PS: Advanced Probability Theory
Sheet 8 Solutions

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Exercise 1.

a) Let $y > 0$, and set for all N, $k = k(y, N) = \lfloor y\sqrt{N} \rfloor$. Then, for all N,

$$\mathbb{P}(T_N > k) = \mathbb{P}(X_1, \ldots, X_k \text{ are all different}) = \frac{N}{N}\frac{N-1}{N}\cdots\frac{N-k+1}{N} = \frac{N!}{(N-k)!N^k}.$$

Using Stirling's formula $n! \underset{n\to\infty}{\sim} \sqrt{2\pi n}(n/e)^n$ and after reordering the different terms,

$$\mathbb{P}(T_N > k) \underset{N\to\infty}{\sim} \left(1 - \frac{k}{N}\right)^{-1/2} e^{-k} \left(1 - \frac{k}{N}\right)^{k-N} \underset{N\to\infty}{\sim} e^{-k} e^{\ln(1-\frac{k}{N})(k-N)}.$$

Finally, using that $\ln(1-x) = -x - \frac{x^2}{2} + o_{x\to 0}\left(x^2\right)$, and after some reordering again,

$$\mathbb{P}(T_N > k) \xrightarrow[N\to\infty]{} e^{-y^2/2},$$

which proves that $T_N/\sqrt{N} \to Y$ in distribution (via convergence of the cdf).

Remark. You should be careful when you compose equivalents. For example, for all $x \in \mathbb{R}$, $(1 + \frac{x}{n})^n \xrightarrow[n\to\infty]{} e^x$, but $(1 + \frac{x}{n})^{n^2} = \left((1 + \frac{x}{n})^n\right)^n$ is NOT equivalent to $(e^x)^n$. More precisely, for all fixed $x \in \mathbb{R}$, as $n \to \infty$, $\log(1 - \frac{x}{n}) = \frac{x}{n} - \frac{1}{2}\frac{x^2}{n^2} + o\left(\frac{1}{n^2}\right)$, and so

$$\left(1 + \frac{x}{n}\right)^{n^2} = e^{n^2 \log(1-\frac{x}{n}} = e^{n^2\left(\frac{x}{n} - \frac{1}{2}\frac{x^2}{n^2} + o\left(\frac{1}{n^2}\right)\right)} \sim e^{nx}e^{-x^2/2}.$$

b) Let $x \in \mathbb{R}$. For all N,

$$\begin{aligned}
\mathbb{P}\left(M_N - \frac{1}{\lambda}\log(N) \le x\right) &= \mathbb{P}\left(\forall 1 \le i \le N,\ X_i \le x + \frac{1}{\lambda}\log(N)\right) \\
&= \mathbb{P}\left(X_1 \le x + \frac{1}{\lambda}\log(N)\right)^N \quad \text{by independence} \\
&= \left(1 - e^{-\lambda\left(x + \frac{1}{\lambda}\log(N)\right)}\right)^N \\
&= \left(1 - \frac{e^{-\lambda x}}{N}\right)^N \xrightarrow[N\to\infty]{} e^{-e^{-\lambda x}},
\end{aligned}$$

which allows to conclude, as in a).

Exercise 2.

a) First, let us compute the characteristic function of X, which will give us the general formula for Poisson random variables. Let $t \in \mathbb{R}$, then

$$\phi_X(t) = \sum_{k=0}^{\infty} \mathbb{P}(X=k)e^{ikt} = \sum_{k=0}^{\infty} \frac{\lambda^k}{e^{\lambda}k!}e^{ikt} = \frac{1}{e^{\lambda}} \sum_{k=0}^{\infty} \frac{\left(\lambda e^{it}\right)^k}{k!} = e^{\lambda\left(e^{it}-1\right)}.$$

Hence, for all $t \in \mathbb{R}$,

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = e^{\lambda\left(e^{it}-1\right)}e^{\mu\left(e^{it}-1\right)} = e^{(\lambda+\mu)\left(e^{it}-1\right)} = \phi_Z(t),$$

where, Z is a Poisson random variable of parameter $\lambda+\mu$. As the characteristic function characterises the law, we conclude that $X+Y \sim Z$.

b) By Lévy continuity theorem, it is enough to prove that for all $t \in \mathbb{R}$, $\phi_{X_n}(t) \xrightarrow[n\to\infty]{} \phi_X(t)$.

Let $t \in \mathbb{R}$. Then for all $n \geq 1$, noting that $X_n \sim Y_1 + \ldots + Y_n$, where the Y_i's are i.i.d. Bernoulli(λ/n) variables,

$$\phi_{X_n}(t) = \phi_{Y_1}(t)^n = \left(\left(1-\frac{\lambda}{n}\right)+\frac{\lambda}{n}e^{it}\right)^n = \left(1 - \frac{\lambda\left(e^{it}-1\right)}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-\lambda\left(e^{it}-1\right)} = \phi_X(t).$$

