$ and $\dot{s}_t < 0$, $\dot{y}_t < 0$, $\dot{z}_t < 0$;

c) $\dot{s}_t - \beta\dot{s}_{t-1} < 0$, where $\beta = \frac{2a-(1-\delta)}{2a\delta}$, and $\theta_t - \theta_S < \varepsilon_\gamma \frac{1}{\delta} s_t$, where $\varepsilon_\gamma = \frac{\delta}{2a-1}$.

Next, we show that there exist θ_1 and θ_2 such that those conditions are satisfied. Finally, we show that for some T_0 we get $\theta_{T_0}(\theta_1) > \theta_S$ and, therefore, $\theta_t(\theta_1) > \theta_S$ for all $t > T_0$ and all $\theta_1 \in U_t^0 \subset U_{T_0}^0$. □

Proof of Proposition 3. The indifference equation (2) can be rewritten as $v\eta_t\left(\theta_{t-1}, \theta_t\right) = \frac{1}{\delta}s_{t-1} + \theta_{t-1}$. Function $\eta_t\left(\theta_{t-1}, \theta_t\right)$ strictly increases w.r.t. θ_t, hence, there exists an inverse function which determines θ_t as a function of θ_{t-1} and $\frac{1}{\delta}s_{t-1}$, $\theta_t = \theta_t\left(\theta_{t-1}, \frac{1}{\delta}s_{t-1}\right)$. This function is defined for all θ_1 as long as $v\eta_t\left(\theta_{t-1}(\theta_1), \overline{\theta}\right) \geqslant \frac{1}{\delta}s_{t-1}(\theta_1) + \theta_{t-1}(\theta_1)$. Using (A.4) we can write:

$$v\frac{t\eta\left(\theta_{t-1}, \theta_t\right)\mu\left(\theta_{t-1}, \theta_t\right) + \eta\left(\underline{\theta}, \theta_{t-1}\right)\mu\left(\underline{\theta}, \theta_{t-1}\right)}{t\mu\left(\theta_{t-1}, \theta_t\right) + \mu\left(\underline{\theta}, \theta_{t-1}\right)} = \gamma_{t-1} + \theta_{t-1},$$

and

$$vy_t^2K_t + y_t\varphi_{t-1} - \frac{\mu\left(\underline{\theta}, \theta_{t-1}\right)}{tF_t}\left(\gamma_{t-1} + g\left(\theta_{t-1}\right)\left(\theta_{t-1} - \theta_S\right)\right) = 0, \qquad \text{(A.6)}$$

where F_t, K_t and g were defined in (A.2) and (A.3).

Now let us take any $\varepsilon_S > 0$, $\varepsilon_\theta > 0$ such that $\varepsilon_\theta < \min\left\{\varepsilon_\gamma, \frac{1}{2}\left(\overline{\theta} - \theta_S\right)\right\}$, $\varepsilon > 0$ such that $\varepsilon < \frac{1-\delta}{v\delta}\min\left\{1, \frac{\varepsilon_S}{2(v-1)\overline{\theta}}, \frac{\varepsilon_\theta}{3\overline{\theta}}\right\}$, $\varepsilon_1 > 0$ such that $\varepsilon_1 < \frac{1}{2}\min\left\{\varepsilon_S, (v-1)\theta_S\right\}$ and T such that

$$T > \frac{\left(1 + \overline{g}\varepsilon_\gamma\right)\mu\left(\underline{\theta}, \overline{\theta}\right)}{m_\mu}\max\left\{\frac{1}{\delta\varepsilon\varepsilon_1}, \frac{4vM_\eta(v-1)\overline{\theta}}{\varepsilon_1^2}, \frac{4\left(\overline{\theta} - \underline{\theta}\right)}{m_\eta\left(\overline{\theta} - \theta_S\right)^2}\right\}.$$

By the assumption of the Proposition 3, for that T there exist $U_T^0 = \left(\theta_1^0(T), \theta_S\right)$ and $\Theta_T\left(U_T^0\right)$. Now we take a subset $\hat{U}_T^0 = \left(\hat{\theta}_1^0(T), \theta_S\right) \subset U_T^0$ such that for all $\theta_1 \in \hat{U}_T^0$ $\theta_1 - \theta_S + \frac{1}{\delta}s_1(\theta_1) < \frac{1}{3}\varepsilon_\theta$, $\theta_T - \theta_S < \varepsilon_\theta$, $\varphi_T(\theta_1) > \varepsilon_1$ (it is always possible as $\varphi_T(\theta_S) = (v-1)\theta_S > \varepsilon_1$) and $\max\limits_{\tau=1,\ldots,T}\sup\limits_{\theta_1 \in \hat{U}_T^0} s_\tau(\theta_1) < \frac{\varepsilon_\theta}{3}\frac{\delta}{(1-\delta)(T-1)}$.

