PS: Advanced Probability Theory
Sheet 9

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Exercise 1. Let $X_1, \dots, X_n$ be i.i.d. Cauchy with parameter $a > 0$, i.e. they have a density which is equal to

$$x \in \mathbb{R} \mapsto \frac{1}{\pi} \frac{a}{x^2 + a^2}.$$

a) Show that for all $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} e^{ixt} e^{-a|t|} dt = \frac{2a}{x^2 + a^2}. \tag{1}$$

b) Using the Fourier inversion theorem, prove that $\phi_{X_1}(t) = e^{-a|t|}$.
Deduce $(X_1 + \dots + X_n)/n$ is again Cauchy with parameter a.

This gives an example of distribution for which the law of large numbers fails to be true.

Exercise 2. Let $X_1, \dots$ be i.i.d. random variables with mean zero and $\mathbb{E}(X_i^2) = \sigma^2 < \infty$. Let $S_n = X_1 + \ldots + X_n$.

a) Show that for any $K > 0$, $\mathbb{P}(S_n/\sqrt{n} \geq K)$ has a limit as $n \to \infty$.

b) Deduce that $\limsup S_n/\sqrt{n} = +\infty$, almost surely.

Exercise 3. a) Let X, Y be two independent $\mathcal{N}(0,1)$ variables. Show that $X+Y$ and $X-Y$ are independent.

b) We propose to prove the converse. Consider X, Y i.i.d such that $X+Y$ and $X-Y$ are independent and $\mathbb{E}[X] = 0$ and $\mathbb{E}\left[X^2\right] = 1$. Prove that the characteristic function φ of X satisfies $\varphi(2t) = \varphi(t)^3 \varphi(-t)$.

c) Deduce that $\psi(t) := \varphi(t)/\varphi(-t)$ satisfies $\psi(t) = \psi(t/2)^2$.

d) Show that $\psi(t) = 1 + o(t)$[1] as $t \to 0$ and deduce that $\psi(t) = 1$ for all $t \in \mathbb{R}$.

e) Conclude.

Exercise 4. a) Let $\{X_t : t \in T\}$ be a collection of random variables. Show that it is uniformly integrable (UI) if and only if it is bounded in L^1 and $J(\delta) \to 0$, where

$$J(\delta) = \sup\{\mathbb{E}(|X_t| 1_A) : t \in T, A \in \mathcal{F} \text{ with } \mathbb{P}(A) \leq \delta\}.$$

In other words, this condition says that events of small probability do not contribute significantly to the expectation, uniformly over the collection of R.V.

[1] We can actually prove directly that $\psi(t) = 1 + o(t^2)$, but we it is no necessary here.

b) Now let $X_1, \ldots,$ be i.i.d. integrable random variables with mean $\mathbb{E}(X) = m$. Show that $\{X_n, n \geq 1\}$ is uniformly integrable. Deduce that if $S_n = \sum_{i=1}^{n} X_i$, then $\{S_n/n, n \geq 1\}$ is also uniformly integrable. Conclude that $S_n/n \to m$ in L^1 as $n \to \infty$.