Exercise 3. (Moritz Dober)

a) Let $a<b$. Since $\left|(e^{-ita}-e^{-itb})/(it)\right| = \left|\int_a^b e^{-ity}dy\right| \leq b-a$ we get that

$$\int_{\mathbb{R}} \left|\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\right| dt \leq (b-a)\int_{\mathbb{R}} |\phi(t)|\, dt < \infty$$

by assumption and hence the occurring integrand is integrable over $\mathbb{R}$. Moreover

$$\begin{aligned}\frac{1}{2\pi}\int_a^b \int_{\mathbb{R}} e^{-itx}\phi(t)\, dtdx &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_a^b e^{-itx}\phi(t)\, dxdt = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita}-e^{-itb}}{it}\phi(t)\, dt \\ &= \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\, dt = \mu((a,b)) + \frac{1}{2}\mu(\{a,b\}),\end{aligned}$$

where we used Fubini's theorem, dominated convergence and the Fourier inversion theorem. In particular $\mu((a,b)) + \frac{1}{2}\mu(\{a,b\}) \leq (b-a)/(2\pi)\int |\phi(t)|\, dt$ and letting $a \nearrow b$ we obtain that μ has no point masses. From this and the above calculation we conclude that f given by $f(x) = 1/(2\pi)\int e^{-itx}\phi(t)dt$ is a density for the law of X.

b) Let $n \geq 2$. The characteristic function of $Y_n := X_1 + ... + X_n$ is given by $\phi_{Y_n}(t) = \phi_{X_1}(t)^n = (\sin(t)/t)^n$ and

$$\int_{\mathbb{R}}\left|\frac{\sin t}{t}\right|^n dt = 2\int_0^{\infty}\left|\frac{\sin t}{t}\right|^n dt \leq 2\left(\int_0^1\left|\frac{\sin t}{t}\right| dt + \int_1^{\infty}\frac{1}{t^2}dt\right) < \infty,$$

since $|\sin(t)/t| \leq 1$ for all t and $n \geq 2$. Hence ϕ_{Y_n} is integrable over $\mathbb{R}$ and we get by a) that Y_n has a density f_n given by

$$\begin{aligned}f_n(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\left(\frac{\sin t}{t}\right)^n dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\cos(tx)\left(\frac{\sin t}{t}\right)^n dt - \frac{1}{2\pi}\int_{\mathbb{R}}\sin(tx)\left(\frac{\sin t}{t}\right)^n dt,\end{aligned}$$

but $t \mapsto \sin(tx)(\sin(t)/t)^n$ is odd and thus the second integral vanishes.

Exercise 4. (Daniel Bäumer)

a) Let μ_n be tight. Then, for some compact $K \subset \mathbb{R}$, $\mu_n(K^c) < \epsilon$ for all n. We can therefore write

$$|\phi_n(t) - \phi_n(0)| = |\int_{\mathbb{R}} e^{itx}\mu_n(dx) - 1| \leq \int_{\mathbb{R}} |e^{itx} - 1|\mu_n(dx) =$$

$$= \int_K |e^{itx} - 1|\mu_n(dx) + \int_{K^c} |e^{itx} - 1|\mu_n(dx) \leq \int_K |e^{itx} - 1|\mu_n(dx) + 2\epsilon.$$

Since K is compact, we can find a $\delta > 0$ such that $|e^{itx} - 1| \leq \epsilon \forall t \in [-\delta, \delta] \forall x \in K$; this yields

$$|\phi_n(t) - \phi_n(0)| \leq 3\epsilon$$

for all n.
If conversely ϕ_n is equicontinuous at zero, we can use the following estimate:

$$\int_{-k}^{k} 1 - \phi_n(t)dt = \int_{\mathbb{R}} \mu_n(dx) \int_{-k}^{k} (1 - e^{itx})dt = 2k \int_{\mathbb{R}} (1 - \frac{\sin kx}{kx})\mu_n(dx) \geq k\mu_n(\{|kx| \geq 2\}).$$

With the substitution $l = \frac{2}{k}$, this gives

$$\mu_n(\{|x| \geq l\}) \leq \frac{l}{2} \int_{-\frac{2}{l}}^{\frac{2}{l}} 1 - \phi_n(t)dt \leq 2 \max_{[-\frac{2}{l}, \frac{2}{l}]} |\phi_n(t) - \phi_n(0)|,$$

which yields the result.

b) By Cauchy-Schwarz, we see that equicontinuity at 0 actually implies uniform equicontinuity; taking X_n as a random variable with the distribution ϕ_n:

$$|\phi_n(t) - \phi_n(s)|^2 = |\mathbb{E}[e^{itX_n} - e^{isX_n}]|^2 = |\mathbb{E}[e^{isX_n}(e^{i(t-s)X_n} - 1)]|^2 \leq$$
$$\leq \mathbb{E}[|e^{isX_n}|^2]\mathbb{E}[|e^{i(t-s)X_n} - 1|^2] = \mathbb{E}[1 - e^{i(t-s)X_n} + 1 - e^{i(s-t)X_n}] =$$
$$\phi_n(0) - \phi_n(t-s) + \phi_n(0) - \phi_n(s-t).$$

Now, let $\delta > 0$ be such that $|\phi_n(t) - \phi_n(s)| < \epsilon$ and $|\phi(t) - \phi(s)| < \epsilon$ for all t, s with $|t-s| < \delta$; for any compact set K, there exist $N \in \mathbb{N}$ and $t_1, \ldots, t_N \in K$ such that $K \subset \bigcup_{i=1}^{N} B_\delta(t_i)$. If $n_0 \in \mathbb{N}$ is such that $|\phi_n(t_i) - \phi(t_i)| < \epsilon \forall i \forall n \geq n_0$, it holds:

$$|\phi_n(t) - \phi(t)| \leq |\phi_n(t) - \phi_n(t_i)| + |\phi_n(t_i) - \phi(t_i)| + |\phi(t_i) - \phi(t)| < 3\epsilon,$$

if t_i is chosen such that $|t - t_i| < \delta$. $\square$