Now we will prove that if $\varphi_{t-1} > \varepsilon_1$ for all $\theta_1 \in \hat{U}_T^0$ and some $t \geqslant T+1$ then there exist functions θ_t and s_t such that $y_t = \theta_t - \theta_{t-1}$ is determined by (A.6) and $s_t > s_{t-1}\left(\frac{1}{\delta} - \varepsilon\right) > 0$. First, we prove the existence of θ_t showing that $v\eta_t\left(\theta_{t-1}, \overline{\theta}\right) \geqslant \frac{1}{\delta}s_{t-1} + \theta_{t-1}$:

$$\begin{aligned} v\eta_t\left(\theta_{t-1},\overline{\theta}\right) - \\ \frac{1}{\delta}s_{t-1}+\theta_{t-1} &\geqslant v\left(\frac{t\eta\left(\theta_{t-1},\overline{\theta}\right)\mu\left(\theta_{t-1},\overline{\theta}\right)+\eta\left(\underline{\theta},\theta_{t-1}\right)\mu\left(\underline{\theta},\theta_{t-1}\right)}{t\mu\left(\theta_{t-1},\overline{\theta}\right)+\mu\left(\underline{\theta},\theta_{t-1}\right)}-\theta_{t-1}\right) \\ &> v\frac{tm_\eta m_\mu\left(\overline{\theta}-\theta_{t-1}\right)^2-\left(\overline{\theta}-\underline{\theta}\right)\mu\left(\underline{\theta},\overline{\theta}\right)}{t\mu\left(\theta_{t-1},\overline{\theta}\right)+\mu\left(\underline{\theta},\theta_{t-1}\right)} \\ &> v\frac{\frac{1}{4}Tm_\eta m_\mu\left(\overline{\theta}-\theta_S\right)^2-\left(\overline{\theta}-\underline{\theta}\right)\mu\left(\underline{\theta},\overline{\theta}\right)}{t\mu\left(\theta_{t-1},\overline{\theta}\right)+\mu\left(\underline{\theta},\theta_{t-1}\right)} > 0. \end{aligned}$$

Thus, there exist $\theta_t(\theta_1)$ and $s_t(\theta_1)$ such that $y_t = \theta_t - \theta_{t-1}$ is determined by (A.6). Using the fact that for $t = T+1 \varphi_{t-1}(\theta_1) > \varepsilon_1$ we solve (A.6) for the positive root y_t:

$$y_t = \frac{\varphi_{t-1}}{2vK_t}\left(-1+\sqrt{1+\frac{4vK_t\mu\left(\underline{\theta},\theta_{t-1}\right)}{tF_t\varphi_{t-1}^2}\left(\gamma_{t-1}+g\left(\theta_{t-1}\right)\left(\theta_{t-1}-\theta_S\right)\right)}\right). \tag{A.7}$$

Then, applying for (A.7) the well-known inequality $\sqrt{1+x} < 1+\frac{1}{2}x$, $x > 0$, yields

$$\begin{aligned} y_t &< \frac{\mu\left(\underline{\theta},\theta_{t-1}\right)}{tF_t\varphi_{t-1}}\left(\gamma_{t-1}+g\left(\theta_{t-1}\right)\left(\theta_{t-1}-\theta_S\right)\right) \\ &< \frac{\mu\left(\underline{\theta},\overline{\theta}\right)}{Tm_\mu\varepsilon_1}\left(1+\overline{g}\varepsilon_\gamma\right)\gamma_{t-1} < \delta\varepsilon\gamma_{t-1}, \end{aligned}$$

and, therefore, $s_t = \frac{1}{\delta}s_{t-1} - y_t > \frac{1}{\delta}s_{t-1} - \varepsilon s_{t-1} = s_{t-1}\left(\frac{1}{\delta}-\varepsilon\right)$.

Now we will prove that $\exists T_S > T$, $\exists U_{T_S}^S = \left(\theta_1^S(T_S), \theta_S\right) \subset \hat{U}_T^0$ and $\exists \Theta_{T_S}$ such that $s_t > \left(\frac{1}{\delta}-\varepsilon\right)s_{t-1}$ and $\varphi_{t-1} > \varepsilon_1$ for all $t = T+1,\ldots,T_S-1$ and $\theta_1 \in U_{T_S}^S$, and $\varphi_{T_S-1}\left(\theta_1^S\right) = \varepsilon_1$. Suppose not, then $\varphi_{t-1} > \varepsilon_1$ for all $t \geqslant T+1$ and $\theta_1 \in \hat{U}_T^0$. But in this case we have an induction: for all $t \geqslant T+1$ $\exists\theta_t > \theta_{t-1}$ and $\exists s_t > \left(\frac{1}{\delta}-\varepsilon\right)s_{t-1} > 0$. Let us fix any $\theta_1 \in \hat{U}_T^0$ and consider the sequences $\{\theta_t\}_{t=T+1}^{\infty}$ and $\{s_t\}_{t=T+1}^{\infty}$. The former is increasing and bounded, hence, $\exists \lim_{t\to\infty}\theta_t = \theta_\infty$. The later is also increasing and $\lim_{t\to\infty} s_t = +\infty$ as $\frac{1}{\delta}-\varepsilon > \frac{1}{\delta}-\frac{1-\delta}{v\delta} > 1$. But taking a limit $\lim_{t\to\infty}\varphi_t = -\infty$ contradicts $\varphi_{t-1} > \varepsilon_1$.

