PS: Advanced Probability Theory
Sheet 9 Solutions

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Exercise 1. (Moritz Dober)

a) We have for $x \in \mathbb{R}$

$$\int_{\mathbb{R}} e^{ixt} e^{-a|t|} dt = \int_{-\infty}^{0} e^{(ix+a)t} dt + \int_{0}^{\infty} e^{(ix-a)t} dt = \frac{1}{ix+a} - \frac{1}{ix-a} = \frac{2a}{x^2+a^2}.$$

b) Let Y be a r.v. whose law has density $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (a/2)e^{-a|x|}$. Using a) we get that the characteristic function of Y is

$$\varphi_Y(t) = \frac{a}{2} \int_{\mathbb{R}} e^{ixt} e^{-a|x|} dx = \frac{a^2}{t^2+a^2},$$

which is integrable. We have seen in exercise 3 on sheet 8 that as a consequence of the Fourier inversion theorem, the law of Y then has a density g given by

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} \frac{a^2}{t^2+a^2} dt = \frac{a}{2} \varphi_{X_1}(x).$$

We deduce that $f = g$ almost everywhere (w.r.t. the Lebesgue measure) but since f and g are continuous we get that $f = g$ pointwise ($\{f \neq g\}$ is open and of measure zero, so it must be the empty set). We conclude that $\varphi_{X_1}(t) = e^{-a|t|}$ for all t.
Now, since $X_1, ..., X_n$ are i.i.d. Cauchy with parameter a, we have for $t \in \mathbb{R}$

$$\varphi_{\frac{1}{n}(X_1+...+X_n)}(t) = \varphi_{X_1+...+X_n}(t/n) = \varphi_{X_1}(t/n)^n = \varphi_{X_1}(t).$$

We deduce by Fourier inversion that $(X_1 + ... + X_n)/n$ is again Cauchy with parameter a.

Exercise 2. (Vivian Obernosterer) Let X_i and S_n be defined as on the exercise sheet.

a) By the CLT, we have for $K > 0$,

$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq K\right) \xrightarrow{n\to\infty} \mathbb{P}\left(\mathcal{N}(0,1) \geq K\sigma\right) > 0.$$

b) By Fatou's lemma, we have

$$0 < \lim_n \mathbb{P}\left(\frac{S_n}{n} \geq K\right) \leq \mathbb{P}\left(\limsup_n \frac{S_n}{\sqrt{n}} \geq K\right).$$

Because $\left\{\limsup_n \frac{S_n}{\sqrt{n}} \geq K\right\}$ is a tail event, Kolmogorov's 0-1-law implies that it has probability 1. By intersecting over $K \in \mathbb{N}$, we deduce that $\limsup_n \frac{S_n}{\sqrt{n}} = \infty$ a.s.

Exercise 3. (M.D. for a), b), c))

a) By exercise 4 on sheet 2 it suffices to prove that $(X+Y, X-Y)$ has a joint density which factorizes. Since X, Y are independent, (X, Y) has a joint density given by $f(x,y) = (2\pi)^{-1} exp(-(x^2+y^2)/2)$. Set

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

then $(X+Y, X-Y)^t = A(X,Y)^t$. Now let $U \subseteq \mathbb{R}^2$ be open, then we have by substitution

$$\begin{aligned} \mathbb{P}\left((X+Y, X-Y) \in U\right) &= \mathbb{P}\left((X,Y) \in A^{-1}U\right) = \int_{A^{-1}U} f(x,y) dxdy \\ &= \int_U f(A^{-1}(x,y)^t) \left|\det A^{-1}\right| dxdy = \frac{1}{4\pi} \int_U e^{-\frac{x^2+y^2}{4}} dxdy. \end{aligned}$$

We deduce that the law of $(X+Y, X-Y)$ has a density given by $(4\pi)^{-1} exp(-(x^2+y^2)/4)$, which clearly factorizes. Using characteristic functions it is easy to see that $X+Y$ as well as $X-Y$ are $\mathcal{N}(0,2)$ but we do not need this.

