PS: Advanced Probability Theory
Sheet 8

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Exercise 1. a) (Birthday problem). Let $N \geq 1$ and $(X_i, i \geq 1)$ i.i.d. uniform r.v. on $\{1, \ldots, N\}$. Define $T_N = \inf\{j \geq 2 : \exists i < j, X_i = X_j\}$. Show that $T_N/\sqrt{N} \to Y$ in distribution where $\mathbb{P}(Y > y) = \exp(-y^2/2)$ for any $y > 0$.

b) Let $\lambda > 0$, $X_1, \ldots, X_N$ i.i.d. Exp(λ) variables and consider $M_N = \max(X_1, \ldots, X_N)$. Show that

$$M_N - \frac{1}{\lambda} \log N \xrightarrow[N \to \infty]{} X \text{ in distribution where } \forall x \in \mathbb{R}, \mathbb{P}(X \leq x) = \exp\left(-e^{-\lambda x}\right).$$

X is called the Gumbel distribution.

Exercise 2. a) Let X, Y be independent r.v. with the Poisson distribution with parameters λ and μ. Using characteristic functions, show that $X + Y$ has the Poisson distribution with parameter $\lambda + \mu$.

b) Let X_n be Binomial with parameters n and $p_n = \lambda/n$. In a previous example sheet (Sheet 7, Exercise 3) you showed that X_n converges to X in distribution. Give another proof of this fact based on characteristic functions.

Exercise 3. a) Suppose X is a random variable and let ϕ be its characteristic function. Suppose that $\int |\phi(t)| dt < \infty$. Show that the law of X has a density f given by

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi(t) dt$$

Hint. Start from the Fourier inversion theorem, and observe that under the above assumption the function appearing in the inversion formula is in fact integrable over $\mathbb{R}$, so there is no need to take a limit.

b) Let $X_1, \ldots, X_n$ be i.i.d. uniform on $[-1, 1]$. Show that the density of $X_1 + \ldots + X_n$ is given by

$$f_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin t}{t}\right)^n \cos(tx) dt.$$

Exercise 4. Let μ, μ_n be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and ϕ, ϕ_n be their characteristic functions.

a) Show that (μ_n) is tight iff (ϕ_n) is equicontinuous at 0 [It means that for all $\varepsilon > 0$, there exists δ, such that for all $x \in [-\delta, \delta]$ and $n \geq 1$, $|\phi_n(x) - \phi_n(0)| \leq \varepsilon$].

b) Deduce that if $\mu_n \Rightarrow \mu$, then $\phi_n \to \phi$ uniformly on compact sets.

Optional exercises

Exercise A. Denote by $\mathcal{P}$ the set of probability measures on $\mathbb{R}$ and $\mathcal{B}(\mathbb{R})$ the collection of Borel sets of $\mathbb{R}$. For any $A \in \mathcal{B}(\mathbb{R})$ and $\varepsilon > 0$, we consider the ε-neighbourhood of A:

$$A^\varepsilon = \{x \in \mathbb{R} : \operatorname{dist}(x, A) < \varepsilon\}.$$

For any $\mu, \nu \in \mathcal{P}$, we define

$$\pi(\mu, \nu) = \inf \left\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(\mathbb{R})\right\}.$$

a) Show that π is a metric on $\mathcal{P}$.

b) In the following questions, we consider $\mu, \mu_n \in \mathcal{P}$. Show that if $\pi(\mu_n, \mu) \to 0$, then $\mu_n \Rightarrow \mu$.

c) (Warning: difficult question) Show the converse: if $\mu_n \Rightarrow \mu$ then $\pi(\mu_n, \mu) \to 0$.
Hint: Use the tightness of $\{\mu, \mu_n, n \geq 1\}$.

This exercise shows that π is a metrization of the topology of weak convergence on $\mathcal{P}$.