Now we will prove by induction that for all $t \geqslant T_S$ $\exists U_t^S = \left(\theta_1^S(t), \theta_S\right) \subset U_{t-1}^S$ and $\exists\Theta_t \supset \Theta_{t-1}$ such that $\varphi_{t-1}\left(\theta_1^S(t)\right) = \varepsilon_1$ and $s_t(\theta_1) > \left(\frac{1}{\delta}-\varepsilon\right)s_{t-1}(\theta_1)$ for all $\theta_1 \in U_t^S$. Suppose that for some $t \geqslant T_S$ $\exists U_t^S = \left(\theta_1^S(t), \theta_S\right) \subset U_{T_S}^S$ and $\exists\Theta_t$ such that $\varphi_{\tau-1}(\theta_1) > \varepsilon_1$ and $s_\tau(\theta_1) > \left(\frac{1}{\delta}-\varepsilon\right)s_{\tau-1}(\theta_1) > 0$ for all $\tau = T+1,\ldots,t-1$ and $\theta_1 \in U_t^S$, and $\varphi_{t-1}\left(\theta_1^S(t)\right) = \varepsilon_1$. It implies that $(v-1)\theta_t\left(\theta_1^S(t)\right) - s_t\left(\theta_1^S(t)\right) = \varphi_{t-1} + vy_t < \frac{1}{2}\varepsilon_S + v\varepsilon\delta(v-1)\theta_{t-1} < \frac{1}{2}\varepsilon_S + \frac{1}{2}\varepsilon_S = \varepsilon_S$, hence, $s_t\left(\theta_1^S\right) > (v-1)\theta_t\left(\theta_1^S\right) - \varepsilon_S$. Then, summing up the indifference Equation (2) in a form $\delta\left(\theta_\tau - \theta_{\tau-1}\right) = s_{\tau-1} - \delta s_\tau$ from $\tau = 2$ to $\tau = t$ we get $\delta(\theta_t - \theta_1) = (s_1 - s_t) + (1-\delta)\sum_{\tau=2}^{t} s_\tau$, that leads to $\theta_t - \theta_S < \varepsilon_\theta$.

Finally, as $\varphi_t\left(\theta_1^S\right) < \varepsilon_1$, $\varphi_t(\theta_S) = (v-1)\,\theta_S > \varepsilon_1$ and $\varphi_t(\theta_1)$ is continuous then $\exists\theta_1^S(t+1) \in U_t^S = \left(\theta_1^S(t), \theta_S\right)$ such that $\varphi_t\left(\theta_1^S(t+1)\right) = \varepsilon_1$, that ends the induction.

But if $\varphi_t\left(\theta_1^S(t+1)\right) = \varepsilon_1$ then $s_{t+1}\left(\theta_1^S(t+1)\right) > (v-1)\,\theta_{t+1}\left(\theta_1^S(t+1)\right) - \varepsilon_S$ as was shown above, and $\exists U_{t+1}^S = \left(\theta_1^S(t+1), \theta_S\right) \subset U_t^S$ such that $\theta_{t+1} - \theta_S < \varepsilon_\theta$ for all $\theta_1 \in U_{t+1}^S$. □

Proof of Proposition 4. So far we were considering θ_t as a function of θ_1, $\theta_t = \theta_t(\theta_1)$. Now we will consider θ_t as a function of θ_{t-1} and γ_{t-1}, $\theta_t = \theta_t\left(\theta_{t-1}, \gamma_{t-1}\right)$. We define the following limit function:

$$\hat{\theta}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \lim_{\tau\to\infty} \theta_\tau\left(\hat{\theta}_0, \hat{\gamma}_0\right). \tag{A.8}$$