b) Since $2X = (X+Y) + (X-Y)$ we get

$$\begin{aligned} \varphi(2t) &= \mathbb{E}\left[e^{it((X+Y)+(X-Y))}\right] \\ &= \mathbb{E}\left[e^{it(X+Y)}\right] \mathbb{E}\left[e^{it(X-Y)}\right] && \text{(since } X+Y, X-Y \text{ independent)} \\ &= \mathbb{E}\left[e^{itX}\right] \mathbb{E}\left[e^{itY}\right] \mathbb{E}\left[e^{itX}\right] \mathbb{E}\left[e^{-itY}\right] && \text{(since } X, Y \text{ independent)} \\ &= \varphi(t)^3 \varphi(-t) && \text{(since } X \sim Y\text{).} \end{aligned}$$

c) Since $\varphi(-t) = \overline{\varphi(t)}$ we have that $\varphi(t) = 0$ iff $\varphi(-t) = 0$. In this case we set $\psi(t) = 1$ and by the formula in b) we also have $\varphi(t/2) = 0$, so $\psi(t/2) = 1$ too. Now let $t \in \mathbb{R}$ such that $\varphi(t) \neq 0$. Applying the formula from b) to both the numerator and denominator we get

$$\psi(t) = \frac{\varphi(t/2)^3 \varphi(-t/2)}{\varphi(-t/2)^3 \varphi(t/2)} = \frac{\varphi(t/2)^2}{\varphi(-t/2)^2} = \psi(t/2)^2.$$

d) Since X has finite second moment we have that φ is twice differentiable and therefore ψ is also twice differentiable on a neighbourhood of 0. Furthermore $\varphi'(0) = i\mathbb{E}[X] = 0$ and hence $\psi'(0) = 2\varphi'(0) = 0$. We deduce that $\psi(t) = \psi(0) + t\psi'(t) + o(t) = 1 + o(t)$ as $t \to 0$. Now, fix $t \in \mathbb{R}$. By c) and induction, we get that for all n,

$$\psi(t) = \psi(t/2^n)^{2^n} = (1 + o(t/2^n))^{2^n}.$$

As $(1 + o(1/x))^x = e^{x \log(1+o(1/x))} = e^{o(1)} = 1 + o(1)$ as $x \to \infty$, we deduce that

$$(1 + o(t/2^n))^{2^n} \xrightarrow[n \to \infty]{} 1.$$

Hence, we have proven that for all $t \in \mathbb{R}$, $\psi(t) = 1$.

e) We know from b) that $\varphi(t) = \varphi(t/2)^3 \varphi(-t/2)$ and from d) that $\varphi(t) = \varphi(-t)$ for all t. This gives $\varphi(t) = \varphi(t/2)^4$. By induction, for all n,

$$\varphi(t) = \varphi(t/2^n)^{2^{2n}}.$$

Furthermore we have by Taylor's theorem, that as $t \to 0$,

$$\varphi(t) = 1 + t\varphi'(0) + t^2 \frac{\varphi''(0)}{2} + o(t^2) = 1 - t^2 \frac{\mathbb{E}\left[X^2\right]}{2} + o(t^2) = 1 - \frac{t^2}{2} + o(t^2).$$

As in d), manipulating little o's, for all fixed t and for every n,

$$\varphi(t) = \varphi(t/2^n)^{2^{2n}} = \left(1 - \frac{t^2/2}{2^{2n}}(1 + o(1))\right)^{2^{2n}} \xrightarrow[n\to\infty]{} e^{-t^2/2}.$$

Hence, X has the same characteristic function as $\mathcal{N}(0,1)$. We conclude by Fourier inversion that $X, Y \sim \mathcal{N}(0,1)$.

Remark. In e), when you see the functional equation $\varphi(2t) = \varphi(t)^4$. You might want to make it simpler. It is possible. Set $f(x) = \log(\varphi(x))/x^2$. We then get the simple funtional equation $f(2x) = f(x)$. (If f is continuous at 0, it is easy to see that it is constant, equal to $f(0)$.)
However, proving that f is well defined and continuous at 0 is equivalent to what we actually proved with Taylor expansions.