Then, in the same spirit as before, we introduce functions $\hat{y}_1 = \hat{\theta}_1 - \hat{\theta}_0$, $\hat{s}_1 = \hat{\gamma}_0 - \hat{y}_1$, $\hat{\gamma}_1 = \frac{1}{\delta}\hat{s}_1$ and $\hat{\varphi}_0 = (v-1)\,\hat{\theta}_0 - \hat{\gamma}_0$. Taking the limit (A.8) explicitly yields that it exists for all $\hat{\theta}_0 \in [\theta_S, \overline{\theta} - \varepsilon)$ and $\hat{\gamma}_0 \in \left(\varepsilon, v\eta\left(\hat{\theta}_0, \overline{\theta}\right) - \hat{\theta}_0 - \varepsilon\right)$, where $\varepsilon > 0$ is an arbitrarily small number. Convergence is uniform, hence $\hat{\theta}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right)$ is continuous and it follows that $\hat{\theta}_1 = \hat{\theta}_0$ and $\hat{\gamma}_1 = \frac{1}{\delta}\hat{\gamma}_0$ when $\varepsilon < \hat{\gamma}_0 \leqslant (v-1)\,\hat{\theta}_0$, and $v\eta\left(\hat{\theta}_0, \hat{\theta}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right)\right) = \hat{\theta}_0 + \hat{\gamma}_0$ when $(v-1)\,\hat{\theta}_0 \leqslant \hat{\gamma}_0 < v\eta\left(\hat{\theta}_0, \overline{\theta}\right) - \hat{\theta}_0 - \varepsilon$. Then we define $\hat{\theta}_1$ for $\hat{\gamma}_0 = 0$, $\hat{\gamma}_0 = v\eta\left(\hat{\theta}_0, \overline{\theta}\right) - \hat{\theta}_0$ and $\hat{\theta}_0 = \overline{\theta}$ by taking corresponding limits of the function $\hat{\theta}_1\left(\hat{\theta}_0, \hat{\gamma}_0\right)$ when $\varepsilon \to +0$, that yields $\hat{\theta}_1\left(\hat{\theta}_0, 0\right) = \hat{\theta}_0$, $\hat{\theta}_1\left(\hat{\theta}_0, v\eta\left(\hat{\theta}_0, \overline{\theta}\right) - \hat{\theta}_0\right) = \overline{\theta}$ and $\hat{\theta}_1\left(\overline{\theta}, \hat{\gamma}_0\right) = \overline{\theta}$.

Finally we define $\hat{\theta}_{t+1}\left(\hat{\theta}_0, \hat{\gamma}_0\right)$ for all $t > 1$ as follows. If for some $\hat{\theta}_0 \in [\theta_S, \overline{\theta}]$, $\hat{\gamma}_0 \in [0, \frac{v-1}{\delta}\hat{\theta}_0]$ and for all $\tau = 0, \ldots, t$ there exist functions $\hat{\theta}_\tau\left(\hat{\theta}_0, \hat{\gamma}_0\right)$ and $\hat{\gamma}_\tau\left(\hat{\theta}_0, \hat{\gamma}_0\right)$ such that $0 \leqslant \hat{\gamma}_\tau \leqslant v\eta\left(\hat{\theta}_\tau, \overline{\theta}\right) - \hat{\theta}_\tau$ and $\hat{\theta}_\tau\left(\hat{\theta}_0, \hat{\gamma}_0\right) \leqslant \overline{\theta}$, then we take $\hat{\theta}_{t+1}\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \hat{\theta}_1\left(\hat{\theta}_t, \hat{\gamma}_t\right)$. It can be easily seen that if $0 < \hat{\gamma}_\tau < v\eta\left(\hat{\theta}_\tau, \overline{\theta}\right) - \hat{\theta}_\tau$ and $\hat{\theta}_\tau\left(\hat{\theta}_0, \hat{\gamma}_0\right) < \overline{\theta}$ then $\hat{\theta}_{t+1}\left(\hat{\theta}_0, \hat{\gamma}_0\right)$ has the following limit representation:

$$\hat{\theta}_{t+1}\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \lim_{\tau\to\infty} \theta_{\tau+t+1}\left(\theta_{\tau+t}\left(\ldots\left(\hat{\theta}_0, \hat{\gamma}_0\right)\right), \gamma_{\tau+t}\left(\ldots\left(\hat{\theta}_0, \hat{\gamma}_0\right)\right)\right).$$

The main use of that trick is to substitute complex functions $\theta_\tau(\ldots)$ by their limit analogs for very large t when the measure of “low quality goods” becomes negligible compare to the measure of “high quality goods”. Limit functions $\hat{\theta}_t\left(\hat{\theta}_0, \hat{\gamma}_0\right)$ would have been exactly the same as $\theta_\tau(\ldots)$ if there had been no entry of new sellers.

Now let us fix $\hat{\theta}_0 = \theta^{(k)}$ and take any $\hat{\gamma}_0 \in \left(0, \frac{v-1}{\delta}\theta^{(k)}\right)$. If for some $\tau \geqslant 0$ we have obtained the functions $\hat{\theta}_\tau\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \hat{\theta}_\tau(\hat{\gamma}_0)$ and $\hat{\gamma}_\tau\left(\hat{\theta}_0, \hat{\gamma}_0\right) = \hat{\gamma}_\tau(\hat{\gamma}_0)$, and at the same time $0 < \hat{\gamma}_\tau \leqslant v\eta\left(\hat{\theta}_\tau, \overline{\theta}\right) - \hat{\theta}_\tau$ then there exists the next function $\hat{\theta}_{\tau+1}(\hat{\gamma}_0) = \hat{\theta}_1\left(\hat{\theta}_\tau(\hat{\gamma}_0), \hat{\gamma}_\tau(\hat{\gamma}_0)\right)$ such that $\hat{\theta}_{\tau+1}(\hat{\gamma}_0) \in [\hat{\theta}_\tau, \overline{\theta}]$ for all $\hat{\gamma}_0 \in \left(0, \frac{v-1}{\delta}\theta^{(k)}\right)$.

We will show that $\exists\hat{t} \geqslant 0$ and $\exists\hat{\gamma} \in \left(0, \frac{v-1}{\delta}\theta^{(k)}\right)$ such that either $\hat{\gamma}_{\hat{t}}(\hat{\gamma}) = 0$, or $\hat{\gamma}_{\hat{t}}(\hat{\gamma}) > v\eta\left(\hat{\theta}_{\hat{t}}(\hat{\gamma}), \overline{\theta}\right) - \hat{\theta}_{\hat{t}}(\hat{\gamma})$. Suppose not, that means that for any $t \geqslant 0$ and $\hat{\gamma}_0 \in \left(0, \frac{v-1}{\delta}\theta^{(k)}\right)$ $\exists\hat{\theta}_t(\hat{\gamma}_0)$ and $\exists\hat{\gamma}_t(\hat{\gamma}_0)$ such that $0 < \hat{\gamma}_t \leqslant v\eta\left(\hat{\theta}_t, \overline{\theta}\right) - \hat{\theta}_t$ and $\theta^{(k)} \leqslant \hat{\theta}_t \leqslant \overline{\theta}$. Let us fix any $\hat{\gamma} \in \left(0, \frac{v-1}{\delta}\theta^{(k)}\right)$ and get infinite sequences $\left\{\hat{\theta}_t\right\}_{t=0}^{\infty}$ and $\{\hat{\gamma}_t\}_{t=0}^{\infty}$. The former is weakly increasing and bounded, hence, $\exists \lim_{t\to\infty}\hat{\theta}_t = \hat{\theta}_\infty$. But this implies that the later has a limit either: $\lim_{t\to\infty}\hat{\gamma}_t = \lim_{t\to\infty}\left(v\eta\left(\hat{\theta}_t, \hat{\theta}_{t+1}\right) - \hat{\theta}_t\right) = (v-1)\hat{\theta}_\infty > 0$. Taking a limit of the indifference equation $\delta\hat{\gamma}_{t+1} = \hat{\gamma}_t - \left(\hat{\theta}_{t+1} - \hat{\theta}_t\right)$ gives rise to a contradiction: $\lim_{t\to\infty}\delta\hat{\gamma}_{t+1} = \lim_{t\to\infty}\hat{\gamma}_t$.

Hence, only two possibilities are left:

a) Case 1. $\exists\hat{t}$ and $\exists\hat{\gamma} \in \left(0, \frac{v-1}{\delta}\theta^{(k)}\right)$ such that $\hat{\gamma}_t > 0$, $v\eta\left(\hat{\theta}_{t-1}, \overline{\theta}\right) \geqslant \hat{\theta}_{t-1} + \hat{\gamma}_{t-1}$ for all $t = 1, \ldots, \hat{t}-1$ and $\hat{\gamma}_0 \in (0, \hat{\gamma})$, $\hat{\gamma}_{\hat{t}} > 0$ and $v\eta\left(\hat{\theta}_{\hat{t}-1}(\hat{\gamma}), \overline{\theta}\right) < \hat{\theta}_{\hat{t}-1}(\hat{\gamma}) + \hat{\gamma}_{\hat{t}-1}(\hat{\gamma})$; and

b) Case 2. $\exists\hat{t}$ and $\exists\hat{\gamma} \in \left(0, \frac{v-1}{\delta}\theta^{(k)}\right)$ such that $\hat{\gamma}_t > 0$ and $v\eta\left(\hat{\theta}_{t-1}, \overline{\theta}\right) \geqslant \hat{\theta}_{t-1} + \hat{\gamma}_{t-1}$ for all $t = 1, \ldots \hat{t}$ and $\hat{\gamma}_0 \in (0, \hat{\gamma})$, and $\hat{\gamma}_{\hat{t}}(\hat{\gamma}) = 0$.