Exercise 4. (M.D.)

a) Suppose that $\{X_t : t \in T\}$ is UI. As in the lecture we have for $t \in T$ and $K > 0$

$$\mathbb{E}\left[|X_t|\right] = \mathbb{E}\left[|X_t| \mathbb{1}_{\{|X_t|<K\}}\right] + \mathbb{E}\left[|X_t| \mathbb{1}_{\{|X_t|\geq K\}}\right] \leq K + I(K),$$

where $I(K) = \sup\{\mathbb{E}\left[|X_t| \mathbb{1}_{\{X_t \geq K\}}\right] : t \in T\}$ tends to 0 as $K \to \infty$. We deduce that $\{X_t : t \in T\}$ is bounded in L^1. Now let $\varepsilon > 0$. We can find $K > 0$ s.t. $\mathbb{E}\left[|X_t| \mathbb{1}_{\{|X_t|\geq K\}}\right] < \varepsilon$ for all t. Then we have for $A \in \mathcal{F}$ with $\mathbb{P}(A) \leq \varepsilon/K$

$$\mathbb{E}\left[|X_t| \mathbb{1}_A\right] = \mathbb{E}\left[|X_t| \mathbb{1}_{A\cap\{|X_t|<K\}}\right] + \mathbb{E}\left[|X_t| \mathbb{1}_{A\cap\{|X_t|\geq K\}}\right] < K\mathbb{P}(A) + \varepsilon \leq 2\varepsilon$$

and hence $J(\delta) \to 0$ as $\delta \to 0$.
Now suppose that $\{X_t : t \in T\}$ is bounded in L^1 by some $C > 0$ and that $J(\delta) \to 0$ as $\delta \to 0$. Let $\varepsilon > 0$ and choose $\delta > 0$ s.t $J(\delta) < \varepsilon$. We have by Markov's inequality that

$$\mathbb{P}(|X_t| \geq K) \leq \frac{\mathbb{E}\left[|X_t|\right]}{K} \leq \frac{C}{K}$$

and therefore for $K \geq C/\delta$ that $\mathbb{E}\left[|X_t| \mathbb{1}_{\{|X_t|\geq K\}}\right] \leq J(\delta) < \varepsilon$ for all t.

b) Since the X_n are identically distributed we have that $\{X_n\}_{n\geq 1}$ is UI iff $\{X_1\}$ is UI. By integrability of X_1 we have that $|X_1| < \infty$ a.s. and thus $|X_1| \mathbb{1}_{\{|X_1|\geq K\}}$ converges to 0 a.s. as $K \to \infty$ dominated by $|X_1|$. The DCT implies $\mathbb{E}\left[|X_1| \mathbb{1}_{\{|X_1|\geq K\}}\right] \to 0$ as $K \to \infty$.
To prove that $\{S_n/n\}_{n\geq 1}$ is UI too, we show the equivalent condition from a). Let $\varepsilon > 0$ and choose $\delta > 0$ such that $\mathbb{E}\left[|X_i| \mathbb{1}_A\right] \leq \varepsilon$ for all $i \geq 1$ and all $A \in \mathcal{F}$ with $\mathbb{P}(A) \leq \delta$. Then we also have for such an A that

$$\mathbb{E}\left[|S_n/n| \mathbb{1}_A\right] \leq \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[|X_i| \mathbb{1}_A\right] \leq \varepsilon$$

for all n, so $\{S_n/n\}_{n\geq 1}$ is UI. We already know from the weak LLN that $S_n/n \to m$ in probability as $n \to \infty$ and we proved in class that if $Y_n \to Y$ in probability and $\{Y_n : n \geq 1\}$ is UI, then $Y_n \to Y$ in L^1. Altogether we deduce that $S_n/n \to m$ in L^1 as $n \to \infty$.