It can be shown that in Case 1 $\exists\overline{\varepsilon_S} > 0$ such that for any T $\exists\overline{t} > T$, $\exists\tilde{\theta}_1\left(\overline{t}\right) \in \left(\theta_S - \varepsilon_\mu, \theta_S\right)$ and $\exists\Theta_{\overline{t}}\left(\tilde{\theta}_1\right)$ such that $\theta_{\overline{t}}\left(\tilde{\theta}_1\right) = \overline{\theta}$ and $s_{\overline{t}}\left(\tilde{\theta}_1\right) > \overline{\varepsilon_S}$. In other words in this case there exist infinite number equilibrium sequences such that all goods are traded in the last period. In Case 2 we define $\theta^{(k+1)}$ as $\theta^{(k+1)} = \hat{\theta}_{\hat{t}}(\hat{\gamma})$. It is easily seen that $\theta^{(k+1)} \in (v\theta^{(k)}, \overline{\theta}]$ and then either we have the same result as in Case 1, or for any $\overline{\varepsilon_S} > 0$ and $\overline{\varepsilon_\theta} > 0$ $\exists T_S^{(k+1)} > T_S^{(k)}$ such that for all $t > T_S^{(k+1)}$ $\exists U_t^{(k+1)} = \left(\underline{\theta}_1(t, k+1), \overline{\theta}_1(t, k+1)\right) \subset U_t^{(k)}$ and $\exists\Theta_t\left(U_t^{(k+1)}\right)$ such that $\left|\theta_t(\theta_1) - \theta^{(k+1)}\right| < \overline{\varepsilon_\theta}$ for all $\theta_1 \in U_t^{(k+1)}$, $0 \leqslant s_t\left(\underline{\theta}_1\right) < \overline{\varepsilon_S}$ and $s_t\left(\overline{\theta}_1\right) > (v-1)\theta_t\left(\overline{\theta}_1\right) - \overline{\varepsilon_S}$. □

Proof of Theorem 1. Consequently applying Proposition 2 and Proposition 3 we conclude that for any $\varepsilon_S > 0$ and $\varepsilon_\theta > 0$ $\exists T_S$ such that for all $t \geqslant T_S$ $\exists U_t^S = \left(\theta_1^S(t), \theta_S\right)$ and $\exists\Theta_t\left(U_t^S\right)$ such that for any $\theta_1 \in U_t^S$ $\theta_t(\theta_1) \in (\theta_S, \theta_S + \varepsilon_\theta)$ and $s_t\left(\theta_1^S\right) > (v-1)\theta_t\left(\theta_1^S\right) - \varepsilon_S$. Now we are under the conditions of Proposition 4 if we take $\theta^{(1)} = \theta_S < \overline{\theta}$. Here we distinguish three cases.

a) Case 1. For any $k = 1, \ldots \infty$ there exists $\theta^{(k+1)} \in \left(v\theta^{(k)}, \overline{\theta}\right)$ such that for any $\overline{\varepsilon_S} > 0$ and $\overline{\varepsilon_\theta} > 0$ $\exists T_S^{(k+1)}$ such that for all

$$t > T_S^{(k+1)} \ \exists U_t^{(k+1)} = [\underline{\theta}_1(t, k+1), \overline{\theta}_1(t, k+1)] \subset U_t^{(k)}$$

and $\exists\Theta_t\left(U_t^{(k+1)}\right)$ such that:

- $\left|\theta_t(\theta_1) - \theta^{(k+1)}\right| < \overline{\varepsilon_\theta}$ for all $\theta_1 \in U_t^{(k+1)}$;
- $0 \leqslant s_t\left(\underline{\theta}_1(t, k+1)\right) < \overline{\varepsilon_S}$ and $s_t\left(\overline{\theta}_1(t, k+1)\right) > (v-1)\,\theta_t\left(\overline{\theta}_1(t, k+1)\right) - \overline{\varepsilon_S}$.

But in this case we get infinite sequence $\left\{\theta^{(k)}\right\}_{k=1}^{\infty}$, where $\theta^{(k+1)} > v\theta^{(k)}$, that contradicts $\theta^{(k)} < \overline{\theta}$. Hence, after some steps $\hat{k}$ we must meet either Case 2 or 3.

b) Case 2. There exists $\theta^{(\hat{k})}$ such that there does not exist $\theta^{(k+1)} \in (v\theta^{(k)}, \overline{\theta}]$. In accordance with Proposition 4, we can make a conclusion: $\exists \overline{\varepsilon_S} > 0$ such that for any T $\exists \bar{t} > T$, $\exists \tilde{\theta}_1\left(\bar{t}\right)$ and $\exists \Theta_{\bar{t}}\left(\tilde{\theta}_1\right)$ such that $\theta_{\bar{t}}\left(\tilde{\theta}_1\right) = \overline{\theta}$ and $s_{\bar{t}}\left(\tilde{\theta}_1\right) > \overline{\varepsilon_S}$. In other words, there are infinite number of equilibrium sequences such that all goods are sold in the last period and the last marginal surplus is strictly positive and separated from zero, $s_{\bar{t}}\left(\tilde{\theta}_1\right) > \overline{\varepsilon_S} > 0$. In this case we can construct infinite number of dynamic equilibria by concatenating equilibrium sequences, e.g. we take $\Theta_t = \{\theta_\tau\}_{\tau=1}^{t}$ and let a dynamic equilibrium be the following sequence of marginal sellers: $\left\{\overline{\theta}_\tau\right\}_{\tau=1}^{\infty}$ such that $\overline{\theta}_\tau = \theta_\tau$ for $\tau \leqslant t$ and $\overline{\theta}_\tau = \theta_{\tau-t}$ for $\tau > t$.

c) Case 3. There exists $\theta^{(\hat{k})}$ such that there exists $\theta^{(k+1)} = \overline{\theta}$. Note here that $\theta^{(k+1)}$ is determined in terms of the previous point $\theta^{(\hat{k})}$, the measure function $\mu(\theta)$ and parameters $\underline{\theta}$, v and δ. In other words, $\theta^{(k+1)} = \Omega\left(\theta^{(k)}, \mu(\theta), \underline{\theta}, v, \delta\right)$. Therefore the case $\theta^{(k+1)} = \overline{\theta}$ is non-generic. □

Proof of Proposition 5. We begin with the system (3) that can be written as follows:

$$p_\tau - \theta_\tau\left(1-\delta^{t-\tau}\right) = \frac{1}{\delta^{\tau-1}}\left(p_1 - \left(1-\delta^{t-1}\right)\theta_1\right), \quad \tau = 1, \ldots, t-1, \quad \text{(A.9)}$$

and look for a sequence of functions $\{\theta_\tau(\theta_1)\}_{\tau=1}^{t}$ satisfying (A.9) such that $\theta_{\tau+1} < \theta_\tau$ for all $\tau = 1, \ldots, t-2$. In this case we have $p_\tau(\theta_\tau) = \frac{v}{\mu(\underline{\theta}, \theta_\tau)}\int_{\underline{\theta}}^{\theta_\tau}\theta d\mu$, and $\dot{p}_\tau = a$. Substituting this into the first differential of (A.9) we get $\dot{\theta}_\tau = \frac{1}{\delta^{\tau-1}}\frac{1-a-\delta^{t-1}}{1-a-\delta^{t-\tau}}$. It follows that if $a < 1-\delta$ then $\dot{\theta}_\tau > 1$. Thus, there exists a neighborhood $U_{t-1}^0 = \left(\theta_1^0(t-1), \theta_S\right)$ such that $\theta_{\tau+1} < \theta_\tau < \theta_S$ and $s_\tau > 0$ for all $\theta_1 \in U_{t-1}^0$. Therefore, there exists a sequence $\{\theta_\tau(\theta_1)\}_{\tau=1}^{t-1}$ such that all conditions to be proved are satisfied except the last one and we only have to show that if $a \in (0, \tilde{a}(\delta))$ then there exists $\theta_t(\theta_1)$ and $U_t^0 = \left(\theta_1^0(t), \theta_S\right) \in U_{t-1}^0$such that $0 < \theta_t(\theta_1) - \theta_S < \varepsilon_\gamma \frac{1}{\delta} s_t(\theta_1)$ for all $\theta_1 \in U_t^0$.

Given the structure of $\{\theta_\tau(\theta_1)\}_{\tau=1}^{t}$ we can write:

$$p_t(\theta_1) = p_t(\theta_1(\theta_1), \theta_2(\theta_1), \ldots, \theta_t(\theta_1)) = v\frac{t\int_{\underline{\theta}}^{\theta_t}\theta d\mu - \sum_{\tau=1}^{t-1}\int_{\underline{\theta}}^{\theta_\tau}\theta d\mu}{t\mu(\underline{\theta}, \theta_t(\theta_1)) - \sum_{\tau=1}^{t-1}\mu(\underline{\theta}, \theta_\tau)},$$

and, consequently, $d_S p_t = at\, d_S\theta_t - d_S\theta_1\frac{1-a-\delta^{t-1}}{\delta^{t-1}}a\sum_{\tau=1}^{t-1}\frac{\delta^\tau}{1-a-\delta^\tau}$. Substituting this into the first differential of (3) yields $\dot{\theta}_t = \frac{1-a-\delta^{t-1}}{at\delta^{t-1}}\left(a\sum_{\tau=1}^{t-1}\frac{\delta^\tau}{1-a-\delta^\tau} - 1\right)$. This implies that there exists a neighborhood $\overline{U_t^0} = \left(\overline{\theta_1^0(t)}, \theta_S\right) \in U_{t-1}^0$, such that $\theta_t > \theta_S$ for all $\theta_1 \in \overline{U_t^0}$ as long as $\sum_{\tau=1}^{\infty} \frac{a\delta^\tau}{1-a-\delta^\tau} \leqslant 1$. Note here that $\sum_{\tau=1}^{\infty} \frac{a\delta^\tau}{1-a-\delta^\tau} < 1 + \frac{a-(1-\delta)^2}{(1-a-\delta)(1-\delta)}$. Hence, if $a \leqslant (1-\delta)^2$ then $a \sum_{\tau=1}^{\infty} \frac{\delta^\tau}{1-a-\delta^\tau} < 1$. Then we check whether $s_t > 0$:

$$\begin{aligned} d_S s_t &= d_S p_t - d_S \theta_t \\ &= -\frac{1-a-\delta^{t-1}}{\delta^{t-1}} \left(1 - \frac{1}{at}\left(1 - a\sum\nolimits_{\tau=1}^{t-1} \frac{\delta^\tau}{1-a-\delta^\tau}\right)\right) d_S\theta_1. \end{aligned}$$

Hence, $\dot{s}_t < 0$ when $t > \frac{1}{a}$. So, there exists $\overline{\overline{T_0}}$ such that for all $t > \overline{\overline{T_0}}$ $\exists \overline{\overline{U_t^0}} = \left(\overline{\overline{\theta_1^0(t)}}, \theta_S\right) \subset \overline{U_t^0}$ and $\exists \Theta_t^2\left(\overline{\overline{U_t^0}}\right)$ such that $\theta_t(\theta_1) > \theta_S$, $s_t(\theta_1) > 0$ for all $\theta_1 \in \overline{\overline{U_t^0}}$. $\theta_\tau(\theta_S) = \theta_S$ and $s_\tau(\theta_S) = 0$ for all $\tau = 1, \ldots, t$ by construction.

Finally, let us consider the following ratio:

$$\frac{s_t(\theta_1)}{\theta_t(\theta_1) - \theta_S} = \frac{\dot{s}_t(\theta_1 - \theta_S) + o(\theta_1 - \theta_S)}{\dot{\theta}_t(\theta_1 - \theta_S) + o(\theta_1 - \theta_S)} = \frac{at + o\left((\theta_1 - \theta_S)^0\right)}{1 - a\sum_{\tau=1}^{t-1} \frac{\delta^\tau}{1-a-\delta^\tau}} - 1.$$

Hence, $\lim_{t\to\infty} \lim_{\theta_1 \to \theta_S - 0} \frac{s_t(\theta_1)}{\theta_t(\theta_1) - \theta_S} = \lim_{t\to\infty} \frac{at}{1 - a\sum_{\tau=1}^{t-1} \frac{\delta^\tau}{1-a-\delta^\tau}} - 1 = +\infty$. This implies that for any $\varepsilon_\gamma > 0$ $\exists T_0 > \overline{\overline{T_0}}$ such that for all $t > T_0$ $\exists U_t^0 = \left(\theta_1^0(t), \theta_S\right)$ and $\exists \Theta_t^2\left(U_t^0\right)$ such that $0 < \theta_t(\theta_1) - \theta_S < \varepsilon_\gamma \frac{1}{\delta} s_t(\theta_1)$. □

Proof of Proposition 6. Suppose we have obtained an equilibrium sequence $\Theta_{k_1}\left(U_{k_1}^0\right)$, where $U_{k_1}^0 = \left(\theta_1^0(k_1), \theta_S\right)$ such that θ_τ and s_τ are differentiable at θ_S for all $\tau = 1, \ldots, k_1$, $\dot{s}_\tau < 0$, $\dot{\theta}_{k_1} > 0$, and $\theta_\tau(\theta_1) < \theta_{k_1}(\theta_1) < \theta_S$ for all $\theta_1 \in U_{k_1}^0$. There exists at least one of such a sequence, namely $\{\theta_1\}$, where $k_1 = 1$.

Now we will construct a new equilibrium sequence $\Theta_t\left(\Theta_{k_1}\right)$ in the following way. We will repeat the whole structure of Θ_{k_1} t times. In other words, for all $\tau = 1, \ldots, t-1$ and for all $l = 1, \ldots, k_1-1$ we put $\theta_{(\tau-1)k_1+l+1} < \theta_{(\tau-1)k_1+l}$ if $\theta_{l+1} < \theta_l$ and vice versa. And we do that such that $\theta_{\tau k_1} < \theta_{(\tau-1)k_1}$ for all $\tau = 2, \ldots, t$. Having done this we can see that each of the subsequences $\{\theta_l\}_{l=(\tau-1)k_1+1}^{l=\tau k_1}$ for all $\tau = 1, \ldots, t$ is an equilibrium sequence Θ_{k_1}. Now we have to find θ_{tk_1+1} such that for all $\tau = 1, \ldots, t$ $\delta^{\tau k_1 - 1}\left(p_{\tau k_1} - \theta_{\tau k_1}\right) = \delta^{tk_1}\left(p_{tk_1+1} - \theta_{\tau k_1}\right)$, in other words, the seller of quality $\theta_{\tau k_1}$ is indifferent between selling in time period τk_1 and $tk_1 + 1$. Loosely speaking, we try to construct a sort of equilibrium sequence of type II using Θ_{k_1} instead of θ_τ as a single component. In the full version of the paper we show how this can be done and why the result follows. □

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